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STABILIZATION FOR SMALL MASS IN A QUASILINEAR PARABOLIC–ELLIPTIC–ELLIPTIC ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH DENSITY-DEPENDENT SENSITIVITY: BALANCED CASE

Y. CHIYO - T. YOKOTA

This paper is concerned with the Neumann initial-boundary problem for the quasilinear parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with $q = p$ and $\chi\alpha - \xi\gamma = 0$:

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big), \\ 0 = \Delta v + \alpha u - \beta v, \\ 0 = \Delta w + \gamma u - \delta w \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ with smooth boundary $\partial\Omega$, where $m, p, q \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants. In the case that $m \neq 1$, $p \neq 2$ and $q \neq 2$ boundedness and finite-time blow-up have been classified by the sizes of p, q and the sign of $\chi\alpha - \xi\gamma$ (Z. Angew. Math. Phys.; 2022; 73; 61), where the critical case $\chi\alpha - \xi\gamma = 0$ has been excluded. The purpose of this paper is to prove boundedness and stabilization in the case $\chi\alpha - \xi\gamma = 0$.

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1. Introduction

We consider the following initial-boundary value problem for the quasilinear parabolic–elliptic–elliptic attraction-repulsion chemotaxis system with $q = p$ and $\chi\alpha - \xi\gamma = 0$:

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big), \\ 0 = \Delta v + \alpha u - \beta v, \\ 0 = \Delta w + \gamma u - \delta w, \\ \nabla u \cdot \nu|_{\partial\Omega} = \nabla v \cdot \nu|_{\partial\Omega} = \nabla w \cdot \nu|_{\partial\Omega} = 0, \\ u(\cdot,0) = u_0 \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ with smooth boundary $\partial\Omega$, where

$$m, p, q \in \mathbb{R}, \quad \chi, \xi, \alpha, \beta, \gamma, \delta > 0$$

are constants, ν is the outward normal vector to $\partial\Omega$,

$$u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0 \text{ in } \overline{\Omega} \quad \text{and} \quad u_0 \neq 0. \tag{1.2}$$

The fully parabolic version of (1.1) with $m = 1$ and $p = q = 2$ has been proposed by Luca et al. [11] in order to describe the aggregation of microglial cells in Alzheimer's disease, and has been studied mathematically as will be explained later. This original problem is also a specialized one introduced by Painter and Hillen [12, Section 3.3] to represent the quorum sensing effect that cells keep away from a repulsive chemical substance. One can observe that (1.1) is regarded as a simplified problem of parabolic–elliptic–elliptic type and is generalized problem to the quasilinear version. In these systems the functions u, v and w idealize the density of the cells, the concentration of the chemoattractant and chemorepellent, respectively. To the best of our knowledge, quasilinear attraction-repulsion chemotaxis systems as in (1.1) were studied firstly by Frassu, van der Mee and Viglialoro [5] and also by Frassu, Li and Viglialoro [4], where the second and third equations have consumption and nonlinear production terms, respectively.

Before stating our main results, we briefly review previous works related to the subjects in this paper. Liu and Wang [10] established the first result on global existence and steady states in the fully parabolic version of the problem (1.1) with $m = 1$ and $p = q = 2$ as well as $\chi = \xi = \alpha = 1$ in the one-dimensional setting. After that, Tao and Wang [13] derived boundedness in the problem (1.1) with $m = 1$ and $p = q = 2$ by assuming $\chi\alpha - \xi\gamma < 0$ in two or more space dimensions, and proved finite-time blow-up in this problem when $\chi\alpha - \xi\gamma > 0$, $\beta = \delta$, $\|u_0\|_{L^1(\Omega)} > \frac{8\pi}{\chi\alpha - \xi\gamma}$ and $\int_\Omega u_0(x)|x - x_0|^2\,dx$ $(x_0 \in \Omega)$ is sufficiently small in the two-dimensional setting. In the literature, it was also shown that the problem (1.1) possesses only one constant equilibrium $(\overline{u_0}, \frac{\alpha}{\beta}\overline{u_0}, \frac{\gamma}{\delta}\overline{u_0})$, where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$, under the condition $\chi\alpha - \xi\gamma \le 0$ and $\beta = \delta$, and that solutions of the problem (1.1) stabilize toward this constant equilibrium under the condition $\chi\alpha - \xi\gamma < 0$ and $\beta = \delta$. We note that boundedness under some condition including $\chi\alpha - \xi\gamma = 0$ was established by Jin and Wang [6, 7] in the parabolic–parabolic–elliptic and fully parabolic versions in two dimensions. After that, Li, Lin and Mu [8] showed boundedness in this problem under the condition $\chi\alpha - \xi\gamma = 0$ in the two- and three-dimensional settings. Also, stabilization was derived in the literature under the condition $\chi\alpha - \xi\gamma = 0$ and some smallness condition for u_0; note that the fully parabolic version was investigated by Lin, Mu and Wang [9]. On the other hand, in the case that $m \ne 1$, $p \ne 2$ and $q \ne 2$ boundedness and finite-time blow-up were classified by the sign of $\chi\alpha - \xi\gamma$ in [2]. Also, stabilization was shown in [1] under the condition $p < q$, or $p = q$ and $\chi\alpha - \xi\gamma < 0$.

In summary, boundedness, finite-time blow-up and stabilization in the problem (1.1) were obtained under conditions for the sign of $\chi\alpha - \xi\gamma$. However, in the critical case $\chi\alpha - \xi\gamma = 0$ the problem (1.1) has not been studied yet. The purpose of this paper is to establish boundedness and stabilization in the problem (1.1) in the critical case $\chi\alpha - \xi\gamma = 0$.

The main results read as follows.

Theorem 1.1 (Boundedness). *Let $n \in \mathbb{N}$. Let $q = p$ and $\chi\alpha - \xi\gamma = 0$. Assume that $m \ge \max\{1, p - \frac{2}{n}\}$. Then for all u_0 satisfying* (1.2) *there exists a unique triplet (u, v, w) of nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v, w \in \bigcap_{\vartheta > n} C^0([0,\infty); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

which solves the problem (1.1) *classically, and is bounded, that is,*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

for all $t > 0$ with some constant $C > 0$.

Throughout the sequel we denote by

$$\overline{f} := \frac{1}{|\Omega|}\int_\Omega f$$

the spatial average of arbitrary functions $f \in L^1(\Omega)$.

Theorem 1.2 (Stabilization). *Let $n \in \mathbb{N}$. Let $q = p$ and $\chi\alpha - \xi\gamma = 0$. Assume that $m \geq 1$ and $0 \leq p - m \leq \frac{2}{n}$. Suppose that u_0 satisfies* (1.2) *and*

$$\chi\alpha \|u_0\|_{L^1(\Omega)}^{p-m} < \frac{1}{2C_{\langle p-m \rangle}},$$

where $C_{\langle p-m \rangle} > 0$ is a constant appearing in the Poincaré–Sobolev inequality $\|\varphi - \overline{\varphi}\|_{L^2(\Omega)} \leq C_{\langle p-m \rangle} \|\nabla\varphi\|_{L^{\frac{2}{p-m+1}}(\Omega)}$ for all $\varphi \in W^{1,\frac{2}{p-m+1}}(\Omega)$. Then the solution (u, v, w) of the problem (1.1)*, provided by Theorem* 1.1*, fulfills*

$$u(\cdot,t) \to \overline{u_0} \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty \tag{1.3}$$

and

$$v(\cdot,t) \to \frac{\alpha}{\beta}\overline{u_0} \ \text{ in } L^\infty(\Omega) \quad \text{as } t \to \infty \tag{1.4}$$

as well as

$$w(\cdot,t) \to \frac{\gamma}{\delta}\overline{u_0} \ \text{ in } L^\infty(\Omega) \quad \text{as } t \to \infty. \tag{1.5}$$

Theorem 1.3 (Exponential stabilization). *Let $n = 2$ and let $m = 1$. Let $q = p$ and $\chi\alpha - \xi\gamma = 0$. Let $\kappa \in (0, \lambda_1)$, where $\lambda_1 > 0$ is the first nonzero eigenvalue of the Neumann Laplacian in Ω. Assume that $1 < p \leq 2$. Suppose that u_0 satisfies* (1.2)*. Then one can find $t_0 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, whenever u_0 fulfills*

$$\|u_0\|_{L^1(\Omega)} \leq \varepsilon,$$

the solution (u, v, w) of the problem (1.1)*, provided by Theorem* 1.1*, satisfies*

$$\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \leq \varepsilon e^{-\kappa(t-t_0)} \tag{1.6}$$

and

$$\left\| v(\cdot,t) - \frac{\alpha}{\beta}\overline{u_0} \right\|_{L^\infty(\Omega)} \leq \frac{\alpha}{\beta}\varepsilon e^{-\kappa(t-t_0)} \tag{1.7}$$

as well as

$$\left\| w(\cdot,t) - \frac{\gamma}{\delta}\overline{u_0} \right\|_{L^\infty(\Omega)} \leq \frac{\gamma}{\delta}\varepsilon e^{-\kappa(t-t_0)} \tag{1.8}$$

for all $t > t_0$.

The strategy for showing boundedness (Theorem 1.1) is to derive the differential inequality

$$\frac{d}{dt}\int_\Omega u^\sigma + \int_\Omega u^\sigma \le c_1$$

for some $\sigma > n$ and $c_1 > 0$. The key to the construction of this inequality is to estimate the term $J_1 := c_2 \int_\Omega u^{\sigma+p-1}$ with $c_2 > 0$. In [2] the term J_1 can be removed by taking advantage of the effect of the repulsion. On the other hand, in our case, we cannot handle the term J_1 by the same way as in the literature. Hence, we shift our perspective to the diffusion instead of the repulsion. Specifically, we cope with J_1 by using the effect of the diffusion via the Gagliardo–Nirenberg inequality. Once boundedness is established, stabilization (Theorem 1.2) follows directly from boundedness and [1, Remark 1.1]. We next explain the strategy for proving exponential stabilization (Theorem 1.3). We first obtain the estimate

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \|u_0\|_{L^1(\Omega)}^{1+c_4}$$

with $c_3, c_4 > 0$, where $U(x,t) := u(x,t) - \overline{u_0}$ for $x \in \Omega$, $t > 0$ (see Lemma 4.2), which implies that there exists $t_0 > 0$ such that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \|u_0\|_{L^1(\Omega)}^{1+c_4} \tag{1.9}$$

for all $t > t_0$. We next take $\varepsilon_0 > 0$ small enough, and for each $\varepsilon \in (0, \varepsilon_0)$, fix u_0 such that $\|u_0\|_{L^1(\Omega)} \le \varepsilon$. We also define the set

$$S^* := \left\{ T^* \ge t_0 \mid \|U(\cdot,t)\|_{L^\infty(\Omega)} \le \varepsilon e^{-\kappa(t-t_0)} \text{ for all } t \in [t_0, T^*] \right\}$$

and put $T := \sup S^*$. Since the power of $\|u_0\|_{L^1(\Omega)}^{1+c_4}$ in (1.9) is greater than 1, we obtain the sharper estimate $\|U(\cdot,t)\|_{L^\infty(\Omega)} < \frac{\varepsilon}{2} e^{-\kappa(t-t_0)}$ on $[t_0, T]$. This entails that $T = \infty$, which derives exponential decay of U (see Lemma 4.3). This argument is based on that in [8], which deals with the case $p = 2$. However, since in our case the problem (1.1) includes $(u+1)^{p-2}$, we need to modify the argument slightly.

This paper is organized as follows. In Section 2 we give a result on local existence in (1.1) and a lemma such that an L^{σ_0}-estimate for u with some $\sigma_0 > n$ yields an L^∞-estimate for u. In addition, we state a lemma, which guarantees that $\int_\Omega w^\ell$ is controlled by $\int_\Omega u^\ell$ for $\ell > 1$. Section 3 is devoted to the proofs of boundedness (Theorem 1.1) and stabilization (Theorem 1.2). In Section 4 we show exponential stabilization (Theorem 1.3).

Throughout this paper, we denote by c_i generic positive constants, which will be sometimes specified by $c_i(\varepsilon)$ and $c_i(M)$ depending on small parameter $\varepsilon > 0$ and the mass $M := \int_\Omega u_0$, respectively.

2. Preliminaries

We first give a result on local existence in (1.1), which can be proved by standard arguments based on the contraction mapping principle (see e.g., [3, 14, 15]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with smooth boundary and let $m \geq 1$, $p,q \in \mathbb{R}$, $\chi,\xi,\alpha,\beta,\gamma,\delta > 0$. Then for all u_0 satisfying the condition* (1.2) *there exists $T_{\max} \in (0,\infty]$ such that* (1.1) *admits a unique classical solution (u,v,w) such that*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ v,w \in \bigcap_{\vartheta > n} C^0([0,T_{\max});W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})). \end{cases}$$

Moreover,

$$\textit{if } T_{\max} < \infty, \quad \textit{then } \lim_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

We next give a lemma, which provides a strategy to prove global existence and boundedness. This lemma can be derived from the proof of [14, Lemma A.1].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded domain with smooth boundary and let $m \geq 1$, $p,q \in \mathbb{R}$, $\chi,\xi,\alpha,\beta,\gamma,\delta > 0$. Assume that u_0 satisfies* (1.2)*. Denote by (u,v,w) the local classical solution of* (1.1) *given in Lemma* 2.1 *and by $T_{\max} \in (0,\infty]$ its maximal existence time. Then there are $\sigma_0 > \max\{n,-p+3\}$ and constants $C_1,C_2 > 0$ independent of $M = \int_\Omega u_0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_1 \Big(\sup_{s \in (0,T_{\max})} \|u(\cdot,s)\|_{L^{\sigma_0}(\Omega)}^{C_2} \Big) + \|u_0\|_{L^\infty(\Omega)} \tag{2.2}$$

for all $t \in (0,T_{\max})$.

We next recall a lemma, which asserts that $\int_\Omega w^\ell$ is dominated by $\int_\Omega u^\ell$ for $\ell > 1$. This lemma can be shown by clarifying the part containing $M = \int_\Omega u_0$ in [2, (3.3)].

Lemma 2.3. *Let $\ell > 1$. Denote by (u,v,w) the local classical solution of* (1.1) *given in Lemma* 2.1 *and by $T_{\max} \in (0,\infty]$ its maximal existence time. Then the first and third components of the solution satisfy that for all $\varepsilon > 0$,*

$$\int_\Omega w^\ell(\cdot,t) \leq \varepsilon \int_\Omega u^\ell(\cdot,t) + c(\varepsilon) M^{C_1}$$

for all $t \in (0,T_{\max})$ with some constants $c(\varepsilon) > 0$ and $C_1 > 0$ independent of $M = \int_\Omega u_0$.

3. Boundedness and stabilization

In these next sections we assume that $q = p$ and $\chi\alpha - \xi\gamma = 0$, and that u_0 satisfies (1.2). Then we denote by (u,v,w) the local classical solution of the problem (1.1) given in Lemma 2.1 and by $T_{\max} \in (0,\infty]$ its maximal existence time.

We employ the transformation

$$z = \chi v - \xi w$$

which was originally introduced by [13]. Noting that $q = p$ and $\chi\alpha - \xi\gamma = 0$, we see from the transformation that the triplet (u,z,v) satisfies

$$\begin{cases} u_t = \nabla\cdot\big((u+1)^{m-1}\nabla u - u(u+1)^{p-2}\nabla z\big) & \text{in } \Omega\times(0,T_{\max}), \\ 0 = \Delta z - \delta z + \chi(\delta-\beta)v & \text{in } \Omega\times(0,T_{\max}), \\ 0 = \Delta v + \alpha u - \beta v & \text{in } \Omega\times(0,T_{\max}), \\ \nabla u\cdot\nu = \nabla z\cdot\nu = \nabla v\cdot\nu = 0 & \text{on } \partial\Omega\times(0,T_{\max}), \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume that $m \geq 1$ and $p - m \leq \frac{2}{n}$. Then the first component of the solution (u,z,v) to* (3.1) *satisfies that for all $\sigma > \max\{n, -p+3\}$ there exist constants $C_1, C_2, C_3 > 0$ independent of $M = \int_\Omega u_0$ such that*

$$\|u(\cdot,t)\|_{L^\sigma(\Omega)} \leq \Big\{C_1(M^{C_2}+M^{C_3}) + e^{-t}\cdot\big[\|u_0\|^\sigma_{L^\sigma(\Omega)} - C_1(M^{C_2}+M^{C_3})\big]\Big\}^{\frac{1}{\sigma}} \tag{3.2}$$

for all $t \in (0,T_{\max})$.

Proof. Let $\sigma > \max\{n, -p+3\}$. Then we verify that the asserted estimate (3.2) holds on $(0,T_{\max})$; note that we omit the specification of the range of t in the proof. The first equation in (3.1) and the condition $m \geq 1$ as well as integration by parts imply

$$\begin{aligned} \frac{1}{\sigma}\frac{d}{dt}\int_\Omega u^\sigma &= \int_\Omega u^{\sigma-1}\nabla\cdot\big((u+1)^{m-1}\nabla u - u(u+1)^{p-2}\nabla z\big) \\ &= -(\sigma-1)\int_\Omega u^{\sigma-2}(u+1)^{m-1}|\nabla u|^2 \\ &\quad + (\sigma-1)\int_\Omega u^{\sigma-1}(u+1)^{p-2}\nabla u\cdot\nabla z \\ &\leq -(\sigma-1)\int_\Omega u^{\sigma+m-3}|\nabla u|^2 + (\sigma-1)\int_\Omega \nabla f(u)\cdot\nabla z \\ &= -\frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \big|\nabla u^{\frac{\sigma+m-1}{2}}\big|^2 + (\sigma-1)\int_\Omega \nabla f(u)\cdot\nabla z, \end{aligned} \tag{3.3}$$

where $f(u) := \int_0^u s^{\sigma-1}(s+1)^{p-2}\,ds$. Also, multiplying the second equation in (3.1) by $f(u)$, integrating by parts and using $z = \chi v - \xi w$, we obtain

$$\begin{aligned}
0 &= \int_\Omega f(u)\Delta z - \delta \int_\Omega f(u)z + \chi(\delta-\beta)\int_\Omega f(u)v \\
&= -\int_\Omega \nabla f(u)\cdot\nabla z - \delta\int_\Omega f(u)(\chi v - \xi w) + \chi(\delta-\beta)\int_\Omega f(u)v \\
&= -\int_\Omega \nabla f(u)\cdot\nabla z + \xi\delta\int_\Omega f(u)w - \chi\beta\int_\Omega f(u)v \\
&\le -\int_\Omega \nabla f(u)\cdot\nabla z + \xi\delta\int_\Omega f(u)w,
\end{aligned}$$

that is,

$$\int_\Omega \nabla f(u)\cdot\nabla z \le \xi\delta\int_\Omega f(u)w,$$

which combined with (3.3) entails

$$\frac{d}{dt}\int_\Omega u^\sigma + \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 \le \sigma(\sigma-1)\xi\delta\int_\Omega f(u)w.$$

Here, noting from the choice $\sigma > -p+3$ that

$$\begin{aligned}
f(u) &= \int_0^u s^{\sigma-1}(s+1)^{p-2}\,ds \\
&\le \int_0^u (s+1)^{\sigma+p-3}\,ds \\
&\le \frac{1}{\sigma+p-2}(u+1)^{\sigma+p-2}
\end{aligned}$$

and using the fact $(A+1)^{\sigma+p-2} \le 2^{\sigma+p-3}(A^{\sigma+p-2}+1)$ for $A>0$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^\sigma + \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 \\
&\quad\le \frac{\sigma(\sigma-1)\xi\delta}{\sigma+p-2}\int_\Omega (u+1)^{\sigma+p-2}w \\
&\quad\le \frac{2^{\sigma+p-3}\sigma(\sigma-1)\xi\delta}{\sigma+p-2}\left[\int_\Omega u^{\sigma+p-2}w + \int_\Omega w\right]. \qquad (3.4)
\end{aligned}$$

Moreover, from the third equation in (3.1) and the mass conservation property, we derive

$$\int_\Omega w = \frac{\gamma}{\delta}\int_\Omega u = \frac{\gamma}{\delta}\int_\Omega u_0 = \frac{\gamma}{\delta}M.$$

Substituting this into (3.4) yields

$$\frac{d}{dt}\int_\Omega u^\sigma + \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 \le c_1\int_\Omega u^{\sigma+p-2}w + c_2 M. \tag{3.5}$$

We now estimate $\int_\Omega u^{\sigma+p-2}w$. Employing the Hölder inequality, we have

$$\int_\Omega u^{\sigma+p-2}w \le \Big(\int_\Omega u^{\sigma+p-1}\Big)^{\frac{\sigma+p-2}{\sigma+p-1}}\Big(\int_\Omega w^{\sigma+p-1}\Big)^{\frac{1}{\sigma+p-1}}.$$

Here, by virtue of Lemma 2.3 with $\ell = \sigma+p-1$, we infer that for all $\varepsilon > 0$,

$$\int_\Omega w^{\sigma+p-1} \le \varepsilon^{\sigma+p-1}\int_\Omega u^{\sigma+p-1} + c_3(\varepsilon)M^{c_4}.$$

Combining the above two inequalities implies

$$\int_\Omega u^{\sigma+p-2}w \le \varepsilon\int_\Omega u^{\sigma+p-1} + c_5(\varepsilon)M^{\frac{c_4}{\sigma+p-1}}. \tag{3.6}$$

Thus we see from (3.5) and (3.6) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^\sigma &+ \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2\\ &\le c_1\varepsilon\int_\Omega u^{\sigma+p-1} + c_1c_5(\varepsilon)M^{\frac{c_4}{\sigma+p-1}} + c_2M. \end{aligned}\tag{3.7}$$

Here, the Gagliardo–Nirenberg inequality ensures

$$\begin{aligned}\int_\Omega u^{\sigma+p-1} &= \|u^{\frac{\sigma+m-1}{2}}(\cdot,t)\|^{\frac{2(\sigma+p-1)}{\sigma+m-1}}_{L^{\frac{2(\sigma+p-1)}{\sigma+m-1}}(\Omega)}\\ &\le c_6\Big(\|\nabla u^{\frac{\sigma+m-1}{2}}(\cdot,t)\|^{\frac{2(\sigma+p-1)}{\sigma+m-1}\theta_1}_{L^2(\Omega)}\|u^{\frac{\sigma+m-1}{2}}(\cdot,t)\|^{\frac{2(\sigma+p-1)}{\sigma+m-1}(1-\theta_1)}_{L^{\frac{2}{\sigma+m-1}}(\Omega)}\\ &\qquad + \|u^{\frac{\sigma+m-1}{2}}(\cdot,t)\|^{\frac{2(\sigma+p-1)}{\sigma+m-1}}_{L^{\frac{2}{\sigma+m-1}}(\Omega)}\Big)\\ &\le c_6\Big[\Big(\int_\Omega|\nabla u^{\frac{\sigma+m-1}{2}}|^2\Big)^{\frac{\sigma+p-1}{\sigma+m-1}\theta_1}M^{(\sigma+p-1)(1-\theta_1)} + M^{\sigma+p-1}\Big], \end{aligned}\tag{3.8}$$

where $\theta_1 = \theta_1(p) := \frac{\frac{\sigma+m-1}{2} - \frac{\sigma+m-1}{2(\sigma+p-1)}}{\frac{\sigma+m-1}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$. Indeed, since $\frac{n}{2}(p-m)-p+1 < -p+3 < \sigma$ due to the condition $p-m \le \frac{2}{n}$ and the choice $\sigma > -p+3$, we can verify that

$$\frac{\sigma+m-1}{2(\sigma+p-1)} > \frac{1}{2}-\frac{1}{n}.$$

Noticing from the condition $p-m \le \frac{2}{n}$ that

$$\frac{\sigma+p-1}{\sigma+m-1}\theta_1 = \frac{\frac{\sigma+p-1}{2}-\frac{1}{2}}{\frac{\sigma+m-1}{2}+\frac{1}{n}-\frac{1}{2}} \le \frac{\frac{\sigma+m-1}{2}+\frac{1}{n}-\frac{1}{2}}{\frac{\sigma+m-1}{2}+\frac{1}{n}-\frac{1}{2}} = 1,$$

we see that

$$\Big(\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\Big)^{\frac{\sigma+p-1}{\sigma+m-1}\theta_1} \le \int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + 1.$$

Hence we have from (3.8) that

$$\begin{aligned}\int_\Omega u^{\sigma+p-1} &\le c_6 M^{(\sigma+p-1)(1-\theta_1)}\Big(\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + 1\Big) + c_6 M^{\sigma+p-1}\\ &\le c_6 M^{(\sigma+p-1)(1-\theta_1)}\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + c_6\big(M^{(\sigma+p-1)(1-\theta_1)} + M^{\sigma+p-1}\big),\end{aligned}$$

which combined with (3.7) entails

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^\sigma + \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2\\ &\quad\le c_1 c_6 M^{(\sigma+p-1)(1-\theta_1)}\varepsilon \int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 + c_7(M,\varepsilon),\end{aligned}$$

where $c_7(M,\varepsilon) := c_1 c_5(\varepsilon) M^{\frac{c_4}{\sigma+p-1}} + c_2 M + c_1 c_6 \varepsilon\big(M^{(\sigma+p-1)(1-\theta_1)} + M^{\sigma+p-1}\big)$. We now add $\int_\Omega u^\sigma$ on the both sides of this inequality. Then we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^\sigma + \int_\Omega u^\sigma + \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2\\ &\quad\le \int_\Omega u^\sigma + c_1 c_6 M^{(\sigma+p-1)(1-\theta_1)}\varepsilon\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + c_7(M,\varepsilon).\end{aligned} \tag{3.9}$$

Here, using (3.8) with $p=1$, we infer

$$\int_\Omega u^\sigma \le c_8\Big[\Big(\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2\Big)^{\frac{\sigma}{\sigma+m-1}\theta_2} M^{\sigma(1-\theta_2)} + M^\sigma\Big],$$

where $\theta_2 := \theta_1(1) = \frac{\frac{\sigma+m-1}{2}-\frac{\sigma+m-1}{2\sigma}}{\frac{\sigma+m-1}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$, because the relation

$$\frac{\sigma+m-1}{2\sigma} \ge \frac{1}{2} > \frac{1}{2} - \frac{1}{n}$$

holds by the condition $m \ge 1$. Also, since the condition $m \ge 1$ again ensures

$$\frac{\sigma}{\sigma+m-1}\theta_2 = \frac{\frac{\sigma}{2}-\frac{1}{2}}{\frac{\sigma+m-1}{2}+\frac{1}{n}-\frac{1}{2}} \le \frac{\frac{\sigma}{2}-\frac{1}{2}}{\frac{\sigma}{2}+\frac{1}{n}-\frac{1}{2}} < 1,$$

the Young inequality derives that for all $\varepsilon' > 0$,

$$\begin{aligned}\int_\Omega u^\sigma &\le c_8 M^{\sigma(1-\theta_2)}\Big(\varepsilon' \int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + c_9(\varepsilon')\Big) + c_8 M^\sigma \\ &\le c_8 M^{\sigma(1-\theta_2)}\varepsilon' \int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + c_{10}(\varepsilon')(M^{\sigma(1-\theta_2)} + M^\sigma).\end{aligned}$$

Applying this inequality to the right-hand side of (3.9), we see that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^\sigma + \int_\Omega u^\sigma + \frac{4\sigma(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \big|\nabla u^{\frac{\sigma+m-1}{2}}\big|^2 \\ &\quad\le \Big(c_1 c_6 M^{(\sigma+p-1)(1-\theta_1)}\varepsilon + c_8 M^{\sigma(1-\theta_2)}\varepsilon'\Big)\int_\Omega |\nabla u^{\frac{\sigma+m-1}{2}}|^2 + c_{11}(M,\varepsilon,\varepsilon'),\end{aligned}$$

where $c_{11}(M,\varepsilon,\varepsilon') := c_7(M,\varepsilon) + c_{10}(\varepsilon')(M^{\sigma(1-\theta_2)} + M^\sigma)$. Hence, choosing $\varepsilon, \varepsilon' > 0$ small enough, we obtain

$$\frac{d}{dt}\int_\Omega u^\sigma + \int_\Omega u^\sigma \le c_{12}(M). \tag{3.10}$$

Here we put

$$\omega_* := \min\Big\{1,\ \frac{c_4}{\sigma+p-1},\ (\sigma+p-1)(1-\theta_1),\ \sigma+p-1,\ \sigma(1-\theta_2),\ \sigma\Big\}$$

and

$$\omega^* := \max\Big\{1,\ \frac{c_4}{\sigma+p-1},\ (\sigma+p-1)(1-\theta_1),\ \sigma+p-1,\ \sigma(1-\theta_2),\ \sigma\Big\},$$

which is the smallest and largest power of M appearing in $c_{12}(M)$, respectively. Then, noting from the choice of σ that $(\sigma+p-1)(1-\theta_1)$ and $\sigma(1-\theta_2)$ are possibly smaller than 1, we can estimate $c_{12}(M)$ as $c_{12}(M) \le c_{13}(M^{\omega_*} + M^{\omega^*})$. We thereby infer from the inequality (3.10) that

$$\frac{d}{dt}\int_\Omega u^\sigma + \int_\Omega u^\sigma \le c_{13}(M^{\omega_*} + M^{\omega^*}).$$

Therefore we have

$$\int_\Omega u^\sigma \le c_{13}(M^{\omega_*} + M^{\omega^*}) + e^{-t}\cdot\big[\|u_0\|^\sigma_{L^\sigma(\Omega)} - c_{13}(M^{\omega_*} + M^{\omega^*})\big],$$

which leads to the conclusion. □

We are now in a position to complete the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. A combination of (2.2) and (3.2) with $\sigma = \sigma_0$ ensures that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_1$. Therefore, by virtue of the extensibility criterion (2.1), we arrive at the conclusion. □

Proof of Theorem 1.2. Thanks to boundedness established by Theorem 1.1, the stabilization properties (1.3)–(1.5) result from [1, Remark 1.1]. □

4. Exponential stabilization

In this section, assuming that $n = 2$ and $m = 1$, we prove Theorem 1.3. To this end we further rewrite the system (3.1) reduced by (1.1). Setting $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$, we define the functions $U = U(x,t)$, $Z = Z(x,t)$ and $V = V(x,t)$ as

$$U(x,t) := u(x,t) - \overline{u_0},$$
$$Z(x,t) := z(x,t) - \chi\alpha\Big(\frac{1}{\beta} - \frac{1}{\delta}\Big)\overline{u_0},$$
$$V(x,t) := v(x,t) - \frac{\alpha}{\beta}\overline{u_0}$$

for $x \in \Omega \subset \mathbb{R}^2$, $t > 0$, where $z = \chi v - \xi w$. Then we see from (3.1) with $m = 1$ that the triplet (U,Z,V) satisfies

$$\begin{cases} U_t = \nabla\cdot\big(\nabla U - u(u+1)^{p-2}\nabla Z\big) & \text{in } \Omega\times(0,\infty),\\ 0 = \Delta Z - \delta Z + \chi(\delta-\beta)V & \text{in } \Omega\times(0,\infty),\\ 0 = \Delta V + \alpha U - \beta V & \text{in } \Omega\times(0,\infty),\\ \nabla U\cdot\nu = \nabla Z\cdot\nu = \nabla V\cdot\nu = 0 & \text{on } \partial\Omega\times(0,\infty),\\ U(\cdot,0) = u_0 - \overline{u_0} & \text{in } \Omega. \end{cases} \tag{4.1}$$

We first present the following lemma which can be proved by well-known estimates for solutions of elliptic equations.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Let $\psi \in C^0(\overline{\Omega})$ and let $a,b > 0$. Then the solution φ of the boundary value problem*

$$\begin{cases} 0 = \Delta\varphi + a\psi - b\varphi & \text{in } \Omega,\\ \nabla\varphi\cdot\nu = 0 & \text{on } \partial\Omega \end{cases}$$

fulfills

$$\|\varphi\|_{L^\theta(\Omega)} \le aC_{\langle\theta\rangle}\|\psi\|_{L^1(\Omega)},$$
$$\|\nabla\varphi\|_{L^\mu(\Omega)} \le aC_{\langle\mu\rangle}\|\psi\|_{L^2(\Omega)}$$

for all $\theta,\mu > 1$ with some $C_{\langle\theta\rangle}, C_{\langle\mu\rangle} > 0$ independent of $\|\psi\|_{L^1(\Omega)}$ and $\|\psi\|_{L^2(\Omega)}$.

We next prove an estimate for U, which is the key to the derivation of L^∞-convergence of u. The proof is parallel to [8, Proof of Lemma 4.3], however, we confirm it because (4.1) is the quasilinear system including $(u+1)^{p-2}$.

Lemma 4.2. *Assume that $1 < p \le 2$. Then the first component of the solution (U,Z,V) to (4.1) satisfies that for all $\sigma > 2$ there exist constants $C_1, C_2, C_3 > 0$ independent of $M = \int_\Omega u_0$ such that*

$$\limsup_{t\to\infty}\|U(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 M(M^{C_2} + M^{C_3}). \tag{4.2}$$

Proof. Let $\sigma > 2$. Then we infer from (3.2) that there exists $t_1 > 0$ such that for all $t > t_1$,

$$\|u(\cdot,t)\|_{L^\sigma(\Omega)} \le c_1(M^{c_2} + M^{c_3}). \tag{4.3}$$

Also, we can show that the second component Z of the solution (U,Z,V) to (4.1) satisfies

$$\|\nabla Z(\cdot,t)\|_{L^{\theta_3}(\Omega)} \le c_4 M \tag{4.4}$$

for all $t > t_1$ and all $\theta_3 > 1$. Indeed, by the identity $\nabla Z = \nabla z$ and the second equation of (3.1), we see from Lemma 4.1 that

$$\begin{aligned}\|\nabla Z(\cdot,t)\|_{L^{\theta_3}(\Omega)} &= \|\nabla z(\cdot,t)\|_{L^{\theta_3}(\Omega)}\\ &\le \chi|\delta-\beta|c_5\cdot\|v(\cdot,t)\|_{L^2(\Omega)}\end{aligned}$$

for all $t > t_1$. Moreover, from the third equation of (3.1), again by Lemma 4.1, we have

$$\begin{aligned}\|v(\cdot,t)\|_{L^2(\Omega)} &\le \alpha c_6\|u(\cdot,t)\|_{L^1(\Omega)}\\ &= \alpha c_6 M\end{aligned}$$

for all $t > t_1$. The above two estimates yield (4.4). We now rewrite the first equation in (4.1) as

$$\begin{aligned}U(\cdot,t) &= e^{(t-t_1)\Delta}U(\cdot,t_1) - \int_{t_1}^t e^{(t-s)\Delta}\nabla\cdot\left[u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\right]ds\\ &=: I_1(\cdot,t) + I_2(\cdot,t) \quad \text{for } t > t_1.\end{aligned} \tag{4.5}$$

In order to prove (4.2) we first show that

$$\|I_1(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{4.6}$$

We infer from [16, Lemma 1.3 (i)] with $n = 2$ that

$$\begin{aligned}\|I_1(\cdot,t)\|_{L^\infty(\Omega)} &= \|e^{(t-t_1)\Delta}U(\cdot,t_1)\|_{L^\infty(\Omega)}\\ &\le c_7(1+(t-t_1)^{-1})e^{-\lambda_1 t}\|U(\cdot,t)\|_{L^1(\Omega)}\\ &\le c_7(1+(t-t_1)^{-1})e^{-\lambda_1 t}\cdot 2M\end{aligned}$$

for all $t > t_1$, where $\lambda_1 > 0$ is the first nonzero eigenvalue of the Neumann Laplacian in Ω. Hence we derive (4.6). We next estimate $\|I_2(\cdot,t)\|_{L^\infty(\Omega)}$. For $k > 2$, we observe from [16, Lemma 1.3 (iv)] with $n = 2$ that

$$\begin{aligned}&\|I_2(\cdot,t)\|_{L^\infty(\Omega)}\\ &\quad\le c_8\int_{t_1}^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)}\|u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\|_{L^k(\Omega)}\,ds\end{aligned} \tag{4.7}$$

for all $t > t_1$. Here, owing to the Hölder inequality and the condition $1 < p \leq 2$, we deduce

$$\begin{aligned}
&\|u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\|_{L^k(\Omega)} \\
&\leq \|u^{p-1}(\cdot,s)\nabla Z(\cdot,s)\|_{L^k(\Omega)} \\
&= \|u(\cdot,s)\|^{p-1}_{L^\sigma(\Omega)}\|\nabla Z(\cdot,s)\|_{L^{\frac{k\sigma}{\sigma-k(p-1)}}(\Omega)}
\end{aligned}$$

for all $s > t_1$. Since $\frac{k\sigma}{\sigma-k(p-1)} > 1$ due to the facts $k > 2$ and $\frac{\sigma}{\sigma-k(p-1)} > 1$, the estimates (4.3) and (4.4) imply that

$$\|u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\|_{L^k(\Omega)} \leq c_9 M(M^{c_2}+M^{c_3}) \tag{4.8}$$

for all $s > t_1$. A combination of (4.7) and (4.8) yields

$$\|I_2(\cdot,t)\|_{L^\infty(\Omega)} \leq c_9 M(M^{c_2}+M^{c_3})\int_{t_1}^{t}(1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)}\,ds. \tag{4.9}$$

Also, noting from the condition $k > 2$ that $-\frac{1}{2}-\frac{1}{k} > -1$, we derive that

$$\begin{aligned}
\int_{t_1}^{t}(1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)}\,ds &\leq \int_0^\infty(1+\eta^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1\eta}\,d\eta \\
&= \frac{1}{\lambda_1}+\frac{1}{\lambda_1^{\frac{1}{2}-\frac{1}{k}}}\Gamma\Big(\frac{1}{2}-\frac{1}{k}\Big) =: c_{10},
\end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. Thus we see from (4.9) that

$$\|I_2(\cdot,t)\|_{L^\infty(\Omega)} \leq c_{11} M(M^{c_2}+M^{c_3}) \tag{4.10}$$

for all $t > t_1$. Combining (4.6) and (4.10) with (4.5), we arrive at (4.2). $\square$

In light of Lemma 4.2 we infer that there exist $t_2 > 0$ and $c_1, c_2, c_3 > 0$ such that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq c_1 M(M^{c_2}+M^{c_3}) \tag{4.11}$$

for all $t > t_2$. We now pick $\varepsilon_0 > 0$ such that

$$2c_1(\varepsilon_0^{c_2}+\varepsilon_0^{c_3}) \leq 1, \tag{4.12}$$

and for each $\varepsilon \in (0,\varepsilon_0)$, fix $M = \int_\Omega u_0$ such that $0 < M \leq \varepsilon$. Then we deduce from (4.11), the fact $M \leq \varepsilon < \varepsilon_0$ and (4.12) that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq 2c_1(\varepsilon^{c_2}+\varepsilon^{c_3})\cdot\frac{1}{2}\varepsilon \leq \frac{1}{2}\varepsilon \tag{4.13}$$

for all $t > t_2$. Thus we have that

$$S^* := \left\{T^* \geq t_2 \mid \|U(\cdot,t)\|_{L^\infty(\Omega)} \leq \varepsilon e^{-\kappa(t-t_2)} \text{ for all } t \in [t_2, T^*]\right\} \tag{4.14}$$

is nonempty, where $\kappa \in (0, \lambda_1)$ and $\lambda_1 > 0$ is the first eigenvalue of the Neumann Laplacian in Ω. Indeed, noting that if $t = t_2$, then $\varepsilon e^{-\kappa(t-t_2)} = \varepsilon\, (> \frac{1}{2}\varepsilon)$, we derive from the continuity of the function $t \mapsto \varepsilon e^{-\kappa(t-t_2)}$ that there exists $T^* > t_2$ such that $\varepsilon e^{-\kappa(t-t_2)} > \frac{1}{2}\varepsilon$ for all $t \in [t_2, T^*]$, which in conjunction with (4.13) implies $\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq \varepsilon e^{-\kappa(t-t_2)}$ for all $t \in [t_2, T^*]$.

We put

$$T := \sup S^* \in (t_2, \infty] \tag{4.15}$$

and note that

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq \varepsilon e^{-\kappa(t-t_2)} \quad \text{for all } t \in [t_2, T] \tag{4.16}$$

holds by the definition of S^*. In the following lemma we derive $T = \infty$ which yields that u converges to $\overline{u_0}$ at an exponential rate as $t \to \infty$. The argument in [8] based on [16, Lemma 1.2], however, this is not applicable directly to our case because the system (4.1) includes $(u+1)^{p-2}$. So, we go back to the proof of [16, Lemma 1.2].

Lemma 4.3. *Let $\varepsilon_0 > 0$ satisfy* (4.12). *Let $\kappa \in (0, \lambda_1)$. Assume that $1 < p \leq 2$. Then for all $\varepsilon \in (0, \varepsilon_0)$, whenever u_0 fulfills that $0 < M = \int_\Omega u_0 \leq \varepsilon$, the first component of the solution (U, Z, V) to* (4.1) *satisfies that*

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} \leq \varepsilon e^{-\kappa(t-t_2)} \tag{4.17}$$

for all $t > t_2 + 1$, where $t_2 > 0$ is the time appearing in (4.11).

Proof. We first rewrite the first equation in (4.1) as

$$\begin{aligned} U(\cdot,t) &= e^{(t-t_2)\Delta}U(\cdot,t_2) - \int_{t_2}^{t} e^{(t-s)\Delta}\nabla \cdot \left[u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\right] ds \\ &=: I_3(\cdot,t) + I_4(\cdot,t) \quad \text{for } t \in (t_2, T) \end{aligned} \tag{4.18}$$

with $T = \sup S^*$, where S^* is defined in (4.14). We then estimate $\|I_3(\cdot,t)\|_{L^\infty(\Omega)}$. Using [16, Lemma 1.3 (i)] with $n = 2$ and the fact $e^{-\lambda_1(t-t_2)} \leq e^{-\kappa(t-t_2)}$, we have

$$\begin{aligned} \|I_3(\cdot,t)\|_{L^\infty(\Omega)} &\leq c_1(1 + (t-t_2)^{-1})e^{-\lambda_1(t-t_2)}\|U(\cdot,t_2)\|_{L^\infty(\Omega)} \\ &\leq 2c_1 e^{-\kappa(t-t_2)}\|U(\cdot,t_2)\|_{L^\infty(\Omega)} \end{aligned}$$

for all $t \in (t_2, T)$. Moreover, by virtue of the estimate (4.11) and the condition $M \le \varepsilon$, we obtain

$$\begin{aligned} \|I_3(\cdot,t)\|_{L^\infty(\Omega)} &\le 2c_1 e^{-\kappa(t-t_2)} \cdot c_2\varepsilon(\varepsilon^{c_3}+\varepsilon^{c_4}) \\ &= c_5\varepsilon(\varepsilon^{c_3}+\varepsilon^{c_4})e^{-\kappa(t-t_2)} \end{aligned} \tag{4.19}$$

for all $t \in (t_2, T)$. We next estimate $\|I_4(\cdot,t)\|_{L^\infty(\Omega)}$. Taking $k > 2$, we see from [16, Lemma 1.3 (iv)] with $n = 2$ that

$$\begin{aligned} &\|I_4(\cdot,t)\|_{L^\infty(\Omega)} \\ &\quad \le c_6 \int_{t_2}^{t} (1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)} \|u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\|_{L^k(\Omega)}\,ds \end{aligned} \tag{4.20}$$

for all $t \in (t_2, T)$. Here, we infer from the condition $1 < p \le 2$ that

$$\|u(\cdot,s)(u(\cdot,s)+1)^{p-2}\nabla Z(\cdot,s)\|_{L^k(\Omega)} \le \|u(\cdot,s)\|_{L^\infty(\Omega)}^{k(p-1)} \|\nabla Z(\cdot,s)\|_{L^k(\Omega)} \tag{4.21}$$

for all $s \in (t_2, T)$. Let us estimate the right-hand side of this inequality. In view of the definition of U (see the beginning of Section 4), the estimate (4.16) and $M \le \varepsilon$, we derive

$$\begin{aligned} \|u(\cdot,s)\|_{L^\infty(\Omega)} &= \|U(\cdot,s)+\overline{u_0}\|_{L^\infty(\Omega)} \\ &\le \|U(\cdot,s)\|_{L^\infty(\Omega)} + \frac{M}{|\Omega|} \\ &\le c_7\varepsilon e^{-\kappa(s-t_2)} + \frac{\varepsilon}{|\Omega|} \end{aligned}$$

for all $s \in (t_2, T)$, which means that

$$\|u(\cdot,s)\|_{L^\infty(\Omega)}^{k(p-1)} \le \left(c_7\varepsilon e^{-\kappa(s-t_2)} + \frac{\varepsilon}{|\Omega|}\right)^{k(p-1)} \tag{4.22}$$

for all $s \in (t_2, T)$. Also, by virtue of the second and third equations in (4.1) and the estimate (4.16) as well as Lemma 4.1, we obtain

$$\begin{aligned} \|\nabla Z(\cdot,s)\|_{L^k(\Omega)} &\le \chi|\delta-\beta|c_8 \cdot \|V(\cdot,s)\|_{L^2(\Omega)} \\ &\le \chi|\delta-\beta|c_8 \cdot \alpha c_9 \|U(\cdot,s)\|_{L^1(\Omega)} \\ &\le c_{10}|\Omega| \cdot \|U(\cdot,s)\|_{L^\infty(\Omega)} \\ &\le c_{11}\varepsilon e^{-\kappa(s-t_2)} \end{aligned} \tag{4.23}$$

for all $s \in (t_2, T)$. Collecting (4.21), (4.22) and (4.23) in (4.20), we have

$$\|I_4(\cdot,t)\|_{L^\infty(\Omega)} \leq c_{12}\varepsilon \int_{t_2}^{t} (1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)} \cdot \Big(c_7\varepsilon e^{-\kappa(s-t_2)} + \frac{\varepsilon}{|\Omega|}\Big)^{k(p-1)} \cdot e^{-\kappa(s-t_2)}\,ds \tag{4.24}$$

for all $t \in (t_2, T)$. Combining (4.19) and (4.24) with (4.18) yields

$$\begin{aligned}\|U(\cdot,t)\|_{L^\infty(\Omega)} &\leq c_5\varepsilon(\varepsilon^{c_3}+\varepsilon^{c_4})e^{-\kappa(t-t_2)} \\ &\quad + c_{12}\varepsilon\int_{t_2}^{t}(1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)} \\ &\quad \cdot \Big(c_7\varepsilon e^{-\kappa(s-t_2)} + \frac{\varepsilon}{|\Omega|}\Big)^{k(p-1)} \cdot e^{-\kappa(s-t_2)}\,ds\end{aligned} \tag{4.25}$$

for all $t \in (t_2, T)$. We next estimate the integral appearing in the right-hand side of (4.25). We first estimate it as

$$\begin{aligned}&\int_{t_2}^{t}(1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)} \cdot \Big(c_7\varepsilon e^{-\kappa(s-t_2)} + \frac{\varepsilon}{|\Omega|}\Big)^{k(p-1)} \cdot e^{-\kappa s}\,ds \\ &\leq c_{13}\varepsilon^{k(p-1)}\int_{t_2}^{t}(1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)} \cdot (e^{-k(p-1)\cdot\kappa(s-t_2)}+1)\cdot e^{-\kappa s}\,ds \\ &\leq 2c_{13}\varepsilon^{k(p-1)}\int_{t_2}^{t} e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds \\ &\quad + c_{13}\varepsilon^{k(p-1)}\int_{t_2}^{t}(t-s)^{-\frac{1}{2}-\frac{1}{k}}e^{-k(p-1)\cdot\kappa(s-t_2)}e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds \\ &=: 2c_{13}\varepsilon^{k(p-1)}I_5(\cdot,t) + c_{13}\varepsilon^{k(p-1)}I_6(\cdot,t)\end{aligned} \tag{4.26}$$

for all $t \in (t_2, T)$. From a straightforward calculation we rewrite $I_5(\cdot,t)$ as

$$\begin{aligned}I_5(\cdot,t) &= e^{-\lambda_1 t}\cdot\frac{1}{\lambda_1-\kappa}(e^{(\lambda_1-\kappa)t}-e^{(\lambda_1-\kappa)t_2}) \\ &= \frac{1}{\lambda_1-\kappa}(e^{-\kappa t}-e^{\kappa t_2}e^{-\lambda_1(t-t_2)})\end{aligned} \tag{4.27}$$

for all $t \in (t_2, T)$. We next estimate $I_6(\cdot,t)$ by dividing the interval (t_2,t) into $(t_2,t-1)$ and $(t-1,t)$ for $t \in (t_2+1,T)$. Namely, we rewrite $I_6(\cdot,t)$ as

$$\begin{aligned}I_6(\cdot,t) &= \int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{1}{k}}e^{-k(p-1)\cdot\kappa(s-t_2)}e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds \\ &\quad + \int_{t_2}^{t-1}(t-s)^{-\frac{1}{2}-\frac{1}{k}}e^{-k(p-1)\cdot\kappa(s-t_2)}e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds \\ &=: I_6^{(1)}(\cdot,t)+I_6^{(2)}(\cdot,t) \quad \text{for } t \in (t_2+1,T),\end{aligned} \tag{4.28}$$

and estimate $I_6^{(1)}(\cdot,t)$ and $I_6^{(2)}(\cdot,t)$. As to $I_6^{(1)}(\cdot,t)$, we see from the condition $k>2$ and the fact $e^{-\kappa s}\le e^{-\kappa(t-1)}$ for all $s\in(t-1,t)$ that

$$\begin{aligned}
I_6^{(1)}(\cdot,t) &= \int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{1}{k}}e^{-k(p-1)\cdot\kappa(s-t_2)}e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds\\
&\le e^{-\kappa(t-1)}\int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{1}{k}}\,ds\\
&= \frac{2ke^{\kappa}}{k-2}e^{-\kappa t}
\end{aligned}\tag{4.29}$$

for all $t\in(t_2+1,T)$. Also, as to $I_6^{(2)}(\cdot,t)$, we observe that

$$\begin{aligned}
I_6^{(2)}(\cdot,t) &= \int_{t_2}^{t-1}(t-s)^{-\frac{1}{2}-\frac{1}{k}}e^{-k(p-1)\cdot\kappa(s-t_2)}e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds\\
&\le \int_{t_2}^{t-1}e^{-\lambda_1(t-s)}e^{-\kappa s}\,ds\\
&= \frac{e^{-\lambda_1 t}}{\lambda_1-\kappa}(e^{(\lambda_1-\kappa)(t-1)}-e^{(\lambda_1-\kappa)t_2})\\
&= \frac{1}{\lambda_1-\kappa}(e^{-(\lambda_1-\kappa)}e^{-\kappa t}-e^{\kappa t_2}e^{-\lambda_1(t-t_2)})
\end{aligned}\tag{4.30}$$

for all $t\in(t_2+1,T)$. Hence, combining (4.29) and (4.30) with (4.28) asserts that

$$I_6(\cdot,t)\le\frac{2ke^{\kappa}}{k-2}e^{-\kappa t}+\frac{1}{\lambda_1-\kappa}(e^{-(\lambda_1-\kappa)}e^{-\kappa t}-e^{\kappa t_2}e^{-\lambda_1(t-t_2)})\tag{4.31}$$

for all $t\in(t_2+1,T)$. Collecting (4.27) and (4.31) in (4.26) ensures

$$\begin{aligned}
&\int_{t_2}^{t}(1+(t-s)^{-\frac{1}{2}-\frac{1}{k}})e^{-\lambda_1(t-s)}\cdot\Big(c_7\varepsilon e^{-\kappa(s-t_2)}+\frac{\varepsilon}{|\Omega|}\Big)^{k(p-1)}\cdot e^{-\kappa s}\,ds\\
&\le 2c_{13}\varepsilon^{k(p-1)}I_5(\cdot,t)+c_{12}\varepsilon^{k(p-1)}I_6(\cdot,t)\\
&\le \frac{2c_{13}\varepsilon^{k(p-1)}}{\lambda_1-\kappa}(e^{-\kappa t}-e^{\kappa t_2}e^{-\lambda_1(t-t_2)})\\
&\quad+\frac{2c_{13}\varepsilon^{k(p-1)}ke^{\kappa}}{k-2}e^{-\kappa t}+\frac{c_{13}\varepsilon^{k(p-1)}}{\lambda_1-\kappa}(e^{-(\lambda_1-\kappa)}e^{-\kappa t}-e^{\kappa t_2}e^{-\lambda_1(t-t_2)})\\
&\le c_{13}\varepsilon^{k(p-1)}\Big[\frac{2e^{-\kappa t_2}}{\lambda_1-\kappa}+\frac{2ke^{\kappa(1-t_2)}}{k-2}+\frac{e^{-\lambda_1+\kappa(1-t_2)}}{\lambda_1-\kappa}\Big]e^{-\kappa(t-t_2)}
\end{aligned}\tag{4.32}$$

for all $t \in (t_2+1,T)$. Thus, a combination of (4.32) and (4.25) implies

$$\begin{aligned}
&\|U(\cdot,t)\|_{L^\infty(\Omega)} \\
&\quad \le c_5\varepsilon(\varepsilon^{c_3}+\varepsilon^{c_4})e^{-\kappa(t-t_2)} \\
&\qquad + c_{12}\varepsilon\cdot c_{13}\varepsilon^{k(p-1)}\Big[\frac{2e^{-\kappa t_2}}{\lambda_1-\kappa}+\frac{2ke^{\kappa(1-t_2)}}{k-2}+\frac{e^{-\lambda_1+\kappa(1-t_2)}}{\lambda_1-\kappa}\Big]e^{-\kappa(t-t_2)} \\
&\quad = c_{14}\varepsilon(\varepsilon^{c_3}+\varepsilon^{c_4}+\varepsilon^{k(p-1)})e^{-\kappa(t-t_2)}
\end{aligned}$$

for all $t \in (t_2+1,T)$. Taking ε_0 such that $c_{14}(\varepsilon_0^{c_3}+\varepsilon_0^{c_4}+\varepsilon_0^{k(p-1)}) < \frac{1}{2}$, we have

$$\|U(\cdot,t)\|_{L^\infty(\Omega)} < \frac{\varepsilon}{2}e^{-\kappa(t-t_2)}$$

for all $t \in (t_2+1,T)$. Therefore, in view of the definition of T (see (4.15) together with (4.14)), we conclude from the continuity of U that $T=\infty$, which completes the proof. $\square$

Proof of Theorem 1.3. We put $U(x,t) := u(x,t)-\overline{u_0}$, $V(x,t) := v(x,t)-\frac{\alpha}{\beta}\overline{u_0}$ and $W(x,t) := w(x,t)-\frac{\gamma}{\delta}\overline{u_0}$ for $x\in\Omega\subset\mathbb{R}^2$, $t>0$. Then the second and third equations and boundary conditions in (1.1) are rewritten as

$$\begin{aligned}
0 &= \Delta V+\alpha U-\beta V, \quad \nabla V\cdot\nu|_{\partial\Omega}=0,\\
0 &= \Delta W+\gamma U-\delta W, \quad \nabla W\cdot\nu|_{\partial\Omega}=0.
\end{aligned}$$

Thus the maximum principle warrants that

$$\begin{aligned}
\frac{\alpha}{\beta}\min_{x\in\overline{\Omega}}U(x,t) \le V(\cdot,t) \le \frac{\alpha}{\beta}\max_{x\in\overline{\Omega}}U(x,t),\\
\frac{\gamma}{\delta}\min_{x\in\overline{\Omega}}U(x,t) \le W(\cdot,t) \le \frac{\gamma}{\delta}\max_{x\in\overline{\Omega}}U(x,t)
\end{aligned}$$

for all $t>0$. Under the assumption of Lemma 4.3, this along with (4.17) yields

$$\begin{aligned}
\|V(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\alpha}{\beta}\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\alpha}{\beta}\varepsilon e^{-\kappa(t-t_2)},\\
\|W(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\gamma}{\delta}\|U(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\gamma}{\delta}\varepsilon e^{-\kappa(t-t_2)}
\end{aligned}$$

for all $t>t_2+1$. Therefore we arrive at (1.6)–(1.8). $\square$

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Y. CHIYO
Department of Mathematics
Tokyo University of Science
e-mail: ycnewssz@gmail.com

T. YOKOTA
Department of Mathematics
Tokyo University of Science
e-mail: yokota@rs.tus.ac.jp