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Mathematical analysis of recent analytical approximations to the collapse of an empty spherical bubble

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We analyze the Rayleigh equation for the collapse of an empty bubble and provide an explanation for some recent analytical approximations to the model. We derive the form of the singularity at the second boundary point and discuss the convergence of the approximants. We also give a rigorous proof of the asymptotic behavior of the coefficients of the power series that are the basis for the approximate expressions. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4793217]

I. INTRODUCTION

Hydrodynamic cavitation plays a major role in many technological applications and industrial systems (see, for example the summary in Refs. 1 and 2 and the references therein). For this reason there has been great interest in experiments where spherical bubbles collapse in microgravity environments.[1–3] It has been shown that the remarkably simple and well-known Rayleigh equation for an empty bubble[4] proves suitable for fitting experimental data in the case of millimetric bubbles in extended water volumes.[1,3]

In a recent paper Obreschkow *et al.*[3] derived simple and accurate analytical approximations to the solution of the Rayleigh equation in order to have a better insight into the mathematical nature of the collapse of a spherical bubble. Their approximants are based on the expansion of the solution about the origin of time and can be improved systematically. They showed that those simple analytical expressions are suitable for the analysis of cavitation data obtained in microgravity.

Each approximant is the partial sum of the power series times a function that takes into account the algebraic singularity at the other boundary point. The authors also derived an approximate limit of the sequence of partial sums in terms of the polylogarithm or Jonquière's function. To this end, they resorted to a linear fit of the logarithm of the expansion coefficients.

The results obtained by Obreschkow *et al.*[3] are partly analytical and partly numerical. In this paper we analyze them in a somewhat more rigorous way with the purpose of providing a sound analytical foundation and explanation of the main expressions.

II. APPROXIMATE SOLUTIONS

The Rayleigh equation for the motion of a collapsing bubble is

$$\frac{3}{2}\left[\frac{dR(T)}{dT}\right]^2 + R(T)\frac{d^2R(T)}{dT^2} + k = 0,$$
$$R(0) = R_0, \ \left.\frac{dR}{dT}\right|_{T=0} = 0, \tag{1}$$

where $R(T)$ is the radius of the bubble at time T and R_0 is the initial radius. The last term reads $k = [p_\infty - p(R)]/\rho$, where ρ is the liquid density, p_∞ is the pressure at infinity, and $p(R)$ is the pressure in the liquid at the bubble interface.[5,6] If we neglect liquid compressibility, surface tension, viscosity, and thermal effects[5,6] we can consider k to be a constant.[1,3] The bubble is supposed to collapse at $T = T_c$: $R(T_c) = 0$.

In order to facilitate the calculation it is convenient to convert the Rayleigh equation (1) into a dimensionless differential equation in terms of the new independent and dependent variables $t = T/T_c$ and $r = R/R_0$, respectively. We thus have[3]

$$r(t)\ddot{r}(t) + \frac{3}{2}\dot{r}(t)^2 + \xi^2 = 0,$$
$$r(0) = 1, \ \dot{r}(0) = 0, \tag{2}$$

where

$$\xi^2 = \frac{kT_c^2}{R_0^2}. \tag{3}$$

The collapse of the bubble is now given by the condition $r(1) = 0$.

Taking into account the initial conditions and the fact that $r(-t)$ is also a solution we conclude that $r(-t) = r(t)$. If we multiply Eq. (2) by $r^2\dot{r}$ and integrate with respect to the dimensionless time t we obtain

$$\frac{3}{2}r^3\dot{r}^2 + \xi^2(r^3 - 1) = 0. \tag{4}$$

Note that $r(t) \equiv 1$ is a solution to Eq. (4) that satisfies the boundary conditions at $t = 0$. Since this solution does not satisfy Eq. (2) then both equations are not identical. If we solve Eq. (4) for dt/dr and integrate between $r = 0$ and $r = 1$

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we obtain the value of ξ:[4]

$$\xi = \sqrt{\frac{3}{2}} \int_0^1 \frac{r^{3/2}\, dr}{\sqrt{1-r^3}} = \sqrt{\frac{3\pi}{2}} \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} \approx 0.914681. \tag{5}$$

If, on the other hand, we solve Eq. (4) for $\dot{r}^2$ and differentiate the result with respect to t we obtain another useful equation[3]

$$\ddot{r} = -\frac{\xi^2}{r^4}. \tag{6}$$

The solution to the Rayleigh equation (2) can be expanded in a Taylor series about the origin as follows:

$$r(t) = \sum_{j=0}^{\infty} c_j t^{2j} = 1 - \frac{\xi^2 t^2}{2} - \frac{\xi^4 t^4}{6} - \frac{19\xi^6 t^6}{180} - \frac{59\xi^8 t^8}{720} - \frac{4571\xi^{10} t^{10}}{64800} - \cdots. \tag{7}$$

Although Eqs. (2), (4) and (6) are not identical this series can be obtained from any of them and it converges for all $0 \leq t \leq 1$ as discussed below.

Taking into account the initial conditions, the behavior of $r(t)$ about $t = 1$ and the symmetry of the solution Obreschkow *et al.*[3] obtained the first and simplest approximant $r_0(t) = (1 - t^2)^{2/5}$. This expression is considerably accurate in a neighborhood of $t = 0$ because $\ddot{r}_0(0) = -0.4$ is quite close to $c_1 \approx -0.418$. In addition to it, the authors found that the error of this expression is smaller than 1% for all t. For this reason they proposed the modified power-series approximants

$$r_n(t) = (1-t^2)^{2/5} S_n(t),$$
$$S_n(t) = \sum_{j=0}^{n} a_j t^{2j} = 1 + \frac{4-5\xi^2}{10} t^2 + \frac{42 - 30\xi^2 - 25\xi^4}{150} t^4 + \ldots, \tag{8}$$

where the functions $S_n(t)$ are the partial sums for the Taylor expansion of $S(t) = r(t)/r_0(t)$ about $t = 0$. Numerical calculation suggests that $a_j < 0$ and $|a_{j+1}| < |a_j|$ for all j. Based on these results Obreschkow *et al.*[3] concluded that the approximants $r_n(t)$ converge monotonically towards $r_\infty(t)$ as $n \to \infty$. The accuracy of these approximants increases with n, but according to Obreschkow *et al.*[3] $r_\infty(t)$ is not identical to $r(t)$ because $\ddot{r}(t)$ in Eq. (6) and $\ddot{r}_\infty(t)$ derived from Eq. (8) do not obey the same asymptotic behavior as $t \to 1$.

Obreschkow *et al.*[3] realized that $\ln(a_j)$ vs $\ln(j)$ is an almost straight line from which they estimated that $a_j \approx a_1 j^{-2.21}$. Based on this approximate relationship they derived the following quite accurate analytical approximation to $r(t)$:

$$r_*(t) = (1-t^2)^{2/5}[1 + a_1 Li_{2.21}(t^2)], \tag{9}$$

where $Li_s(z) = \sum_{j=1}^{\infty} z^j/j^s$ is the polylogarithm or Jonquière's function.

III. MATHEMATICAL ANALYSIS OF THE APPROXIMATIONS

In what follows we will discuss the following points: first, why $r_0(t)$ and the approximants of greater order $r_n(t)$ are so accurate, second, if $r_\infty(t)$ is equivalent to $r(t)$ for all t, and third, why $r_\infty(t)$ is approximately given by Eq. (9). In order to answer these questions we need the actual behavior of $r(t)$ as $t \to 1$.

We can obtain the asymptotic behavior of $r(t)$ as $t \to 1$ most easily from Eq. (4); the result is

$$r(t) = \frac{[180\xi(1-t)]^{2/5}}{6} - \frac{5 \times 750^{1/5}[\xi(1-t)]^{8/5}}{66} + O((1-t)^{14/5}). \tag{10}$$

It is worth noting that the leading term

$$r_a(t) \equiv \frac{[180\xi(1-t)]^{2/5}}{6} \approx 1.28371(1-t)^{2/5} \tag{11}$$

is an exact solution to Eq. (6) that does not satisfy the initial conditions. The function (11) does not satisfy the other two alternative equations (2) and (4).

If we substitute $r(t) = r_0(t)S(t)$ into either of the equations (2) and (4), or (6) and take the limit $t \to 1^-$ then we obtain

$$S(1) = \frac{(90\xi)^{2/5}}{6} \approx 0.972867 \tag{12}$$

that is consistent with the analytical asymptotic expression (10) as follows from

$$\lim_{t \to 1^-} \frac{r(t)}{r_0(t)} = \lim_{t \to 1^-} \frac{r_a(t)}{r_0(t)} = \frac{(90\xi)^{2/5}}{6}. \tag{13}$$

Obreschkow *et al.*[3] already proved that $r_0(t)$ is a reasonably good approximation to $r(t)$ in the neighborhood of $t = 0$. Equation (13) tells us that $r_0(t)$ is also quite close to $r(t)$ in the neighborhood of $t = 1$. For this reason $r_0(t)$ is so accurate for all t and the approach of Obreschkow *et al.*[3] is remarkably successful even when the sequence of partial sums $S_n(t)$ converges slowly.

Let us now go into the question whether $r_n(t)$ actually gives $r(t)$ when $n \to \infty$. To begin with, note that the sequence of partial sums converges for all $t < 1$ because the singular point closest to the origin is located at $t = 1$. Therefore it is clear that if $S_n(1)$ converges towards $S(1)$ as $n \to \infty$ then $S_n(t)$ converges towards $S(t)$ for all t and $r_n(t)$ converges towards the actual solution $r(t)$ of the dimensionless Rayleigh equation. Our numerical analysis suggests that $S_n(1) \to S(1)$ as $n \to \infty$; compare, for example, $S_{200}(1) \approx 0.972892$ with Eq. (12). If we accept that $S_\infty(1) = S(1)$ then we can easily prove that $r_\infty(t)$ satisfies any of the equations (2) and (4), or (6) as $t \to 1$. Consider, for example, Eq. (6). If we substitute $r(t) = r_0(t)S(t)$ then $\lim_{t\to 1^-} r(t)^4 \ddot{r}(t) = -24S(1)^5/25 = -\xi^2$. On the other hand $\lim_{t \to 1^-} r_n(t)^4 \ddot{r}_n(t) = -24S_n(1)^5/25$ which proves the point. Therefore, if $r_\infty(t)$ satisfies Eq. (6) for the most unfavorable case $t = 1$ then it satisfies that equation for all t.

Approximant (9) is quite accurate in the neighborhood of $t = 1$ because

$$\lim_{t\to 1}\frac{r_*(t)}{(1-t)^{2/5}} = 2^{2/5}[1 + a_1 Li_{2.21}(1)] \approx 1.28363 \quad (14)$$

is very close to the exact asymptotic behavior given by Eq. (11). In what follows we show how the form of $r_*(t)$ emerges from the asymptotic behavior of the coefficients a_j.

To begin with, note that if $f(x)$ exhibits a branch point at $x = x_0$ with exponent α $(1 - x/x_0)^\alpha$ then the coefficients c_j of the Taylor expansion about $x = 0$ for $f(x)$ behave asymptotically as $|c_j| \sim c|x_0|^{-j}j^{-\alpha-1}$, where c is a constant. Obviously, we are assuming that there is no other singularity closer to the origin or in the vicinity of x_0. For this reason, the coefficients of the original series (7) behave approximately as $c_j \approx cj^{-7/5}$ reflecting the branch-point singularity at $t = 1$ with exponent $\alpha = 2/5$. Note that the Taylor series about $x = 0$ for $f(x)$ converges for all $0 \le x \le x_0$ if $\alpha > 0$ and that the rate of convergence increases with α. The function $S(t)$ exhibits a branch-point singularity at $t = 1$ with exponent $\alpha = 6/5$ as shown by the asymptotic expansion

$$S(t) = \frac{r(t)}{r_0(t)} = \frac{90^{2/5}\xi^{2/5}}{6} + \frac{90^{2/5}\xi^{2/5}}{30}(1-t) - \frac{5\times 6000^{1/5}\xi^{8/5}}{132}(1-t)^{6/5} + O((1-t)^2). \quad (15)$$

Therefore, the coefficients a_j behave asymptotically as $|a_j| \sim aj^{-6/5-1} = aj^{-2.2}$, where a is a constant. This theoretical result clearly explains the outcome of the linear fitting by which Obreschkow *et al.*[3] obtained the approximant $r_*(t)$. The slight discrepancy between the theoretical and numerical exponents is due to the fact that those authors fitted all the coefficients a_j and the asymptotic behavior is determined by those of sufficiently large j. If, for example, we fit the coefficients a_j for $100 \le j \le 150$ then we obtain a much better agreement between theory and numerical approximation: $|a_j| \approx 0.017j^{-2.20}$. Obviously, the reason for fitting all the coefficients is the practical purpose of obtaining a suitable approximation for all t.[3] In the present case we are mainly interested in explaining the form of approximant (9) and for that reason we resort to fitting the coefficients with the largest available orders that reflect the asymptotic behavior of $r(t)$ close to $t = 1$. We also appreciate that the sequence of partial sums $S_n(t)$ converges for all $0 \le t \le 1$ because $1 + \alpha = 11/5 > 1$ and that the rate of convergence of the series (8) is greater than the one of (7).

Finally, we want to discuss an alternative power series with much better convergence properties. It is well known that in some cases the inverted series exhibits better convergence properties than the original one.[7] The series inversion is the basis for the parametric perturbation theory.[8] In the present case we define the new variable ρ

$$\rho = \frac{r-1}{c_1} = \frac{2}{\xi^2}(1-r) = z + \frac{\xi^2 z^2}{3} + \frac{19\xi^4 z^3}{90} + \frac{59\xi^6 z^4}{360} + \frac{4571\xi^8 z^5}{32\,400} + \ldots, \quad (16)$$

where $z = t^2$, and invert the series to obtain $z(\rho)$:

$$z = \rho + \sum_{j=2}^{\infty} b_j\rho^j = \rho - \frac{\rho^2\xi^2}{3} + \frac{\rho^3\xi^4}{90} + \frac{\rho^4\xi^6}{360} + \frac{7\rho^5\xi^8}{10800} + \ldots. \quad (17)$$

It follows from the asymptotic expansion

$$t = 1 - \frac{\sqrt{6}r^{5/2}}{5\xi} - \frac{\sqrt{6}r^{11/2}}{22\xi} + \ldots \quad (18)$$

that $z(\rho)$ exhibits a singularity of the form $(1 - \rho/\rho_0)^{5/2}$, where $\rho_0 = 2/\xi^2 \approx 2.3905$. Therefore, the coefficients b_j behave asymptotically as $|b_j| \sim b|\rho_0|^{-j}j^{-7/2}$, where b is a positive constant. It follows from fitting $\ln(|b_j|)$ for $80 \le j \le 100$ that $|b_j| \sim 1.78 \times 2.39^{-j} \times j^{-3.6}$, where $3.6 \approx 7/2$ and $2.39 \approx 2/\xi^2$ which confirm the theoretical result.

Clearly, the coefficients of the inverted series decrease more rapidly than the coefficients of either (7) or (8). Therefore, from a numerical point of view it is convenient to build approximants based on the inverted series. The price we have to pay is that the inverted series does not yield $r(t)$ directly, which may not be a serious drawback for some purposes. We can improve the convergence of the inverted series by means of an appropriate summation method such as the Padé approximants and thus obtain $t(\rho) = \sqrt{z(\rho)}$ which together with $r(\rho) = 1 - \frac{\xi^2\rho}{2}$ yields the parametric representation for $r(t)$.

IV. CONCLUSIONS

Summarizing: the most important results of this paper are

- The particular form and remarkable accuracy of the approximation $r_*(t)$ of Obreschkow *et al.* is now explained by the fact that the coefficients a_j behave asymptotically as $j^{-11/5}$ for large j.
- Present analysis strongly suggests that $r_\infty(t)$ is identical to $r(t)$ for all $0 \le t \le 1$ in disagreement with the statement of Obreschkow *et al.*
- Alternative approximations to $r(t)$ in terms of the inverted power series exhibit faster convergence than the approach of Obreschkow *et al.*, although the inverted power series does not yield $r(t)$ directly.

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