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Cover Page Footnote

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Tiling Rectangles and 2-Deficient Rectangles with L-Pentominoes

By *Monica Kane*

Abstract. We investigate tiling rectangles and 2-deficient rectangles with L-pentominoes. First, we determine exactly when a rectangle can be tiled with L-pentominoes. We then determine locations for pairs of unit squares that can always be removed from an $m \times n$ rectangle to produce a tileable 2-deficient rectangle when $m \equiv 1 \pmod 5$, $n \equiv 2 \pmod 5$ and when $m \equiv 3 \pmod 5$, $n \equiv 4 \pmod 5$.

1 Introduction

A *polyomino* is a two-dimensional shape made of one or more unit squares connected at their edges, as in Figure 1. The shapes used in the game Tetris, for example, are polyominoes made of four squares (called tetrominoes), and the game involves trying to fit the pieces together, almost like a jigsaw puzzle, as they fall in the space provided.

The game of Tetris can lead to further puzzles to explore: Given a rectangle of a particular size, could it be completely filled with Tetris pieces? Or, if the rectangle could be almost filled, but there were a hole or two inside, at what locations could the holes be? What if we use polyominoes made of five squares (called pentominoes) instead of four? There is mathematics behind puzzles like this which can be studied.

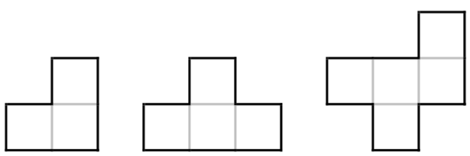

Figure 1: Some polyominoes.

The goal of such a puzzle is to tile a region. We say that polyominoes *tile* a region (that is, a region is *tileable*) when the region can be covered by polyominoes such that a) every unit square in the region is covered by a unit square of exactly one polyomino, and b) every unit square of the polyominoes used covers a unit square in the region.

Polyominoes were first introduced to mathematics by Solomon W. Golomb [4] in 1954. Ash and Golomb [2] and Chu and Johnsonbaugh [3] discuss tiling rectangles and 1-deficient rectangles, which are missing 1 unit square, with L-trominoes (made of 3 unit squares), and Nitica [6] studies the problem for L-tetrominoes. There is less available

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research about tiling 2-deficient rectangles, which are missing 2 unit squares, although Aanjaneya [1] discusses this topic for trominoes, as do Ash and Golomb [2] briefly.

Since much of the existing research focuses on trominoes and tetrominoes, we will study tiling with a different polyomino, the L-pentomino, shown in Figure 2. Furthermore, since there is little available research on tiling 2-deficient rectangles, we will focus on using the L-pentomino to tile rectangles and 2-deficient rectangles. An example of such a tiling is given in Figure 3.

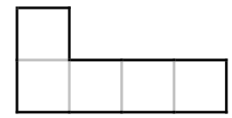

Figure 2: The L-pentomino.

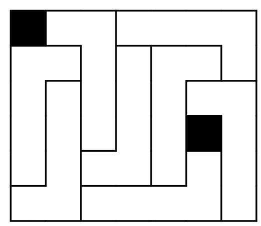

Figure 3: A 2-deficient rectangle tiled with L-pentominoes.

In section 2, we will discuss tiling rectangles with L-pentominoes and will determine exactly when a rectangle can be tiled with L-pentominoes. In section 3, we will discuss tiling 2-deficient rectangles with L-pentominoes. We will introduce two new approaches for tiling 2-deficient rectangles, and our main focus in both will be on finding pairs of squares whose removal from a rectangle yields a 2-deficient rectangle that can be tiled.

We denote an $m \times n$ rectangle (m rows, n columns) by $R(m,n)$, where m and n are non-negative integers. We denote a 1-deficient $m \times n$ rectangle by $R(m,n)^{-}$, and we denote a 2-deficient $m \times n$ rectangle by $R(m,n)^{--}$. We use (i,j) to refer to the unit square in the ith row and the jth column of a rectangle.

We use 8 orientations of the L-pentomino, which are given and named in Figure 4. Any further mention of pentominoes refers to L-pentominoes, and any further mention of tiling refers to tiling with L-pentominoes.

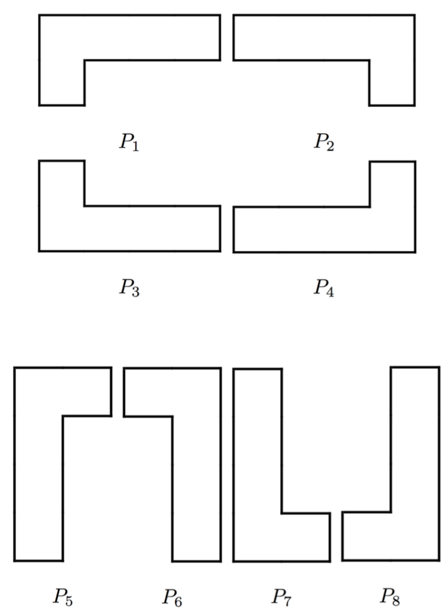

Figure 4: The 8 orientations of the L-pentomino.

2 Tiling Rectangles with L-Pentominoes

A preliminary result is that $R(2,5)$ can be tiled with pentominoes, as shown in Figure 5.

As a result of this, any rectangle of the form $R(2a,5b)$ (or $R(5a,2b)$) can be tiled with pentominoes, since any such rectangle can be divided into ab copies of $R(2,5)$ (or $R(5,2)$). As a result, we have the following lemma.

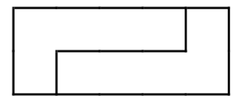

Figure 5: A tiling of $R(2,5)$.

Lemma 2.1. *A rectangle of the form* $R(2a,5b)$ *is tileable.*

If $R(m,n)$ is tileable, then its area must be divisible by 5, the area of a pentomino; that is, $5|mn$. Therefore, m or n must be divisible by 5.

However, meeting this requirement is not enough to guarantee that a rectangle is tileable. Any rectangle with a side length of 1 cannot be tiled, since each L-pentomino is 2 units wide. For rectangles with a side length of 3, we have the following theorem.

Theorem 2.2. $R(3,n)$ *is not tileable for any positive value of* n*.*

We can suppose, to the contrary, that there is a tiling for some rectangle $R(3,n)$. Then some pentomino must cover the top left square, (1,1). There are 3 orientations of the pentomino that can do so: P_1, P_2, and P_3.

If P_3 is used to cover (1,1), then the square (3,1) cannot be covered, as shown in Figure 6.

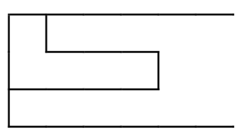

Figure 6: P_3 is used to cover (1,1).

Suppose P_2 is used to cover (1,1). Some pentomino must cover the square (2,1), and P_3 is the only orientation that can do so. But this renders squares (2,2) and (2,3) impossible to cover, as shown in Figure 7.

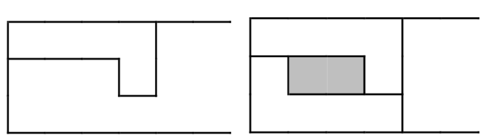

Figure 7: P_2 is used to cover (1,1).

Similarly, suppose P_1 is used to cover (1,1). Some pentomino must cover the square (3,1), and P_4 is the only orientation that can do so. But this renders squares (2,2) and (2,3) again impossible to cover, as shown in Figure 8.

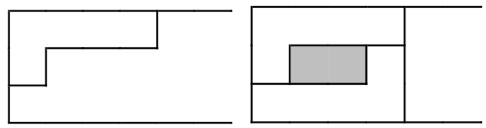

Figure 8: P_1 is used to cover (1,1).

Therefore, $R(3,n)$ cannot be tiled.

We find a similar result when attempting to tile a rectangle with one side length of 5 and the other side length odd.

Theorem 2.3. *$R(5,n)$ is tileable if and only if n is even.*

If n is even, then $R(5,n)$ is of the form $R(5a,2b)$ and therefore is tileable. We now need to show that if n is odd, then $R(5,n)$ is not tileable. Suppose, to the contrary, that there exists a tiling for some $R(5,2k+1)$ with $k \in \mathbb{Z}$. Then some pentomino must cover the square (1,1). Then, by the enumeration of positions as in the previous argument, it can be shown that the only workable option for tiling the leftmost edge of the rectangle is to use $R(5,2)$, as shown in Figures 9 and 10. (See the appendix for the full enumeration of positions for this theorem.)

Then the portion of the rectangle that remains to be tiled is $R(5,2k-1)$. By the same reasoning, the only workable option is to place another $R(5,2)$, reducing the remaining portion of the rectangle to $R(5,2k-3)$. After repeating this process of placing a $R(5,2)$ $k-2$ more times, the remaining portion of the rectangle is $R(5,1)$. By our assumption, a tiling for $R(5,2k+1)$ exists, but $R(5,1)$ cannot be tiled. This is a contradiction.

Thus, if n is odd, there is no tiling for $R(5,n)$.

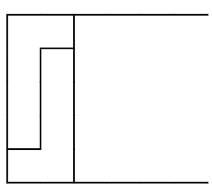

Figure 9: P_5 is used to cover (1,1) and P_8 is used to cover (5,1).

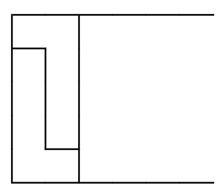

Figure 10: P_6 is used to cover (1,1) and P_7 is used to cover (2,1).

We now have enough information to determine exactly when any $m \times n$ rectangle is tileable. If $m = 0$ or $n = 0$, we say that $R(m,n)$ is trivially tileable. If both m and n are positive, then the following theorem applies.

Theorem 2.4. *Let $m, n > 0$, with $m = 5k$ for some $k \in \mathbb{Z}$.*

- *If n is even, then $R(5k, n)$ is tileable.*
- *If n is odd:*
  - *If $n = 5$, then $R(5k, n)$ is tileable if and only if k is even.*
  - *If $n \neq 5$, then $R(5k, n)$ is tileable if and only if $k \geq 2$ and $n \geq 7$.*

Proof. Suppose n is even. Then $R(5k, n)$ is of the form $R(5a, 2b)$ and can therefore be tiled.

Now suppose n is odd.

Consider $n = 5$. If $R(5k, n)$ is tileable, then by **theorem 2.3**, $5k$ is even, and so k must be even. Conversely, if k is even, then $R(5k, n)$ is of the form $R(2a, 5b)$ and can therefore be tiled.

Consider $n \neq 5$. Suppose $R(5k, n)$ is tileable. Then $k \neq 1$ by **theorem 2.3**, and so $k \geq 2$. Furthermore, $n \neq 1$ since $R(5k, 1)$ cannot be tiled, and $n \neq 3$ by **theorem 2.2**, and so $n \geq 7$. Conversely, suppose $k \geq 2$ and $n \geq 7$. We will show by induction that $R(5k, 7)$ is tileable, and use this to show that $R(5k, n)$ is tileable.

If k is even, then the base case is $R(10, 7)$ (a tiling of $R(7, 10)$ is given in Figure 11). As inductive hypothesis, suppose $R(5(2h), 7)$ can be tiled for some $h \in \mathbb{Z}$. Then we can use a tiling of $R(5(2h), 7)$ and a tiling of $R(10, 7)$ to obtain a tiling of $R(5(2h) + 10, 7) = \mathrm{R}(5(2(h+1)), 7)$.

If k is odd, then the base case is $R(15, 7)$ (a tiling of $R(7, 15)$ is given in Figure 12). As inductive hypothesis, suppose $R(5(2h+1), 7)$ can be tiled for some $h \in \mathbb{Z}$. Then we can use a tiling of $R(5(2h+1), 7)$ and a tiling of $R(10, 7)$ to obtain a tiling of $R(5(2h+1)+10, 7) = \mathrm{R}(5(2(h+1)+1), 7)$.

Therefore, for all $k \geq 2$, $R(5k, 7)$ is tileable.

Now, consider $R(5k, n)$ with $k \geq 2$ and $n \geq 7$. We can decompose this rectangle with vertical cuts into the following subrectangles:

$$R(5k, 7) + m \cdot R(5k, 2)$$

for some $m \geq 0$. $R(5k, 7)$ is tileable by the inductions above, and each copy of $R(5k, 2)$ is tileable by **lemma 2.1**. Thus, $R(5k, n)$ is tileable. □

Now that we know exactly when a rectangle can be tiled with L-pentominoes, we can use these results to study tiling 2-deficient rectangles.

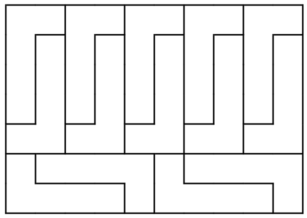

Figure 11: A tiling of $R(7,10)$.

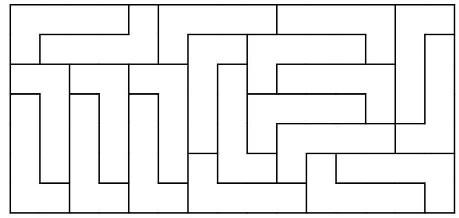

Figure 12: A tiling of $R(7,15)$.

3 Tiling 2-Deficient Rectangles with L-Pentominoes

As before, a region is tileable only if its area is divisible by 5, and so the 2-deficient rectangle $R(m,n)^{--}$ is tileable only if $mn-2 \equiv 0 \pmod 5$. As a result, we will focus on 2-deficient rectangles with $m \equiv 1 \pmod 5$, $n \equiv 2 \pmod 5$ or with $m \equiv 3 \pmod 5$, $n \equiv 4 \pmod 5$. Our goal is to find locations where the missing unit squares can be so that the 2-deficient rectangle is tileable. Pairs of squares in such locations are called *good* [2]. Meanwhile, a pair of squares is called *bad* if its removal from a rectangle produces a 2-deficient rectangle that is impossible to tile [2].

We now focus on showing that certain pairs of squares are good. The following example introduces one approach to this kind of problem.

3.1 Splitting a 2-Deficient Rectangle into Subrectangles

Consider a rectangle of the form $R(1+5k,2+5l)^{--}$, $k \geq 1$, $l \geq 0$, that is missing a horizontal domino from its corner. Without loss of generality, let the horizontal domino be in the upper left corner.

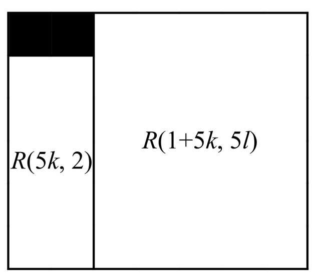

Figure 13: Splitting $R(1+5k,2+5l)^{--}$ into subrectangles when a horizontal domino is missing from its corner.

We can split the rectangle into three pieces: the missing domino $\{(1,1),(1,2)\}$, the rectangle $R(5k,2)$ underneath the domino, and the rectangle $R(1+5k,5l)$, as shown in Figure 13. We denote this splitting by the following equation:

$$R(1+5k,2+5l)^{--} = \{(1,1),(1,2)\} + R(5k,2) + R(1+5k,5l). \tag{1}$$

We now must determine when each of these subrectangles is tileable.

We know that $R(5k,2)$ is always tileable because it is of the form $R(5a,2b)$.

Consider $R(1+5k,5l)$. If k is odd, then $R(1+5k,5l)$ is tileable because it is of the form $R(2a,5b)$. If k is even and $l=1$, then the rectangle is of the form $R(m,5)$ with m odd, which is not tileable by **theorem 2.3**. But if k is even and $l \geq 2$, then $R(1+5k,5l)$ is tileable by **theorem 2.4**, and if k is even and $l=0$, then $R(1+5k,5l)$ is trivially tileable.

Thus, we can conclude that a rectangle of the form $R(1+5k,2+5l)^{--}$, $k \geq 1$, $l \geq 0$, is tileable when missing a horizontal domino from its corner, except when k is even and $l=1$.

Splitting a 2-deficient rectangle into tileable subrectangles in this way is one method of showing that a 2-deficient rectangle is tileable when a certain pair of unit squares is missing. In the following theorem, we use this method in a more generalized way that gives multiple good pairs of squares. This next theorem involves tiling 2-deficient $m \times n$ rectangles with $m \equiv 3 \pmod 5$, $n \equiv 4 \pmod 5$.

Theorem 3.1. *In a rectangle of the form* $R(3+5k,4+5l)^{--}$, $k \geq 1$, $l \geq 0$, *good horizontal dominoes include* $\{(2+5i,2+5j),(2+5i,3+5j)\}$, $0 \leq i \leq k$, $0 \leq j \leq l$, *excluding* $j=1$, $j=l-1$ *when k is even.*

Proof. First, note that $R(3,4)^{--}$ is tileable, as shown in Figure 14.

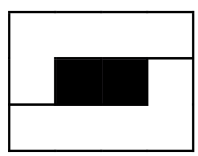

Figure 14: A tiling of $R(3,4)^{--}$.

We define $R_{ij}(3,4)^{--}$ to be the tileable rectangle $R(3,4)^{--}$ positioned (within a rectangle $R(3+5k,4+5l)^{--}$) so that it is missing the domino $\{(2+5i,2+5j),(2+5i,3+5j)\}$, $0 \leq i \leq k$, $0 \leq j \leq l$. Figure 15 shows these different locations for $R_{ij}(3,4)^{--}$ within the rectangle $R(13,19)^{--}$.

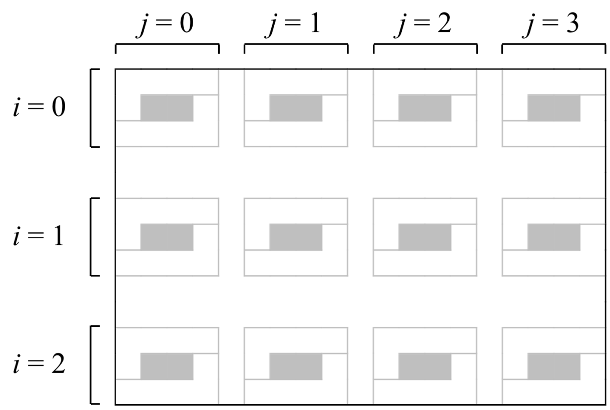

Figure 15: Locations of $R_{ij}(3,4)^{--}$ in $R(13,19)^{--}$.

Let $k \geq 1$ and $l \geq 0$.

When k is odd, we split the rectangle into subrectangles according to the following equation:

$$\begin{aligned} &R(3+5k, 4+5l)^{--} \\ &= R_{ij}(3,4)^{--} + R(5i,4) + R(5(k-i),4) + R(3+5k,5j) + R(3+5k,5(l-j)) \end{aligned} \tag{2}$$

for $0 \leq i \leq k$ and $0 \leq j \leq l$.

This splitting of the rectangle is shown in Figure 16.

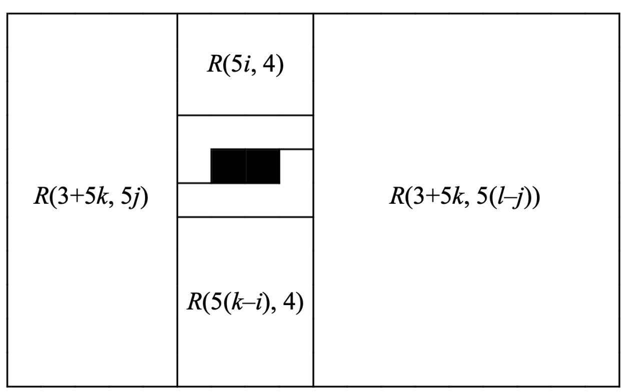

Figure 16: Splitting $R(3+5k, 4+5l)^{--}$ into tileable subrectangles according to Equation 2.

In order to find good locations for the domino, we need to determine when these subrectangles are tileable.

The subrectangles $R(5i,4)$ and $R(5(k-i),4)$ are always tileable since they are of the form $R(5a,2b)$. Since k is odd, then $m = 3+5k$ is even, and so $R(3+5k,5j)$ and $R(3+5k,5(l-j))$ are of the form $R(2a,5b)$ and therefore are tileable.

Thus, $\{(2+5i, 2+5j), (2+5i, 3+5j)\}$ for $0 \le i \le k$, $0 \le j \le l$ are good horizontal dominoes when k is odd.

When k is even, $R(3+5k, 4+5l)^{--}$ can be split into tileable subrectangles as in Equation 2 but with two exceptions. As before, $R(5i, 4)$ and $R(5(k-i), 4)$ are tileable since they are of the form $R(5a, 2b)$. However, since k is even, then $m = 3+5k$ is odd. Then $R(3+5k, 5j)$ is tileable when $j = 0$ or when $j \ge 2$ and is not tileable only when $j = 1$ (by **theorem 2.4**). Similarly, $R(3+5k, 5(l-j))$ is tileable when $l-j = 0$ or when $l-j \ge 2$ and is not tileable only when $j = l-1$ (by **theorem 2.4**). As a result, we exclude the cases $j = 1$ and $j = l-1$ when k is even.

Thus, $\{(2+5i, 2+5j), (2+5i, 3+5j)\}$ for $0 \le i \le k$, $0 \le j \le l$, excluding dominoes with $j = 1$ or $j = l-1$ when k is even, are good horizontal dominoes. □

It is important to note that excluding these dominoes from our set of good dominoes does not necessarily mean that these dominoes are bad, since a 2-deficient rectangle that is missing one of these dominoes may be tileable some other way. However, we exclude them because this particular splitting does not prove that these dominoes are good.

We can use a similar method to study tiling 2-deficient $m \times n$ rectangles with $m \equiv 1 \pmod 5$, $n \equiv 2 \pmod 5$.

Theorem 3.2. *In a rectangle of the form* $R(1+5k, 2+5l)^{--}$, $k \ge 1$, $l \ge 0$, *good horizontal dominoes include* $\{(1+5i, 1+5j), (1+5i, 2+5j)\}$ *and* $\{(6+5i, 1+5j), (6+5i, 2+5j)\}$, $0 \le i \le k-1$, $0 \le j \le l$, *excluding the following cases when both k is even and l is odd:*

- $i = 1$, $j = 1$
- $i = 1$, $j = l-1$
- $i = (k-1)-1$, $j = 1$
- $i = (k-1)-1$, $j = l-1$.

Proof. First, note that $R(6,2)^{--}$ is tileable, as in Figures 17 and 18.

Define $R_{Aij}(6,2)^{--}$ to be the tileable rectangle $R_A(6,2)^{--}$, shown in Figure 17, positioned (in a rectangle $R(1+5k, 2+5l)^{--}$) so that it is missing the domino $\{(1+5i, 1+5j), (1+5i, 2+5j)\}$, $0 \le i \le k-1$, $0 \le j \le l$.

Define $R_{Bij}(6,2)^{--}$ to be the tileable rectangle $R_B(6,2)^{--}$, shown in Figure 18, positioned (in a rectangle $R(1+5k, 2+5l)^{--}$) so that it is missing the domino $\{(6+5i, 1+5j), (6+5i, 2+5j)\}$, $0 \le i \le k-1$, $0 \le j \le l$.

Note that $R_{Aij}(6,2)^{--}$ and $R_{Bij}(6,2)^{--}$ with the same values of i and j occupy the same position in $R(1+5k, 2+5l)^{--}$; only the locations of the missing domino are different.

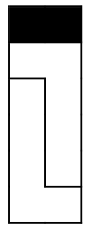

Figure 17: A tiling of $R_A(6,2)^{--}$.

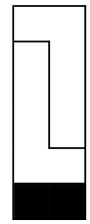

Figure 18: A tiling of $R_B(6,2)^{--}$.

Let $k \geq 1$ and $l \geq 0$.

This time, we will consider two different ways of splitting the rectangle into subrectangles.

The first option is to split the rectangle into subrectangles according to the following equation:

$$\begin{aligned} &R(1+5k, 2+5l)^{--} \\ &= R_{ij}(6,2)^{--} + R(5i,2) + R(5(k-1-i),2) + R(1+5k,5j) + R(1+5k,5(l-j)) \end{aligned} \tag{3}$$

for $0 \leq i \leq k-1$ and $0 \leq j \leq l$, where $R_{ij}(6,2)^{--}$ represents either tiling, $R_{Aij}(6,2)^{--}$ or $R_{Bij}(6,2)^{--}$.

This first splitting is shown in Figure 19.

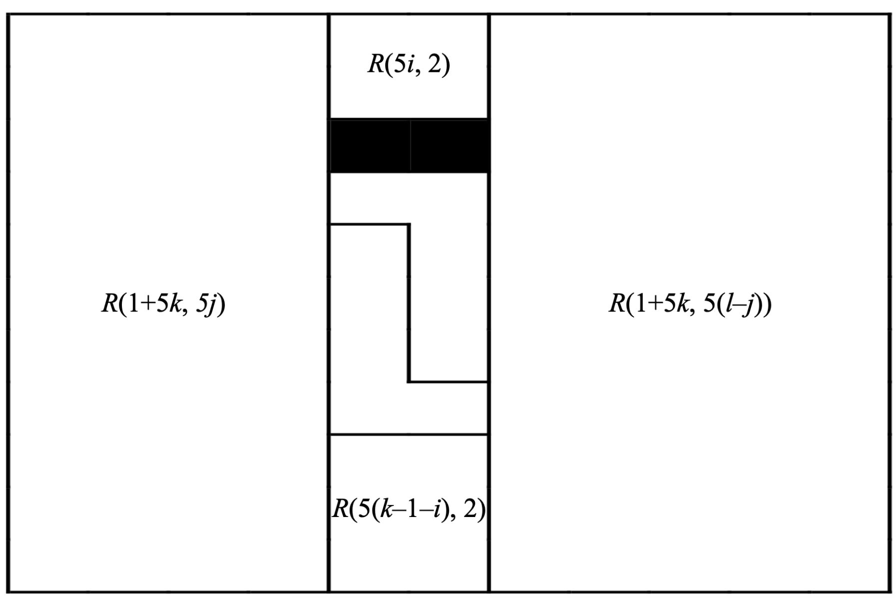

Figure 19: Splitting $R(1+5k, 2+5l)^{--}$ vertically into subrectangles according to Equation 3.

When k is odd, all of the subrectangles in Figure 19 are of the form $R(2a,5b)$ or $R(5a,2b)$ and therefore are tileable by **lemma 2.1**.

Thus, $\{(1+5i,1+5j),(1+5i,2+5j)\}$ and $\{(6+5i,1+5j),(6+5i,2+5j)\}$, $0 \le i \le k-1$, $0 \le j \le l$ are good dominoes when k is odd.

The second option is to split the rectangle into subrectangles according to the following equation:

$$\begin{aligned} &R(1+5k,2+5l)^{--} \\ &= R_{ij}(6,2)^{--} + R(5i,2+5l) + R(5(k-1-i),2+5l) + R(6,5j) + R(6,5(l-j)) \end{aligned} \tag{4}$$

for $0 \le i \le k-1$ and $0 \le j \le l$, where $R_{ij}(6,2)^{--}$ represents either tiling, $R_{Aij}(6,2)^{--}$ or $R_{Bij}(6,2)^{--}$.

This second splitting is shown in Figure 20.

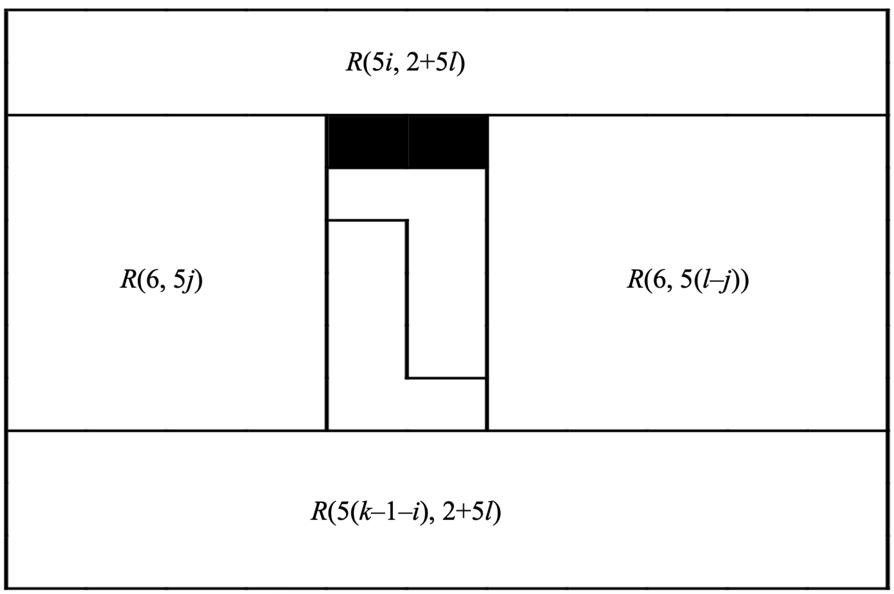

Figure 20: Splitting $R(1+5k,2+5l)^{--}$ horizontally into subrectangles according to Equation 4.

When l is even, all of the subrectangles in Figure 20 are of the form $R(2a,5b)$ or $R(5a,2b)$ and therefore are tileable by **lemma 2.1**.

Thus, $\{(1+5i,1+5j),(1+5i,2+5j)\}$ and $\{(6+5i,1+5j),(6+5i,2+5j)\}$, $0 \le i \le k-1$, $0 \le j \le l$ are good dominoes when l is even.

When both k is even and l is odd, $R(1+5k,2+5l)^{--}$ can be split into tileable subrectangles as in Figure 19 but with two exceptions. Since k is even, $R(1+5k,5j)$ is tileable when $j=0$ or when $j \ge 2$ and is not tileable only when $j=1$ (by **theorem 2.4**). Similarly, $R(1+5k,5(l-j))$ is tileable when $l-j=0$ or when $l-j \ge 2$ and is not tileable only when

$j = l-1$ (by **theorem 2.4**). In a similar way, we can split $R(1+5k, 2+5l)^{--}$ as in Figure 20, but again there are two exceptions. Since l is odd, $R(5i, 2+5l)$ is not tileable only when $i = 1$, and $R(5(k-1-i), 2+5l)$ is not tileable only when $i = (k-1)-1$ (by **theorem 2.4**).

Therefore, when these conditions on i and j occur simultaneously, neither Equation 3 nor Equation 4 provides a way to split $R(1+5k, 2+5l)^{--}$ into tileable subrectangles. Then, when both k is even and l is odd, we exclude the following cases:

- $i = 1,\ j = 1$
- $i = 1,\ j = l-1$
- $i = (k-1)-1,\ j = 1$
- $i = (k-1)-1,\ j = l-1.$

The excluded locations for $R_{ij}(6,2)^{--}$ are shown in Figure 21.

Thus, $\{(1+5i, 1+5j), (1+5i, 2+5j)\}$ and $\{(6+5i, 1+5j), (6+5i, 2+5j)\}$ for $0 \le i \le k-1$, $0 \le j \le l$, excluding the four cases listed above when both k is even and l is odd, are good dominoes. □

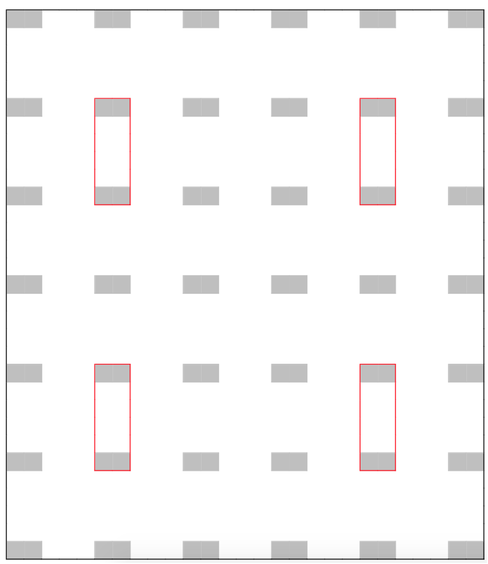

Figure 21: Excluded locations (outlined) for $R_{ij}(6,2)^{--}$ in $R(31,27)^{--}$. (Shown in gray are the horizontal dominoes $\{(1+5i, 1+5j), (1+5i, 2+5j)\}$ and $\{(6+5i, 1+5j), (6+5i, 2+5j)\}$ for $0 \le i \le 5$ and $0 \le j \le 5$.)

Thus far, we have discussed only domino-deficient rectangles, but we can use this same splitting technique to tile generally 2-deficient rectangles, whose two missing squares may or may not form a domino.

For example, to find good pairs of squares for generally 2-deficient rectangles with $m \equiv 1$, $n \equiv 2 \pmod 5$, we can recreate the previous proof using tilings of a generally 2-deficient $R(6,2)^{--}$ (such as $R_C(6,2)^{--}$ and $R_D(6,2)^{--}$, shown in Figures 22 and 23).

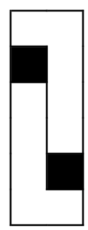

Figure 22: A tiling of $R_C(6,2)^{--}$.

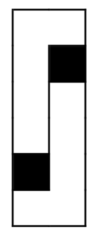

Figure 23: A tiling of $R_D(6,2)^{--}$.

To find good pairs of squares for generally 2-deficient rectangles with $m \equiv 3$, $n \equiv 4 \pmod 5$, we can use tilings such as those for a generally 2-deficient $R(8,4)^{--}$ (such as $R_A(8,4)^{--}$ and $R_B(8,4)^{--}$, shown in Figures 24 and 25) and determine the splitting equations for those rectangles.

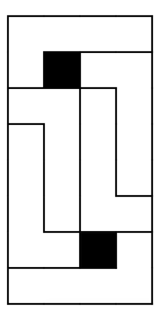

Figure 24: A tiling of $R_A(8,4)^{--}$.

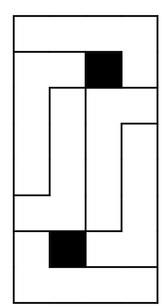

Figure 25: A tiling of $R_B(8,4)^{--}$.

3.2 Using Two 1-Deficient Rectangles

While this splitting technique is useful, it is not the only method for tiling 2-deficient rectangles. Another approach is to use two tileable 1-deficient rectangles. For example, Figure 26 shows a tiling of the 1-deficient rectangle $R(6,6)^-$. From this tiling we have 8 different locations for the missing unit square, given in Figure 27, which are obtained from rotations and reflections of the tiling of $R(6,6)^-$. Using two copies of $R(6,6)^-$, we can obtain the 2-deficient rectangle $R(6,12)^{--}$. Figure 28 shows the pairs of missing squares in $R(6,12)^{--}$ that we know to be good: Any 1 square from the leftmost $R(6,6)^-$ and any 1 square from the rightmost $R(6,6)^-$ is a good pair.

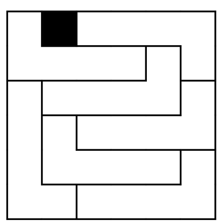

Figure 26: A tiling of $R(6,6)^-$.

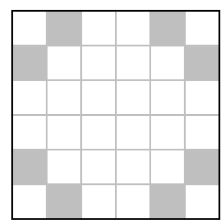

Figure 27: Some good unit squares in $R(6,6)^-$.

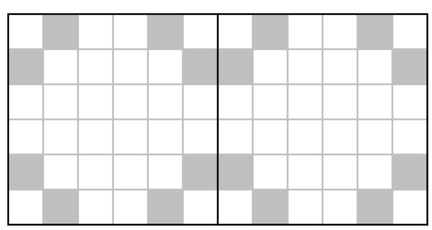

Figure 28: Some good pairs of unit squares in $R(6,12)^{--}$.

We can use other tileable 1-deficient rectangles to get results in the same way. Figure 29 shows five tilings of the 1-deficient rectangle $R(9,9)^-$. From these tilings we have 21 different locations for the missing unit square, given in Figure 30, which are obtained from rotations and reflections of the tilings of $R(9,9)^-$. Using two copies of $R(9,9)^-$, we can obtain the 2-deficient rectangle $R(9,18)^{--}$. Figure 31 shows the pairs of missing squares in $R(9,18)^{--}$ that we know to be good: Any 1 square from the leftmost $R(9,9)^-$ and any 1 square from the rightmost $R(9,9)^-$ is a good pair.

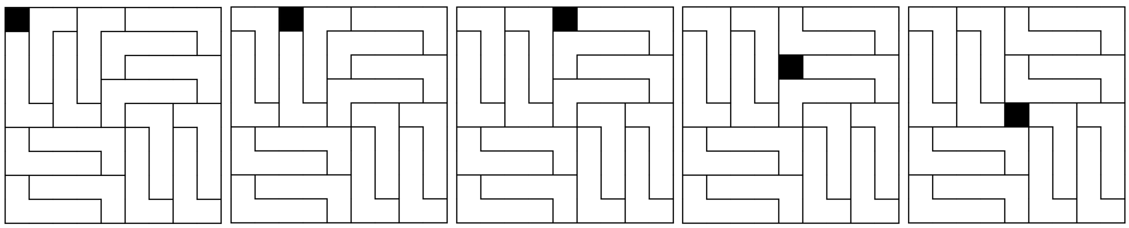

Figure 29: Tilings of $R(9,9)^-$.

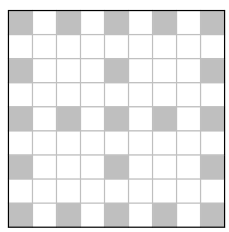

Figure 30: Some good unit squares in $R(9,9)^-$.

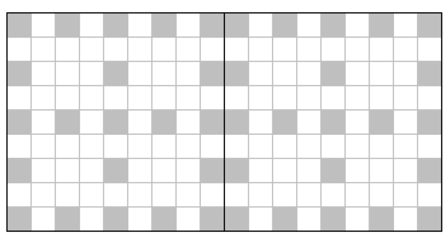

Figure 31: Some good pairs of unit squares in $R(9,18)^{--}$.

In this way, given a tiling of a 1-deficient rectangle $R(m,n)^-$, we can obtain good pairs of squares in the 2-deficient rectangles $R(m,2n)^{--}$ and $R(2m,n)^{--}$.

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4 Appendix

Theorem 2.3. $R(5,n)$ is tileable if and only if n is even.

Proof. If n is even, then $R(5,n)$ is a rectangle of the form $R(5a,2b)$ and therefore is tileable.

We now need to show that if n is odd, then $R(5,n)$ is not tileable. Suppose, on the contrary, that there exists a tiling for some $R(5,2k+1)$ with $k \in \mathbb{Z}$. Then some pentomino must cover the square (1,1).

The pentominoes P_4 and P_8 cannot be used to cover (1,1) due to their orientation. Any of the other 6 L-pentominoes could potentially cover (1,1), and so we have the following 6 cases.

Case 1: Suppose P_1 is used to cover (1,1), as in Figure 32. Some pentomino must cover the square (3,1).

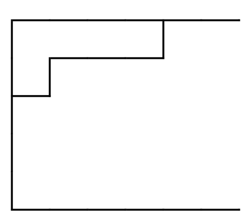

Figure 32: P_1 is used to cover (1,1).

Suppose P_1 or P_2 is used to cover (3,1). Then some pentomino must cover the square (5,1), which renders squares (4,2), (4,3) unable to be covered. See Figures 33 and 34.

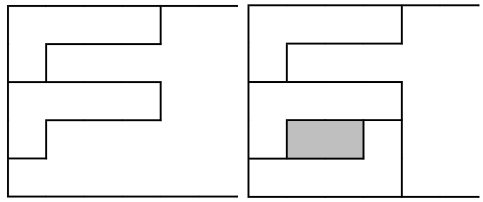

Figure 33: P_1 is used to cover (3,1).

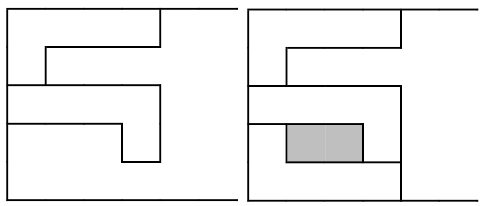

Figure 34: P_2 is used to cover (3,1).

If P_3 is used to cover (3,1), then square (5,1) cannot be covered, as shown in Figure 35.

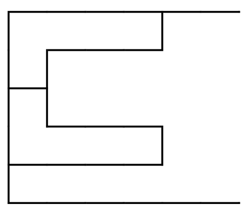

Figure 35: P_3 is used to cover (3,1).

If P_4 is used to cover (3,1), then squares (2,2), (2,3) cannot be covered, as shown in Figure 36.

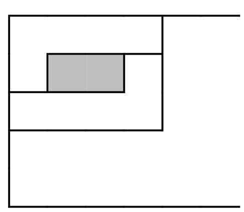

Figure 36: P_4 is used to cover (3,1).

Therefore, the pentomino P_1 must not be used to cover square (1,1).

Case 2: Suppose P_2 is used to cover (1,1), as in Figure 37. Some pentomino must cover square (2,1).

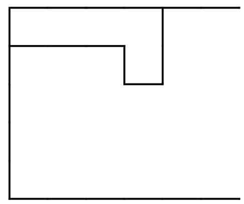

Figure 37: P_2 is used to cover (1,1).

If P_3 is used to cover (2,1), then squares (2,2), (2,3) cannot be covered, as shown in Figure 38.

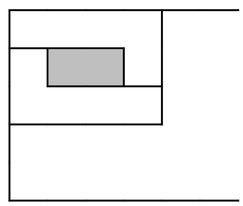

Figure 38: P_3 is used to cover (2,1).

Suppose P_5 is used to cover (2,1), as in Figure 39. Some pentomino must cover (3,2).

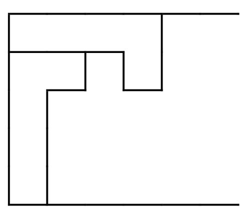

Figure 39: P_5 is used to cover (2,1).

If P_1, P_2, or P_4 is used to cover (3,2), then square (2,3) cannot be covered, as shown in Figure 40.

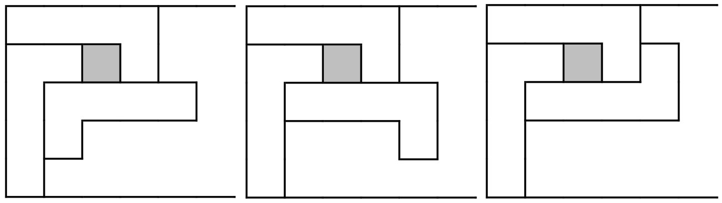

Figure 40: P_1, P_2, or P_4 is used to cover (3,2).

If P_3 is used to cover (3,2), then square (5,2) cannot be covered, as shown in Figure 41.

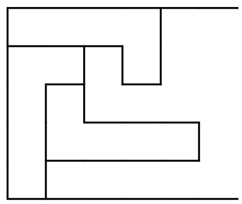

Figure 41: P_3 is used to cover (3,2).

If P_6 is used to cover (2,1), then squares (3,1), (4,1), (5,1) cannot be covered, as shown in Figure 42.

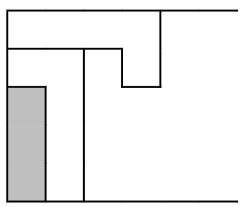

Figure 42: P_6 is used to cover (2,1).

Suppose P_7 is used to cover (2,1). Then P_3 must be used to cover (2,2), which renders (2,3) unable to be covered. See Figure 43.

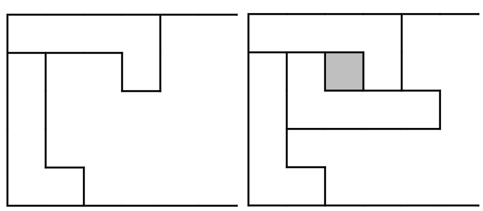

Figure 43: P_7 is used to cover (2,1).

Therefore, P_2 must not be used to cover (1,1).

Case 3: Suppose P_3 is used to cover (1,1), as in Figure 44. Some pentomino must cover square (3,1).

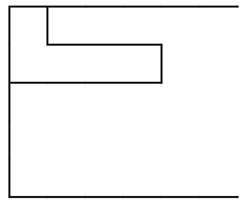

Figure 44: P_3 is used to cover (1,1).

Suppose P_1 is used to cover (3,1). Then P_4 must be used to cover (5,1), which renders (4,2) and (4,3) unable to be covered, as shown in Figure 45.

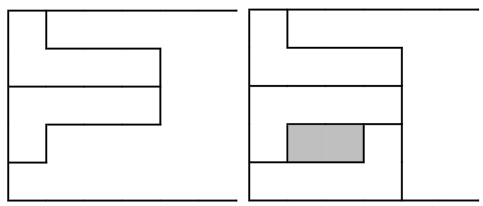

Figure 45: P_1 is used to cover (3,1).

Suppose P_2 is used to cover (3,1). Then P_3 must be used to cover (5,1), which renders (4,2) and (4,3) again unable to be covered, as shown in Figure 46.

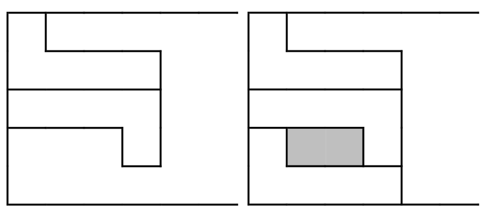

Figure 46: P_2 is used to cover (3,1).

If P_3 is used to cover (3,1), then (5,1) cannot be covered, as shown in Figure 47.

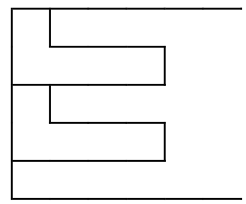

Figure 47: P_3 is used to cover (3,1).

Therefore, P_3 must not be used to cover (1,1).

Case 4: Suppose P_7 is used to cover (1,1), as in Figure 48. Then P_4 must cover (5,1). By horizontal symmetry, this case is equivalent to the case in Figure 39, and so we may omit it here.

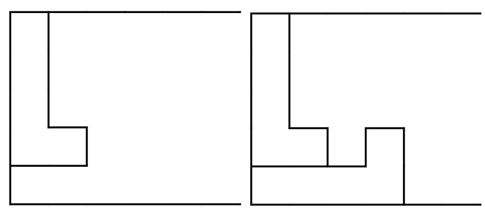

Figure 48: P_7 is used to cover (1,1).

Case 5: Suppose P_5 is used to cover (1,1), as in Figure 49. Some pentomino must cover (5,1).

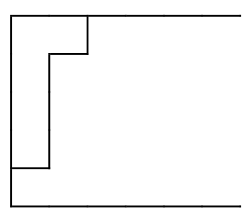

Figure 49: P_5 is used to cover (1,1).

Suppose P_4 is used to cover (5,1), as in Figure 50. By horizontal symmetry, this case is equivalent to the case in Figure 43, and so we may omit it here.

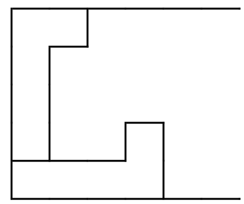

Figure 50: P_4 is used to cover (5,1).

Suppose P_8 is used to cover (5,1), as in Figure 51. By horizontal symmetry, this case is equivalent to the only other remaining case, Case 6, in which P_6 is used to cover (1,1) and P_7 is used to cover (2,1), as in Figure 52.

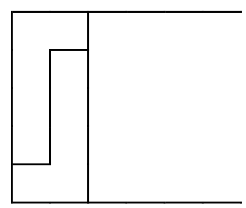

Figure 51: P_5 is used to cover (1,1) and P_8 is used to cover (5,1).

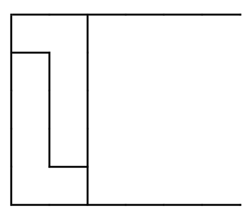

Figure 52: P_6 is used to cover (1,1) and P_7 is used to cover (2,1).

These are the final cases to consider. Since the original rectangle was $R(5,2k+1)$, then the portion of the rectangle that remains to be tiled is $R(5,2k-1)$. By the same reasoning, the only useful option is to place another $R(5,2)$ (as we did in Figure 51 or Figure 52), reducing the remaining portion of the rectangle to $R(5,2k-3)$. After repeating this process of placing a $R(5,2)$ $k-2$ more times, the remaining portion of the rectangle is $R(5,1)$. By our assumption, a tiling for $R(5,2k+1)$ exists, but we have already established that $R(5,1)$ cannot be tiled. This is a contradiction.

Thus, if n is odd, there is no tiling for $R(5,n)$. $\square$

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