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Varieties of Nonassociative Rings of Bol-Moufang Type

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VARIETIES OF NONASSOCIATIVE RINGS OF BOL-MOUFANG TYPE

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ABSTRACT

Varieties of Nonassociative Rings of Bol-Moufang Type

By

Ronald Edsel White

In this paper we investigate Bol-Moufang identities in a more general and very natural setting, *nonassociative rings*.

We first introduce and define common algebras. We then explore the varieties of nonassociative rings of Bol-Moufang type. We explore two separate cases, the first where we consider binary rings, rings in which we make no assumption of it's structure. The second case we explore are rings in which, $2x = 0$ implies $x = 0$.

DEDICATION

For my family, friends and *The Darling Oranges*. Most of all my father, thank you for all of your assistance. Rock on.

ACKNOWLEDGEMENTS

I would like to thank all of the NMU mathematics faculty. Most of all Dr. Rowe and Dr. J.D. Phillips who worked on this project with me and Davin Hemmila. Dr. Lawton for all of your guidance. You are appreciated. Jason Haskell, thank you for all of your help. Anthony Webb, thank you for making me take breaks.

All citations are done in Chicago Style.

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SYMBOLS AND ABBREVIATIONS

$\forall$	for all
$\in$	in
$\mathbb{F}_n$	finite field of n elements
$s.t.$	such that
Φ'	duel Bol-Moufang identity of Φ
$\leftrightarrow$	equivalent
$k[M]$	The span of M over ring k
$\mathbb{Z}$	the set of all integers
$span_{\mathbb{Z}}(\alpha,\beta)$	span of α,β over $\mathbb{Z}$
$\neq$	not equal
$\mathbb{Z}/n$	$\mathbb{Z}$ modulo n
$\binom{n}{k}$	choose k elements from a set with n elements

1 Introduction to Algebra

The purpose of this introduction is to explain the relevant background knowledge to understand the work in the next two chapters.

1.1 Algebra

To understand the setting we work in we build from scratch the idea of an algebra and what structures impose different classes of algebras. At the end of the chapter a Hasse diagram is included that should sum up the relations between common algebras.

Definition 1.1 (Algebra) *An **algebra** is a set that admits a collection of operations.*

A classic example of algebras is that of groups. We become very comfortable with the group of integers under addition long before we understand underlying group structure. Groups are something of a wonder, their immense amount of structure leads to a countless number of theorems and beauty to appreciate.

The final result we see is proper vector spaces that satisfy one of the Bol-Moufang conditions having underling magma rings that classify or 'form a basis' of the vector space. We slowly build up to the structure of groups, careful to appreciate each bit of structure along the way.

1.2 Magmas

Definition 1.2 (Magma) *A **magma** is a set that is closed under a binary operation.*

Magmas are the most general form of an algebra. There is almost no structure and thus makes it normally not an area of interest for study. Recall, for us these emerge as a 'basis' for our spaces of interest. Shown below are actually two of the magma 'bases' for some of our distinguishing examples. Notice how Figure 2 is a unital magma, that is, it is just a set with a binary operation that admits an identity element.

$\cdot$	α	β
α	β	1
β	β	β

Figure 1: An example of a magma

$\cdot$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	0	1
β	0	β	1	0

Figure 2: An example of a unital magma

1.3 Quasigroups and Loops

Definition 1.3 (Quasigroup) *A **quasigroup** is a set ($\boldsymbol{Q}$) in which $\forall a, b \in \boldsymbol{Q}$ $ax = b$ and $ya = b$ have unique solutions.*

Quasigroups are the next natural expansion of structure on a set. It should be clear from the definition that we are now imposing some sort of invertibility of elements in the set. Notice that the left and right inverse need not be the same. From the definition above we would say that x is the left inverse of a relative to b; and that y is the right inverse of a relative to b. They are not too far from groups as they might seem. In fact every quasigroup is isotopic to a group. this can be seen easily noticing that in each element appears exactly once in each row and column. Thus, rearranging rows and columns in the correct manner results in a group.

$\cdot$	a	b	c
a	a	c	b
b	b	a	c
c	c	b	a

$\cdot$	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

Figure 3: Two examples of quasigroups

The next imposition of structure allows a two sided identity element to the set. This addition forms what we call a *loop*.

Definition 1.4 (Loop) *A **loop** ($\boldsymbol{L}$) is a quasigroup in which there is an element (e) s.t. $ex = xe = x$; $\forall x \in \boldsymbol{L}$.*

Loops, though seemingly not having much more structure have been studied extensively. Research by Ruth Moufang and her four identities has motivated a large amount of work in the area of loops. The most famous of which was most likely Doro, which inspired work on groups with triality from the likes of Phillips and Hall.[1,2] Below are the four famous Moufang identities, which are interestingly equivalent in the context of loops.

$$x(y(xz)) = ((xy)x)z$$

$$(x((yz)x)) = ((xy)(zx))$$

$$(xy)(zx) = (x(yz))x$$

$$x(y(zy)) = ((xy)z)y$$

These identities occur in many unexpected places. For example, the octonions are an eight dimensional division algebra over the real numbers. But if we take the non-zero octonions, notice that we get a 16 element Moufang loop. The multiplication is defined by the Cayley table below.

$e_0 = 1$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	e_5
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Figure 4: Multiplication table for non-zero octonions

1.4 Semigroups and Monoids

1. J.D. PHILLIPS, "MOUFANG LOOP MULTIPLICATION GROUPS WITH TRIALITY," *The Rocky Mountain Journal of Mathematics* 29, no. 4 (1999): 1483–1490, ISSN: 00357596, 19453795, http://www.jstor.org/stable/44238444.

2. J. I. Hall, "Moufang Loops and Groups with Triality are Essentially the Same Thing," *Memoirs of the American Mathematical Society*, 2019,

Definition 1.5 (Semigroup) *A semigroup (**S**) is a set closed under a binary operation satisfying* $x(yz) = (xy)z; \forall x, y, z \in \boldsymbol{S}$.

Semigroups take a different approach to imposing structure on a set. They impose associativity on the set. Hence the name semigroups.

$\cdot$	α	β	γ
α	β	γ	β
β	γ	β	γ
γ	β	γ	β

$\cdot$	α	β	γ
α	β	γ	γ
β	γ	γ	γ
γ	γ	γ	γ

$\cdot$	α	β	γ
α	γ	α	α
β	α	γ	γ
γ	α	γ	γ

Figure 5: Three examples of semigroups

Then as a quasigroup is to a loop a semigroup is to a monoid.

Definition 1.6 (Monoid) *A **monoid** (M) is a semigroup in which there is an element (e) s.t.* $ex = xe = x; \forall x \in \boldsymbol{M}$.

Though monoids and semigroups are well studied, they rarely are investigated as a pure object. Rather there is some sort of artificial imposed to use in a different context. For example using semigroups for the sake of functional analysis. That being said it provides a good symmetry for the Hasse diagram at the end of the chapter.

$\cdot$	α	β	γ
α	γ	α	γ
β	α	β	γ
γ	γ	γ	γ

$\cdot$	α	β	γ
α	γ	α	α
β	α	β	γ
γ	α	γ	γ

$\cdot$	α	β	γ
α	γ	β	α
β	β	β	β
γ	α	β	γ

Figure 6: Three examples of monoids

1.5 Groups

Definition 1.7 (Group) *A **group** (G) is a non-empty set with a binary operation that satisfies:*

1. *The operation is associative;* $\forall x, y, z \in G$; $x(yz) = (xy)z$
2. *There is an identity element;* $\exists e \in G$ *s.t.* $xe = ex = x; \forall x \in G$
3. *Each element in the group is invertible;* $\forall x \in G \exists a$ *s.t.* $ax = xa = e$

Groups may be the most well classified area of mathematics today. They have a massive amount of structure and have long been the interest of many great minds. They are generally the main focus of an undergraduate algebra class. Our result is in *nonassociative rings*, which are an abelian group under (+). To be an abelian group it must also satisfy commutativity; $\forall x, y \in G; xy = yx$.

$\cdot$	1	a	b	c
1	1	a	b	c
a	a	b	c	1
b	b	c	1	a
c	c	1	a	b

Figure 7: This is the structure of $\mathscr{C}_4$

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Figure 8: This is the structure of $\mathscr{K}_4$

Above we have an example of the cyclic group of order four ($\mathscr{C}_4$) and the Klein 4 ($\mathscr{K}_4$) group. Notice that these are the only possible groups with four elements and are both abelian groups.

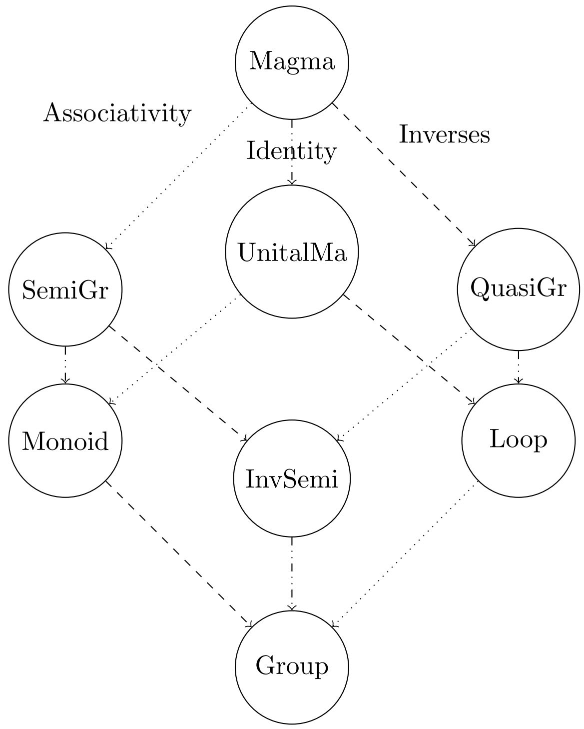

Figure 9: Hasse diagram for the structures of common algebras with a single binary operation.

1.6 Sets with Two Binary Operations (Rings)

In our result we are working in *nonassociative rings*. So we need introduce the idea of a ring. A ring extends the idea of a group by adding a second binary operation.

Definition 1.8 (Ring) *A **ring** (R) is a set equipped with two binary operations such that the set is an abelian group under the first and a semigroup under the second. The classic example of a ring is to consider the integers ($\mathbb{Z}$) under (+) and (*).*

For our result we relax the condition of a semigroup on the second operation. We refer to these as nonassociative rings. We require the set satisfy one of the 60 Bol-Moufang equations in place of associativity (though some of the identities imply associativity). This is further discussed in the next chapter.

2 Notation and Bol-Moufang Identities

Throughout the rest of this paper we use the term *nonassociative* to mean *not necessarily associative* when referring to a binary operation defined on a set. An identity is said to be of *Bol-Moufang type* if it is an equality of two degree 4 monomial expressions of monomial degree $\{2,1,1\}$, where the ordering of the variables is identical on each side. For example, $((xy)x)z = (xy)(xz)$ and $(x(yz))y = x(y(zy))$ are identities of Bol-Moufang type, while the identities $((xy)x)z = (yx)(xz)$ and $(x(xy))x = x(x(yx))$ are not of Bol-Moufang type.

The identities of Bol-Moufang type have been well studied in the context of loops, and quasigroups. It seems a perfectly natural idea to study their properties in other algebras. In this paper we consider them in the context of nonassociative rings.

2.1 Dual Identities and Systematic Notation

If Φ is an identity of Bol-Moufang type, then its dual Φ' is another identity of Bol-Moufang type obtained by reflection, i.e. reading the identity backwards. For example, the dual of the identity Ψ: $((xy)x)z = (xy)(xz)$ is Ψ': $(zx)(yx) = z(x(yx))$. Notice that Ψ' is simply equivalent to $(xy)(zy) = x(y(zy))$.

Identities of Bol-Moufang type can be indexed using the following scheme. This is the same scheme laid out by Phillips, Vojtěchovský in the context of loops.[3]

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	$o(o(oo))$
2	$o((oo)o)$
3	$(oo)(oo)$
4	$(o(oo))o$
5	$((oo)o)o$

3. J. D. Phillips and Petr Vojtechovský, "The varieties of loops of Bol-Moufang type," *algebra universalis* 54 (2005): 259–271.

For example, the identity $x((yz)x) = (xy)(zx)$ may be called $D23$. Duality interacts nicely with this notation, i.e. $A \leftrightarrow F$, $B \leftrightarrow E$, $C \leftrightarrow C$, $D \leftrightarrow D$, and $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $3 \leftrightarrow 3$. For example: $B34' = E23$, $F15' = A15$, and $D24' = D24$. There is a total of $6 \cdot \binom{5}{2} = 60$ Bol-Moufang identities. Four of the sixty are self-dual identities: $C15, C24, D15, D24$.

2.1.1 Linearized Bol-Moufang Identities

When in the context of non-binary, nonassociative rings (Chapter 3), we have an equivalent way to express the Bol-Moufang identities. The method of construction of linearized identities comes from Dart, Goodaire (2009). The linearized versions of these identities come from the idea that, since a second binary operation is being considered (+), we now have to consider sums of elements. The example below shows the construction of the linearized identity of the flexible law.

$$(xy)x = x(yx); \forall x, y \in R = k[M]$$

$$\text{Let } x = a + c,\ y = b$$

$$((a+c)b)(a+c) = (a+c)(b(a+c))$$

$$(ab+cb)(a+c) = (a+c)(ba+bc)$$

$$(ab)a + (ab)c + (cb)a + (cb)c = a(ba) + a(bc) + c(ba) + c(bc)$$

$$(ab)c + (cb)a = a(bc) + c(ba)$$

Notice that the flexible identity is expressible in two variables and the linearized version steps the identity up to three variables. Using the same construction the Bol-Moufang identities that are necessarily expressed with three variables will be linearized to four variables. Shown below is the identity B45 and it's linearized version.

$$\text{B45: } (x(yx))z = ((xy)x)z$$

$$\mathscr{L}\text{B45: } (x(yz))w + (z(yx))w = ((xy)z)w + ((zy)x)w$$

When generating magma rings that satisfy Bol Moufang identities these identities must be realized through a different lens. When attempting to construct such rings, it is sufficient to consider just the underlying magma. But magmas only have one binary operation. To insure that the overarching *magma ring* inherits the linearized property we realize the linearized identity as the magma satisfying an equality of unordered pairs. Shown below is the unordered pair necessary to lift a magma that is flexible up to a magma ring that will satisfy $\mathscr{L}$B14.

$$\{x*(y*(z*u)), z*(y*(x*u))\} = \{(x*(y*z))*u, (z*(y*x))*u\}$$

2.1.2 Definitions of varieties of nonassociative rings Bol-Moufang type

variety	abbrev.	defining identity	its name
associative	A	(xy)z = x(yz)	
extra	E	x(y(zx)) = ((xy)z)x	D15
Moufang	M	(xy)(zx) = (x(yz))x	D34
left Bol	LB	x(y(xz)) = (x(yx))z	B14
right Bol	RB	x((yz)y) = ((xy)z)y	E25
C	C	x(y(yz)) = ((xy)y)z	C15
left C	LC	(xx)(yz = (x(xy)))z	A34
right C	RC	x((yz)z) = (xy)(zz)	F23
left alternative	LA	x(xy) = (xx)y	
right alternative	RA	x(yy) = (xy)y	
flexible	FL	x(yx) = (xy)x	
left nuclear square	LN	(xx)(yz) = ((xx)y)z	A35
middle nuclear square	MN	x((yy)z) = (x(yy))z	C24
right nuclear square	RN	x(y(zz)) = (xy)(zz)	

2.2 Definitions

Definition 2.1 *(Nonassociative Ring) A nonassociative ring is a set R with two binary operations* $+$ *and* $\cdot$ *possessing identities* 0 *and* 1 *respectively, such that* $(R,+,0)$ *forms an Abelian group, and the operations obey the distributive law:*

$$(a+b)(c+d) = ac+ad+bc+bd \quad \forall a,b,c,d \in R.$$

Notice how this relaxes the semigroup condition on the second binary operation. A natural source of examples comes from *magma rings*, which are rings of the form $R=\mathbb{Z}[M]$ where M is a *unital magma*, or more generally $R=k[M]$ where k is a commutative ring or field.

Definition 2.2 *(Magma Ring) Let* M *be a unital magma and* k *a commutative ring. The magma ring* $k[M]$ *is defined to be the set of all finite linear combinations* $k_1a_1+...+k_na_n$ *where* $k_i \in k$ *and* $a_i \in M$*. The set* $R=k[M]$*, with its natural addition and multiplication structures, forms a nonassociative ring.*

Example 2.1 *Consider* $\mathbb{Z}[M] = span_{\mathbb{Z}}(1,\beta,\gamma)$*, where the unital magma* $M=\{\alpha,\beta,\gamma\}$ *is given below.*

$\cdot$	0	1	β	γ
0	0	0	0	0
1	0	1	β	γ
β	0	β	γ	1
γ	0	γ	γ	β

Figure 10: A non-communicative, non-associative magma

Notice that the unital magma M is not commutative nor associative, for example, $\gamma\cdot\beta \neq \beta\cdot\gamma$ and $(\gamma\cdot\beta)\cdot\gamma \neq \gamma\cdot(\beta\cdot\gamma)$. The ring $\mathbb{Z}[M]$ inherits these properties from M.

As an aid in understanding the arithmetic in $\mathbb{Z}[M]$, consider the elements $\gamma+\beta$ and $1-3\beta$. Addition and multiplication behave in the natural ways,

$$(\gamma+\beta)+(1-3\beta)=1+\gamma-2\beta$$

$$\begin{aligned}(\gamma+\beta)\cdot(1-3\beta)&=(\gamma\cdot 1)-3(\gamma\cdot\beta)+(\beta\cdot 1)-3(\beta\cdot\beta)\\&=\gamma-3\gamma+\beta-3\gamma\\&=-5\gamma+\beta\end{aligned}$$

Of course, one is free to consider the magma ring $R=k[M]$ when k is any commutative ring, for example $R=(\mathbb{Z}/n)[M], R=\mathbb{R}[M], R=\mathbb{C}[M], R=\mathbb{F}_p[M]$, etc.

Example 2.2 *(a magma ring where the magma contains zero) Consider the magma ring* $R=\mathbb{Z}[M]=span_{\mathbb{Z}}(0,1,\alpha,\beta)=span_{\mathbb{Z}}(1,\alpha,\beta)$*, where the unital magma* $M=\{0,1,\alpha,\beta\}$ *is given below:*

$\cdot$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	0	1
β	0	β	1	0

Figure 11: A unital magma that is both non-communicative and non-associative magma

One can see that this unital magma M is commutative and not associative. Therefore the magma ring $R=\mathbb{Z}[M]$ is also commutative and not associative. The ring R is actually of Bol-Moufang type, since it is flexible (for all $x,y\in R$ we have $(xy)x=x(yx)$, which is equivalent to B45, D24, E12). It happens to inherit that property from its underlying magma, but as mentioned above, is an atypical phenomenon.

In Example 4.4 we use the finite 27-element ring $\mathbb{F}_3[M]$ as an example of a nonassociative ring that is flexible (FL) but not left or right alternative (LA,RA).

We clearly know the behavior of the elements 0 and 1, so the very same magma M can be more compactly described as:

$\cdot$	α	β
α	β	1
β	1	0

Figure 12: Simplified Cayley table of magma ring in Example 2.2.

Throughout the paper, we employ these compact expressions whenever convenient.

Example 2.3 *(a magma ring where the magma contains zero) Consider the magma ring* $R = \mathbb{Z}[M] = span_{\mathbb{Z}}(0,1,\alpha,\beta) = span_{\mathbb{Z}}(1,\alpha,\beta)$*, where the unital magma* $M = \{0,1,\alpha,\beta\}$ *is given below:*

$\cdot$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	0	1
β	0	β	1	0

Figure 13: A magma that contains zero

In the first chapter we examine structures of nonassociative rings that are *binary*. This structure admits a finer structure than the non-binary rings.

Definition 2.3 *(Binary Ring) Let B be a nonassociative ring as described above in which* $x+x=0$ *is satisfied for all x in the ring.*

Definition 2.4 *(Non-binary rings) We call a ring non-binary if the kernel of multiplication by* 2 *is trivial, i.e.* $\mathscr{K}=0$*. Equivalently:* $2x=0$ *implies* $x=0$*.*

3 Varieties of Binary Nonassociative Rings of Bol-Moufang Type

The main result of this chapter is a classification of nonassociative rings of Bol-Moufang type. The term binary comes from the fact that elements in the nonassociative ring are allowed to satisfy $x+x=0$ for some nonzero x.

This case has a finer structure than we observe in the next chapter. In the next chapter we also discuss how some of these distinguishing examples from the binary case were used to construct counterexamples for the non-binary case as well.

3.1 Main Result 1

Theorem 3.1 *(Hemmila, Phillips, Rowe, White) The variety of binary nonassociative rings of Bol-Moufang type consists of thirteen classes: and finally the associative class (A) consisting of the remaining thirty Bol-Moufang identities, all equivalent to associativity.*

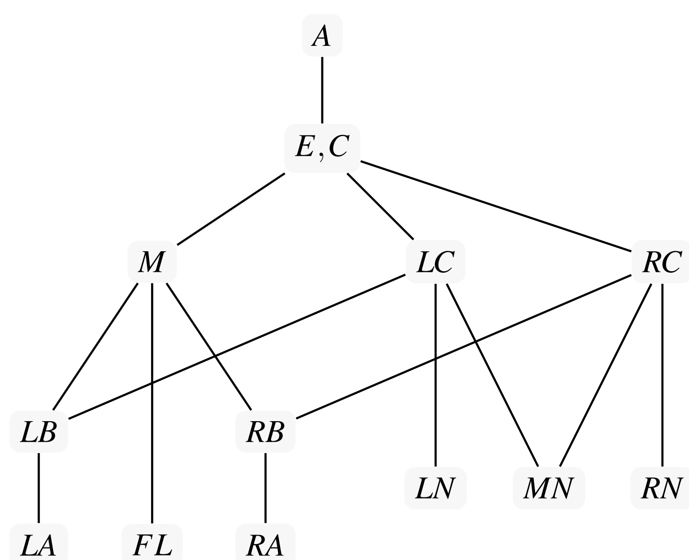

Figure 14: Hasse diagram of implications for binary nonassociative rings of Bol-Moufang type

This diagram can also be realized through the lens of duality. In the Hasse diagram below imagine that the mirror plane is duality and the right identities are coming out of the plane and the left identities are behind the plane.

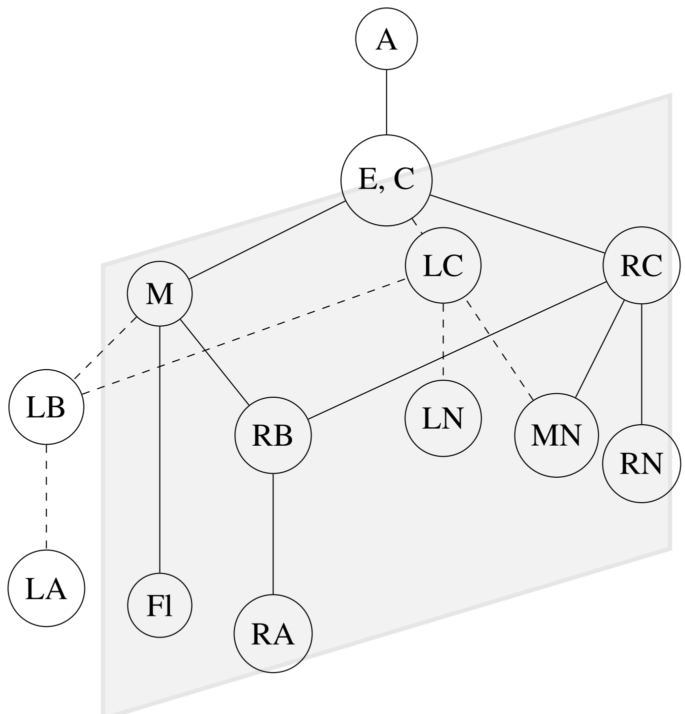

Figure 15: Mirror Hasse diagram of implications in Binary Nonassociative Rings of Bol-Moufang Type

3.2 Equivalences

Proposition 3.1 *The following Bol-Moufang identities are equivalent to the defining identity for associativity: A12, A23, A24, A25, B12, B13, B24, B25, B34, B35, C13, C23, C34, C35, D12, D13, D14, D25, D35, D45, E13, E14, E23, E24, E35, E45, F14, F24, F34, F45.*

Proposition 3.2 *The following Bol-Moufang identities are equivalent to the defining identity for (E,C): B23, D15, E34, C15.*

Proposition 3.3 *The following Bol-Moufang identities are equivalent to the defining identity for Moufang: B15, D23, D34, E15.*

Proposition 3.4 *The following Bol-Moufang identities are equivalent to the defining identity for RC: C25, F15, F23, F25.*

Proposition 3.5 *The following Bol-Moufang identities are equivalent to the defining identity for LC: A14, A15, A34, C14.*

Proposition 3.6 *The following Bol-Moufang identities are equivalent to the defining identity for RA: C45, F12, F35.*

Proposition 3.7 *The following Bol-Moufang identities are equivalent to the defining identity for LA: A13, A45, C12.*

Proposition 3.8 *The following Bol-Moufang identities are equivalent to the defining identity for LA: B45, D24, E12.*

Identities not shown by Propositions 3.1-3.8 are A35, B14, C15, C24, E25 and F13. These are the defining identities of LN, LB, CL, MN, RB and RN, respectively.

3.3 Select Proofs

All of the proofs of implications and equivalence have been verified using the automated theorem prover9.[4] In this chapter and the next we include some select proofs.

Proof 3.1 *(LC implies LA) Notice if we take A14, with* $y = 1$*, A15 with* $z = 1$*, A34 with* $z = 1$*, C14 with* $x = 1$*. Then we see that they are all LA.*

□

Proof 3.2 *(The A14, A15, A34, C14 are equivalent) First show that C14 implies to A14. Start with* $\mathscr{L}$*C14 as C14 implies* $\mathscr{L}$*C14 by construction.* $(x(yu))z + (x(uy))z = x(y(uz)) + x(u(yz))$*. Let* $u = 1$*. We have,* $(x(xy))z + (x(yx))z = x(x(yz)) + x(y(xz))$*. Since LC is LA we can re-associate and cancel to have* $(x(xy))z = x(x(yz))$*. We are done.*

4. William McCune, *Prover9/Mace4/LADR*, Download at https://www.cs.unm.edu/~mccune/prover9/, November 2009.

Now A14 implies C14 start with $\mathscr{L}$LA14. x(y(zu)) + y(x(zu)) = (x(yz))u + (y(xz))u. Let z = y to get, x(y(yu)) + y(c(yu)) = (x(yy))u + (y(xy))u. Since LC is LA we can re-associate and cancel to have x(y(yu)) = (x(yy))u. We are done.

Notice if we take A14, with $y = 1$, A15 with $z = 1$ and A34 with $z = 1$. Then we see that they are all LA. We can re-associate to show all are equivalent.

□

Proof 3.3 *(LC implies LB) Start with C14, (x(y(yz))) = ((x (yy))z). Notice if we let $x = 1$ then this we have left alternative. Then take $\mathscr{L}$C14. $(x(yz))u + (x(zy))u = x(z(yu)) + x(y(zu))$ We use the fact we are LA and make the substitutions $y = x$, $z = y$, and $u = z$. Then we get $x(x(yz)) + (x(yx))z = x(y(xz)) + x(x(yz))$ notice that the outside terms cancel as they are the A14 (other LC identity) identity. We are left with $(x(yx))z = x(y(xz))$, which is B14 (LB).*

□

3.4 Distinguishing Examples

The following section shows the distinctness of the different classes of binary nonassociative rings of Bol-Moufang type. Below is given a table to guide the reader quickly to the desired counterexample. The row is be the class that the example satisfies and the column is all of the classes that it is not. For instance Example 3.3 shows an model that is LC but not A, (E,C), M, RB, FL, or RA. the $'$ symbol is to indicate the example is to be duel to the given example. The duel example is simply the transpose of the given model. So for an example that is RC not A, (E,C), M, LC, LB, LA , FL, RA or LN. You could take the transpose of Example 3.3.

	A	E,C	M	LC	RC	LB	RB	LA	FL	RA	LN	MN	RN
A	○	○	○	○	○	○	○	○	○	○	○	○	○
E,C	1	○	○	○	○	○	○	○	○	○	○	○	○
M	2	2	○	2	2	○	○	○	○	○	2	2	2’
LC	3	3	3	○	3	○	3	○	3	3	○	○	○
RC	3’	3’	3’	3’	○	3’	○	3’	3’	3’	3’	○	○
LB	4	4	4	4	4	○	4	○	4	4	4	4	4
RB	4’	4’	4’	4’	4’	4’	○	4’	4’	○	4’	4’	4’
LA	5	5	5	5	5	5	5	○	5	5	5	5	5
FL	6	6	6	6	6	6	6	6	○	6	6	6	6
RA	5’	5’	5’	5’	5’	5’	○	5’	5’	○	5’	5’	5’
LN	7	7	7	7	7	7	7	7	7	7	○	7	7
MN	8	8	8	8	8	8	8	8	8	8	8	○	8
RN	7’	7’	7’	7’	7’	7’	7’	7’	7’	7’	7’	7’	○

Figure 16: Distinguishing examples of Binary Nonassociative Rings of Bol-Moufang Type

Example 3.1 *((E,C) ring that is not A) Consider the 256-element magma ring* $R = \mathbb{F}_2[1, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta]$ *where:*

	α	β	γ	δ	ε	ζ	η
α	0	0	ζ	ε	0	0	0
β	0	0	δ	0	0	ε	0
γ	η	δ	0	0	0	0	0
δ	ε	0	0	0	0	0	0
ε	0	0	0	0	0	0	0
ζ	0	0	0	0	0	0	0
η	0	ε	0	0	0	0	0

Figure 17: Ring that is (E, C) not A

One can verify that the ring R is in fact (E,C) (B23, D15, E34, C15). However it is not associative since $(\beta\gamma)\alpha \neq \beta(\gamma\alpha)$.

This could be the most interesting example in all of the paper. This is the only example that could not be stepped up to the non-binary case. This is because in the instance of the magma ring being non-binary, (E, C) collapses in to associativity.

Example 3.2 *((Moufang ring that is not LN, MN, RN) Consider the* 32*-element magma ring* $R = \mathbb{F}_2[1, \alpha, \beta, \gamma, \delta]$ *where:*

$\cdot$	α	β	γ	δ
α	α	0	0	δ
β	β	0	δ	0
γ	γ	δ	0	0
δ	0	0	0	0

Figure 18: Moufang ring that is not LN, MN, RN

One can verify that the ring R is in fact Moufang (B15, D23, D34, E15). However, it is not left nuclear-square since $((\alpha\alpha)\beta)\gamma \neq (\alpha\alpha)(\beta\gamma)$; it is not middle nuclear-square since $(\gamma(\alpha\alpha))\beta \neq \gamma((\alpha\alpha)\beta)$; and finally it is not right nuclear-square since $\gamma(\beta(\alpha\alpha)) \neq (\gamma\beta)(\alpha\alpha)$.

This example does not push through, as this model was constructed using the binary assumption. I conjecture that there is in fact an underlying magma that could be muscled up to the non-binary case.

Example 3.3 *(LC ring that is not FL, RA, RN) Consider the* 32*-element magma ring* $R = \mathbb{F}_2[1, \alpha, \beta, \gamma, \delta]$ *where:*

$\cdot$	α	β	γ	δ
α	γ	δ	γ	0
β	0	0	δ	0
γ	γ	0	γ	0
δ	δ	0	δ	0

Figure 19: LC ring that is not FL, RA, RN

One can verify that the ring R is in fact LC (A14,A15,A34,C14). However, it is not flexible since $(\alpha\beta)\alpha \neq \alpha(\beta\alpha)$; it is not right alternative since $(\beta\alpha)\alpha \neq \beta(\alpha\alpha)$; and finally it is not right nuclear-square since $\alpha(\beta(\gamma\gamma)) \neq (\alpha\beta)(\gamma\gamma)$.

Example 3.4 *(LB ring that is not FL, RA, LN, MN, RN) Consider the* 16*-element magma ring* $R = \mathbb{F}_2[1,\alpha,\beta,\gamma]$ *where:*

$\cdot$	α	β	γ
α	α	β	0
β	γ	γ	0
γ	0	0	0

Figure 20: LB ring that is not FL, RA,, LA ,MN, RN

One can verify that the ring R is in fact left Bol (B14). However, it is not flexible since $(\alpha\beta)\alpha \neq \alpha(\beta\alpha)$; it is not right alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$; it is not left nuclear-square since $((\alpha\alpha)\beta)\beta \neq (\alpha\alpha)(\beta\beta)$; it is not middle nuclear-square since $(\beta(\alpha\alpha))\beta \neq \beta((\alpha\alpha)\beta)$; and finally it is not right nuclear-square since $\alpha(\beta(\alpha\alpha)) \neq (\alpha\beta)(\alpha\alpha)$.

Example 3.5 *(LA ring that is not LB, FL, RA, LN, MN, RN) Consider the* 16*-element magma ring* $R = \mathbb{F}_2[1,\alpha,\beta,\gamma]$ *where:*

$\cdot$	α	β	γ
α	α	β	β
β	1	γ	0
γ	β	0	0

Figure 21: LA ring that is not FL, RA, LN, MN, RN

One can verify that the ring R is in fact left alternative. However, it is not left Bol since $\alpha(\gamma(\alpha\beta)) \neq (\alpha(\gamma\alpha))\beta$; it is not flexible since $(\alpha\beta)\alpha \neq \alpha(\beta\alpha)$; it is not right alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$; it is not left nuclear-square since $((\alpha\alpha)\beta)\alpha \neq (\alpha\alpha)(\beta\alpha)$; it is not middle nuclear-square since $(\alpha(\beta\beta))\alpha \neq \alpha((\beta\beta)\alpha)$; and finally it is not right nuclear-square since $\beta(\alpha(\alpha\alpha)) \neq (\beta\alpha)(\alpha\alpha)$.

This example also does not push through, as this model was constructed using the binary assumption. We conjecture that there is in fact an underlying magma that could be muscled up to the non-binary case. We are actively searching for this example to complete chapter 3.

Example 3.6 *(FL ring that is not LA, RA, LN, MN, RN) Consider the* 8*-element magma ring* $R = \mathbb{F}_2[1, \alpha, \beta]$ *where:*

$\cdot$	α	β
α	β	0
β	0	1

Figure 22: FL ring that is not LA, RA, LN, MN, RN

One can verify that the ring R is in fact flexible. However, it is not left alternative since $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$; it is not right alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$; it is not left nuclear-square since $((\alpha\alpha)\beta)\alpha \neq (\alpha\alpha)(\beta\alpha)$; it is not middle nuclear-square since $(\alpha(\alpha\alpha))\beta \neq \alpha((\alpha\alpha)\beta)$; and finally it is not right nuclear-square since $\alpha(\beta(\alpha\alpha)) \neq (\alpha\beta)(\alpha\alpha)$.

Example 3.7 *(LN ring that is not MN, RN, LA, FL, RA) Consider the* 8*-element magma ring* $R = \mathbb{F}_2[1, \alpha, \beta]$ *where:*

$\cdot$	α	β
α	β	1
β	β	β

Figure 23: LN ring that is not MN,RN, LA, FL, RA

One can verify that the ring R is in fact left nuclear-square. However, it is not middle nuclear-square since $(\alpha(\alpha\alpha))\beta \neq \alpha((\alpha\alpha)\beta)$; it is not right nuclear-square since $\alpha(\alpha(\beta\beta)) \neq (\alpha\alpha)(\beta\beta)$; it is not left alternative since $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$; it is not flexible since $(\alpha\beta)\alpha \neq \alpha(\beta\alpha)$; and finally it is not right alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$.

Example 3.8 *(MN ring that is not LN, RN, LA, FL, RA) Consider the* 8*-element magma ring* $R = \mathbb{F}_2[1, \alpha, \beta]$ *where:*

$\cdot$	α	β
α	β	β
β	0	β

Figure 24: MN ring that is not, LN, RN, LA, FL, RA

One can verify that the ring R is in fact middle nuclear-square. However, it is not left nuclear-square since $((\alpha\alpha)\alpha)\beta \neq (\alpha\alpha)(\alpha\beta)$; it is not right nuclear-square since $\beta(\alpha(\alpha\alpha)) \neq (\beta\alpha)(\alpha\alpha)$; and finally it is not flexible nor left/right alternative since $(\alpha\alpha)\alpha \neq \alpha(\alpha\alpha)$.

4 Varieties of Non-Binary Nonassociative Rings of Bol-Moufang Type

The main result of this chapter is a classification of nonassociative rings of Bol-Moufang type subject to the condition: 2x = 0 implies x = 0, for every x. We refer to these rings as non-binary, and they form a suitably generic case, as most nonassociative rings are non-binary, in a certain sense.

4.1 Main Result 2

Conjucture 4.1 *(Hemmila, Phillips, Rowe, White) The variety of non-binary, nonassociative rings of Bol-Moufang type consists of seven equivalence classes: the left alternative class and its dual right alternative class, the flexible class, the left Bol class and its dual right Bol class, the Moufang class , and finally the associative class*

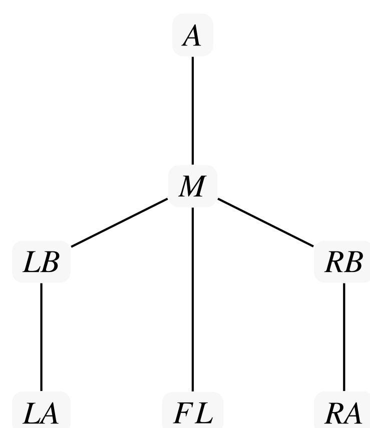

Figure 25: Hasse diagram of implications for nonassociative rings of Bol-Moufang type

4.2 Equivalences

Proposition 4.1 *The following Bol-Moufang identities are equivalent to the defining identity for Moufang: B15, D23, D34, E15.*

Proposition 4.2 *The following Bol-Moufang identities are equivalent to the defining identity for LA: A13, A45, C12.*

Proposition 4.3 *The following Bol-Moufang identities are equivalent to the defining identity for FL: B45, D24, E12.*

Proposition 4.4 *The following Bol-Moufang identities are equivalent to the defining identity for RA: C45, F12, F35.*

Two identities not shown by Propositions 4.1-4.4 are B14 and E25 which are the defining identities for LB and RB respectively. The remaining forty-five identities are all equivalent to associativity.

4.3 Select Proofs

Proof 4.1 *(Moufang implies Flexible) Start with* $\mathscr{L}B15$, $x(y(zw)) + z(y(xw)) = ((xy)z)w + ((zy)x)w$. *Set* $z = x$ *to get* $2(x(y(xw))) = 2(((xy)x)w)$. *Then since this is the non binary case* $(x(y(xw))) = (((xy)x)w)$. *We can set* $w=1$ *to see that Moufang implies flexible.* □

Proof 4.2 *This has the same beginning as the last proof.*

Start with $\mathscr{L}B15$, $x(y(zw)) + z(y(xw)) = ((xy)z)w + ((zy)x)w$. *Set* $z = x$ *to get* $2(x(y(xw))) = 2(((xy)x)w)$. *Then since this is the non binary case* $(x(y(xw))) = (((xy)x)w)$. *Since above we showed Moufang is Flexible* $x(y(xw)) = (x(yx))w$. *Now let* $x=x+z$ *and distribute to get* $x(y(xw)) + x(y(zw)) + z(y(xw)) + z(y(zw)) = (x(yx))w + (x(yz))w + (z(yx))w + (z(yz))w$. *Since we know that* $x(y(xw)) = (x(yx))w$ *we cancel to get* $x(y(zw)) + z(y(xw)) = (x(yz))w + (z(yx))w$, *which is* $\mathscr{L}B14$. □

Proof 4.3 *(LB implies LA) Start with $\mathscr{L}B14$ $x(y(zw))+z(y(xw)) = ((x(yz))w+((z(yx))w$. Multiply on the right by u, $(x(y(zw)))u+(z(y(xw)))u = (((x(yz))w)u+(((z(yx))w)u$. Now set x=1. We get $(y(zw))u+(z(yw))u = ((yz)w)u+((zy)w)u$ which is $\mathscr{L}A45$.*

□

4.4 Distinguishing Examples

The following section shows the distinctness of the different classes of non-binary nonassociative rings of Bol-Moufang type. Below is given a table to guide the reader quickly to the desired counterexample. Read this table just as Figure 16.

	A	M	LB	RB	LA	FL	RA
A	○	○	○	○	○	○	○
M	1	○	○	○	○	○	○
LB	2	2	○	2	○	2	2
RB	2’	2’	2’	○	2’	2’	○
LA	3	3	○	3	○	3	3
FL	4	4	4	4	4	○	4
RA	3’	3’	3’	○	3’	3’	○

Figure 26: Distinguishing examples of Binary Nonassociative Rings of Bol-Moufang Type

Example 4.1 *((Moufang ring that is not A) Consider the 6,561-element magma ring $R = \mathbb{F}_3[1,\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta]$ where:*

	α	β	γ	δ	ε	ζ	η
α	0	α	ε	0	0	0	ζ
β	δ	1	η	α	ε	ζ	γ
γ	ζ	γ	0	ε	0	0	0
δ	0	δ	ζ	0	0	0	ε
ε	ζ	0	0	0	0	0	0
ζ	0	ε	0	0	0	0	0
η	ε	η	0	ζ	0	0	0

Figure 27: Moufang ring that is not A

One can verify that the ring R is in fact Moufang (B15, D23, D34, E15). However, it is not associative since $(\alpha\beta)\gamma \neq \alpha(\beta\gamma)$.

Example 4.2 *(LB ring that is not FL, RA) Consider the* 81*-element magma ring* $R = \mathbb{F}_3[1, \alpha, \beta, \gamma]$ *where:*

$\cdot$	α	β	γ
α	α	β	0
β	γ	γ	0
γ	0	0	0

Figure 28: LB ring that is not FL, RA

One can verify that the ring R is in fact left Bol (B14). However, it is not flexible since $(\alpha\beta)\alpha \neq \alpha(\beta\alpha)$ and it is not right alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$.

Notice how this is the same underlying magma as Example 3.4. This is because this model was found using the underlying magma table and stepped up to the span using the previously discussed equality of unordered pairs.

Example 4.3 *(LA ring that is not FL, LB, RA)*

We do have a model of order 5 that is LA not FL and RA. Sadly, as of now we have not yet found a counterexample for this case. We are actively searching for a model. This is currently the only missing piece to definitively classify all non-binary nonassociative rings of Bol-Moufang type.

Example 4.4 *(FL ring that is not LA, RA, LN, MN, RN) Consider the* 27*-element magma ring* $R = \mathbb{F}_3[1, \alpha, \beta]$ *where:*

$\cdot$	α	β
α	β	0
β	0	1

Figure 29: FL ring that is not LA, RA, LN, MN, RN

One can verify that the ring R is in fact flexible. However, it is not left alternative since $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$ and it is not right alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$.

Notice how this is the same underlying magma as Example 3.6. This is because this model was found using the underlying magma table and stepped up to the span using the previously discussed equality of unordered pairs.

5 Summary and Conclusions

To summarize we gave a brief background regarding common algebras of a single binary operation, hopefully this would be enough to help a motivated undergraduate understand the rest of the paper.

We then offered a complete classification of binary nonassociative rings of Bol-Moufang type, and with the exception of a single counterexample, we offer a classification of non-binary nonassociative rings of Bol-Moufang type.

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