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A Survey of the Representations of Rational Ruled Surfaces*

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Abstract The rational ruled surface is a typical modeling surface in computer aided geometric design. A rational ruled surface may have different representations with respective advantages and disadvantages. In this paper, the authors revisit the representations of ruled surfaces including the parametric form, algebraic form, homogenous form and Plücker form. Moreover, the transformations between these representations are proposed such as parametrization for an algebraic form, implicitization for a parametric form, proper reparametrization of an improper one and standardized reparametrization for a general parametrization. Based on these transformation algorithms, one can give a complete interchange graph for the different representations of a rational ruled surface. For rational surfaces given in algebraic form or parametric form not in the standard form of ruled surfaces, the characterization methods are recalled to identify the ruled surfaces from them.

Keywords Birational transformation, characterization, implicitization, parametrization, rational ruled surface, reparametrization.

1 Introduction

A ruled surface is generated by sweeping a line along the directrix curve. This type of surfaces is widely used in computer aided geometric design, geometric modeling and computer

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numerical control. There are many papers discussing the ruled surface and their applications (see [1–20]). Using the μ-bases method, Chen and Wang[7] gave an implicitization algorithm for rational ruled surface. The univariate resultant has also been used to compute these implicit equations efficiently[13, 14]. Wang and Goldman[21] presented a new implicitization method for ruled translational surfaces. For this purpose, they used two linearly independent vectors that are perpendicular to the generating line of the surface. For a given parametrization of a rational ruled, people could find a simplified reparametrization that did not contain any non-generic base point and had a pair of directrices with the lowest possible degree[7]. Busé, et al. and Dohm studied ruled surface using μ-bases[5, 9]. Li, et al.[12] computed a proper reparametrization of an improperly parametrized ruled surface. Andradas, et al.[1] presented an algorithm to decide whether a proper rational parametrization of a ruled surface could be properly reparametrized over a real field.

Ruled surfaces were studied further for applications. The collision and intersection of the ruled surfaces were discussed in [8, 15] and self-intersection was studied in [22]. Izumiya and Takeuchi[10] studied the cylindrical helices and Bertrand curves on ruled surfaces. The offset of ruled surfaces was discussed in [23]. These surfaces had been used for geometric modeling of architectural freeform design in [11, 19, 20]. Ruled surfaces were also studied in the context of approximation with modeling surfaces since they have conveniences in NC flank milling[16, 17].

A rational ruled surface is usually given in the standard parametric form $\boldsymbol{P}(s,t) = \boldsymbol{P}_1(s) + t\boldsymbol{P}_2(s) \in \mathbb{K}(s,t)^3$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. The parametric form is not unique since one rational parametrization can be transformed to another by a rational parameter transformation, and the parameter transformation does not change the algebraic surface defined by the parametrization. The properness of the parametrization is preserved if and only if the transformation is birational. There are several classic problems dealing with the parametric form, for instance, finding a standard parametrization for a given non standard one; finding a proper parametrization for a given improper one or identifying a ruled surface from a general rational parametrization.

In contrast to the parametric form, the square free algebraic form of a rational ruled surface is unique. In computer aided geometric design and computer graphics, people prefer the rational parametric form in modeling design[24]. On the other hand, in computer algebra and algebraic geometry, people generally use the algebraic form. Since there are different advantages of parametric and implicit forms, the natural problems are to transform the forms from one to another.

Finding a parametric form from the implicit form is known as the parametrization problem. Conversely, finding the implicit equation from the parametric one is called implicitization. There are lots of papers focusing on the implicitization problem. Some typical methods are proposed using Gröbner bases[25, 26], characteristic sets[27, 28], resultants[13, 14, 29] and μ-bases[6, 7, 9]. The parametrization problem is more difficult than the implicitization problem. Only certain special algebraic curves and surfaces have rational parametric representations. For a general surface given in algebraic form, no efficient symbolic parametrization algorithm has yet been given. However, to meet practical demands, people had to design parametrization algorithms for some commonly used surfaces, such as quadric algebraic surfaces[30] and cubic algebraic surfaces [31, 32]. Recently, Shen and Pérez-Díaz determined and parameterized rational ruled surfaces based on algebraic computations[33].

In this paper, we review the representation of ruled surfaces for both symbolic and numeric considerations. More importantly, transformations between different representations are proposed. Based on these transformations, we give a complete interchange graph for the different representations of a rational ruled surface (see Figure 1). The discussions benefit from the intrinsic property of rational ruled surfaces, i.e., the ruled surfaces are linear in one direction. Thus in the standard rational parametric form, the parameter t is linear. Then t is always solvable such that there exists a reparametrization with one coordinate only involving t and then the ruled surface can be projected as a rational parametric curve. On the other hand, it is possible that the surface is given in algebraic form or parametric form not in the standard form of ruled surfaces, in this case people cannot tell that whether this surface is a rational ruled surface or not according to the representation. We then recall the characterization methods to identify the ruled surfaces from these general forms. For practical applications, one has to deal with numerical equations that are given approximately, probably perturbed under engineering design. In the numeric case, we need to analyze ruled surfaces close to an input (even not necessarily ruled) surface. The recent approaches concerning symbolic-numeric reparametrization of ruled surfaces are then proposed.

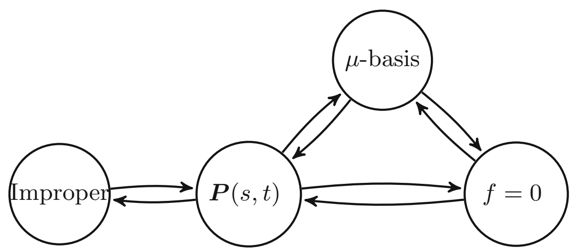

Figure 1 Interchange graph for the representations of a rational ruled surface

The paper is organized as follows. First, some typical representations are presented in Section 2. In Section 3, we give the proper reparametrization for an improper one, and furthermore, a simplification for the proper one. In Section 4, we focus on the implicitization of parametric ruled surface. In Section 5, we identify the ruled surface from a given algebraic surface or a rational parametric surface and give a standard parametrization. In addition, we adapt the symbolic algorithms to numerical situations by involving numerical techniques. The numeric algorithms are designed to find ruled surfaces close to an input (not necessarily ruled) surface. In Section 6, we briefly conclude the paper.

2 Representations of Ruled Surfaces

A standard parametrization of a rational ruled surface $\mathcal{P}$ is given by a parametrization of the form

$$(x,y,z)=\boldsymbol{P}(s,t)=\boldsymbol{P}_1(s)+t\boldsymbol{P}_2(s)\in\mathbb{K}(s,t)^3, \tag{1}$$

where $\mathbb{K}$ is an algebraically closed field of characteristic zero and the parametrizations $\boldsymbol{P}_i(s)=(p_{i1}(s),p_{i2}(s),p_{i3}(s))\in\mathbb{K}(s)^3, i=1,2$ define two rational space curves. $\boldsymbol{P}_1(s)$ is the directrix and $\boldsymbol{P}_2(s)$ is the indicatrix of $\mathcal{P}$. We assume that the rational parametrization (1) is nontrivial, that is, it defines a real surface, not a space curve. A rational ruled surface can also be defined by an algebraic variety

$$\{(x,y,z)|f(x,y,z)=0,f(x,y,z)\in\mathbb{K}[x,y,z]\}. \tag{2}$$

A rational parametrization $\boldsymbol{P}(s,t)$ of a variate $\mathcal{P}$, defines a rational map $\phi_{\boldsymbol{P}}:\mathbb{K}^2\to\mathcal{P}$ given by $(s,t)\to\boldsymbol{P}(s,t)$. A rational parametrization is proper (resp. improper) if $\phi_{\boldsymbol{P}}$ is one-to-one (resp. many-to-one). For every generic point $\boldsymbol{P}(s,t)$ on $\mathcal{P}$ where $(s_0,t_0)\in\mathbb{K}^2$, denote

$$\mathcal{F}_{\boldsymbol{P}}(s_0,t_0)=\{(s,t)\in\mathbb{K}^2\mid\boldsymbol{P}(s,t)=\boldsymbol{P}(s_0,t_0)\},$$

i.e., $\mathcal{F}_{\boldsymbol{P}}(s_0,t_0)$ is the fibre of $\boldsymbol{P}(s_0,t_0)$ via $\phi_{\boldsymbol{P}}$. The cardinality of this fibre is defined to be the improper index of $\boldsymbol{P}(s,t)$ and denoted by $\mathrm{IX}(\boldsymbol{P})$. A parametrization $\boldsymbol{P}(s,t)$ is proper (resp. improper) if $\mathrm{IX}(\boldsymbol{P})=1$ (resp. $\mathrm{IX}(\boldsymbol{P})>1$).

A proper $\boldsymbol{P}(s,t)$ of the form (1) is well known as the standard form parametrization of $\mathcal{P}$. Observe that $\mathcal{P}$ always admits a proper parametrization of the form

$$\boldsymbol{Q}(s,t)=(q_{11}(s)+q_{21}(s)\,t,q_{12}(s)+q_{22}(s)\,t,t)\in\mathbb{K}(s,t)^3, \tag{3}$$

where $q_{i,j}, i=1,2;k=1,2$ are rational functions. Such a parametrization is obtained by performing the birational transformation $(s,t)\to\left(s,\frac{t-p_{13}(s)}{p_{23}(s)}\right)$. In the following discussion, we refer to parametrization $\boldsymbol{Q}(s,t)$ as the standard reduced form parametrization of $\mathcal{P}$.

In some situations, the parametrization of rational ruled surface is written in the homogenous form as

$$(x,y,z,w)=\boldsymbol{P}^h(s,t)=\boldsymbol{P}_1^h(s)+t\boldsymbol{P}_2^h(s)\in\mathbb{K}[s,t]^4, \tag{4}$$

where $\boldsymbol{P}_i^h(s)=(p_{i1}^h(s),p_{i2}^h(s),p_{i3}^h(s),p_{i4}^h(s))\in\mathbb{K}[s]^4, i=1,2$. The associated affine form of representation (4) is $(\frac{x}{w},\frac{y}{w},\frac{z}{w})$.

The homogenous form (4) and the affine form (1) are equivalent, i.e., there exists a birational transformation between these two representations. In fact, we only need to show the equivalence of representation (3) and representation (4) by the following proposition.

Proposition 2.1 *There exists a birational transformation between the representation* (3) *and the representation* (4) *of a ruled surface.*

Proof Since the representation (4) defines a ruled surface, $p_{14}^h(s)$ and $p_{24}^h(s)$ cannot be the zero vector simultaneously. If $p_{24}^h(s)=0$ then the birational transformation is $(s,t)\to$

$\left(s, \frac{p^h_{14}(s)\,t-p^h_{13}(s)}{p^h_{23}(s)}\right)$. Otherwise, let $(s,t) \to \left(s, \frac{1-p^h_{14}(s)\,t}{p^h_{24}(s)\,t}\right)$ be the birational transformation. With each birational transformation, the reparametrization is written as $(p_{11}(s)+p_{21}(s)\,t, p_{12}(s)+p_{22}(s)\,t, p_{13}(s)+p_{23}(s)\,t, p_{14}(s))$ and its affine form is exactly of the form (4). ∎

2.1 Moving Planes and μ-Bases

A moving plane $\boldsymbol{L}(s,t) := (A(s,t), B(s,t), C(s,t), D(s,t))$ is a family of planes $L(s,t) := A(s,t)x + B(s,t)y + C(s,t)z + D(s,t)w = 0$ corresponding to the parameters s and t. A moving plane $\boldsymbol{L}(s,t)$ is said to follow the rational ruled surface $\boldsymbol{P}(s,t)$ if

$$\boldsymbol{L}(s,t)\cdot\boldsymbol{P}(s,t) = A(s,t)a(s,t) + B(s,t)b(s,t) + C(s,t)c(s,t) + D(s,t)d(s,t) \equiv 0. \tag{5}$$

The moving planes $\boldsymbol{L}(s,t)$ form a module $\boldsymbol{M}(s,t) := \{\boldsymbol{L}(s,t)|\boldsymbol{L}(s,t)\cdot\boldsymbol{P}(s,t) = 0\}$ which is a free module[34, 35]. And a basis of $\boldsymbol{M}(s,t)$ is called a μ-basis of the surface $\boldsymbol{P}(s,t)$.

The papers [6, 9] studied the μ-bases of rational ruled surfaces and gave the moving planes that involve only the parameter value s, that is, $\boldsymbol{L}(s) := (A(s), B(s), C(s), D(s))$ for which $\boldsymbol{L}(s)\cdot\boldsymbol{P}(s,t) \equiv 0$. These moving planes form a free module $\boldsymbol{M}(s) := \{\boldsymbol{L}(s)|\boldsymbol{L}(s)\cdot\boldsymbol{P}(s,t) = 0\}$ with rank two. A basis of $\boldsymbol{M}(s)$ is defined as a μ-basis of the ruled surface in [6, 9]. This μ-basis is extended in [7] by adding one more moving plane $\boldsymbol{r}(s,t)$ which is linear in t. We review some properties of μ-bases needed for our further discussions.

Proposition 2.2 *For a ruled surface parametrically defined by $\boldsymbol{P}(s,t)$, there is a μ-basis, $\boldsymbol{p}(s), \boldsymbol{q}(s), \boldsymbol{r}(s,t)$, of $\boldsymbol{P}(s,t)$. Furthermore, let $f(x,y,z,w) = 0$ be the implicit equation of $\boldsymbol{P}(s,t)$. Then*

1) $\deg(\boldsymbol{p}) + \deg(\boldsymbol{q}) = \deg(f)$.

2) $[\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}] = k\boldsymbol{P}$, *where $[\cdot]$ returns the outer product of three vectors and k is a nonzero constant.*

3) $f(x,y,z,w) = k\mathrm{Res}(\boldsymbol{p}\cdot\boldsymbol{X}, \boldsymbol{q}\cdot\boldsymbol{X}, s)$, *where k is a nonzero constant and $\boldsymbol{X} = (x,y,z,w)$.*

2.2 Plücker Coordinates

A ruled surface consists of lines. This property leads the line geometry study of ruled surfaces[18, 36]. Using Plücker coordinates, each line can be projected to a point in projective space $\mathbb{P}^5$.

Let $p^p_{i,j} = \begin{vmatrix} p^h_{1,i} & p^h_{2,i} \\ p^h_{1,j} & p^h_{2,j} \end{vmatrix}, 1 \le i < j \le 4$, and consider the quadratic equation $p^p_{12}p^p_{34} - p^p_{13}p^p_{24} + p^p_{14}p^p_{23} = 0$. Then a representation of $\mathcal{P}$ in Plücker coordinates is

$$\boldsymbol{P}^p = (p^p_{12}, p^p_{13}, p^p_{14}, p^p_{23}, p^p_{24}, p^p_{34}). \tag{6}$$

Therefore, a rational ruled surface can be regarded as a rational curve in $\mathbb{P}^5$ with the parametrization of the form (6).

2.3 Discussions

We review five representations of the rational ruled surface including standard form, reduced standard form, μ-bases, Plücker form and implicit form. The standard form is classical parametric form used in text books. With a birational transformation, one can get the reduced form firstly defined in [12] and this form brings convenience in study of ruled surface[33, 37, 38]. The μ-basis is a new representation of ruled surfaces which can play a bridge role between the parametric form and the implicit form, since one can recover the parametric form by outer product of the μ-bases and find the implicit form by computing the resultant of μ-bases. The Plücker form projects a line to a point in the new space and then the ruled surface is projected to a quadratic curve in the new space. The technique is not so popular but can be used in some research problem[5, 36]. Finally, the implicit form is another classic form with respect to the parametric form.

3 Simplification of the Parametrization

The rational parametrization of a surface is not unique, thus a natural question arises on how to simplify the parametrization, by which we mean to find rational functions with degrees as small as possible. This problem is usually divided into two sub problems as proper reparametrization and degree reduction of the proper parametrization. Both of these problems are still open for general surfaces, but for the ruled surfaces, we have settled these problems.

3.1 Proper Reparametrization

Although the representation (1) is referred as the standard representation for ruled surfaces, a given parametrization of the form (1) may be improper. For a given parametrization of the form (1), one can determine whether a surface is proper using a u-resultant[39] or a Gröbner basis[27]. If a rational parametrization is improper, we should find a proper reparametrization. However, the problem of finding a proper reparametrization for an improper rational parametrization of a general algebraic surface is open[40, 41].

Fortunately, we can give an effective solution to the proper reparametrization problem for rational ruled surfaces. For a parametrization of the form (1), we compute its reduced form (3) with a birational transformation. The reduced form has the same properness as the origin one and is proper with respect to the variable t. Then, by considering the improper parameter only, the rational parametrization can be treated as a rational parametrization of an algebraic curve with the proper parameter in the coefficients. The proper reparametrization for curves is well solved based on Lüroth's theorem[42] and various proper reparametrization algorithms have been developed, such as the Gröbner basis method, the characteristic set method, and the greatest common division method[27, 32, 40, 41]. Finally, we find a proper reparametrization for this curve with known methods and show that this reparametrization also provides a proper reparametrization for the ruled surface.

Before reviewing the following theorem from [12], we introduce two functions: numer$(\cdot)$ returns the numerator of an input rational function and Res$(\cdot)$ returns the resultant of two polynomials.

Theorem 3.1 *Consider a ruled surface defined in the form of* (3). *Let*

$$\begin{aligned} H_1(s,\overline{s}) &= \operatorname{numer}((q_{11}(s)+q_{21}(s)t) - ((q_{11}(\overline{s})+q_{21}(\overline{s})t)),\\ H_2(s,\overline{s}) &= \operatorname{numer}((q_{12}(s)+q_{22}(s)t) - ((q_{12}(\overline{s})+q_{22}(\overline{s})t)),\\ H(s,\overline{s}) &= \gcd(H_1,H_2). \end{aligned}$$

If $H = c(s-\overline{s})$ *for* $c\in\mathbb{K}[t]$, *then* (3) *is proper; otherwise, write* H *as a polynomial in* $\overline{s}$*:*

$$H = c_d\,\overline{s}^{\,d} + \cdots + c_1\overline{s} + c_0, \quad c_d \neq 0,$$

where $c_i\in\mathbb{K}[t][s], i=0,1,\cdots,d$. *Then there exists* $k\neq l$ *such that* $\frac{c_k}{c_l}\notin\mathbb{K}(t)$, *and a set of new parameters for the surface are*

$$\overline{s} = \frac{c_k(s,t)}{c_l(s,t)}, \quad \overline{t} = t.$$

Furthermore, let $L_1(\overline{s},x) = \operatorname{Res}(G_1(s,x), c_l\overline{s}-c_k, s)$ *and* $L_2(\overline{s},y) = \operatorname{Res}(G_2(s,y), c_l\overline{s}-c_k, s)$; $G_1(s,x) = \operatorname{numer}(x-q_{11}(s)-q_{21}(s)t)$ *and* $G_2(s,y) = \operatorname{numer}(y-q_{12}(s)-q_{22}(s)t)$. *Then*

$$\begin{aligned} L_1 &= (Q_{12}(\overline{s},t)x - Q_{11}(\overline{s},t))^{\deg(H(\overline{s},s))},\\ L_2 &= (Q_{22}(\overline{s},t)y - Q_{21}(\overline{s},t))^{\deg(H(\overline{s},s))}, \end{aligned}$$

where $Q_{ij}\in\mathbb{K}[\overline{s},t]$. *A proper reparametrization of* (3) *using the new parameters* $\overline{s},\overline{t}$ *is* $(\frac{Q_{11}(\overline{s},\overline{t})}{Q_{12}(\overline{s},\overline{t})}, \frac{Q_{21}(\overline{s},\overline{t})}{Q_{22}(\overline{s},\overline{t})}, \overline{t})$.

Example 3.2 The parametrization $\boldsymbol{P}(s,t) = (1+3s-s^2+(s+1)t, 3s-s^2+st, t)$, can be treated as a rational parametrization of an algebraic curve with parameter s with coefficients in the field $\mathbb{K}(t)$. Using Theorem 3.1, we have

$$\begin{aligned} H_1(s,\overline{s}) &= (1+3\overline{s}-\overline{s}^2+(\overline{s}+1)t) - (1+3s-s^2+(s+1)t),\\ H_2(s,\overline{s}) &= (3\overline{s}-\overline{s}^2+\overline{s}t) - (3s-s^2+st),\\ H(s,\overline{s}) &= \overline{s}^2+(-t-3)\overline{s}-s^2+st+3s. \end{aligned}$$

$\boldsymbol{P}(s,t)$ is an improper parametrization since $\deg(H)=2$. Then, rewrite $H = c_2\overline{s}^2+c_1\overline{s}+c_0$ where $c_2=1, c_1=(-t-3), c_0=-s^2+st+3s$. Then we obtain the new parameters $\overline{s} = c_0/c_2 = 3s-s^2+ts$, $\overline{t}=t$ and a proper reparametrization is

$$\boldsymbol{Q}(\overline{s},\overline{t}) = (\overline{s}+1+\overline{t}, \overline{s}, \overline{t}).$$

The paper [5] tried to simplify the proper reparametrization, the authors consider the proper reparametrization of the curve in $\mathbb{P}^5$ projected by the ruled surface using Plücker coordinates. However, an improper ruled surface may have a proper parametrization of the Plücker coordinates. Considering the improper ruled surface in Example 3.2, one finds that the projected curve is

$$(s^2-2\,s, -s^2+3\,s+1, -1-s, -s^2+3\,s, -s, -1),$$

which is proper since there exist linear coordinates. So this example is a counterexample for Proposition 2.2 of [5]. In fact, the statement of Proposition 2.2 is true only for the improper ruled surfaces having a separated improper parameter transformation for s and proper parameter for t.

3.2 Simplifying a Proper Parametrization

The proper rational parametrization of a ruled surface is not unique since any parametrization represents the same surface up to a birational parameter transformation. Hence, people would like to reparametrize a rational surface to simplify the parametrization as needed. For instance, it is a cumbersome task to make the parametrization contain no base points, even for a ruled surface.

An affine base point of a rational surface parameterized by $\boldsymbol{P}(s,t)$ is a parameter pair (s_0,t_0) so that the numerator and denominator of each component of $\boldsymbol{P}(s,t)$ at (s_0,t_0) are zero. For the homogenous form $\boldsymbol{P}^h(s,t)$, all components are zeros at the parameter pair (s_0,t_0). The μ-basis technique in [6] provides a simple and elegant way to reparameterize a rational ruled surface such that it does not contain any non-generic base points. Furthermore, the directrices of the reparameterized surface have the lowest possible degree. Thus there are both geometric and computational advantages to be gained from such a reparametrization. Here we refine the parametrization using the μ-basis method in [6]. A more efficient algorithm to compute μ-basis can be found in [35]. The main result of [6] (also in [5]) is recalled in the following theorem.

Theorem 3.3 *Let* $(\boldsymbol{p},\boldsymbol{q})$ *be a* μ*-basis for the rational ruled surface* $\mathcal{P}$ *and* $(\widetilde{\boldsymbol{p}},\widetilde{\boldsymbol{q}})$ *is a* μ*-basis for the ruled surface* $\boldsymbol{p}(s)+t\boldsymbol{q}(s)$*. Then* $\widetilde{\boldsymbol{p}}(s)+t\widetilde{\boldsymbol{q}}(s)$ *is a base point free reparameterization of* $\mathcal{P}$*, and the directrices of* $\widetilde{\boldsymbol{p}}(s)+t\widetilde{\boldsymbol{q}}(s)$ *have the lowest possible degree.*

Example 3.4 Let $(x,y,z,w)=\boldsymbol{P}^h(s,t)$ be a ruled surface given in homogenous form

$$\boldsymbol{P}^h(s,t)=\left(1-s^2-2\,st, 2\,s+t(1-s^2),(1+s^2)t,1+s^2\right).$$

A μ-basis of $\mathcal{P}$ is $\boldsymbol{p}=(s,-1,1,s),\boldsymbol{q}=(1,s,s,-1)$ and a μ-basis of $\boldsymbol{p}(s)+t\boldsymbol{q}(s)$ is $\widetilde{\boldsymbol{p}}=(s,-1,-1,-s),\widetilde{\boldsymbol{q}}=(1,s,-s,1)$. According to Theorem 3.3, we get a reparametrization

$$(s+t,-1+st,-1-st,-s+t),$$

which has no base points and its directrices have the lowest degree.

3.3 Discussions

A basic property of a rational parametrization is whether it is proper or improper. Improper parametrizations are undesirable because the parametric degree could be unnecessarily high. A rational parametric ruled surface can be improper even the parameter t is linear. There are some methods can deal with proper reparametrization of ruled surfaces such as [41], we here review a simple and efficient method specially designed for the ruled surfaces.

For a proper rational ruled surface $\boldsymbol{P}(s,t)$ with bidegree $(m,1)$ w.r.t. parameters (s,t), its implicit degree is fixed and can be computed as $2m-n_b$ where n_b is the number of all base points of P, multiplicities counted. Note that there may be complex base points in the parametrization since the parametrization is not unique. It is a challenge problem to find the simplest parametrization for arbitrary rational surfaces. However, Theorem 3.3 gives a novel way to find the simplest parametrization for a ruled surface.

4 Implicitization of Rational Ruled Surfaces

As mentioned in the introduction, different methods are used to implicitize a given rational surface. The methods based on Gröbner bases or characteristic sets are complete in theory but not suitable for practical computation, since both of these methods have double exponential time complexity. Here, we prefer complete and more effective methods for ruled surfaces. At first, we develop an implicitization method using elements of the μ-basis (see Proposition 2.2). The implicit equation is the resultant of the two μ-basis.

Example 4.1 Continuation to Example 3.4. By Proposition 2.2, the implicit equation of the ruled surface, up to a nonzero scalar, is $\mathrm{Res}(\boldsymbol{p}\cdot\boldsymbol{X},\boldsymbol{q}\cdot\boldsymbol{X},s)=\mathrm{Res}(xs-y+z+ws,x+ys+zs-w,s)=y^2-z^2+x^2-w^2$. The implicit equation in affine space is then $y^2-z^2+x^2-1=0$.

Without computing the μ-basis, we here introduce a simpler implicitization method still based on the univariate resultant computation. The surface parametrically defined by (3) can be treated as a collection of the following planar curves with specified parameter t:

$$(x,y)=\left(\frac{q_{1n}(s,t)}{q_{1d}(s,t)},\frac{q_{2n}(s,t)}{q_{2d}(s,t)}\right), \tag{7}$$

where $q_{in}(s,t)$ and $q_{id}(s,t)$ are the numerator and denominator of q_i. Assuming $\gcd(q_{in},q_{id})=1$ and $\max\{\deg(q_{in}),\deg(q_{id})\}\geq 1, i=1,2$, then the resultant

$$\mathrm{Res}(q_{1d}x-q_{1n},q_{2d}y-q_{2n},s)=l(t)L(x,y,t) \tag{8}$$

is not identically zero, where $l(t)\in\mathbb{Q}[t]$ is the content of the resultant with respect to the parameters x,y. Hence, $L(x,y,t)$ is the primitive part of the resultant.

If $\deg_t(L)=0$, the surface (3) is a cylindrical surface over the xy-plane with the irreducible implicit equation $L(x,y)=0$. To determine whether a rational surface is cylindrical over the coordinate plane is not difficult[13, 43], hence we consider only the non-degenerate case with $\deg_t(L)\geq 1$. We have the following theorem.

Theorem 4.2 *Let $\boldsymbol{P}(s,t)$ be a rational projective surface of the form* (3) *with implicit equation $f(x,y,z)\notin\mathbb{Q}[x,y]$. Then up to a constant multiple*

$$f(x,y,z)=L(x,y,t)|_{t=z}. \tag{9}$$

This theorem is a simplified version of Theorem 2 in [14]. This implicitization method is more efficient than other existing approaches. Readers are referred to [14] for further details.

Example 4.3 Continuation to Example 3.4. The affine form of the ruled surface is $(x,y,z)=\boldsymbol{P}(s,t)$ given by

$$\boldsymbol{P}(s,t)=\left(\frac{1-s^2-2\,st}{1+s^2},\frac{2\,s+t(1-s^2)}{1+s^2},t\right).$$

By Theorem 4.2, we compute a univariate resultant and get

$$L(x,y;t)=4\,y^2+4\,t^2x^2+4\,t^2y^2-8\,t^2-4\,t^4+4\,x^2-4.$$

By removing the content $(4t^2+4)$, i.e., the gcd of the coefficients of $L(x,y,t)$, we get the primitive part $-t^2+y^2-1+x^2$. Then the implicit equation of the ruled surface is

$$-z^2+y^2-1+x^2=0.$$

The implicitization of a ruled surface is also based on resultants in [13], for the same ruled surface, they computed the gcd of three resultants.

There are different methods can find the implicit equation of a rational ruled surface. The method based on the univariate resultant can only succeed on some types of rational surfaces but it is still the most efficient method for the ruled surface comparing with other methods.

5 Characterization of Rational Ruled Surfaces

The above discussions begin from a parametrization of the standard (reduced) form. However, a ruled surface can have rational parameterizations other than the standard form. The surface may even be given or designed by the implicit equation while there are not standard form for the implicit forms. Therefore it is necessary to determine whether a given general rational parametrization or an implicit equation defines a ruled surface or not. Moreover, we would like to find a standard parametric form of a given implicit surface if it defines a ruled surface, since the parametrization is well used in rendering, curvature computation and the control of position or tangency. We unify these problems as the characterization of the ruled surface from a given rational parametrization or an algebraic equation. The complete results are proposed in [33]. The discussions benefit from the standard representation of the ruled surface. Using the linearity of t, the surface can be projected to a rational parametric curve after a certain birational transformation. We will review the main theorems and the algorithms in this section, both for symbolic and numeric situations.

5.1 From Algebraic Surfaces

For a surface $\mathcal{P}$ defined implicitly by a polynomial $f(x,y,z)\in\mathbb{K}[x,y,z]$, we analyze whether $\mathcal{P}$ is a rational ruled surface. In the affirmative case, we compute a rational proper parametrization of $\mathcal{P}$ in standard reduced form (3).

Suppose $\mathcal{P}$ is a ruled surface, then $\mathcal{P}$ admits a parametrization of the form (3)

$$\boldsymbol{Q}(s,t)=(q_{11}(s)+q_{21}(s)\,t, q_{12}(s)+q_{22}(s)\,t, t).$$

In the following approach, we assume that $\mathcal{P}$ is not the plane $x-c=0$, $c\in\mathbb{K}$ (or $y-c=0$ or $z-c=0$) and $\mathcal{P}$ is not a cylinder over any of the coordinate planes of $\mathbb{K}^3$. That is, $\deg_x(f)>0, \deg_y(f)>0$ and $\deg_z(f)>0$. If $\deg_z(f)=0$ (similarly if $\deg_x(f)=0$ or $\deg_y(f)=0$), we may compute a proper parametrization $(p(s),q(s))$ of the plane curve defined by the polynomial $f(x,y)=0$. Then, $\boldsymbol{P}(s,t)=(p(s),q(s),t)\in\mathbb{K}(s,t)^3$ is a proper parametrization of $\mathcal{P}$. Except for the coordinate cylinder surfaces, we can assume that $q_{2k}\neq 0$ for some $k=1,2$ and use the projected planar curve.

$$\boldsymbol{P}^{12}(s,0)=(q_{11},q_{12},0),\quad \boldsymbol{P}^{23}\left(s,-\frac{q_{11}}{q_{21}}\right)=\left(0,q_{12}-\frac{q_{11}}{q_{21}}q_{22},-\frac{q_{11}}{q_{21}}\right),$$

$$\boldsymbol{P}^{13}\left(s,-\frac{q_{12}}{q_{22}}\right)=\left(q_{11}-\frac{q_{12}}{q_{22}}q_{21},0,-\frac{q_{12}}{q_{22}}\right).$$

The implicit equations of these three rational planar curves are factors of the polynomials

$$f^{12}(x,y)=f(x,y,0),\quad f^{23}(y,z)=f(0,y,z),\quad f^{13}(x,z)=f(x,0,z).$$

Denote the three plane algebraic curves by $\mathcal{C}^{ij}$, $ij\in\{12,13,23\}$. We give the following criterion theorem for implicit ruled surfaces.

Theorem 5.1 *A surface $\mathcal{P}$ defined by a polynomial $f(x,y,z)\in\mathbb{K}[x,y,z]$ is a rational ruled surface if and only if*

1) *At least two of the plane algebraic curves $\mathcal{C}^{ij}$, say $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, are rational with proper parametrizations $\boldsymbol{P}^{12}=(p_1,p_2)\in\mathbb{K}(s)^2$, $\boldsymbol{P}^{23}=(\widetilde{p}_1,\widetilde{p}_2)\in\mathbb{K}(s)^2$.*

2) *There exists $(R(s),S(s))\in(\mathbb{K}(s)\setminus\mathbb{K})^2$ such that*

$$\boldsymbol{P}(s,t)=\left(p_1(S(s))-t\frac{p_1(S(s))}{\widetilde{p}_2(R(s))},p_2(S(s))+t\frac{\widetilde{p}_1(R(s))-p_2(S(s))}{\widetilde{p}_2(R(s))},t\right)$$

is a rational proper parametrization of $\mathcal{P}$, and (R,S) is proper.

Suppose the statement 1 holds, then a ruled surface must have a rational parametrization in the statement 2 which is construed by the solutions from the statement 1. The statement 1 requires computing two planar parametrizations that will be used to determine a rational planar base curve of the ruled surface and the ruling direction of the ruled surface in the statement 2. The functions S and R are for coordinating the parameterization of the base curve. To simplify the computation, we prove that the statement 2 is equivalent to checking the rationality of a plane curve. Let $N(x,y)=\mathrm{Content}_t(g)$, where $g(x,y,t)=\mathrm{numer}\left(f\left(p_1(y)-t\frac{p_1(y)}{\widetilde{p}_2(x)},p_2(y)+t\frac{\widetilde{p}_1(x)-p_2(y)}{\widetilde{p}_2(x)},t\right)\right)$. The function $\mathrm{Content}_t(\cdot)$ returns the content of a polynomial with respect to the variable t.

Corollary 5.2 *Let $\mathcal{P}$ be a surface defined by a polynomial $f(x,y,z)$ such that statement* 1 *in Theorem* 5.1 *holds. $\mathcal{P}$ is a rational ruled surface if and only if there exists a factor of $N(x,y)$ defining a rational plane curve $\mathcal{C}_N$. In this case, $(R(s),S(s))\in\mathbb{K}(s)^2$, where $S\notin\mathbb{K}$, is a rational proper parametrization of $\mathcal{C}_N$.*

Remark 5.3 Theorem 5.1 and Corollary 5.2 are simplified versions of Theorem 2 and Corollary 3 in [33] with the assumptions that the surface admits a parametrization of (3) and $p_1\neq 0$. For other cases with certain degeneracies, we have similar discussions and results.

Determining if a surface is a ruled surface is reduced to finding a rational parametrization of the algebraic plane curves successively. There are algorithms to find a rational parameterization of a plane algebraic curve (see [44–46]).

Theorem 5.1 and Corollary 5.2 were proved constructively and an algorithm for characterizing ruled surfaces was naturally proposed.

Here we provide an example to illustrate the identification and the parametrization for ruled surfaces from an implicit equation.

Example 5.4 Consider an algebraic surface over $\mathbb{C}$ defined by $f(x,y,z) = -x^2yz^2 + 2\,xy^2z^2 - 2\,xyz^3 - 5\,y^3z^2 + 2\,y^2z^3 - x^2z^2 - 8\,xy^2z + 4\,xyz^2 - 23\,y^2z^2 + 4\,yz^3 + 4\,z^4 - 4\,x^2y - 16\,xyz + 2\,xz^2 - 10\,yz^2 + 4\,z^3 - 4\,x^2 + 8\,xy + 16\,yz + 16\,z^2 + 8\,x - 20\,y + 8\,z + 12$. We first find it is not a cylinder over any of the coordinate planes since the degrees of x, y and z are positive. Then we compute

$$\begin{aligned} f^{12} &= -4\,x^2y - 4\,x^2 + 8\,xy + 8\,x - 20\,y + 12, \\ f^{23} &= -5\,y^3z^2 + 2\,y^2z^3 - 23\,y^2z^2 + 4\,yz^3 + 4\,z^4 - 10\,yz^2 \\ &\quad +4\,z^3 + 16\,yz + 16\,z^2 - 20\,y + 8\,z + 12. \end{aligned}$$

These equations define two plane curves $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, and we find that these curves have rational parametrizations

$$\begin{aligned} \boldsymbol{P}^{12} &= (p_1, p_2) = \left(s, -\frac{s^2 - 2\,s - 3}{s^2 - 2\,s + 5}\right), \\ \boldsymbol{P}^{23} &= (\widetilde{p}_1, \widetilde{p}_2) = \left(\frac{s^4 + 4\,s^3 + 256}{2\,(8 - s)\,(s^2 + 16)}, \frac{s^3 + 2\,s^2 + 16\,s + 32}{8(8 - s)}\right). \end{aligned}$$

Checking whether there exists a rational curve $\mathcal{C}_N$ defined by a factor of the polynomial $N(x,y) = \mathrm{Content}_t(g_1)$. We find that

$$N(x,y) = 2\ (x - 8)^3\,(2 + x - 2\,y)\,.$$

Since we look for $(R,S) \in (\mathbb{C}(s) \setminus \mathbb{C})^2$, we consider the curve $\mathcal{C}_N$ defined by the irreducible polynomial $2 + x - 2\,y$. This curve has a rational parametrization

$$(R(s), S(s)) = \left(s, 1 + \frac{s}{2}\right).$$

Therefore we conclude that the given surface is ruled and get a parametrization of the reduced standard form as

$$\begin{aligned} \boldsymbol{P}(s,t) &= \left(p_1(S(s)) - t\frac{p_1(S(s))}{\widetilde{p}_2(R(s))}, p_2(S(s)) + t\frac{\widetilde{p}_1(R(s)) - p_2(S(s))}{\widetilde{p}_2(R(s))}, t\right) \\ &= \left(\frac{s^3 + 2\,s^2 + 16\,s + 32 + t\,(8\,s - 64)}{2\,s^2 + 32}, \frac{-s^2 + 16 + 4\,ts}{s^2 + 16}, t\right). \end{aligned}$$

An alternative method is to find a μ-basis for a rational ruled surface, starting from its implicit representation. A parametrization for this ruled surface is then derived from this μ-basis (see [47]).

Algorithm 5.5 [47] **Computing a μ-basis for an algebraic ruled surface**
Input: An algebraic surface $\mathcal{P}$ defined by $f(x,y,z,w) = 0$.
Output: A parametrization $\boldsymbol{P}$ (without non-generic base points, with a directrix having lowest degree) and a mu-basis of $f(x,y,z,w) = 0$ if $\mathcal{P}$ is ruled.

1. Select a nonsingular point of $f(x,y,z,w) = 0$ and a plane $\mathcal{L}$ passing through the point which is not a tangential plane.

2. Find a rational parametrization $\boldsymbol{C}(s)$ of the intersection curve of $\mathcal{P}$ and $\mathcal{L}$ if it exists, otherwise return " $\mathcal{P}$ is not a ruled surface".

3. Compute a μ-basis $\boldsymbol{p_c}, \boldsymbol{q_c}, \boldsymbol{r_c}$ of $\boldsymbol{C}(s)$, where $\boldsymbol{p_c}$ is a representation of $\mathcal{L}$.

4. Let $\boldsymbol{p}(s) = \alpha\boldsymbol{p_c} + \boldsymbol{q_c}$ and $\boldsymbol{q}(s) = \beta\boldsymbol{p_c} + \boldsymbol{r_c}$. Solve for α, β from $f(\boldsymbol{p}(s)\bigcap\boldsymbol{q}(s)) \equiv 0$ and update $\boldsymbol{p}(s), \boldsymbol{q}(s)$. Otherwise if there is no solution for α, β then return " $\mathcal{P}$ is not a ruled surface".

5. Compute a μ-basis $\widetilde{\boldsymbol{p}}$ and $\widetilde{\boldsymbol{q}}$ of the dual ruled surface $\boldsymbol{p} + t\boldsymbol{q}$ and compute the moving plane $\boldsymbol{r}(s,t)$ of the ruled surface $\widetilde{\boldsymbol{p}} + t\widetilde{\boldsymbol{q}}$. Output the parametrization $\widetilde{\boldsymbol{p}} + t\widetilde{\boldsymbol{q}}$ and its μ-basis $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\}$.

5.2 From Parametric Surfaces

Consider a surface $\mathcal{P}$ defined by a parametrization (not necessarily proper) over $\mathbb{K}$,

$$\mathbf{M}(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)) \in \mathbb{K}(s,t)^3.$$

We shall identify whether $\mathcal{P}$ is a ruled surface not given in the standard form (1) (or (3), (4)), and in the affirmative case we compute a proper reduced reparametrization of the form (3).

A direct approach to this problem is to implicitize the parametrization and apply the results of the previous subsection. But the implicitization of a general parametrization is not easy and we would like to approach the problem without implicitizing. Precisely, we will find a linear parameter transformation to reparameterized the given parametrization. Note that any reparametrization of a rational parametrization is again a parametrization of the same variety.

Similar to the implicit case, we first assume that $\mathcal{P}$ is neither a plane nor a cylinder surface. Note that this assumption of not being a plane is still general, since one can easily deduce whether a parametrically given surface is a plane. For the cylinder case, we can apply the result presented in [13] (Theorem 5) which gives a criterion characterizing the cylinder surface from a given rational parametrization.

We now give the following theorem for the parametric case. For this purpose, we need to compute a rational proper parametrization of $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$ (or $\mathcal{C}^{13}$ if needed, similar to the statement 1 of Theorem 5.1). The computation is mainly based on the computing resultants successively.

Theorem 5.6 *A surface $\mathcal{P}$ defined by the parametrization $\boldsymbol{M}(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)) \in \mathbb{K}(s,t)^3$ is a rational ruled surface if and only if*

1) *At least two of the three plane algebraic curves $\mathcal{C}^{ij}$, say $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, are rational with proper parametrizations $\boldsymbol{P}^{12} = (p_1, p_2) \in \mathbb{K}(s)^2$, $\boldsymbol{P}^{23} = (\widetilde{p}_1, \widetilde{p}_2) \in \mathbb{K}(s)^2$ respectively.*

2) *There exists $(U,V) \in \mathbb{K}(s,t)\backslash\mathbb{K}$ such that $p_1(V) - m_3\frac{p_1(V)}{\widetilde{p}_2(U)} - m_1 = p_2(V) + m_3\frac{\widetilde{p}_1(U) - p_2(V)}{\widetilde{p}_2(U)} - m_2 = 0$. In this case,*

$$\mathbf{P}(s,t) = \left(p_1(S(t_1)) - t_2\frac{p_1(S(t_1))}{\widetilde{p}_2(R(t_1))}, p_2(S(t_1)) + t_2\frac{\widetilde{p}_1(R(t_1)) - p_2(S(t_1))}{\widetilde{p}_2(R(t_1))}, t_2\right),$$

is a rational proper parametrization of $\mathcal{P}$, where $(R,S)\in(\mathbb{K}(t_1)\setminus\mathbb{K})^2$ is a rational proper parametrization of the curve $\mathcal{C}_N$ defined parametrically by (U,V).

Remark 5.7 Theorem 5.6 is a simplified version of Theorem 5[33] with the assumptions that the surface admits a reparametrization of (3) and $p_1\neq 0$. For other cases with certain degeneracies, we have similar discussions and results.

In Theorem 5.6, one important task is to solve for (U,V) from the equation system $p_1(x)-m_3p_1(x)/\widetilde{p}_2(y)-m_1=p_2(x)+m_3(\widetilde{p}_1(y)-p_2(x))/\widetilde{p}_2(y)-m_2=0$. The solution is zero dimensional if $\boldsymbol{M}(s,t)$ defines a ruled surface and then the computation of (U,V) is efficient.

Similar to Theorem 5.1, the proof of Theorem 5.6 is also constructive and so an algorithm for the parametric surface was proposed.

We illustrate the computation with the following example.

Example 5.8 Consider the surface $\mathcal{P}$ defined by the parametrization

$$\begin{aligned}\boldsymbol{M}(s,t)&=(m_1(s,t),m_2(s,t),m_3(s,t))\\&=\left(\frac{(t+1)\,s^3+\left(3t^2+5t+10\right)s^2+\left(3t^3+7t^2+28t-48\right)s+t^4+3t^3+18t^2+48t+32}{2\,(1+t)\,(s^2+2ts+t^2+16)},\right.\\&\qquad\left.-\frac{s^2t+2\,st^2+t^3-3\,s^2-2\,ts+t^2-16\,t-16}{(1+t)\,(s^2+2\,ts+t^2+16)},\frac{s}{1+t}\right)\in\mathbb{R}(s,t)^3.\end{aligned}$$

We first find it is not a cylinder over any of the coordinate planes using the cylinder criterion. Then we compute

$$\begin{aligned}f^{12}&=-4\,x^2y-4\,x^2+8\,xy+8\,x-20\,y+12,\\f^{23}&=-5y^3z^2+2y^2z^3-23y^2z^2+4yz^3+4z^4-10yz^2+4z^3+16yz+16z^2-20y+8z+12.\end{aligned}$$

These two equations define two plane curves $\mathcal{C}^{12}$ and $\mathcal{C}^{23}$, and we find that they have have rational parametrizations

$$\begin{aligned}\boldsymbol{P}^{12}&=(p_1,p_2)=\left(s,-\frac{s^2-2\,s-3}{s^2-2\,s+5}\right),\\\boldsymbol{P}^{23}&=(\widetilde{p}_1,\widetilde{p}_2)=\left(\frac{s^4+4\,s^3+256}{2\,(8-s)\,(s^2+16)},\frac{s^3+2\,s^2+16\,s+32}{8(8-s)}\right).\end{aligned}$$

Checking whether there exists a rational curve defined by (U,V) we solve from the equation system

$$p_1(V)-m_3\frac{p_1(V)}{\widetilde{p}_2(U)}-m_1=0,\quad p_2(V)+m_3\frac{\widetilde{p}_1(U)-p_2(V)}{\widetilde{p}_2(U)}-m_2=0,$$

and we obtain

$$U(s,t)=s+t,\quad V(s,t)=\frac{s+t+2}{2}.$$

Then (U,V) defines a plane curve $\mathcal{C}_N$ with implicit equation $x-2y+2=0$ and this curve has a rational parametrization

$$(R(s),T(s))=\left(s,1+\frac{s}{2}\right).$$

Therefore, we conclude that the given surface is ruled and get a parametrization of the reduced standard form as

$$\begin{aligned}\boldsymbol{P}(s,t) &= \left(p_1(S(s)) - t\frac{p_1(S(s))}{\widetilde{p}_2(R(s))}, p_2(S(s)) + t\frac{\widetilde{p}_1(R(s)) - p_2(S(s))}{\widetilde{p}_2(R(s))}, t\right)\\ &= \left(\frac{s^3+2\,s^2+16\,s+32+t\,(8\,s-64)}{2\,s^2+32}, \frac{-s^2+16+4\,ts}{s^2+16}, t\right).\end{aligned}$$

5.3 Symbolic-Numeric Approach for Parametrizing Ruled Surfaces

In practical applications, one has to deal with numerical functions that are given approximately, probably because they are derived from exact data which has been perturbed under some previous measuring process or manipulation. For these numerical objects, one can adapt the symbolic algorithms presented by certain numerical techniques. Recently, numerical algorithms have been designed to determine ruled surfaces close to an input (not necessarily ruled) surface, and the distance between the input and the output surface is computed. For further detail, we refer the reader to [38].

The problem of numerical reparametrization for (ruled) surfaces can be looked at from two different points of view: The implicit and parametric. More precisely:

[Numerical Implicit Ruled Surface Problem] Given a polynomial $f(x,y,z) \in \mathbb{C}[x,y,z]$ (with perturbed floating point coefficients) defining an algebraic surface $\mathcal{V}$, find a rational parametrization $\boldsymbol{P}(s,t) \in \mathbb{C}(s,t)^3$ of an algebraic ruled surface $\mathcal{W}$ such that $\mathcal{V}$ and $\mathcal{W}$ are close enough.

[Numerical Parametric Ruled Surface Problem] Given a rational parametrization $\mathcal{M}(s,t) \in \mathbb{C}(s,t)^3$ (with perturbed floating point coefficients) of an algebraic surface $\mathcal{V}$, find a rational parametrization $\boldsymbol{P}(s,t) \in \mathbb{C}(s,t)^3$ of an algebraic ruled surface $\mathcal{W}$ such that $\mathcal{V}$ and $\mathcal{W}$ are close enough.

5.3.1 Numerical Implicit Ruled Surface Problem

Given a surface $\mathcal{V}$ defined implicitly by a polynomial $f(x,y,z) \in \mathbb{C}[x,y,z]$ with perturbed floating point coefficients, we present an algorithm that returns a rational parametrization $\boldsymbol{P}(s,t) = (p_1(r_1(s)) + tq_1(r_2(s)), p_2(r_1(s)) + tq_2(r_2(s)), t)$ that defines a ruled surface $\mathcal{W}$. In this case, we say that $\mathcal{V}$ is an approximate rational ruled surface. In Theorem 5.10, we show how to compute the distance between the input surface $\mathcal{V}$ and the output surface $\mathcal{W}$.

Algorithm 5.9 (see [38]) **Computation of a rational ruled surface from an approximate implicit surface**

Input: An algebraic surface $\mathcal{V}$ defined by $f(x,y,z) = 0$.

Output: A ruled surface $\mathcal{W}$ parametrized by $\boldsymbol{P}(s,t) = (p_1(r_1(s)) + tq_1(r_2(s)), p_2(r_1(s)) + tq_2(r_2(s)), t)$ or the message "$\mathcal{V}$ is not an approximate rational ruled surface".

1. Compute the polynomials $f(x,y,0)$ and $\overline{f}(0,x,y,1)$, and check whether there exist two approximate rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by an approximate factor of these two polynomials. In the affirmative case, go to Step 2. Otherwise, Return "$\mathcal{V}$ is not an approximate rational ruled surface".

2. Compute $\mathcal{P}_1 = (p_1, p_2) \in \mathbb{C}(s)^2$ and $\mathcal{P}_2 = (q_1, q_2) \in \mathbb{C}(s)^2$ approximate proper rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$.

3. Let $g(x, y, t) = \mathrm{numer}(f(p_1(x) + tq_1(y), p_2(x) + tq_2(y), t))$. Check whether there exists an approximate rational plane curve $\mathcal{D}$ defined by an approximate factor $h(x, y)$ of this polynomial. In the affirmative case, go to Step 4. Otherwise, Return "$\mathcal{V}$ is not an approximate rational ruled surface".

4. Compute an approximate proper rational parametrization $R(s) := (r_1(s), r_2(s)) \in (\mathbb{C}(s) \setminus \mathbb{C})^2$ of $\mathcal{D}$).

5. Return "$\mathcal{W}$ is a ruled surface parametrized by

$$\boldsymbol{P}(s,t) = (p_1(r_1(s)) + tq_1(r_2(s)), p_2(r_1(s)) + tq_2(r_2(s)), t)\text{''}.$$

Let $\mathcal{V}$ and $\mathcal{W}$ be the input and output surfaces, respectively, of Algorithm 5.9. In addition, let $f(x, y, z)$ and $g(x, y, z)$ be the defining polynomials of $\mathcal{V}$ and $\mathcal{W}$, respectively, and let $\boldsymbol{P}(s, t) \in \mathbb{C}(s, t)^3$ be the parametrization of $\mathcal{W}$ output by the algorithm. Next, we study the distance between these surfaces. For this purpose, we consider $\boldsymbol{T}(s,t) = \frac{\frac{\partial \boldsymbol{P}}{\partial s} \times \frac{\partial \boldsymbol{P}}{\partial t}}{\|\frac{\partial \boldsymbol{P}}{\partial s} \times \frac{\partial \boldsymbol{P}}{\partial t}\|_2}$, and $\boldsymbol{N}(a, b, c) = \frac{\nabla f(a,b,c)}{\|\nabla f(a,b,c)\|_2}$, and we get the following theorem.

Theorem 5.10 (see [38]) *Let* $(s_0, t_0) \in \boldsymbol{C}^2$, *and* $(a_0, b_0, c_0) \in \mathcal{V}$.

1) *If* $\nabla f(\boldsymbol{P}(s_0, t_0))$ *and* $\boldsymbol{T}(s_0, t_0)$ *are not orthogonal, then*

$$d(\boldsymbol{P}(s_0,t_0), \mathcal{V}) \le n \left| \frac{f(\boldsymbol{P}(s_0,t_0))}{\nabla f(\boldsymbol{P}(s_0,t_0)) \cdot \boldsymbol{T}(s_0,t_0)} \right|.$$

2) *If* $\nabla g(a_0, b_0, c_0)$ *and* $\boldsymbol{N}(a_0, b_0, c_0)$ *are not orthogonal, then*

$$d((a_0,b_0,c_0), \mathcal{W}) \le n \left| \frac{g(a_0,b_0,c_0)}{\nabla g(a_0,b_0,c_0) \cdot \boldsymbol{N}(a_0,b_0,c_0)} \right|.$$

Example 5.11 Let $\mathcal{V}$ be the surface over $\mathbb{C}$ implicitly defined by the polynomial

$$\begin{aligned} f(x,y,z) = & -y - 5z + 8zy - 6z^2 - 2.xy - 5.9999xz + 31z^2y - 42.xz^2 + 10z^2y^2 \\ & +22z^3y + 8zy^2 - 36z^3x + 1.001x^2 + 12x^2z + 36.0001x^2z^2 + 2y^2 \\ & -18zyx - 36z^2yx + 3z^3 + 4z^4 - 0.001y^3 + 0.001. \end{aligned}$$

We apply Algorithm 5.9 to check whether $\mathcal{V}$ is an approximate rational ruled surface and, in the affirmative case, we compute a parametrization of a ruled surface $\mathcal{W}$. Afterwards, we measure the distance between $\mathcal{V}$ and $\mathcal{W}$.

The algorithm returns the ruled surface $\mathcal{W}$ defined by the proper rational parametrization $\boldsymbol{P}(s,t) = (p_1(r_1(s)) + tq_1(r_2(s)), p_2(r_1(s)) + tq_2(r_2(s)), t)$ where

$$\begin{aligned}
&p_1(r_1(s)) + tq_1(r_2(s))\\
&= -0.1966388521\\
&\cdot(2.055378432\cdot 10^7 ts^2 - 2.055377510\cdot 10^4 ts^3 + 1.028717526\cdot\\
&\quad 10ts^4 + 4.117141228\cdot 10^7 ts + 2.059706394\cdot 10^7 t\\
&\quad +2.492580063\cdot 10^6 s^2 + 4.983272946\cdot 10^6 s\\
&\quad +2.491444722\cdot 10^6 + 7.465434870\cdot 10^2 s^3 - 5.279803414 s^4)\\
&/((2.002\cdot 10^6 - 2\cdot 10^3 s + s^2)(1.439788009 s^2 + 2.879792 s + 1.440004)),
\end{aligned}$$

$$\begin{aligned}
&p_2(r_1(s)) + tq_2(r_2(s))\\
&= -3.441179912\\
&\cdot(2.383354895\cdot 10^6 ts^2 - 2.383353821\cdot 10^3 ts^3 + 1.19286984 ts^4\\
&\quad +4.774102602\cdot 10^6 ts + 2.388363165\cdot 10^6 t\\
&\quad -0.3954871614 s^4 + 6.032184769\cdot 10^2 s^3\\
&\quad -2.298544147\cdot 10^5 s^2 - 4.61554819\cdot 10^5 s - 2.310967918\cdot 10^5)\\
&/((2.002\cdot 10^6 - 2000 s + s^2)(1.439788009 s^2 + 2.879792 s + 1.440004)).
\end{aligned}$$

One may check that these surfaces are very close. Actually, $d(\boldsymbol{P}(s_0,t_0),\mathcal{V}) \le 0.042$ and $d((a_0,b_0,c_0),\mathcal{W}) \le 0.00021$.

5.3.2 Numerical Parametric Ruled Surface Problem

Given a surface $\mathcal{V}$ defined by a parametrization $\boldsymbol{M}(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)) \in \mathbb{C}(s,t)^3$ with perturbed floating point coefficients, we present an algorithm that outputs a rational parametrization $\boldsymbol{P}(s,t) = (p_1(r_1(s))+tq_1(r_2(s)), p_2(r_1(s))+tq_2(r_2(s)), t)$ parametrizing a ruled surface $\mathcal{W}$. In this case, we say that the surface $\mathcal{V}$ is an approximate rational ruled surface. In Theorem 5.13, we show how to compute the distance between the input surface $\mathcal{V}$ and the output surface $\mathcal{W}$.

Algorithm 5.12 (see [38]) **Computation of a rational ruled surface from an approximate parametric surface**

Input: A surface $\mathcal{V}$ defined by the parametrization $\boldsymbol{M}(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)) \in \mathbb{C}(s,t)^3$.

Output: A ruled surface $\mathcal{W}$ parametrized by $\boldsymbol{P}(s,t) = (p_1(r_1(s)) + tq_1(r_2(s)), p_2(r_1(s)) + tq_2(r_2(s)), t)$ or the message "$\mathcal{V}$ is not an approximate rational ruled surface".

1. Compute the polynomials $f(x,y,0)$ and $\overline{f}(0,x,y,1)$. Check whether there exist two approximate rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by an approximate factor of these two polynomials. In the affirmative case, let $f_1(x,y)$ and $f_2(x,y)$ be these polynomials and go to Step 2. Otherwise, Return "$\mathcal{V}$ is not an approximate rational ruled surface".

2. Compute $\boldsymbol{P}_1 = (p_1, p_2) \in \mathbb{C}(s)^2$ and $\boldsymbol{P}_2 = (q_1, q_2) \in \mathbb{C}(s)^2$ approximate proper rational parametrizations of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$.

3. Check whether there exists an approximate rational plane curve $\mathcal{D}$ defined by an approximate factor of the polynomial $R(x, y, s) = \mathrm{Res}_t(e_1, e_2)$, where $e_i(x, y, s, t) = \mathrm{numer}(p_i(x) + m_3 q_i(y) - m_i(s, t))$, $i = 1, 2$. In the affirmative case, compute, $R(s) := (r_1(s), r_2(s)) \in (\mathbb{C}(s) \setminus \mathbb{C})^2$, an approximate proper rational parametrization of $\mathcal{D}$, and Return "$\mathcal{W}$ is a ruled surface parametrized by $\boldsymbol{P}(s, t) = (p_1(r_1(s)) + tq_1(r_2(s)),\ p_2(r_1(s)) + tq_2(r_2(s)), t)$". Otherwise, Return "$\mathcal{V}$ is not an approximate rational ruled surface".

Theorem 5.13 (see [38]) *Let* $(s_0, t_0) \in \boldsymbol{C}^2$.
1) *If* $\nabla f(\boldsymbol{P}(s_0, t_0))$ *and* $\boldsymbol{T_P}(s_0, t_0)$ *are not orthogonal, then*

$$d(\boldsymbol{P}(s_0, t_0), \mathcal{V}) \leq n \left| \frac{f(\boldsymbol{P}(s_0, t_0))}{\nabla f(\boldsymbol{P}(s_0, t_0)) \cdot \boldsymbol{T_P}(s_0, t_0)} \right|.$$

2) *If* $\nabla g(\boldsymbol{M}(s_0, t_0))$ *and* $\boldsymbol{T_M}(s_0, t_0)$ *are not orthogonal, then*

$$d(\boldsymbol{M}(s_0, t_0), \mathcal{W}) \leq n \left| \frac{g(\boldsymbol{M}(s_0, t_0))}{\nabla g(\boldsymbol{M}(s_0, t_0)) \cdot \boldsymbol{T_M}(s_0, t_0)} \right|.$$

Example 5.14 Let $\mathcal{V}$ be the surface defined by $\boldsymbol{M}(s, t) = (m_1(s, t), m_2(s, t), m_3(s, t)) \in \mathbb{C}(s, t)^3$, where

$$m_1(s, t) = \frac{0.9999s^2 + 1.9999ts + t^2 - 2t - 2.0003s}{(s + t + 2)s},$$
$$m_2(s, t) = \frac{s^2 + 2ts + t^2 - 4.9999t - 0.00001s}{(s + t + 2)s}, \qquad m_3(s, t) = \frac{1.0001t - 0.0001s}{s}.$$

Let us apply Algorithm 5.12 to check whether $\mathcal{V}$ is an approximate rational ruled surface and, in the affirmative case, compute a parametrization of a ruled surface $\mathcal{W}$. Afterwards, we will measure the distance between $\mathcal{V}$ and $\mathcal{W}$. By the algorithm, we get the new rational ruled surface, $\mathcal{W}$, defined parametrically by $\boldsymbol{P}(s, t) = (p_1(r_1(s)) + tq_1(r_2(s)), p_2(r_1(s)) + tq_2(r_2(s)), t)$, where

$$p_1(r_1(s)) + tq_1(r_2(s)) = 0.000124971881t + 0.9997750481ts + s,$$
$$p_2(r_1(s)) + tq_2(r_2(s)) = -0.7496813123t + 1.749581340ts + 0.4999850244 + 0.5001149706s.$$

One may check that these surfaces are very close. In fact, from Theorem 5.13, we get $d(\boldsymbol{P}(s_0, t_0), \mathcal{V}) \leq 1.915937999 \cdot 10^{-13}$, and $d(\boldsymbol{M}(s_0, t_0), \mathcal{W}) \leq 3.186985570 \cdot 10^{-10}$.

5.4 Discussions

In many applications such as geometric modeling and computer aided design, people can get benefits if the surface is known as the ruled surfaces or can be approximated by ruled surfaces. However, it is difficult to characterize the type of the surface from a given parametric or implicit equation. Only quadratic surfaces are discussed as a classical results. The result reviewed above is the first work to characterize the ruled surfaces either from a given parametric or implicit equation[33]. The theorems are given with constructive proofs using algebraic computations. Then the algorithms can be proposed according to the proofs. Start with an implicit equation, a simpler method is given to characterize the ruled surface by finding the μ-bases with the geometric considerations (see Algorithm 5.5). Moreover, if one focuses on the ruled surfaces which are developable, there is a simpler and more efficient characterization method proposed by [43].

The numerical computations is ineluctable in practical applications. But the most results of the researches for representations are considered and designed with exact mathematical assumption. The recent result proposes a first approach to identify the ruled surface from a numerical consideration[38]. As well as the algorithms, the error estimations are given explicitly. This approach is expected to be helpful in computer aided design for engineering.

6 Conclusion

This paper takes a look inside the representations of a ruled surface other than the traditional standard parametric form. The different forms have respective advantages: The parametric representation is popular in CAD since it is easy to render, the implicit representation brings benefits for collision detection, the homogenous form is suitable for syzygy computation (μ-basis) and the Plücker form leads to simple cases of line geometry. Thus it is a natural problem to analyze the relationships between these representations, and furthermore the ways to transform one expression form to another. Hence, for the convenience of readers, we reviewed the methods focused on these problems and summarized the main results. Some examples are also provided to illustrate the computations.

For practical applications, we should further extend these methods to the numerical computations. An attendant and important problem is to give an error analysis for the numerical algorithms. The latest results of the authors provide numerical transform algorithms with error estimations.

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