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A COMBINATORIAL IDENTITY AND APPLICATIONS

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ABSTRACT. An identity for the finite sum $\sum_1^N \frac{z^n}{q^n - r}$ is given. Related sums (or series) were studied by Scherk, Clausen, Ramanujan, Shanks, Andrews, and others. We use such identity to give new formulas for $\sum_1^\infty \frac{z^n}{q^n - r}$, the Riemann zeta function and the Euler–Mascheroni constant. An irrationality result is also proved.

1. INTRODUCTION AND RESULTS

The aim of this paper is to prove a finite sum identity for the sum $\sum_{n=1}^N \frac{z^n}{q^n - r}$, which seems to be new, and to give some applications of it. Related sums have been investigated by several authors. A related accelerated series has been given by Clausen (proved in print by Scherk): If $|q| > 1$,

$$\sum_{k=1}^{\infty} \frac{1}{q^k - 1} = \sum_{k=1}^{\infty} \frac{q^k + 1}{q^{k^2} (q^k - 1)}.$$

Ramanujan has discovered this and other related ones, see [4], pp. 147–149. D. Shanks developed an acceleration method in [9] in which he considered sums of the above type. In [2] G. E. Andrews used the little q -Jacobi polynomials to explain and extend Shanks' observations.

In another direction, P. Erdős proved in 1948 the irrationality of $\sum_{n=1}^{\infty} 1/(q^n - 1)$, $1 < q \in \mathbb{N}$. In [5, 6] P. Borwein proved irrationality results for $\sum_{n=1}^{\infty} 1/(q^n - r)$ using the Padé approximants and later this was extended by D. Duverney [7] and M. Prevost [8] to other similar series.

The following identity, which we believe is new, is the key to obtain all the results in this paper.

Theorem 1. *Define, for $1 \leq k \leq n$,*

$$\epsilon_{n,k} = \epsilon_{n,k}(q, r, z) := \frac{(-1)^k z^{n+1} r^{k-1}}{(q^n - r) \cdots (q^{n-k+1} - r)} \cdot \frac{(q^{k-1} - 1) \cdots (q - 1)}{(q^k - z) \cdots (q - z)}.$$

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Then, if $2 < N$, one has

$$\begin{aligned}
 & - \sum_{n=2}^{N-1} \frac{z^n}{q^n - r} + \frac{z^N q}{(q^N - r)(z - q)} + \frac{z^2}{(q - z)(q - r)} \\
 & = \sum_{n=2}^{N-1} \frac{(-z)^n r^{n-1} q^n}{(q^n - z) \cdots (q - z)} \cdot \frac{(q^{n-1} - 1) \cdots (q - 1)}{(q^n - r) \cdots (q - r)} - \sum_{k=2}^{N-1} \epsilon_{N,k} \\
 & \quad + \frac{(-z)^N r^{N-1}}{(q^N - r) \cdots (q - r)} \cdot \frac{(q^{N-1} - 1) \cdots (q - 1)}{(q^{N-1} - z) \cdots (q - z)}.
 \end{aligned}$$

For the next corollary we write, as usual, γ for the Euler–Mascheroni constant and $\zeta(s)$ for the Riemann zeta function. We give some easy applications of this theorem. In what follows, as usual, $(q^{n-1} - 1) \cdots (q - 1)$ is equal to 1 if $n = 1$.

Corollary 1. *i) Let $1 < |q|$, $|z| < |q|$ and $q^n \neq r$, $q^n \neq z$ for all $n = 1, 2, 3, \dots$. Then*

$$\sum_{n=1}^{\infty} \frac{z^n}{q^n - r} = - \sum_{n=1}^{\infty} \frac{(-z)^n r^{n-1} q^n}{(q^n - z) \cdots (q - z)} \cdot \frac{(q^{n-1} - 1) \cdots (q - 1)}{(q^n - r) \cdots (q - r)}.$$

ii)

$$1 - \gamma = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{2^n}{(2^n - 1)} \cdot \frac{(-1)^{j+1} j^{n-1}}{(2^n + j)(2^{n-1} + j) \cdots (2 + j)}.$$

iii) *If $\text{Re } s > 1$ and $0 < r < 1$ then*

$$\Gamma(s)\zeta(s) \sum_{n=1}^{\infty} \frac{r^{n-1}}{n^s} = \int_0^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} r^{n-1}}{(1 - e^{-nt})} \cdot \frac{1}{(e^{nt} - r) \cdots (e^t - r)} \right\} t^{s-1} dt.$$

It is immediate using the series in the right hand side of i) that, with fixed q , $1 < |q|$, the sum $\sum_{n=1}^{\infty} \frac{z^n}{q^n - r}$ can be continued analytically for $z \in \mathbb{C}$; $z, r \neq q^n$. This is a well-known fact.

Also from the above formula i) one has the following corollary.

Corollary 2. *Let f be a real or complex function defined at natural numbers. Let $1 < |q|$ and $q^n \neq r, z$ for all $n = 1, 2, 3, \dots$. Also let $|f(n)| = O(\beta^n)$, for some $0 < \beta$. If*

$$g(n) := g(q, r, n) = \frac{r^{n-1} q^n (q^{n-1} - 1) \cdots (q - 1)}{(q^n - r) \cdots (q - r)},$$

then

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{z^n}{q^n - r} \left\{ \sum_{m=1}^{\infty} \frac{f(m) z^m}{q^{nm} (q^m - z) \cdots (q - z)} \right\} \\
 & = \sum_{k=2}^{\infty} \frac{(-z)^k}{(q^k - z) \cdots (q - z)} \left\{ \sum_{m=1}^{k-1} (-1)^{m+1} f(m) g(k - m) \right\}.
 \end{aligned}$$

Finally we apply Theorem 1 to prove an irrationality result. We define for $2 < N < M$:

$$\begin{aligned} \alpha_{N,M} &= \alpha_{N,M}(q, r, z) \\ &:= \sum_{n=N}^{M-1} \frac{z^n}{q^n - r} + \frac{q}{q - z} \left\{ -\frac{z^N}{q^N - r} + \frac{z^M}{q^M - r} \right\} - \sum_{k=2}^{M-1} \epsilon_{M,k} + \sum_{k=2}^{N-1} \epsilon_{N,k}, \end{aligned}$$

where $\epsilon_{n,k}$ is defined in Theorem 1.

Theorem 2. *Let r, q, z be non zero integers such that $1 < |q|, q^n \neq r, z$ for all $n = 1, 2, 3, \dots$. Assume that:*

i) there exists a fixed number $\delta > 0$ such that $2 < N_{j-1} < M_{j-1} < N_j$ and $(\sqrt{2} + \delta)M_{j-1} \leq N_j$, for $j \geq j_0 \geq 2$.

ii) $N_j = o(M_{j-1}^2)$ as $j \rightarrow \infty$.

iii) $f(i) : \mathbb{N} \rightarrow \mathbb{Z}$ is a bounded function such that there exist infinitely many indices i such that $f(i) \neq 0$.

Then

$$\sum_{i=1}^{\infty} f(i) \alpha_{N_i, M_i}$$

is an irrational number.

Example. Let $\delta > 0$ be a fixed number. Let $[\cdot], \mu(\cdot)$ denote the nearest integer function and the Möbius function, respectively. Set

$$\begin{aligned} M_j &:= [(\sqrt{2} + 3\delta)^j], \\ N_j &:= [M_{j-1}(\sqrt{2} + 2\delta)], \\ f(i) &:= \mu(i). \end{aligned}$$

Then it easy to see that the hypotheses of Theorem 2 hold. Thus if r, q, z are non zero integers such that $1 < |q|, q^n \neq r, z$ for all $n = 1, 2, 3, \dots$, then

$$\sum_{i=1}^{\infty} \mu(i) \alpha_{N_i, M_i}$$

is an irrational number.

2. PROOFS

Proof of Theorem 1. One has

$$\frac{1}{x} - \frac{b_1 \cdots b_K}{x(x+a_1) \cdots (x+a_K)} = \sum_{k=1}^K \frac{b_1 \cdots b_{k-1} \{x+a_k - b_k\}}{x(x+a_1) \cdots (x+a_k)}.$$

This follows by writing $A_k = \frac{b_1 \cdots b_k}{x(x+a_1) \cdots (x+a_k)}$; $A_0 = 1/x$, where the left hand side of this formula is $A_0 - A_K$ and each summand on the right is $A_{k-1} - A_k$.

In the above formula set $x := q^n - r$, $a_k := r - rq^k$, $b_k := b_k(q, r, z) = \frac{rq^k(q^k-1)}{z-q^k}$ with the convention that $b_1 \cdots b_{k-1} = 1$ if $k = 1$ and $K := n - 1$. Multiply by z^n and add from $n = 2$ to N to give

$$\begin{aligned} \sum_{n=2}^N \frac{z^n}{q^n - r} - \sum_{n=2}^N \frac{z^n b_1 \cdots b_{n-1}}{(q^n - r)(q^n - qr) \cdots (q^n - q^{n-1}r)} \\ = \sum_{n=2}^N \sum_{k=1}^{n-1} \frac{z^n b_1 \cdots b_{k-1} (q^n - q^k r - b_k)}{(q^n - r)(q^n - qr) \cdots (q^n - q^k r)}. \end{aligned} \tag{1.1}$$

But $\frac{z^n b_1 \cdots b_{k-1} (q^n - q^k r - b_k)}{(q^n - r)(q^n - qr) \cdots (q^n - q^k r)} = \epsilon_{n,k} - \epsilon_{n-1,k}$, ($k > 1$), if $\epsilon_{n,k}$ is defined as in the theorem. Thus

$$\begin{aligned} \sum_{n=2}^N \sum_{k=1}^{n-1} \frac{z^n b_1 \cdots b_{k-1} (q^n - q^k r - b_k)}{(q^n - r)(q^n - qr) \cdots (q^n - q^k r)} \\ = \sum_{n=3}^N \sum_{k=2}^{n-1} \epsilon_{n,k} - \epsilon_{n-1,k} + \sum_{n=2}^N \frac{z^n (q^n - qr - b_1)}{(q^n - r)(q^n - qr)} \\ = \sum_{k=2}^{N-1} \epsilon_{N,k} - \epsilon_{k,k} + \sum_{n=2}^N \frac{z^n (q^n - qr - b_1)}{(q^n - r)(q^n - qr)}. \end{aligned}$$

And (1.1) is equal to

$$\begin{aligned} \sum_{n=2}^N \frac{z^n}{q^n - r} - \sum_{n=2}^N \frac{z^n b_1 \cdots b_{n-1}}{(q^n - r)(q^n - qr) \cdots (q^n - q^{n-1}r)} \\ = \sum_{k=2}^{N-1} \epsilon_{N,k} - \epsilon_{k,k} + \sum_{n=2}^N \frac{z^n (q^n - qr - b_1)}{(q^n - r)(q^n - qr)}. \end{aligned}$$

This gives, after some slight simplification, the identity:

$$\begin{aligned} \frac{qr(1-q)}{(z-q)} \sum_{n=2}^N \frac{z^n}{(q^n - qr)(q^n - r)} \\ = \sum_{n=2}^{N-1} \frac{(-z)^n r^{n-1} q^n}{(q^n - z) \cdots (q - z)} \cdot \frac{(q^{n-1} - 1) \cdots (q - 1)}{(q^n - r) \cdots (q - r)} - \sum_{k=2}^{N-1} \epsilon_{N,k} \\ + \frac{(-z)^N r^{N-1}}{(q^N - r) \cdots (q - r)} \cdot \frac{(q^{N-1} - 1) \cdots (q - 1)}{(q^{N-1} - z) \cdots (q - z)}. \end{aligned}$$

But the left hand side formula is

$$\begin{aligned} \frac{qr(1-q)}{(z-q)} \sum_{n=2}^N \frac{z^n}{(q^n-qr)(q^n-r)} &= \frac{q}{(z-q)} \sum_{n=2}^N \left(\frac{z^n}{(q^n-r)} - \frac{z}{q} \frac{z^{n-1}}{(q^{n-1}-r)} \right) \\ &= - \sum_{n=2}^{N-1} \frac{z^n}{q^n-r} + \frac{z^N q}{(q^N-r)(z-q)} + \frac{z^2}{(q-z)(q-r)}. \end{aligned}$$

This ends our proof. □

Proof of Corollary 1. (i) Our identity follows from Theorem 1 letting $N \rightarrow \infty$ and noticing that $\sum_{k=2}^{N-1} \epsilon_{N,k} \rightarrow 0$, $\frac{(-z)^N r^{N-1}}{(q^N-r)\dots(q-r)} \cdot \frac{(q^{N-1}-1)\dots(q-1)}{(q^{N-1}-z)\dots(q-z)} \rightarrow 0$, if $N \rightarrow \infty$.

(ii) We use the well-known formula:

$$\gamma = \int_0^1 \frac{\sum_1^\infty x^{2^n-1}}{1+x} dx = \sum_{j=0}^\infty \sum_{n=1}^\infty \frac{(-1)^j}{2^n+j} = 1 + \sum_{j=1}^\infty \sum_{n=1}^\infty \frac{(-1)^j}{2^n+j}.$$

In the last double sum use (i) of this corollary, with $q = 2$, $r = -j$, $z = 1$.

(iii) Expanding $\frac{1}{e^{nt}-r}$ into a geometric series and integrating with respect to t yields:

$$\Gamma(s)\zeta(s) \sum_{n=1}^\infty \frac{r^{n-1}}{n^s} = \int_0^\infty \sum_{n=1}^\infty \frac{t^{s-1}}{e^{nt}-r} dt.$$

In the last sum use (i) of this corollary, with $q = e^t$, $z = 1$, $r = r$.

This ends our proof. □

Proof of Theorem 2. Write

$$\beta_n = \beta_n(q, r, z) := \frac{(-z)^n r^{n-1} q^n}{(q^n-z)\dots(q-z)} \cdot \frac{(q^{n-1}-1)\dots(q-1)}{(q^n-r)\dots(q-r)},$$

and

$$\theta := \sum_{i=1}^\infty f(i) \alpha_{N_i, M_i}.$$

If one subtracts the identity in Theorem 1 from the same identity with the parameter N replaced by M one gets

$$\alpha_{N, M} = - \sum_{n=N}^{M-1} \beta_n + \frac{(q^N-z)}{q^N} \beta_N - \frac{(q^M-z)}{q^M} \beta_M.$$

Therefore,

$$\theta = \sum_{i=1}^\infty f(i) \left\{ - \sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i}-z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i}-z)}{q^{M_i}} \beta_{M_i} \right\}.$$

Next we show that for each natural number $j \geq j_0$ such that $f(j) \neq 0$, there exist two integers A_j, B_j such that $0 < |B_j\theta - A_j| = o(1)$ as $j \rightarrow \infty$. This implies the irrationality of θ .

Take

$$B_j := (q^{M_{j-1}} - z) \cdots (q - z)(q^{M_{j-1}} - r) \cdots (q - r).$$

Thus

$$\begin{aligned} B_j\theta &= B_j \sum_{i=1}^{j-1} f(i) \left\{ - \sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i} - z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i} - z)}{q^{M_i}} \beta_{M_i} \right\} \\ &\quad + B_j \sum_{i=j}^{\infty} f(i) \left\{ - \sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i} - z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i} - z)}{q^{M_i}} \beta_{M_i} \right\} =: A_j + \epsilon_j. \end{aligned}$$

From the definitions of B_j, β_n it is immediate that A_j, B_j are integer numbers.

To show that $0 \neq \epsilon_j = B_j\theta - A_j = o(1)$ we need first to observe that

$$\begin{aligned} \beta_n &= (-z)^n r^{n-1} q^{-\frac{n^2}{2} - \frac{n}{2}} \{c_0 + o(1)\}, \\ B_j &= q^{M_{j-1}(M_{j-1}+1)} \{c_1 + o(1)\}, \end{aligned} \tag{1.2}$$

where c_0, c_1 are non-zero constants. Also

$$\begin{aligned} \frac{\beta_{n+1}}{\beta_n} &= -zrq^{-n-1} \{1 + O(q^{-n})\}, \\ \frac{\beta_{n+m}}{\beta_n} &= (-zr)^m q^{-(nm + \frac{m(m+1)}{2})} \{1 + O(q^{-n})\}. \end{aligned} \tag{1.3}$$

Here the O symbol depends on q, r, z but does not depend on n or m .

Therefore using (1.3) one sees that

$$- \sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i} - z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i} - z)}{q^{M_i}} \beta_{M_i} = \beta_{N_i} \frac{z}{q^{N_i}} \left\{ \frac{r}{q} - 1 + O(q^{-N_i}) \right\},$$

(consider the terms $\beta_{N_i}, \beta_{N_i+1}$; other terms go into the O symbol) and therefore

$$\epsilon_j = B_j \sum_{i=j}^{\infty} f(i) \beta_{N_i} \frac{z}{q^{N_i}} \left\{ \frac{r}{q} - 1 + O(q^{-N_i}) \right\}.$$

Recall that f is a bounded function; using again (1.3) yields

$$\begin{aligned} \sum_{i=j}^{\infty} f(i) \frac{\beta_{N_i}}{q^{N_i}} &= f(j) \frac{\beta_{N_j}}{q^{N_j}} \left(1 + \frac{f(j+1)\beta_{N_{j+1}}}{f(j)\beta_{N_j}q^{N_{j+1}-N_j}} + \frac{f(j+2)\beta_{N_{j+2}}}{f(j)\beta_{N_j}q^{N_{j+2}-N_j}} + \dots \right) \\ &= f(j) \frac{\beta_{N_j}}{q^{N_j}} (1 + o(1)), \end{aligned}$$

and in a similar way

$$\sum_{i=j}^{\infty} f(i) \frac{\beta_{N_i}}{q^{N_i}} O(q^{-N_i}) = f(j) \frac{\beta_{N_j}}{q^{N_j}} o(1).$$

Using these expressions and (1.2), our term ϵ_j is

$$\begin{aligned} \epsilon_j &= B_j f(j) \frac{\beta_{N_j}}{q^{N_j}} z \left\{ \frac{r}{q} - 1 + o(1) \right\} \\ &= f(j) (-z)^{N_j} r^{N_j-1} q^{-\frac{N_j^2}{2} - \frac{3N_j}{2} + M_{j-1}(M_{j-1}+1)} z \left\{ c_o c_1 \left(\frac{r}{q} - 1 \right) + o(1) \right\}. \end{aligned}$$

From the hypothesis one has that $(\sqrt{2} + \delta)M_{j-1} < N_j$, which gives

$$M_{j-1}^2 - \frac{N_j^2}{2} < -\delta \left(\sqrt{2} + \frac{\delta}{2} \right) M_{j-1}^2.$$

Therefore the hypothesis $N_j = o(M_{j-1}^2)$ yields

$$(-z)^{N_j} r^{N_j-1} q^{-\frac{N_j^2}{2} - \frac{3N_j}{2} + M_{j-1}(M_{j-1}+1)} = O(q^{-\delta' M_{j-1}^2}),$$

for some positive δ' . Therefore $0 \neq \epsilon_j = o(1)$ as $j \rightarrow \infty$, if $f(j) \neq 0$. This ends the proof. \square

Proof of Corollary 2. Using Corollary 1 (i) one can prove that, for $m = 0, 1, 2, 3, \dots$ one has

$$\sum_{n=1}^{\infty} \frac{z^{n+m}}{q^{nm}(q^n - r)} \cdot \frac{1}{(q^m - z) \cdots (q - z)} = \sum_{n=1}^{\infty} \frac{(-z)^{n+m} (-1)^{m+1}}{(q^{n+m} - z) \cdots (q - z)} g(n).$$

To prove this, notice that for $m = 0$ this equality is Corollary 1 (i). By induction assume that this is true for m , the inductive parameter. Make the change $z \rightarrow z/q$ and multiply everything by $\frac{z}{q-z}$. This gives the above formula for m replaced by $m + 1$.

Finally, to prove our formula, multiply the above by $f(m)$ and add from m equal to 1 to infinity. This yields the formula summing over $k = n + m$ in the second sum, that is,

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-z)^{n+m} (-1)^{m+1}}{(q^{n+m} - z) \cdots (q - z)} g(n) f(m) \\ = \sum_{k=2}^{\infty} \frac{(-z)^k}{(q^k - z) \cdots (q - z)} \left\{ \sum_{m=1}^{k-1} (-1)^{m+1} f(m) g(k - m) \right\}. \quad \square \end{aligned}$$

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