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# A convergence result for unconditional series in $L^p(\mu)$ *

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**Abstract**

*We give sufficient conditions for the convergence almost everywhere of the expansion with respect to an unconditional basis for functions in $L^p$ $p \geq 2$. This result extends the classical theorem of Menchoff and Rademacher for orthogonal series in $L^2$.*

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## 1. Introduction

In this paper we study the almost everywhere convergence of unconditional series in a general $\sigma$-finite $L^p$ space. The main result of this work is a generalization to $L^p$ of the classical result of D. Menchoff and Rademacher ([1], [7],[9]) on the almost everywhere convergence of orthogonal series in $L^2$. Theorem 1.1 below is a generalization of that result. When $p = 2$ and the series are orthogonal this contains the generalization obtained by Moricz and Tandori [10] (Theorem 1). The result of Menchoff has interesting implications in several areas of Analysis: Fourier Series ([1] [14]), and specially in Probability Theory ([2], [4], [7]) where this result is related to the laws of large numbers. On the other hand, unconditional basis are very important in Wavelet Analysis and related fields. One of the difficulties found when trying to generalize this result to $L^p$, $p \neq 2$, is the lack of the notion of orthogonality. Fortunately, it is not orthogonality what is needed but a consequence of it; uncondtional convergence. We will prove that this property together with another similar to the original used by Moricz and Tandori seem to be sufficient to ensure the almost everywhere convergence of certain expansions in $L^p$.

The main result can be stated formally:

**Theorem 1.1.** *Let $\{f_j\}_{j\in\mathbf{N}}$ be a basic sequence in $L^p(X,F,\mu)$, $p \geq 2$.*
*If for some $0 < \epsilon \leq 2$:*

$$\sum_{n=0}^{\infty}\sum_{j\in I_n} a_j log(j)^{\frac{\varepsilon}{2}} \left(log\frac{2A_n^2}{a_j^2}\right)^{1-\frac{\varepsilon}{2}} f_j \ , \quad I_n = \{2^n{+}1,...,2^{n+1}\}\ , A_n^2 = \sum_{i=1}^{n} |a_i|^2\ .$$

*converges in the norm of $L^p(X,F,\mu)$ then: $\sum_{k=1}^{\infty} a_k f_k(x)$ converges for almost all $x \in X$ $[\mu]$.*

## 2. Some definitions and known results

Recall that a Schauder basis or a basic sequence (a basis for a closed subspace) is call unconditional if it verifies one (and hence all) of the equivalences of the following proposition:

**Proposition 2.1.** *A basic sequence $\{x_n\}_{n\in\mathbf{N}}$ in a Banach space $B$ is unconditional if and only if one of the following conditions is fullfilled:*
*i) For every permutation $\pi$ of the integers the sequence $\{x_{\pi(n)}\}_{n\in\mathbf{N}}$ is a basic sequence (is a basis of $span\{x_n\}_{n\in\mathbf{N}}$).*
*ii) For every subset of integers $\sigma$ the convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n\in\sigma} a_n x_n$.*
*iii) The convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n=1}^{\infty} b_n x_n$, whenever $|b_n| \le |a_n|$.*
*iv) The convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies the convergence of $\sum_{n=1}^{\infty} \theta_n a_n x_n$ where $\theta_n = \mp 1$ arbitrarely.*

As a consequence of this proposition using the properties of the Rademacher functions, an alternative characterization can be given for unconditional basic sequences in the particular case of $\sigma$-finite $L^p$ spaces, [13]:

**Theorem 2.1.** *Let $\{f_j\}_{j\in\mathbf{N}}$ be a basic sequence in $L^p(X,F,\mu)$ $(1 \le p \le \infty)$. Then it is unconditional if and only if there exist $A_p, B_p$ positive constants such that:*

$$A_p \left\| \left( \sum_j |a_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(X)} \le \left\| \sum_j a_j f_j \right\|_{L^p(X)} \le B_p \left\| \left( \sum_j |a_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(X)},$$

$\forall \ \sum_j a_j f_j \in L^p(X,F,\mu)$.

This result gives a very useful characterization in terms of the equivalence of norms for $span\{f_j\}_j$. This equivalence will be very important in the sequel, though we will use it several times without refering to it explicitly, as it will become clear from the context.
Our results rely on maximal inequalities, so we need to define several maximal operators:

**Definition 1.** *Let $f \in L^p(X,F,\mu)$, $f = \sum_{k=1}^{\infty} a_k f_k$ , we define:*

$$\mathcal{M}^d f(x) = \sup_{N\in\mathbf{N}} |S_{2^N} f(x)|$$

*where, given* $n \in \mathbf{N}$, $x \in X$ *then* $S_n f(x) = \sum_{k=1}^{n} a_k f_k$.

$$\mathcal{M}f(x) = \sup_{k \in \mathbf{N}} |S_k f(x)|$$

*and*

$$\mathcal{M}_N^{\sharp} f(x) = \sup_{2^{N-1} < n < 2^N} \left| \sum_{k=2^{N-1}+1}^{n} a_k f_k(x) \right|$$

## 3. Auxiliary Results

In the following we will also suppose that $(X, F, \mu)$ is a $\sigma$-finite measure space. We will call absolute constants as $K$, $C$, $c$, $C_p$, etc. Logarithms are taken in base 2. In our results $\{f_j\}_j$ constitutes an unconditional basic sequence, so sometimes we will no mention this fact as in the following:

**Proposition 3.1.** *Let* $f \in L^p(X, F, \mu)$, $f = \sum_{k=1}^{\infty} a_k f_k$ , *then*

$$\left\| \mathcal{M}^d f \right\|_p \leq C_p \left\| \sum_{k=1}^{\infty} a_k log(k+1) f_k \right\|_p \tag{3.1}$$

**Proof.** first, let us bound the difference $\sum_{N=1}^{\infty} \|f - S_{2^N} f\|_p^p$:

$$\sum_{N=1}^{\infty} \|f - S_{2^N} f\|_p^p = \sum_{N=1}^{\infty} \left\| \sum_{k=2^N+1}^{\infty} a_k f_k \right\|_p^p \leq c_p \sum_{N=1}^{\infty} \left\| \left( \sum_{k=2^N+1}^{\infty} |a_k f_k|^2 \right)^{\frac{1}{2}} \right\|_p^p$$

$$= c_p \sum_{N=1}^{\infty} \left\| \left( \sum_{k=2^N+1}^{\infty} |a_k \frac{log(k+1)}{log(k+1)} f_k|^2 \right)^{\frac{1}{2}} \right\|_p^p$$

$$\leq c_p \sum_{N=1}^{\infty} \frac{1}{(log(2^N+1))^p} \left\| \left( \sum_{k=2^N+1}^{\infty} |a_k log(k+1) f_k|^2 \right)^{\frac{1}{2}} \right\|_p^p$$

$$\leq K \sum_{N=1}^{\infty} \frac{1}{(N+1)^p} \left\| \left( \sum_{k=2^N+1}^{\infty} |a_k log(k+1) f_k|^2 \right)^{\frac{1}{2}} \right\|_p^p . \tag{3.2}$$

On the other hand, given $x \in X$:

$$|S_{2^N} f(x)|^p \leq 2^{p-1} \left(|f(x)|^p + |S_{2^N} f(x) - f(x)|^p\right)$$
$$\leq 2^{p-1} \left(|f(x)|^p + \sum_{N=1}^{\infty} |S_{2^N} f(x) - f(x)|^p\right) ,$$

then

$$|\mathcal{M}^d f(x)|^p \leq 2^{p-1} \left(|f(x)|^p + \sum_{N=1}^{\infty} |S_{2^N} f(x) - f(x)|^p\right) .$$

Integrating at both sides of the inequality,

$$\int_X |\mathcal{M}^d f(x)|^p d\mu \leq 2^{p-1} \left(\int_X |f(x)|^p d\mu + \sum_{N=1}^{\infty} \int_X |S_{2^N} f(x) - f(x)|^p d\mu\right) ,$$

then by 3.2

$$\leq K \left\| \left(\sum_{k=1}^{\infty} |a_k f_k|^2\right)^{\frac{1}{2}} \right\|_p^p + K' \left\| \left(\sum_{k=2^N+1}^{\infty} |a_k log(k+1) f_k|^2\right)^{\frac{1}{2}} \right\|_p^p$$

$$\leq K'' \left\| \left(\sum_{k=1}^{\infty} |a_k log(k+1) f_k|^2\right)^{\frac{1}{2}} \right\|_p^p \leq K''' \left\| \sum_{k=1}^{\infty} a_k log(k+1) f_k \right\|_p^p .$$

□

**Proposition 3.2.** *Let $f \in L^p(X,F,\mu)$, $f = \sum_{k=1}^{\infty} a_k f_k$; then*

$$\|\mathcal{M} f\|_p^p \leq C_p \left\| \sum_{k=1}^{\infty} a_k log(k+1) f_k \right\|_p^p + 2^p \sum_{N=1}^{\infty} \left\| \mathcal{M}_N^\sharp f \right\|_p^p . \tag{3.3}$$

**Proof.** Let $x \in X$;

$$S_n f(x) = \sum_{k=1}^{2^{N-1}} a_k f_k(x) + \sum_{k=2^{N-1}+1}^{n} a_k f_k(x) ,$$

and then

$$|S_n f(x)| \leq \left| \sum_{k=1}^{2^{N-1}} a_k f_k(x) \right| + \left| \sum_{k=2^{N-1}+1}^{n} a_k f_k(x) \right| ,$$

$$|S_nf(x)| \leq \underset{N\geq 1}{sup}\, |S_{2^N}f(x)| + \left| \sum_{k=2^{N-1}+1}^{n} a_k f_k(x) \right| , \quad (n < 2^N)$$

$$\leq \underset{N\geq 1}{sup}\, |S_{2^N}f(x)| + \underset{2^{N-1}<n<2^N}{sup} \left| \sum_{k=2^{N-1}+1}^{n} a_k f_k(x) \right| \leq \mathcal{M}^d f(x) + \mathcal{M}^\sharp_N f(x) ,$$

then

$$|S_nf(x)|^p \leq \; 2^{p-1}\left(|\mathcal{M}^d f(x)|^p + |\mathcal{M}^\sharp_N f(x)|^p\right)$$

$$\leq 2^{p-1}\left(|\mathcal{M}^d f(x)|^p + \sum_{N=1}^{\infty} |\mathcal{M}^\sharp_N f(x)|^p\right) ,$$

This last bound is independent of $n \in \mathbf{N}$, then

$$\int_X |\mathcal{M}f(x)|^p d\mu \leq 2^{p-1}\left(\int_X |\mathcal{M}^d f(x)|^p d\mu + \sum_{N=1}^{\infty} \int_X |\mathcal{M}^\sharp_N f(x)|^p d\mu\right) ,$$

but from equation 3.1

$$\int_X |\mathcal{M}f(x)|^p d\mu \leq 2^{p-1}\left(c_p \left\| \sum_{k=1}^{\infty} a_k log(k+1) f_k \right\|_p^p + \sum_{N=1}^{\infty} \int_X |\mathcal{M}^\sharp_N f(x)|^p d\mu\right) .$$

□

In the following we will use a rather classical technique which consists in decomposing the partial sums in dyadic blocks [1], [14]. From this fact we can easily see that if we take $f \in L^p(X, F, \mu)$, $f = \sum_{k=1}^{\infty} a_k f_k$ then

$$\|\mathcal{M}_N f\|_p \leq \sum_{k=0}^{N} \left(\int_X \underset{1\leq i\leq 2^k}{max} |S_{i\,k} f(x)|^p d\mu\right)^{\frac{1}{p}} . \tag{3.4}$$

where $\mathcal{M}_N f(x) = \underset{n\leq 2^N}{max} |S_n f(x)|$ and

$$S_{i\,k} f(x) = \sum_{m=(i-1)2^{N-k}+1}^{i2^{N-k}} a_m f_m(x)$$

This follows immediately from the following fact: Take $x \in X$ then there exists $n^*(x)$ such that $\underset{n\leq 2^N}{max} |S_n f(x)| = |S_{n^*(x)} f(x)|$ but if we decompose this sum in dyadic blocks (of length $2^i$, $i = 0, 1, 2...$) we have;

$$\left|\sum_{k=1}^{n^*(x)} a_k f_k(x)\right| \le |a_1 f_1(x)| + |a_2 f_2(x) + a_3 f_3(x)| + |a_4 f_4(x) + .... + a_7 f_7(x)| + (...)$$

$$\le \sum_{k=0}^{N} \underset{1\le i\le 2^k}{max} |S_{i\,k} f(x)| \ .$$

From this last result we obtain:

**Proposition 3.3.** *Let $p \ge 2$, $f \in L^p(X, F, \mu)$ and $f = \sum_{k=1}^{\infty} a_k f_k$ then*

$$\sum_{N=1}^{\infty} \left\|\mathcal{M}^{\sharp}_N f\right\|_p^p \le K \left\|\sum_{k=1}^{\infty} a_k log(k+1) f_k\right\|_p^p . \tag{3.5}$$

**Proof.** There exists a disjoint family of sets $\{A_i\}_i \subset F$ such that

$$\underset{1\le i\le 2^k}{max} |S_{i\,k} f(x)| = \left|\sum_{i=1}^{2^k} S_{i\,k} f(x) 1_{A_i}\right| ,$$

then

$$\int_X \underset{1\le i\le 2^k}{max} |S_{i\,k} f(x)|^p d\mu \le \sum_{i=1}^{2^k} \int_X |S_{i\,k} f(x)|^p 1_{A_i} d\mu \,.$$

But, if the $f_k$'s form an unconditional basic sequence then

$$\sum_{i=1}^{2^k} \int_X |S_{i\,k} f(x)|^p 1_{A_i} d\mu \le K \sum_{i=1}^{2^k} \int_X \left(\sum_{m=(i-1)2^{N-k}+1}^{i2^{N-k}} |a_m f_m(x)|^2\right)^{\frac{p}{2}} d\mu$$

$$\le K \int_X \left(\sum_{i=1}^{2^k} \sum_{m=(i-1)2^{N-k}+1}^{i2^{N-k}} |a_m f_m(x)|^2\right)^{\frac{p}{2}} d\mu \qquad (p \ge 2)\,, \tag{3.6}$$

then from equation 3.4 we have

$$\|\mathcal{M}_N f\|_p \le K(N+1) \|f\|_p \ .$$

In particular, replacing $f$ with $\sum_{m=2^{N-1}+1}^{2^N-1} a_m f_m(x)$ we get $\mathcal{M}_N f(x) = \mathcal{M}^\sharp{}_N f(x)$ and then

$$\left\|\mathcal{M}_N^\sharp f\right\|_p \leq K(N+1)\left\|\sum_{m=2^{N-1}+1}^{2^N-1} a_m f_m\right\|_p , \tag{3.7}$$

from this we get

$$\sum_{N=1}^{\infty}\left\|\mathcal{M}_N^\sharp f\right\|_p^p \leq K' \sum_{N=1}^{\infty}(N+1)^p \left\|\left(\sum_{m=2^{N-1}+1}^{2^N-1}|a_m f_m|^2\right)^{\frac{1}{2}}\right\|_p^p$$

$$= K' \int_X \sum_{N=1}^{\infty}(N+1)^p \left(\sum_{m=2^{N-1}+1}^{2^N-1}|a_m f_m(x)|^2\right)^{\frac{p}{2}} d\mu$$

$$\leq K' \int_X \sum_{N=1}^{\infty}\left(\sum_{m=2^{N-1}+1}^{2^N-1}|a_m(log(m)+1)f_m(x)|^2\right)^{\frac{p}{2}} d\mu\,.$$

Since $p \geq 2$, using a similar argument as in inequality 3.6, we have

$$\leq K' \int_X \left(\sum_{N=1}^{\infty}\sum_{m=2^{N-1}+1}^{2^N-1}|a_m(log(m)+1)f_m(x)|^2\right)^{\frac{p}{2}} d\mu$$

$$\leq K'' \int_X \left(\sum_{N=1}^{\infty}\sum_{m=2^{N-1}+1}^{2^N-1}|a_m log(m+1)f_m(x)|^2\right)^{\frac{p}{2}} d\mu\,,$$

from which the result follows. □

With all these results we can completely bound the maximal function $\mathcal{M}f$ and hence we can give a proof of the following:

**Theorem 3.1.** *Let $p \geq 2$ and $\{f_k\}_k$ be an unconditional basic sequence such that*

$$\sum_{k=1}^{\infty} a_k log(k+1) f_k$$

*converges in the norm of $L^p(X,F,\mu)$ then:*

*I) $\sum_{k=1}^{\infty} a_k f_k$ converges in $L^p$-norm.*

*II) $\sum_{k=1}^{\infty} a_k f_k(x)$ converges for almost all $x \in X$ $[\mu]$.*

Remark Here we are considering the limit of the partial sum operators $S_k f$. I) and II) in some way recover the result of [2].

**Proof.** Part I) is trivial.
Part II) follows from the fact that the maximal function $\mathcal{M}f$ can be bounded by combining propositions 3.2 and 3.3:

$$\|\mathcal{M}f\|_p \leq K \left\| \sum_{k=1}^{\infty} a_k log(k+1) f_k \right\|_p . \tag{3.8}$$

□

## 4. Main Result

In the following we will consider the case $L^p$, with $p \geq 2$. Let $f \in span\{f_j\}_j$ , $f = \sum_{k=1}^{\infty} a_k f_k$ and consider $m_n : \mathbf{N} \to \mathbf{N}$ an increasing sequence: $m_n < m_{n+1}$, then if we define $g_n := \sum_{k=m_n+1}^{m_{n+1}} a_k f_k = S_{m_{n+1}} f - S_{m_n} f$, $\{g_j\}_j$ is again an unconditional basic sequence, with this in mind, as a direct application of theorem 3.1 we get that if $\sum_{k=1}^{\infty} a_k log(k+1) g_k$ converges in the norm of $L^p(X, F, \mu)$ then $S_{m_n} f(x)$ converges for almost all $x \in X$ $[\mu]$, as $n \to \infty$. In particular, if $m_n = 2^n$, since the $\{f_j\}_j$ are an unconditional basic sequence:

$$\left\| \sum_{n=1}^{\infty} log(n+1) \sum_{k=2^n+1}^{2^{n+1}} a_k f_k \right\|_p \leq C_p \left\| \left( \sum_{n=1}^{\infty} (log(n+1))^2 \sum_{k=2^n+1}^{2^{n+1}} |a_k f_k|^2 \right)^{\frac{1}{2}} \right\|_p$$

$$\leq C_p \left\| \left( \sum_{n=1}^{\infty} (loglog(2^{2n}))^2 \sum_{k=2^n+1}^{2^{n+1}} |a_k f_k|^2 \right)^{\frac{1}{2}} \right\|_p$$

$$\leq 2C_p \left\| \left( \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} |a_k f_k loglog(k+1)|^2 \right)^{\frac{1}{2}} \right\|_p .$$

So we have proved the following:

**Proposition 4.1.** *Let $p \geq 2$ and $\{f_k\}_k$ be an unconditional basic sequence such that*

$$\sum_{k=1}^{\infty} a_k loglog(k+1) f_k$$

*converges in the norm of $L^p(X, F, \mu)$, then $S_{2^n} f(x)$ converges for almost all $x \in X$ $[\mu]$, as $n \to \infty$.*

From this fact, we may prove the main result of this work, but as in as in [10], the proof relies in a maximal inequality which we shall prove first:

**Lemma 4.1.** *Fixed $\{f_j\}_j$ an unconconditional basic sequence, for every $0 < \varepsilon \leq 2$, there exists a constant $C(\varepsilon)$ depending only on $\varepsilon$ such that for all $(a_j)_j \in \mathbf{C}^{\mathbf{N}}$, and all $N \in \mathbf{N}$:*

$$\int_X \max_{1 \leq i \leq N} |S_i f(x)|^p d\mu \leq C(\varepsilon) log(2N)^{\frac{\varepsilon p}{2}} \left\| \left( \sum_{i=1}^{N} |a_i f_i|^2 \left( log \frac{2A^2}{a_i^2} \right)^{2-\varepsilon} \right)^{\frac{1}{2}} \right\|_{L^p}^p , \tag{4.1}$$

*where $A^2 = \sum_{k=1}^{N} a_k^2$.*

**Proof.** Let us proceed as in [10], consider the case $N = 2^{2^r} = n_r$; $r \in \mathbf{N}$; and define $\mathbf{I}_{n_r} = \{1, ..., 2^{2^r}\}$ and for $p = 0, 1, ..., r-1$ set $J_p = \{2^{2^p} + 1, ..., 2^{2^{p+1}}\}$.

Now, we can find a permutation $\pi \in S_{\mathbf{I}_{n_r}}$ such that:

$$|a_{\pi(1)}| \geq |a_{\pi(2)}| \geq ... \geq |a_{\pi(n_r)}| ,$$

and we can also find a permutation $\pi' \in S_{\mathbf{I}_{n_r}}$ such that

$$\pi \circ \pi'(n_p + 1) < ... < \pi \circ \pi'(n_{p+1}) \ ,$$

that is, a permutation such that $\pi(J_p)$ is reordered with the natural order of the indexes, for each $p = 0, ..., r-1$. On the other hand, for some $N(x)$:

$$\max_{1 \leq i \leq N} |S_i f(x)| = |S_{N(x)} f(x)|$$

$$\leq |a_{\pi(1)} f_{\pi(1)}(x)| + |a_{\pi(2)} f_{\pi(2)}(x)| + \sum_{s=0}^{r-1} \left| \sum_{i \in J_s} a_{\pi \circ \pi'(i)} f_{\pi \circ \pi'(i)}(x) 1_{\{k : \pi \circ \pi'(k) \leq N(x)\}} \right|$$

$$\leq |a_{\pi(1)} f_{\pi(1)}(x)| + |a_{\pi(2)} f_{\pi(2)}(x)| + \sum_{s=0}^{r-1} \max_{k \in J_s} \left| \sum_{i=n_s+1}^{k} a_{\pi\circ\pi'(i)} f_{\pi\circ\pi'(i)}(x) \right| .$$

Taking the $p$ norm at each side of the inequality and by the triangle inequality:

$$\left( \int_X \max_{1\leq i\leq N} |S_i f(x)|^p d\mu \right)^{\frac{1}{p}} \leq |a_{\pi(1)}| \left\| f_{\pi(1)} \right\|_p + |a_{\pi(2)}| \left\| f_{\pi(2)} \right\|_p +$$

(4.2)

$$+ \sum_{s=0}^{r-1} \left( \int_X \left( \max_{k\in J_s} \left| \sum_{i=n_s+1}^{k} a_{\pi\circ\pi'(i)} f_{\pi\circ\pi'(i)}(x) \right| \right)^p d\mu \right)^{\frac{1}{p}} .$$

Apply the maximal inequality 3.8 obtained in the proof of theorem 3.1 to the finite sum to get:

$$\left( \int_X \max_{1\leq i\leq N} |S_i f(x)|^p d\mu \right)^{\frac{1}{p}} \leq |a_{\pi(1)}| \left\| f_{\pi(1)} \right\|_p + |a_{\pi(2)}| \left\| f_{\pi(2)} \right\|_p +$$

$$+ K \sum_{s=0}^{r-1} \left( \int_X \left( \sum_{i=n_s+1}^{n_{s+1}} |a_{\pi(i)} f_{\pi(i)}(x) log(i+1)|^2 \right)^{\frac{p}{2}} d\mu \right)^{\frac{1}{p}} .$$

Now, if $i \leq 2^{2^s}$ then $log(i+1)^{\epsilon} \leq 4^{\frac{\epsilon}{2}} 2^{s\epsilon}$, so that:

$$\sum_{s=0}^{r-1} 4^{\frac{\epsilon}{2}} 2^{\frac{s\epsilon}{2}} \left( \int_X \left( \sum_{i=n_s+1}^{n_{s+1}} |a_{\pi(i)} f_{\pi(i)}(x) log(i+1)|^2 \right)^{\frac{p}{2}} d\mu \right)^{\frac{1}{p}}$$

$$\leq \sum_{s=0}^{r-1} \left( \int_X \left( \sum_{i=n_s+1}^{n_{s+1}} |a_{\pi(i)} f_{\pi(i)}(x)|^2 (log(i+1))^{2-\epsilon} \right)^{\frac{p}{2}} d\mu \right)^{\frac{1}{p}}$$

(4.3)

$$\leq \left( \sum_{s=0}^{r-1} 4^{\frac{\epsilon q}{2}} 2^{\frac{s\epsilon q}{2}} \right)^{\frac{1}{q}} \left( \sum_{s=0}^{r-1} \int_X \left( \sum_{i=n_s+1}^{n_{s+1}} |a_{\pi(i)} f_{\pi(i)}(x)|^2 |log(i+1)|^{2-\epsilon} \right)^{\frac{p}{2}} d\mu \right)^{\frac{1}{p}} ,$$

by Hölder's inequality, with $\frac{1}{p}+\frac{1}{q}=1$. Since $p \geq 2$ then 4.3 is less than:

$$(4.4)\quad 4^{\frac{\epsilon}{2}}\left(\frac{2^{\frac{r\epsilon q}{2}}}{2^{\frac{\epsilon q}{2}}-1}\right)^{\frac{1}{q}}\left(\int_X\left(\sum_{i=3}^{n_r}|a_{\pi(i)}f_{\pi(i)}(x)|^2|log(i+1)|^{2-\epsilon}\right)^{\frac{p}{2}}d\mu\right)^{\frac{1}{p}}$$

Since $\forall\ k \in J_r \cup \{1,2\}$: $k|a_{\pi(k)}|^2 \leq \sum_{i=1}^{n_r}|a_{\pi(i)}|^2$, if we write ${A^2}_{n_r} = \sum_{i=1}^{n_r}|a_{\pi(i)}|^2$ then 4.4 is less than:

$$\leq 4^{\frac{\epsilon}{2}}\left(\frac{2^{\frac{r\epsilon q}{2}}}{2^{\frac{\epsilon q}{2}}-1}\right)^{\frac{1}{q}}\left(\int_X\left(\sum_{i=3}^{n_r}|a_{\pi(i)}f_{\pi(i)}(x)|^2\left(log\frac{2{A^2}_{n_r}}{|a_{\pi(i)}|^2}\right)^{2-\epsilon}\right)^{\frac{p}{2}}d\mu\right)^{\frac{1}{p}}. \tag{4.5}$$

Combining inequalities 4.2 and 4.5 we obtain:

$$(4.6)\quad \left(\int_X \max_{1\leq i\leq N}|S_i f(x)|^p d\mu\right)^{\frac{1}{p}} \leq C(\epsilon)log(n_r+1)^{\frac{\epsilon}{2}} \times (...)$$

$$(...)\times\left(\int_X\left(\sum_{i=3}^{n_r}|a_{\pi(i)}f_{\pi(i)}(x)|^2\left(log\frac{2{A^2}_{n_r}}{|a_{\pi(i)}|^2}\right)^{2-\epsilon}\right)^{\frac{p}{2}}d\mu\right)^{\frac{1}{p}}.$$

□

Now we can give a proof for theorem 1.1:

**Proof.** (Theorem 1.1) Write ${A^2}_N = \sum_{i\in I_N}|a_i|^2$, where $I_N = \{2^N + 1, ..., 2^{N+1}\}$. Using lemma 4.1 we can estimate as in 3.3.

$$\left\|\mathcal{M}^{\sharp}_N f\right\|_p^p \leq C(\epsilon)(N+1)^{\frac{\epsilon p}{2}}\left\|\left(\sum_{k\in I_{N-1}}|a_k f_k|^2\left(log\frac{2{A^2}_{N-1}}{|a_k|^2}\right)^{2-\epsilon}\right)^{\frac{1}{2}}\right\|_p^p.$$

$$\sum_{N=1}^{\infty} \left\|\mathcal{M}^{\sharp}_N f\right\|_p^p \le C(\epsilon) \sum_{N=1}^{\infty} \left\| \left( (N+1)^{\epsilon} \sum_{k\in I_{N-1}} |a_k f_k|^2 \left( log \frac{2A^2{}_{N-1}}{|a_k|^2} \right)^{2-\epsilon} \right)^{\frac{1}{2}} \right\|_p^p$$

$$\le 2C(\epsilon) \sum_{N=1}^{\infty} \left\| \left( \sum_{k\in I_{N-1}} |a_k f_k|^2 \left( log \frac{2A^2{}_{N-1}}{|a_k|^2} \right)^{2-\epsilon} log(k)^{\epsilon} \right)^{\frac{1}{2}} \right\|_p^p$$

$$\le 2C(\epsilon) \left\| \left( \sum_{N=1}^{\infty} \sum_{k\in I_{N-1}} |a_k f_k|^2 \left( log \frac{2A^2{}_{N-1}}{|a_k|^2} \right)^{2-\epsilon} log(k)^{\epsilon} \right)^{\frac{1}{2}} \right\|_p^p .$$

($p \ge 2$ and Monotone Convergence)

Then under the hypothesis of theorem 1.1 we have:

$$\sum_{N=1}^{\infty} \left\|\mathcal{M}^{\sharp}_N f\right\|_p^p < \infty . \tag{4.7}$$

Now, we can proceed as in theorem 3.1 or we can use equation 4.7 to obtain:

$$\lim_{N\to\infty} \mathcal{M}^{\sharp}_N f(x) = 0 .$$

On the other hand the (unconditional) convergence in $L^p$ norm of

$$\sum_{n=0}^{\infty} \sum_{j\in I_n} a_j log(j)^{\frac{\epsilon}{2}} \left( log \frac{2A^2{}_n}{a_j^2} \right)^{1-\frac{\epsilon}{2}} f_j ,$$

implies the convergence in $L^p$ norm of

$$\sum_{k=1}^{\infty} a_k loglog(k+1) f_k . \tag{4.8}$$

As a consequence of proposition 4.1, and since given $m \in \mathbf{N}$ and $2^{m-1} < n < 2^m$ for almost all $x \in X$ the following holds:

$$|S_n f(x) - f(x)| \le \mathcal{M}^{\sharp}_{m+1} f(x) + |S_{2^m} f(x) - f(x)|$$

where $f = \sum_{k=1}^{\infty} a_k f_k$ ( $L^p$), then by proposition 4.1, $S_{2^m} f(x)$ converges to $f(x)$ $[\mu]$-a.e. as $m \to \infty$, and on the other hand $\mathcal{M}^{\sharp}_{m+1} f(x) \to 0$, so that the proof is complete. □

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