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A COMPREHENSIVE APPROACH TO GEOMETRIC SIMPLEXES

Mari Mohammed, William Obeng-Denteh, Fred Asante-Mensa

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ABSTRACT

This paper is an expository research to simplexes. The work focuses on how simplexes are created. It also looked at how complete graphs are treated as simplexes. We further present an important theorem and its proof.

KEYWORDS: topological spaces, sub-simplices, pairwise, affine independent, linearly independent, simplicial complex, complete graph.

Introduction

The idea about simplex was introduced by William Kingdon Clifford. He penned on simplexes but named them “prime” “confines” in 1886. This concept was first described by Hendrick Schout in 1902 with the Latin superlative simplicssium “simplest” and same Latin adjective in normal form simplex “simple”. Henri Poincare, named them “generalized tetrahedra” in 1900, when he wrote Analysis Situs [4].

Basically, simplexes are geometric objects such as point, edge, triangle, tetrahedron etc. For instance, a zero-simplex describes a single point, 1−simplex refers to two vertices which is connected by an edge, a 2−simplex is three vertices which is connected pairwise by three edges with one face to produce a triangle, a 3−simplex refers to four vertices which is connected pairwise by edges, attached by four faces and are filled in to create a tetrahedron and so on [2]. These are shown in Figure 1. To generalize a pattern for n, an n−simplex can be formed using $(n + 1)$ vertices.

Mari Mohammed, Department of Mathematics, Kwame Nkrumah University of Science and Technology, Private Mail Bag-Kumasi, Ghana.
William Obeng-Denteh, Department of Mathematics, Kwame Nkrumah University of Science and Technology, Private Mail Bag-Kumasi, Ghana.
Fred Asante-Mensa Department of Mathematics, Kwame Nkrumah University of Science and Technology, Privat Mail Bag-Kumasi, Ghana.

When a collection of simplexes is nicely gummed to each other in a structured manner it produces a topological space called simplicial compl

**Corresponding author:mohammedmari856@gmail.com*

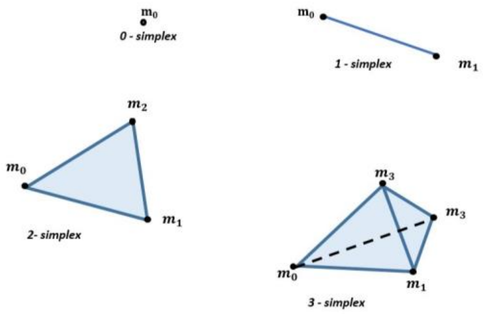

Figure 1: Simplexes

1.1 Basic Definitions

1.1.1 Face of a Simplex

A face of a simplex *S* is a sub-simplex *P* ⊆ *S* whose vertices are also vertices of *S*. A face *P* of *S* is said to be proper face if *P* 6= *S* [5]. Every simplex is a face of itself and all other sub-simplexes of that simplex are faces of the simplex. For instance, all the faces of a 3−dimensional simplex are shown in Figure 2.

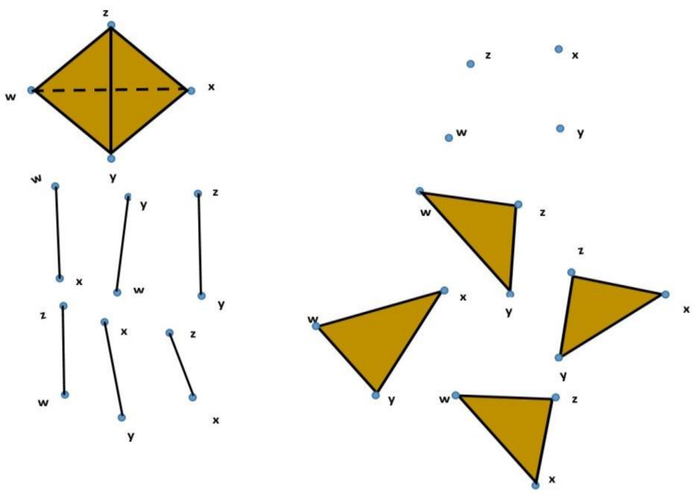

Figure 2: Faces of a 3-simplex

1.1.2 Facet of a Simplex

If a simplex(*S*) has a dimension *k*, i.e., *k*-*S*, then its facets are (k − 1) simplexes or sub simplexes of *S*. For instance, a 2-*S* has 1-simplexes as its facets. Suppose a 2-simplex has (c_0, c_1, c_2) vertices, then the edges (c_0, c_1), (c_0, c_2) and (c_1, c_2) are its facets. Therefore, the total number of facets of 2-simplex is 3 as shown in the table below

Dimension(k) of S	Total Number of facets
1	2
2	3
3	4
k	(k+1)

From the table, it can be deduced that the total number of facets of a k-simplex for $k > 0$ is given by $(k + 1)$, where k is the dimension of the simplex.

1.1.3 Orientation of Simplexes

Orientation of simplex refers to giving the simplex a direction [5]. This is done by ordering the 0-dimensional faces of a simplex. The two possible ways one can orient a simplex is through clockwise and counterclockwise directions [3]. Considering a 1−dimensional simplex in Figure 3.

Figure 3: oriented edge

The oriented edge is (c_0, c_1), taking into consideration the order.
Consider the 2-simplex in Figure 4. Figure 4(a) represents the counterclockwise orientation. The same orientation is represented by three different ordered triples. That is, the oriented face = (c_0, c_1, c_2) = (c_1, c_2, c_0) = (c_2, c_0, c_1). Figure 4(b) also represents the clockwise orientation. The different ordered triples representing this orientation is the oriented face = (c_0, c_2, c_1) = (c_1, c_0, c_2) = (c_2, c_1, c_0). With the 3-simplex, twelve different ordered quadruples denote the same 3-simplex [3].

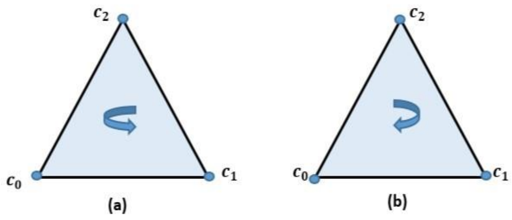

Figure 4: Oriented 2-simplex

1.1.4 Complete Graph

A complete graph refers to a graph consisting of vertices and edges such that any two vertices are joined by an edge [2]. Complete graphs on three vertices and four vertices are shown in Figure 5.

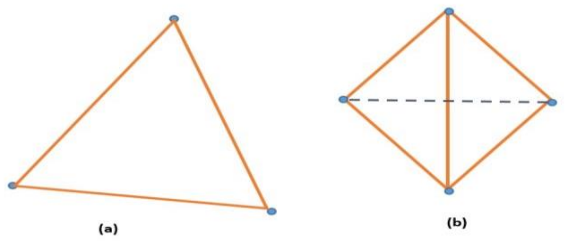

Figure 5: Complete graph

1.2 Preliminaries

This section deals with the definition of some important terminologies relating to this article.

1.2.1 Definition

Let *u, v* ∈ R, the line segment [*u, v*] between *u* and *v* is the set {[1 − *λ*] *u* + *λ v*, where *λ* ∈ [0,1]}. A set Y ⊆ R^n is said to be convex if ∀*u, v* ∈ *Y*, [*u, v*] is contained in Y, otherwise the is concave. For example, in Figure 6, Y is a convex set and X is a concave set.

1.2.2 Definition

If set *Y* ⊆ R^n, then the convex hull denoted by C_x of *Y* is the intersection of all the convex sets contained in *Y* [6]. The convex hull is the set of all convex combinations of points in *Y* [6].

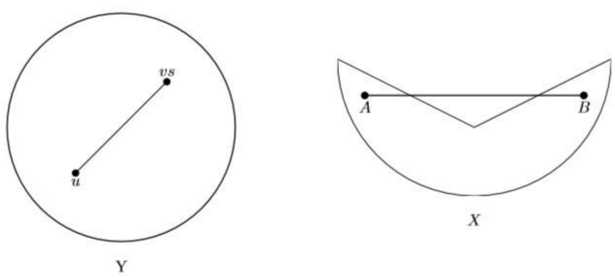

Figure 6: Convex set Y and concave set X

Generally, if the set is convex, then the convex hull is the convex set, since the intersection of any family of convex sets is convex. But with concave set, the convex hull represents the smallest set that is contained in the line segment.

1.2.3 Definition

Given {m_0, m_1, ⋯, m_k} to be a set of points in R^n, the set is said to be affinely independent or geometrically independent provided for any real scalar b_i, the linear system $\sum_{i=0}^{n} b_i\, m_k = 0$

And $\sum_{i=0}^{n} b_i$= 0 is the trivial solution, b_0 = b_1 ⋯ = b_n = 0.

Generally, $\{m_0\ , m_1, \cdots\ m_k\}$ are geometrically independent provided the vectors $m_1 - m_0,\ m_2 - m_0,$ $m_3 - m_0,\ m_k - m_0$ are linearly independent. Any subset of geometrically independent points is also geometrically independent.

1.2.4 Definition

Let $\{m_0\ , m_1, \cdots\ m_k\}$ be a set of vertices in R^n. An *n*−simplex spanned by $m_0, m_1, \cdots\ m_k$

denotes the set of all points *P* in R^n such that *P* = $\sum_{i=0}^{n} b_i\, m_k$: b_i∈ [0, *n*] for i = 0, 1, ⋯, *n* and $\sum_{i=0}^{n} b_i\ = 1$ where the non-negative integer *n* denotes the dimension of the simplex.

The real numbers $b_0,\ b_1, \cdots\ b_n$ which are uniquely determined by *P* are called barycentric coordinates of the point *P* [5],[1]. A convex combination is $\sum_{i=0}^{n} b_i\, m_k$ where $\sum_{i=0}^{n} b_i\ = 1\ and\ b_i \geq 0.$

A simplex is a convex hull by affinely independent points. Any *n*−simplex is produced by taking the convex hull of the previous simplex and adding one more point, positioned in the space of dimension *n* such that it is affine-independent with the previous simplex.

1.2.5 Definition

A standard *n*− simplex (S^n) is obtained by taken the convex hull of the basis vertices in R^{n+1}. The (*n* + 1) vertices of the S^n are the points $a_i \in R^{n+1}$, where each vertex has zero coordinates except for its *ith* position which has 1 as its values. For example,

a_0 = (1, 0, 0, ⋯ ,0)

a_1 = (0, 1, 0, ⋯ ,0)

a_2 = (0, 0, 1, ⋯ ,0)

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.

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a_n= (0, 0, 0, ⋯ ,1)

S^n lies in the hyperplane $x_1 + x_2 + x_3 \cdots + x_n = 1$.

For instance, the standard 0−simplex (S^0) describes the point (1) in one dimensional vector space *R*. S^1 describes an edge connecting the basis vertices (1, 0) and (0, 1) in R^2. S^2 describes an equilateral triangle having the basis vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1) in R^3space [5]. S^3 also describes a regular tetrahedron having (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1) basis vertices in R^4.

S^0 lies in the point a_1 = 1, S^1 lies in the line $a_1 + a_2$ = 1, S^2 lies in the plane $a_1 + a_2 + a_3$ = 1 and S^3 lies in the hyperplane $a_1 + a_2 + a_3 + a_4$ = 1 [5]. Figure 7 represents S^0,S^1,S^2 and S^3respectively in R^n space.

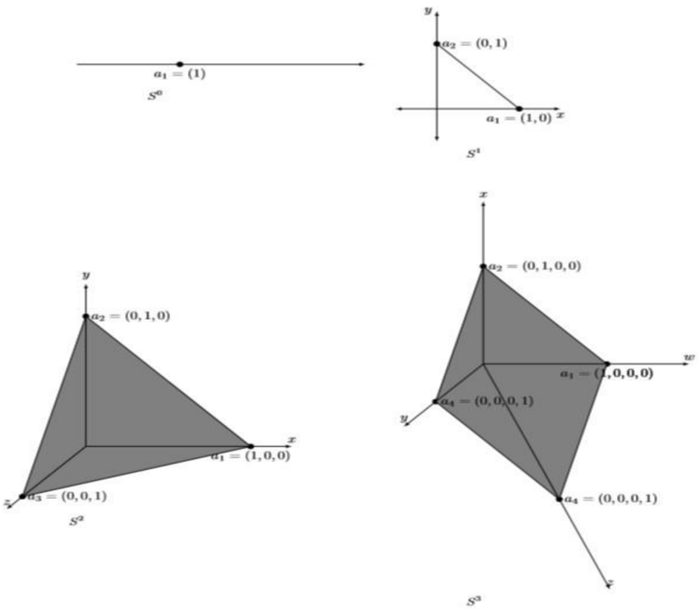

Figure7: S^0, S^1, $S^2, and\ S^3$ in R^n

2 Main Thrust

This section deals with parts of a simplex and how simplexes can be created. It looks at how complete graphs are treated as simplexes. Also, an important theorem which cannot be an oversight when dealing with simplexes is being discussed.

2.1 Interior and Boundary of Simplex

A simplex has two distinct parts. Namely: interior and the boundary.

2.1.1 Definition

The interior of a simplex *S* denoted by Int(*S*) = *S* − *b*(*S*) of *S*, where *b*(*S*) denotes the boundary of the simplex [1]. The interior of *S* is made up of all the vertices of *S* which are not members of any proper face of *S*.

The boundary(b) of a *k*-simplex is the union of all its facets. The boundary of 0-dimensional simplex (S^0) does not exist.
For instance, the boundary of 1-simplex (c_0, c_1) is *b*(*S*) = c_1 - c_0

and the boundary of a 2-simplex (c_0, c_1, c_2) is given by:

$$b\ (S^2) = (c_1, c_2) + (c_2, c_0) + (c_0, c_1)$$

$$= (c_1, c_2) - (c_0, c_2) + (c_0, c_1)$$

$$= \sum_{i=0}^{n} (-1)^i\ (c_0, \cdots \hat{c}_i, \ \cdots, c_2)$$

Where the symbol hat over c_i means omit c_i

Similarly, the boundary a 3-simplex (S^3) with vertices (c_0, c_1, c_2, c_3) can be defined as:

$$b\ (S^3) = \sum_{i=0}^{n} (-1)^i\ (c_0, \cdots \hat{c}_i, \ \cdots, c_3)$$

$$= (c_1, c_2, c_3) - (c_0, c_2, c_3) + (c_0, c_1, c_3) - (c_0, c_1, c_2)$$

2.2 Theorem

The boundary square (b^2) is equal zero.

Proof:

Due to [5] we begin as follows:

Consider S^3 (c_0, c_1, c_2, c_3),

$$B\ (S^3) = (c_1, c_2, c_3) - (c_0, c_2, c_3) + (c_0, c_1, c_3) - (c_0, c_1, c_2)$$

$$B\ (b\ (S^3)) = b\ ((c_1, c_2, c_3) - (c_0, c_2, c_3) + (c_0, c_1, c_3) - (c_0, c_1, c_2))$$

$$= [(c_2, c_3) - (c_1, c_3) + (c_1, c_2)] - [(c_2, c_3) - (c_0, c_3) + (c_0, c_2)] +$$

$$[(c_1, c_3) - (c_0, c_3) + (c_0, c_1)] - [(c_1, c_2) - (c_0, c_2) + (c_0, c_1)]$$

Showing that $b^2(S^3) = 0$.

In general, $b^2(S^n) = 0\ \forall n \geq 1$, where *n* denotes the dimension of the simplex(*S*).

2.3 Creating Simplexes

As said earlier, a simplex begins with a point which is 0−simplex, when two 0−simplexes are joined by an edge, a 1−simplex is formed.
To create a two-simplex, it requires three vertices. Then each pair of vertices is connected with an edge to form a boundary of a triangle, which when filled in with a triangular face produces a 2−simplex. Figure 8 shows the stepwise way to create a 2−simplex and a 3−simplex.

Also, to create a 3−simplex, it requires four vertices and each pair of vertices is joined by an edge to create a boundary of 3−simplex, which when filled in with 2−dimensional triangular faces and the 3−dimensional tetrahedra solid produces a 3−simplex [2].

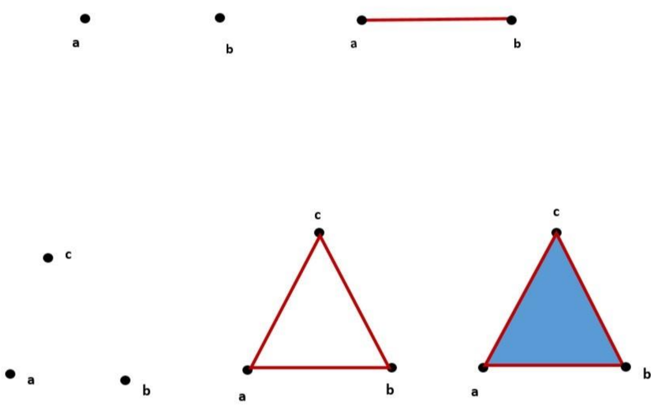

Figure 8: Creating simplexes

2.4 Complete Graphs as Simplexes

Complete graphs can be treated as simplexes. For example, in Figure 5, the complete graph in (a) becomes a 2-simplex when the inside is filled with a 2-dimensional triangular face. Similarly (b) turns to a 3-simplex when the 2-dimensional triangular faces and the 3-dimensional tetrahedra solid are filled [2]. This is represented in Figure 9.

The number of edges each vertex has is its degree. The degree of each vertex of a complete graph (K_m) is determined by (m−1) vertices. Where m denotes the number of vertices of (K_m).

The sum of all the degrees in a complete graph (K_m) is given by m (m − 1) and total number of edges in a complete graph is (K_m) is $\frac{(m(m-1))}{2}$.

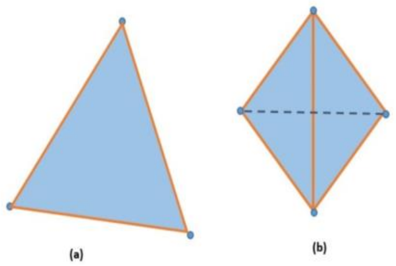

Figure 9: Complete graph as simplexes

3 Conclusion

The paper thoroughly discussed simplexes which are geometric elements such as point, edge, triangle, tetrahedron etc used to build a space called simplicial complex. It highlighted on how simplexes are created. An important theorem and its proof which cannot be overlooked when dealing with simplexes have been discussed. Lastly, complete graphs have also been shown to be simplexes.

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