Multi-objective Optimization Proposal with Fuzzy Coefficients in both Constraints and Objective Functions.

Propuesta de optimización multiobjetivo con coeficientes difusos en restricciones y en funciones objetivo.

Gabriel Jaime Correa-Henao*
Gloria Elena Peña-Zapata**

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Abstract
Fuzzy sets, and more specifically, fuzzy numbers can be a very suitable way to include uncertainty within the formulation and solution of linear problems with multiple goals. Goals in a decision problem do not need to be either maximized, or minimized, as in classical mathematical programming, but they are substituted by aspiration levels, and they need to be met in order to satisfy the decision-maker. Experience shows that it is easier for the decision-maker to formulate both objectives and constraints with fuzzy coefficients, rather than specify a defined quantity for the matrices A, b or g. This paper shows the versatility of a methodology that solves multi-objective linear problems, formulated with fuzzy coefficients. This conception becomes an alternative in contrast with the hard methodologies predominant in Operations Research, since the fuzzy approach allows the decision-maker to make uncertain assumptions both for the formulation and solution of optimization problems.

Keywords: Fuzzy logic, multi-criteria analysis, triangular fuzzy numbers.

Resumen
Los conjuntos difusos y específicamente los números difusos constituyen una manera efectiva de incluir la incertidumbre en la formulación y solución de problemas lineales de optimización multiobjetivo. Las metas en un problema de decisión no necesitan ser maximizadas ni minimizadas, como ocurre en las herramientas clásicas de programación matemática, sino que se pueden sustituir por niveles de aspiración, las cuales constituyen las expectativas para un decisor. La experiencia demuestra que es más fácil para el decisor formular los objetivos y las restricciones en un problema con coeficientes difusos, en vez de simplemente especificar un número concreto en las matrices A, b ó g. Este artículo presenta la versatilidad de una formulación metodológica que permite resolver problemas multiobjetivo de tipo lineal, los cuales son formulados con coeficientes difusos. Esta concepción constituye una alternativa a las metodologías duras que dominan la investigación de operaciones, dado que la aproximación difusa permite que los decisores realicen presunciones inciertas en la formulación y solución en los problemas de optimización.

Palabras Clave: Lógica difusa, análisis multiobjetivo, números triangulares difusos.

* PhD. en Ingeniería. Docente Investigador. Facultad de Ingenierías y Arquitectura. Fundación Universitaria Luis Amigó, Medellín, Colombia. gabriel.correahe@amigo.edu.co.

** Ph.D. en Ingeniería de Organización. Docente Cátedra. Departamento Ingeniería de la Organización. Universidad Nacional de Colombia, Medellín, Colombia. gepena@unal.edu.co.

Introduction

One of the most interesting aspects in fuzzy sets theory is its ability for mathematically representing a kind of decision problems known as multicriteria decision making (Chen et al., 2012; Eiselt & Laporte, 1992; Ekel et al., 2008). These problems usually involve vagueness and uncertainty in both objectives and constraints. This is why a methodology has to aim for the best decision, which has to be efficient in the sense that it needs to meet the goal, as well as simultaneous constraint sets (Wibowo & Deng, 2013). A first approach to fuzzy decision-making was exposed in (Bellman & Zadeh, 1970), and extended in (Yager, 1978). These approaches have proved their effectiveness in many “real world” problems, e.g. resource planning, and production programming (Berredo et al., 2013).

So far, it is possible to identify that one of the most important techniques in Operational Research, is Linear Programming (Luo et al., 2014). Nevertheless, Linear Programming requires well suited information, as well as precise data. This demands higher costs in information processing as exposed in (Hillier & Lieberman, 2002) As a matter of fact, in real world applications, certainty, reliability and preciseness in information are just an illusion. These inconveniences can be overcome with the use of Fuzzy Linear Programming (FLP) (Sakawa et al., 2014; Yang & Lin, 2013).

Optimization models are formulated as follows (Al-Najjar & Malakooti, 2011):

$$\max g(x) \qquad (1)$$
$$\text{subject to } A\text{-}x \leq b$$

with g representing the objective function coefficients, b the resources vector, and A the technical coefficient matrix (Martínez et al., 2006).

Some authors consider that multi-criteria optimization models, being applicable for optimization of Operations Research models could be satisfactorily used in a form of linear programming (Petrovic-Lazarevic & Abraham, 2002). First attemps to model both linear programming and multi-objective programming, using fuzzy sets led to concepts of decision-maker satisfaction and efficient solutions, as it was first proposed by (Zimmermann, 1978). These approaches deal with smaller violations to problem constraints, and consider formulation with crisp coefficients in both objective functions and constraints (Li & Wan, 2013; Ozgen & Gulsun, 2013). Some other authors consider a methodological approach to problems where coefficients are expressed as random variables in formulations for either one objective (Luhandjula, 1987), or several objective decision-making problems, as in (Chanas, 1989).

However, uncertainty treatment in fuzzy programming models is stated in the sense that coefficients are crisp numbers, and some estimations or tolerances in the values of either objective function or resource constraints vectors are allowed, according to the decision-maker’s subjective judgments, as stated in (Zimmermann, 1987), (Chanas & Kuchta, 1996), and (Chen et al., 1992).

Basic assumptions in Operations Research models are supported in the fact that coefficients in objective functions, constraints and resources are well known values. This fact drives the model formulation to be an exact assignment (Ojha & Biswal, 2006). Coefficients in objective functions and constraints can be given by the decision-maker according to his subjective judgements, or by means of statistic inference based on historical data. Hence, in order to reflect this uncertainty, the model formulation requires the use of inexact coefficients (Bit et al., 1992; Zare & Daneshmand, 1995) Many authors have made attempts to solve linear problems with fuzzy coefficients in constraints, as developed by (Joseph, 1995; Mahdavi-Amiri & Nasseri, 2007; Ramík & Ímánek, 1985), or in objective functions, as shown in (Maleki et al., 2000; Sakawa et al., 2014; Sakawa & Matsui, 2013a, b).Considerations of fuzzy coefficients in multi-criteria problems contribute to improve the flexibility and robustness of multiple objective decision-making methodologies (Gupta & Mehlawat, 2009).

The first part of the paper brings up several definitions of fuzzy logic, fuzzy numbers and fuzzy sets, which may support further formulation of multi-objective decision making problems. Furthermore, definitions on uncertainty are gathered into such a formulation. This way, decision-makers are enabled to solve multi-criteria problems keeping into account uncertainty on the basis of optimization algorithms with defuzzification techniques. At the end of the paper, some numerical examples are provided in order to clarify the proposed problem-solving procedure.

Proposal of methodological approaches on multi-objective problems with uncertainty

Fuzzy logic may be an appropriate soft Operations Research tool that may be kept into account for the solution of multi-objective programming problems, since it allows formulation of such models without the necessity to define crisp coefficient matrices on either constraints, or objective functions.

The methodological approach presented in the paper is based upon the first work of (Zimmermann, 1978), which raises the need to calculate the levels of satisfaction of a decision-maker in order to solve linear programming problems with uncertainty for one goal. The same problem is discussed in the literature (Maleki et al., 2000) using fuzzy number coefficients.

As previously exposed in the introduction of the paper, multi-criteria optimization problems may be formulated as noted in Eq. (1) with crisp numbers. In order to establish uncertainty in such a formulation, the reader should relate to Eq. (3), which introduces vagueness in the specification of a number. Note that the solution to such a problem yields decision variables with crisp numbers (xj).

In order to find the solution by means of linear programming with fuzzy numbers (FLP), it is important to implement an orderly procedure in which the problem is solved by parts. The solution implies mapping the most extreme solutions (left and right limits of fuzzy numbers) first, and then, determining the membership functions of triangular numbers, which is also known as defuzzification of the problem (Correa-Henao, 2015). Such a procedure is clearly stated in Eq. (7), where it is possible to identify optimal solutions on the Pareto frontier (zi^+, zi^-).

Finally, it is necessary to determine the minimum level of satisfaction for all membership functions, which is interpreted as the level of aspiration or satisfaction of a decision-maker. This efficient solution corresponds to solving the auxiliary problem posed in Eq. (8).

The procedure that is exposed in the following sections of the paper takes into account the uncertainty in problem formulation by means of fuzzy numbers. Such methodology was a contribution of both authors. It is also illustrated with examples that allow the reader to easily assimilate the methodology. This contribution may also be extended to FLP, as it takes into account the existence of equality constraints.

Fuzzy Logic Definitions

Fuzzy Set: Let X be a universe of discourse. $\tilde{A}$ is a fuzzy subset of X if for all x ϵ X, there is a number $\mu_A(x) \in [0, 1]$ assigned to represent the membership of x to , and $\mu_A(x)$ is called the membership function of A (Petrovic-Lazarevic & Abraham, 2004).

Fuzzy Number: A fuzzy number $\tilde{A}$ is a normal and convex subset of the universe of discourse X. Normally, $\forall X \in R$ and $\mu_A(x) = 1$. Convexity implies: $\forall x_1 \in X, \forall x_2 \in X, \forall \alpha \in [0, 1]$ (Antonsson & Sebastian, 1999).

$$\mu_A(\alpha \cdot x_1 + (1-\alpha) \cdot x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)) \quad \textbf{(2)}$$

Triangular Fuzzy Number: Let $\tilde{A}$ be a fuzzy number. It is considered to be a triangular fuzzy number if its convexity is linear piecewise. It is denoted by: $\bar{a} = (m - \alpha, m, m + \beta)$ (Carlsson & Fuller, 2002; Wang et al., 2013).

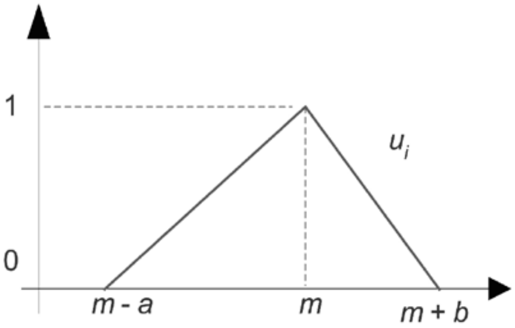

Figure 1. Fuzzy triangular number
Source: Author's own elaboration (2015)

Fuzzy Decision: A decision in "fuzzy environments" was first defined by (Bellman & Zadeh, 1970) as the intersection of the fuzzy sets that represents goals and constraints (Dubois et al., 2000). This implies the existence of a decision set ($\mu_{\tilde{D}}$), $\forall X \in R$, $\mu_{\tilde{D}} = \min\left\{\mu_{\tilde{M}_1},\cdots,\mu_{\tilde{M}_n},\mu_{\tilde{R}_1},\ldots,\mu_{\tilde{R}_m}\right\}$, where $\bar{M}_i$, i = 1 , …,n. and $\bar{R}_j$, j = 1 , ...,m. are the total number of objective functions and constraints respectively (Dubois & Prade, 1979; Sakawa & Matsui, 2013a).

Fuzzy multi-objective problem

Traditional decision-making processes are subject to imprecise data and to subjective judgements. This is the reason why a decision-maker should formulate a problem by means of fuzzy numbers (Fullér & Zimmermann, 1993; Grzegorzewski & Pasternak-Winiarska, 2014). This fact allows the decision-maker to build a model that represents uncertainty in the definition of numbers. For instance, the expression "approximately 100" can be expressed as the fuzzy number <95, 100, 105>.

Formulations of Multi-criteria problems with constraints can be stated as follows (Correa-Henao et al., 2003; Maleki et al., 2000):

$$\max \sum_{j=1}^{n} \bar{g}_{ij} \cdot x_j \ , \quad j = 1, 2, \cdots, n \qquad \textbf{(3)}$$

$$\text{subject to:} \begin{cases} \displaystyle\sum_{j=1}^{m} \bar{A}_{kj} \cdot x_j \leq\sim \bar{b}_k, \quad k = 1, 2, \cdots, m, \\ x_j \geq\sim 0 \end{cases}$$

with $\bar{g}_{ij}$ as the matrix that contains the coefficients of the objective functions. Such numbers can be represented as: $\bar{g}$ = (c-ε, c, c +ω); A_{kj} being the technical coefficient matrix for the constraint functions. Such fuzzy numbers can be represented as a = (m- α, m, m +β);$\bar{b}_k$ represents the resource vector (Berredo et al., 2013). This array is also made up of fuzzy triangular numbers:$\bar{b}$ = (p-γ, p, p + δ) (Luhandjula, 1987). The ≤~ operator implies that soft violations are allowed into the constrained problem in order to find the most efficient solution in the decision variables originally stated using crisp numbers (x_j) (Correa-Henao, 2015; Maleki et al., 2000).

Some authors have worked the notation of fuzzy numbers so they can be then transformed into models that contain the specific information given with the coefficient formulation shown in (Sakawa, 2002). Nonetheless, the basic approach for the solution of this problem needs an equivalent models set, which eventually has to be solved. These authors suggest transforming these formulations into models that implicitly contain the transformed fuzzy numbers. Today's literature has shown that the problem in Eq. (3) can be stated as follows (Grzegorzewski & Pasternak-Winiarska, 2014; Mahdavi-Amiri & Nasseri, 2007; Ramík & Ímánek, 1985):

$$\underline{\mathbf{max}}\ \widetilde{g}_{ij}\cdot x_n=\widetilde{g}_{i1}\cdot x_1+\cdots+\widetilde{g}_{in}\cdot x_n=\left(\sum_j\left(c_{ij}-\varepsilon_{ij}\right)\cdot x_j,\sum_j c_{ij}\cdot x_j,\sum_j\left(c_{ij}+\varphi_{ij}\right)\cdot x_j\right),j=1,\ldots,\mathrm{n}$$

$$\underline{\text{subject}}\text{ to: }\begin{cases}\widetilde{a}_{k1}\cdot x_1+\cdots+\widetilde{a}_{kn}\cdot x_n=\left(\sum_j\left(m_{kj}-\alpha_{kj}\right)\cdot x_j,\sum_j m_{kj}\cdot x_j,\sum_j\left(m_{kj}+\beta_{kj}\right)\cdot x_j\right)\\ \qquad =\widetilde{b}_1+\cdots+\widetilde{b}_n=\left(\sum_j\left(p_{kj}-\gamma_{kj}\right),\sum_j p_{kj},\sum_j\left(p_{kj}+\delta_{kj}\right)\right)\\ x_j\geq\sim 0\end{cases}\qquad\textbf{(4)}$$

Defuzzification of constraints

In the multi-objective problem previously stated in Eq. (4), it is possible to suggest the inclusion of uncertainty by using the most efficient solution to every single objective, and then defining every membership function for each goal (Petrovic-Lazarevic and Abraham, 2002).

Hence, the constraints in Eq. (4) can be stated as follows (Maleki et al., 2000):

$$\varepsilon_L\left(\sum_{j=1}^{n}\alpha_{ij}\cdot x_j-\gamma_i\right)\leq -p_i+\sum_{j=1}^{n}m_{ij}\cdot x_j\qquad\textbf{(5)}$$

$$\delta_L\left(\sum_{j=1}^{n}\alpha_{ij}\cdot x_j-\gamma_i\right)\leq -p_i+\sum_{j=1}^{n}m_{ij}\cdot x_j$$

Once the constraints have been transformed into Eq. (5), it constitutes a system of crisp linear inequalities, which can now be solved by any classical Linear Programming (LP) method (Tanaka et al., 2000). It must be noted that the transformation in (5) can only be applied to the constraints. The objective function needs to be defuzzified with an alternative methodology (Fullér & Zimmermann, 1993; Zimmermann, 1987), which will be described in the following paragraphs.

Objective Function Defuzzification

According to the techniques that are described for the solution of FLP, it is important to verify that the problem have extreme solutions (i.e. left and right bounding of the fuzzy numbers in coefficients of the objective functions) (Sakawa et al., 1996). It is also imperative to find the membership function of the objective functions in order to rewrite the problem as stated in classical LP, i.e. with crisp numbers (Tanaka et al., 2000). The first step is to find the membership function of every objective, which is identified as:

$$\mu_i(x) = \begin{cases} 0 & \text{if } z_i(x) \le z_i^- \\ \left[z_i(x) - z_i^-\right] \big/ \left(z_i^+ - z_i^-\right) & \text{if } z_i^- < z_i(x) \le z_i^+ \\ 1 & \text{if } z_i(x) > z_i^- \end{cases} \quad \textbf{(6)}$$

This definition can be interpreted as a linear membership function as shown in Figure (2)

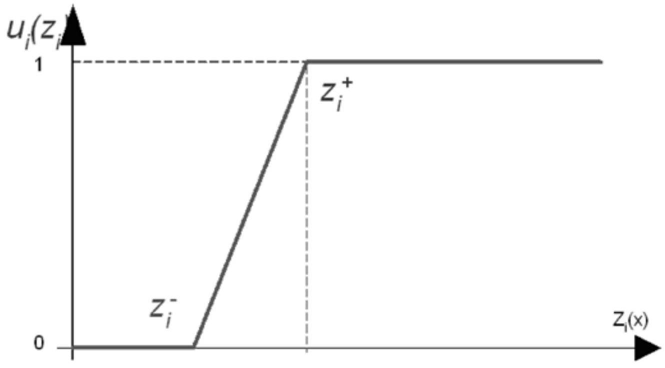

Figure 2. Membership function mapping, for every objective Zi(x)
Source: Author's own elaboration (2015)

Zi$^+$ can be obtained as the optimal value of a particular objective on the feasible solution space (Pareto frontier), and *Zi*$^-$ is the minimal value of the objective on the feasible solution edge (Petrovic-Lazarevic & Abraham, 2004).

In order to defuzzify the objective functions (gi) that are formulated with fuzzy coefficients, it is necessary to divide the multi-criteria problem in three basic sub-problems (Correa-Henao, 2015).

Pessimistic Problem: The one that matches the solution of the crisp problem with the left edges of the objective function coefficients (c_{ij} - ε_{ij}).

Optimal Problem: The one that matches the solution of the crisp problem with the representative coefficients of the objective function whose membership degree is equal to 1 (c_{ij}).
Optimistic Problem: The one that matches the solution of the crisp problem with the right edges of the objective function coefficients (c_{ij} +ω_{ij}).

The following methodological approach allows the decision-maker to find the membership functions for every specific sub-problem (Correa-Henao et al., 2003), as shown in (7):

$$\mu_i^{pesimistic}(x)=\begin{cases} 0 & ; z_i^{pesimistic}(x)\le z_i^-\big|_{pesimistic} \\ \left[(c_i-\varepsilon_i)\cdot x_j - z_i^-\big|_{pesimistic}\right] \Big/ \left(z_i^+\big|_{pesimistic} - z_i^-\big|_{pesimistic}\right) & ; z_i^-\big|_{pesimistic} < z_i^{pesimistic}(x)\le z_i^+\big|_{pesimistic} \\ 1 & ; z_i^{pesimistic}(x) > z_i^+\big|_{pesimistic} \end{cases}$$

$$\mu_i^{optimal}(x)=\begin{cases} 0 & ; z_i^{optimal}(x)\le z_i^-\big|_{optimal} \\ \left[(c_i)\cdot x_j - z_i^-\big|_{optimal}\right] \Big/ \left(z_i^+\big|_{optimal} - z_i^-\big|_{optimal}\right) & ; z_i^-\big|_{optimal} < z_i^{optimal}(x)\le z_i^+\big|_{optimal} \\ 1 & ; z_i^{optimal}(x) > z_i^+\big|_{optimal} \end{cases}$$

$$\mu_i^{optimistic}(x)=\begin{cases} 0 & ; z_i^{optimistic}(x)\le z_i^-\big|_{optimistic} \\ \left[(c_i+\varphi_i)\cdot x_j - z_i^-\big|_{optimistic}\right] \Big/ \left(z_i^+\big|_{optimistic} - z_i^-\big|_{optimistic}\right) & ; z_i^-\big|_{optimistic} < z_i^{optimistic}(x)\le z_i^+\big|_{optimistic} \\ 1 & ; z_i^{optimistic}(x) > z_i^+\big|_{optimistic} \end{cases} \quad \textbf{(7)}$$

In this formulation of equations (8), zi^+ and zi^- are related to the optimal solutions on the Pareto edge for each specific sub-problem.

According to (Bellman & Zadeh, 1970; Zimmermann, 1978), it is necessary to find the minimal degree attained by the membership functions. This is interpreted as an aspiration level, or as the decision-maker's satisfaction (Zimmermann, 1987). This solution can now be found with the solution of the following auxiliary problem:

$$\max \lambda$$

$$\textbf{subject to}: \begin{cases} \left(Z_i^+\big|_{pessimistic} - Z_i^-\big|_{pessimistic}\right)\cdot\lambda - (c_{ij}-\varepsilon_{ij})\cdot x_j \le -Z_i^-\big|_{pessimistic} \\ \left(Z_i^+\big|_{optimal} - Z_i^-\big|_{optimal}\right)\cdot\lambda - (c_{ij})\cdot x_j \le -Z_i^-\big|_{optimal} \\ \left(Z_i^+\big|_{optimistic} - Z_i^-\big|_{optimistic}\right)\cdot\lambda - (c_{ij}+\varphi_{ij})\cdot x_j \le -Z_i^-\big|_{optimistic} \\ (m_{kj}-\alpha_{kj})\cdot x_j \le p_k - \gamma_k \\ m_{kj}\cdot x_j \le p_k \\ (m_{kj}+\beta_{kj})\cdot x_j \le p_k + \delta_k \\ x_i \ge 0 \end{cases} \qquad \begin{matrix} i = 1,\cdots,\#\text{goals} \\ j = 1,\cdots,\#\text{variables} \\ k = 1,\cdots,\#\text{constraint s} \\ \textbf{(8)} \end{matrix}$$

Results

To show the versatility of this method, a numerical example will be shown. Let's consider the following multi-objective problem, which has a formulation of uncertainty in both goals and constraints. The solution of this problem will provide a crisp solution, along with the decision maker's satisfaction.

$$\textbf{max } Z_1(x) = (9.5, 10, 11)\cdot x_1 + (7.3, 8, 8.2)\cdot x_2 + (-2, -1, 0)\cdot x_3 \quad \textbf{(9)}$$
$$Z_2(x) = (-9, -8, -7)\cdot x_1 + (-1, -1, -1)\cdot x_2 + (9, 10, 11.5)\cdot x_3$$

$$\textbf{Subject to:} \quad (-2, 1, 4)\cdot x_1 + (-2, 4, 8)\cdot x_2 + (-3.5, 1, 0.5)\cdot x_3 \sim\le (95, 100, 105)$$
$$(-0.5, 2.5, 5.5)\cdot x_1 + (-4.3, 1, 3.7)\cdot x_2 + (2, 2.4, 2.5)\cdot x_3 \sim\le (240, 250, 265)$$
$$x_1, x_2, x_3 \ge 0$$

As proposed in this paper, the first step in the solution of the problem posed in Eq. (9) requires the constraints to be transformed, as exposed in Eq. (5). The membership functions of the goals will be estimated through the solution of the specific sub-problems (Correa, 2015).

<u>*Pessimistic Problem*</u>

max

$$Z_1(x) = 9.5 \cdot x_1 + 7.3 \cdot x_2 - 2 \cdot x_3$$
$$Z_2(x) = -9 \cdot x_1 - 1 \cdot x_2 + 9 \cdot x_3$$

<u>*Optimal Problem*</u>

max

$$Z_1(x) = 10 \cdot x_1 + 8 \cdot x_2 - 1 \cdot x_3$$
$$Z_2(x) = -8 \cdot x_1 - 1 \cdot x_2 + 10 \cdot x_3$$

<u>*Optimistic Problem*</u>

max

$$Z_1(x) = 11 \cdot x_1 + 8.2 \cdot x_2 + 0.1 \cdot x_3$$
$$Z_2(x) = -7 \cdot x_1 - 1 \cdot x_2 + 11.5 \cdot x_3$$

Subject to: $1 \cdot x_1 + 4 \cdot x_2 + 1 \cdot x_3 \leq 100$

$$-2 \cdot x_1 - 2 \cdot x_2 - 3.5 \cdot x_3 \leq 95$$
$$4 \cdot x_1 + 8 \cdot x_2 + 0.5 \cdot x_3 \leq 105$$
$$2.5 \cdot x_1 + 3 \cdot x_2 + 2.4 \cdot x_3 \leq 250$$
$$-0.5 \cdot x_1 - 4.3 \cdot x_2 + 2 \cdot x_3 \leq 240$$
$$5.5 \cdot x_1 + 3.7 \cdot x_2 + 2.5 \cdot x_3 \leq 265$$
$$x_1 , x_2 , x_3 \geq 0$$

Therefore, it is possible to construct the membership functions for each sub-problem. Through traditional linear programming, it is possible to determine the edges of the related membership functions, whose numerical solutions are as follows, and represented in Figure 3:

Pessimistic	**Optimal**	**Optimistic**
$Z_1^+ = 249.37$	$Z_1^+ =$	$Z_1^+ = 288.75$
$Z_1^- = -212$	262.5	$Z_1^- = 0$
$Z_2^+ = 954$	$Z_1^- = -106$	$Z_2^+ = 1219$
$Z_2^- = -236.25$	$Z_2^+ = 1166$	$Z_2^- = -183.75$
	$Z_2^- = -210$	

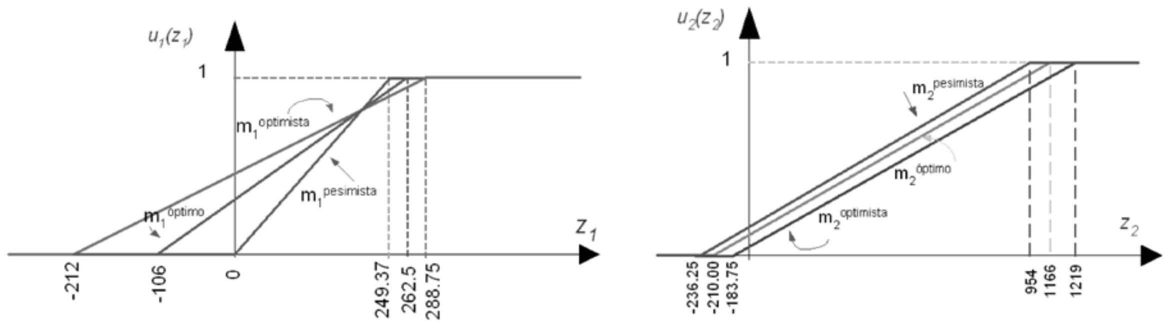

Figure 3. Goal Mapping and Membership Functions.
Source: Author's own elaboration (2015)

The proposed methodology can be now applied in order to find the most efficient solution to the multi-criteria problem, so the same membership degree can be determined, as previously shown in Eq. (8).

$$\begin{gathered}
\underline{\textbf{max}} \quad \lambda \\
\textbf{Subject to:} \\
-0.024398 \cdot x_1 - 0.018748 \cdot x_2 + 0.0051364 \cdot x_3 + \lambda \leq 0 \\
0.01039 \cdot x_1 + 0.001154 \cdot x_2 - 0.0104 \cdot x_3 + \lambda \leq 0.1859 \\
-0.03007 \cdot x_1 - 0.02406 \cdot x_2 + 0.003 \cdot x_3 + \lambda \leq 0.2105 \\
0.0088 \cdot x_1 + 0.0011 \cdot x_2 - 0.011 \cdot x_3 + \lambda \leq 0.23077 \\
-0.03809 \cdot x_1 - 0.0284 \cdot x_2 + 0 \cdot x_3 + \lambda \leq 0.359 \\
0.\ 00708 \cdot x_1 + 0.00101 \cdot x_2 - 0.011631 \cdot x_3 + \lambda \leq 0.2727 \\
1 \cdot x_1 + 4 \cdot x_2 + 1 \cdot x_3 \leq 100 \\
-2 \cdot x_1 - 2 \cdot x_2 - 3.5 \cdot x_3 \leq 95 \\
4 \cdot x_1 + 8 \cdot x_2 + 0.5 \cdot x_3 \leq 105 \\
2.5 \cdot x_1 + 3 \cdot x_2 + 2.4 \cdot x_3 \leq 250 \\
-0.5 \cdot x_1 - 4.3 \cdot x_2 + 2 \cdot x_3 \leq 240 \\
5.5 \cdot x_1 + 3.7 \cdot x_2 + 2.5 \cdot x_3 \leq 265 \\
\underline{x_1}, x_2, x_3 \geq 0
\end{gathered} \qquad \textbf{(10)}$$

The solution to (11) yields the following efficient solution to the multi-objective problem, with a decision–maker satisfaction of $\lambda = 27.7\%$.

Likewise, the decision variables take values of $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{efficient} = \begin{pmatrix} 16.704 \\ 0 \\ 25.457 \end{pmatrix}$, and the goals. $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{efficient} = \begin{pmatrix} 141.6 \\ 121 \end{pmatrix}$.

Similarly, the solution to the numerical example using crisp coefficients (no uncertainty) is as follows:

$$\begin{gathered}
\underline{\textbf{max}}\ Z_1(x) = 10 \cdot x_1 + 8 \cdot x_2 - 1 \cdot x_3 \\
Z_2(x) = -8 \cdot x_1 - 1 \cdot x_2 + 10 \cdot x_3 \\[1em]
\textbf{Subject to:} \quad 1 \cdot x_1 + 4 \cdot x_2 - 1 \cdot x_3 \leq 100 \\
2.5 \cdot x_1 + 1 \cdot x_2 + 2.4 \cdot x_3 \leq 250 \\
\underline{x_1}, x_2, x_3 \geq 0
\end{gathered} \qquad \textbf{(11)}$$

For formulation (11), the crisp solution would give the following values for the decision variables:

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{crisp} = \begin{pmatrix} 50 \\ 0 \\ 50 \end{pmatrix}$ and the goals. $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}_{crisp\ problem} = \begin{pmatrix} 450 \\ 100 \end{pmatrix}$.

The achievement of the objectives can be then measured with respect to the optimal values of Z_1 and Z_2 on the feasible region frontier for this fuzzy problem, as shown in Figure 4.

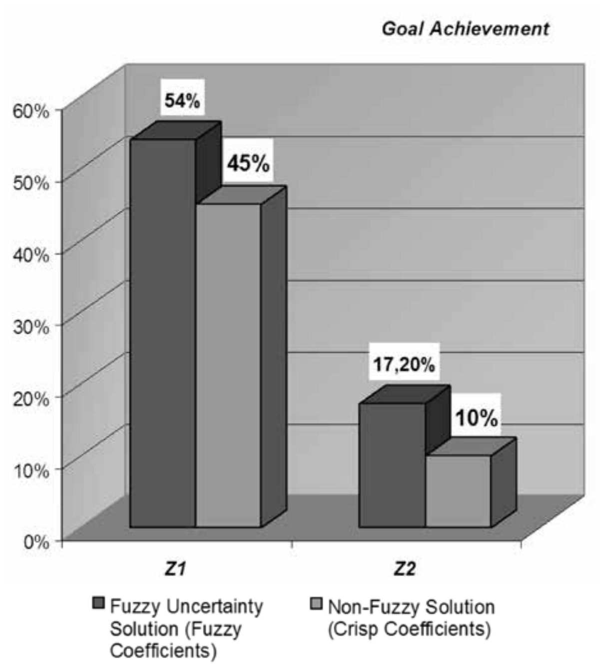

Figure 4. Goals achievement for the multiobjetive problem in (9), and comparison to (11)
Source: Author's own elaboration (2015)

It can be noticed in Figure 4 that formulation and solution of multi-criteria fuzzy problems can be expected to produce better results than deterministic problems, since there are better levels of goal achievement.

Conclusions

A useful methodology approach has been proposed as an alternative to consider vagueness and uncertainty when formulating multi-criteria problems in Operations Research. This method can be considered as complete, since it delivers valuable information (the decision maker's satisfaction) within a complete solution for a continuous decision problem.

A fuzzy approach looks for efficient solutions, which are contained inside a "complete solution", and they can be distinguished by their satisfaction levels.

The technical utility presented in this paper is based in the use of fuzzy sets, fuzzy numbers specifically. It allows to decrease coefficient of uncertainty. In the example, an exact value of "100" has been substituted by a fuzzy number that has been defined as "approximately 100". This way, a new alternative for uncertainty management has been proposed, and it offers a new view of the solution to this kind of problems, which also enriches a future analysis of them.

This methodological approach has been implemented in C++ code. The computer application has easy to use graphical interfaces with high level algorithms that allow any user to solve any kind of fuzzy linear multi-criteria problems.

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"El secreto de la felicidad no es hacer siempre lo que se quiere, sino querer siempre lo que se hace".

León Tolstoi.

Ruellia brittoniana / Autor: Diego Alonso Rivera Vergara