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Non-Standard Discretization of the Advection-Diffusion-Reaction Equation with Logistic Growth Reaction

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Abstract

The goal of this work is to make a comparative analysis between the standard finite difference method and the non-standard finite difference method, then to make a non-standard discretization of the advection-diffusion-reaction equation with a reaction modelling a logistic growth which can be the evolution of the concentration of a microbial population in a medium, the equation will thus model transport and diffusion of this population in the aforementioned medium in one dimension of space and one makes numerical simulations to compare the non-standard scheme and the Euler's scheme, explicit in time, implicit for the first order derivative in $\boldsymbol{x}$ and centered for the second order derivative in $\boldsymbol{x}$. One arrives by constructing a scheme of the advection-reaction equation, then adds the term of diffusion to obtain the non-standard scheme.

Keywords: Non-standard finite differences methods; CFL stability; advection-diffusion-reaction equation; logistic equation; logistic growth; cubic spline.

1. Introduction

Mathematical modelling of problems from the social science and sciences of engineer leads to obtaining partial differential equations and systems of ordinary differential equations. In general, not having analytical solutions, one is led to develop numerical processes to solve or at least find the approximation of solution which express with a right precision the real modelled situation. The finite differences schemes and the Runge-Kutta methods

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are widely used for this purpose, but these present limits such as numerical instabilities linked to the step-size of discretization, the conservation of the positivity of the solution, the approximation of the derivatives. In 1979 [16], show that some finite difference schemes do not converge and have chaos-type instabilities. Then in 1981, Reference [15] publish a paper in which they show the instability of numerical methods for many types of ordinary differential equations. In the early 1990s, Professor Ronald Mickens initiates a research program on the appearance and understanding of numerical instabilities which exist in finite difference schemes. A series of three articles will be published, the third was published in 1994 by [10]. In this article, they construct Preprint submitted to International Journal of Computer February 4, 2022 finite difference schemes which preserve the stability properties of equations studied and this independently of the step-size discretization. The results of these research will lead to a new concept called non-standard finite differences. Mickens publishes in 2000 [9], some rules which will be considered like the rules of construction of a non-standard scheme for some differential equations and partial differential equations, thereafter, several researchers including [11,2, 4, 8], Tchuinté and his colleagues [12, 13], will use the non-standard finite differences in their works. This new way permit to build numerical schemes for which the elementary numerical instabilities are eliminated and it also provides results better compared to traditionally used methods. In addition, unlike standard methods where the non-linear terms are discretized locally, the non-standard scheme requires a non-local approximation of non-linear terms. A non-exhaustive list of some approximations are given. In this work, we want to make a non-standard scheme for the advection-diffusion reaction equation with a reaction modelling a logistics growth. We will use the interpolation by cubic splines for calculate any data which is not available and for the estimation of a bound for the scheme error. This work is organized as follows, we make a reminder of the useful mathematical concepts to section 2. The purpose of section 3 is to make a comparative analysis between the method of standard and non-standard finite differences for the problems of ordinary differential equations and the transport problem in dimensions one and two of space, with some numerical results. Finally, section 4 is devoted to studying the equation of advection-diffusion reaction, with a logistic type reaction, the analysis of the error and the numerical simulation in one dimension of space conclude the paper.

2. Some useful mathematical concept

2.1. Logistic equation

The logistic ordinary equation is given by

$$\frac{d}{dt}u(t) = \lambda_1 u(t) - \lambda_2 \big(u(t)\big)^2 \qquad (2.1)$$

is an (autonomous) equation of Bernoulli with $r = 2,\ P(t) = -\lambda_1$ and $Q(t) = \lambda_2$. Let us pose $v = u^{-1}$, in deriving this equality, one obtains $v' = -u'u^{-2}$ what implies that

$$v' + \lambda_1 v = \lambda_2 \qquad (2.2)$$

The general solution of (2.1) is

$$u(t) = \frac{u(t_0)}{e^{\lambda_1(t-t_0)} + \frac{\lambda_2}{\lambda_1}(1-e^{\lambda_1(t-t_0)})u(t_0)} \qquad (2.3)$$

Now, we introduce some notions about partial differential equations.

2.2. Classification of linear partial differential equation of order 2 in two dimension

The general form of linear partial differential equations of order 2

$$a\frac{\partial^2 u}{\partial x^2}(x,y) + b\frac{\partial^2 u}{\partial y^2}(x,y) + c\frac{\partial^2 u}{\partial x \partial y}(x,y) + d\frac{\partial u}{\partial x}(x,y) + e\frac{\partial u}{\partial y}(x,y) + fu(x,y) = g(x,y) \qquad (2.4)$$

with $a, b, c, d, e, f \in \mathbb{R}$ and $g : \Omega \to \mathbb{R}, \Omega$ open set in $\mathbb{R}^2$. The partial differential equation (2.4) is said to be:

- elliptic if $b^2 - 4ac > 0$,
- parabolic if $b^2 - 4ac = 0$,
- hyperbolic if $b^2 - 4ac < 0$.

It is more significant to have some examples of partial differential equations belonging to each one of these classes, thus give some examples:

- the Laplace (or Poisson) equation posed on Ω, $-\Delta u = f$ is elliptic,
- the heat equation posed on $Q = \mathbb{R}_+ \times \Omega, \partial_t u - \Delta u = f$ is parabolic,
- the waves equation posed on $Q = \mathbb{R}_+ \times \Omega, \partial_{tt} u - \Delta u = f$ is hyperbolic,
- the advection diffusion equation posed on $Q = \mathbb{R}_+ \times \Omega, \partial_t u + c \cdot \nabla u - \mu\Delta u = f$ is parabolic if $\mu > 0$ and hyperbolic if $\mu < 0$.

2.3. Transport equation

Now, let us consider the transport equation given by

$$\partial_t u + v(x,t)\partial_x u = r(u,x,t) \qquad (2.5)$$

with initial data

$$u(x,0) = f(x) \qquad (2.6)$$

and which one can solve by using the method of the characteristics ([3, 7]). Here $v(x,t)$ is a function which represents the speed of transport of the particles in the fluid at the moment t and $r(u,x,t)$ represents the reaction or the source term and u is the unknown function which represents the concentration of the studied matter. Assume that $r(u,x,t) = 0$ and $v(x,t) = c$ (real constant), then solution of (2.5) is given by [3],

$$u(x,t) = f(x - ct). \qquad (2.7)$$

Let us suppose now that the reaction is modelled by a logistic growth i.e $r(u,x,t) = \lambda u(1-u)$. Let Ω be an open bounded domain in ℝ and $T > 0$. One seeks to solve the following transport problem

$$\begin{cases} \partial_t u + c\partial_x u = \lambda u(1-u), \ (x,t) \in \Omega \times]0,T] \\ u(x,0) = f(x), x \in \Omega \end{cases} \quad (2.8)$$

by the characteristics method. Then the solution of the problem (2.8) is given by

$$u(x,t) = \frac{f(x-ct)}{e^{-\lambda t} + (1-e^{-\lambda t})f(x-ct)}. \quad (2.9)$$

2.4. Standards finites differences schemes.

One has several numerical methods for the resolution of the ordinary differential equations and the partial differential equations among which method, the finite differences. In this part, we consider a hyperbolic partial differential equation of order 1: the advection equation given by

$$\partial_t u + c\partial_x u = 0, (x,t) \in \Omega \times \mathbb{R}_+, \Omega \ open\ set\ in\ \mathbb{R} \quad (2.10)$$

with initial data

$$u(x,0) = \phi(x). \quad (2.11)$$

The unknown here is the function u defined for $(x,t) \in \Omega \times \mathbb{R}_+$. The domain of u is discretized by the grid

$$G_{h,k} = \{(x_m,t_n) : x_m = h_m, m \in \mathbb{Z}, t_n = nk, n \in \mathbb{N}\}, h,k > 0.$$

It is said that h is the step-size of space discretization and k the step-size of temporal discretization. The function u which is defined for the continuous variables (x,t), takes the value $u_m^n = u(x_m,t_n)$ at the discrete point $(x_m,t_n) \in G_{h,k}$.

2.5. Construction of the standard finite difference.

Let us consider $h > 0$ and $u \in C^4(\Omega \times \mathbb{R}_+)$. Let us also consider $(x,t) \in (\Omega \times \mathbb{R}_+)$. Under the assumption $h \to 0$, let us make a Taylor expansion with order 1 of u in the neighbourhood of (x,t), one obtains

$$u(x+h,t) = u(x,t) + h\frac{\partial}{\partial x}u(x,t) + O(h^2) \quad (2.12)$$

and

$$u(x-h,t) = u(x,t) - h\frac{\partial}{\partial x}u(x,t) + O(h^2) \quad (2.13)$$

The equations (2.12) and (2.13) give respectively

$$\frac{\partial}{\partial x}u(x,t) = \frac{u(x+h,t) - u(x,t)}{h} + O(h) \tag{2.14}$$

and

$$\frac{\partial}{\partial x}u(x,t) = \frac{u(x,t) - u(x-h,t)}{h} + O(h) \tag{2.15}$$

By subtracting (2.13) to (2.12), one obtains

$$\frac{\partial}{\partial x}u(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} + O(h^2) \tag{2.16}$$

The equations (2.14), (2.15) and (2.16) are used to approximate the first derivative $\frac{\partial}{\partial x}u(x,t)$. In the same way, by making a Taylor expansion with order 4 of $u(x+h,t)$ and $u(x-h,t)$, one obtains the expression

$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{u(x+h,t) - u(x,t) + u(x-h,t)}{h^2} + O(h^2) \tag{2.17}$$

which is used to approximate the second derivative $\frac{\partial^2}{\partial x^2}u(x,t)$.

Let L be a differential operator (linear or non-linear) of order one in t and supposes that the problem

$$\begin{cases} Lu(x,t) = f(x,t),\ (x,t) \in \Omega \times \mathbb{R}_+^* \\ u(x,0) = \phi(x), x \in \Omega \end{cases} \tag{2.18}$$

is well posed. By replacing the partial derivative by finite differences, we obtain a discrete operator $L_{h,k}$ and one applies it to u. To take account of the second member, one uses the discrete operator $I_{h,k}$ which one applies to f.

Definition 2.1. We call finite differences scheme associated to the problem (2.18) the equation given by

$$L_{h,k}u = I_{h,k}\, f \tag{2.19}$$

with the initial condition

$$u_m^0 = \phi(x_m), for\ x_m \in \Omega. \tag{2.20}$$

2.6. Consistency and stability of a numerical scheme

Definition 2.2. (Consistency*)* The scheme (2.19)-(2.20) is said to be consistent with the partial differential equation (2.18) if for all function $\varphi \in C^\infty$, one has

$$\lim_{(h,k)\to(0,0)}(L\varphi - L_{h,k}\varphi) = 0, \forall\, (x_m,t_n) \in G_{h,k}, (h,k) \to (0,0).$$

Consistency involves in particular that a regular solution of the partial differential equation is a solution of the scheme to the finite differences when the steps of discretization tend towards 0. It is a necessary condition of convergence.

Definition 2.3. (Stability) A numerical scheme is stable if the errors (of round-off, of truncation ...) cannot grow during the numerical procedure passing from a step of time to the following.

A numerical scheme is said to be:

- unconditionally stable if whatever the steps of discretization h and k, errors caused by the numerical scheme does not explode with iterations,
- conditionally stable if one must pose a condition on the steps of discretization h and k so that the error caused by the numerical scheme does not explode,
- unconditionally unstable if whatever h and k, the errors develop with the wire of the iterations. This causes results completely false.

2.7. Stability conditions of a numerical scheme for a linear ordinary differential equation

Let us begin by considering a linear ordinary differential equation of order n given by

$$\sum_{i=1}^{n} \alpha_i \frac{d^i}{dt^i} u(t) = f(t). \quad (2.21)$$

The numerical solution satisfied the linear recurring equation with *(n + 1)* levels following

$$a_p u_{k+p} + a_{p-1} u_{k+p-1} + \ldots + a_0 u_k = f_k. \quad (2.22)$$

where the $a_j, j = 0,1\ldots,p$ are coefficients which come from the discrete scheme. Let $r_1, r_2, \ldots, r_p$ be the complex roots of the associated characteristic equation

$$a_p r_p + a_{p-1} r_{p-1} + \ldots + a_1 r_1 + a_0 = 0. \quad (2.23)$$

The stability condition of the numerical scheme (2.22) is written [14],

$$|r_i| \leq 1, \forall i = 1,2\ldots,p\,. \quad (2.24)$$

2.8. Stability conditions of a numerical scheme for a partial differential equation

We firstly resume the *CFL stability condition*. For evolutionary problems, certain schemes are stable under certain value of step-size of the discretization. This inequality constitute the condition of Courant-Friedrichs-Lewy or CFL condition. It is necessary and sufficient to ensure the stability of a numerical scheme. It consists in showing that there exist a $\beta \epsilon\ ]0,1[$, such as for some given α,

$$|\alpha| \frac{\Delta t^p}{\Delta x^p} \leq \beta, \quad (2.25)$$

where Δt and Δx are respectively the steps of temporal and space discretization. We end this section by presenting the *Von Neumann analysis*.

Let $f \in L^2(\mathbb{R})$, we considers the shift or translation operator noted τ_a defined by $(\tau_a f)(x) = f(x + a), \forall\, x \in \mathbb{R}$.

It is shown that [14], $(\widehat{\tau_a f})(x) = e^{ixa}\hat{f}(x), \forall x \in \mathbb{R}$ where $\hat{f}$ is the Fourier transform of f. Let us consider the problem of partial differential equation (2.18) and for a fixed $n \in \mathbb{N}$, let us associate to the vector u_m^n the function $w^n(x)$ defined by $w^n(x) = u_m^n$, for $x \in [x_{m-h}, x_{m+h}]$. The study of stability by the Von-Neumann method consists [14],

- In the case of a one-step numerical scheme, we express a recurrence relation between $\widehat{w}^{n+1}$ and $\widehat{w}^n$ by $\widehat{w}^{n+1} = \rho(x)\widehat{w}^n$, then we determine for which values of Δt and Δx, we have $|\rho(x)| \leq 1$.
- In the case of a two-step scheme, to determine the recurrence relation between $\widehat{w}^{n-1}$, $\widehat{w}^n$ and $\widehat{w}^{n+1}$, we have $\widehat{w}^{n+1} + p(x)\widehat{w}^n + q(x)\widehat{w}^{n-1} = 0$, then to determine Δt and Δx such that $|\rho_1(x)| \leq 1$ and $|\rho_2(x)| \leq 1$ where $\rho_1(x)$ and $\rho_2(x)$ are the roots of the polynomial, $\rho^2 + p(x)\rho + q(x) = 0$.

3. Construction and comparison of non-standard and standard finite difference method

In this section, we make a comparative study between the standards methods of discretization (finite difference, Runge-Kutta) and non-standard finite differences method. This analysis is made for the ordinary differential equations; as regards with the partial differential equation, the method is with some differences close the same one. We begin by showing the construction of the non-standards schemes method, then one builds some examples for the ordinary differential equation, this section is ended by a comparative study of some numerical schemes for the advection equation.

3.1. Construction of non-standard scheme (Rules)

Mickens [9], lays down six rules for the construction of a non-standard scheme.

Rule 1. The order of the discrete representation of derivative must be with the corresponding order of the derivative appearing in the equation. A difference between these orders generally leads to the appearance of the solutions which do not check the problem and thus of numerical instabilities.

Rule 2. The denominator function for the discrete derivative must, in general, according to the step-size, being expressed in functions relatively complex than those used by convention.

Rule 3. The non-linear terms must be replaced by non-local representations.

Rule 4. Any special character of the equation and/or its solution must to be found in the equations of differences and/or its solution, with the case falling due of numerical instabilities occur. Generally, it is of positivity and the boundlessness of solutions of the equations in time continues that one would like to preserve with the scheme which one builds.

Rule 5. The built scheme should not introduce of another solution or false solutions. If not there are numerical instabilities. One example is that of the equation of logistics solved with method of Runge-Kutta of order 4 which introduces two news solutions (balances) constant $x_{3,4} = \frac{2+h\pm\sqrt{h^2-4}}{2h}$ in addition to $x_1 = 0$ and $x_2 = 1$. Stability of these constant solutions strongly depends on the step on discretization h.

Rule 6. If the equation to be solved have $N \geq 3$ derivatives, then it would be prejudicial to apply the five preceding rules to parts having $M (< N)$ derivatives, then gather these results to form a coherent system.

Definition 3.1. A non-standard finite difference scheme is a discrete model of an equation which is built according to the six rules above.

In general, the non-standards scheme are not exact schemes, however, they offer the prospect to obtain the finite differences schemes which do not have usual numerical instabilities. While these rules do not provide a single discrete representation to a particular equation, their use with a knowledge of the characteristics of the solution of the equation reduce considerably in practice the number of possible discrete models.

3.2. Concept of exact finite differences scheme

It is supposed that one wants to solve numerically a problem of ordinary differential equation which one knows the general form of the solutions.

Definition 3.2. One calls exact finite differences scheme a numerical scheme whose solution has the same form as the analytical solution of the studied problem.

Thus, for any problem of ordinary differential equation which one knows the general form of the solution, one can build an exact scheme for this problem. The advantage of building exact numerical schemes comes owing to the fact that the error of the method is almost nil as we will see in the following. The question to which we will give a response in the continuation is: *How to build an exact scheme?*

Let us consider the homogeneous ordinary differential equation of order N given by

$$\sum_{i=1}^{N} a_i \frac{d^i}{dt^i} u(t) = f(t). \qquad (3.1)$$

Its general solution is a linear combination of the independent solutions $\{u^1(t), u^2(t)..., u^3(t)\}$.

Let us set $u_k^i \approx u^i(t_k), \forall k \in \mathbb{N}, i = 1..., N$. Then the exact finite differences scheme for the ordinary differential equation (3.1) is given by [9],

$$\begin{vmatrix} u_k & u_k^1 & \cdots & u_k^N \\ u_{k+1} & u_{k+1}^1 & \cdots & u_{k+N}^N \\ \vdots & \vdots & \ddots & \vdots \\ u_{k+N} & u_{k+N}^1 & \cdots & u_{k+N}^N \end{vmatrix} = 0. \qquad (3.2)$$

A special characteristic of the non-standards scheme is the approximation of the non-linear terms appearing in the equation to discretize. One uses non-local approximations for these terms in order to avoid certain numbers of problems involved in the stability and the convergence of the numerical scheme. Here, we listed below some examples of approximations usually used.

Table 1: Non-local approximation of some non-linear terms.

Non-linear term	Non-local approximation
u^2	$u_k u_{k+1}$, $2u_k^2 - u_k u_{k+1}$, $\left(\frac{u_k + u_{k+1}}{2}\right) u_k$, $\left(\frac{u_{k-1} + u_k + u_{k+1}}{3}\right) u_k$
u^3	$u_k^2 u_{k+1}$, $\frac{1}{2}(3u_{k+1} - u_k)u_k^2$, $2\left(\frac{u_k^2 u_{k+1}^2}{u_k + u_{k+1}}\right)$, $\left(\frac{u_{k-1} + u_{k+1}}{2}\right) u_k^2$, $\left(\frac{u_{k-1} + u_k + u_{k+1}}{3}\right) u_k u_{k-1}$
u^4	$u_k^2 u_{k+1}^2$, $u_k^3 u_{k+1}$

3.3. Non-standard scheme for some ordinary differentials equations

First order ordinary differential equation

A- Exponential decay equation

A quantity is known as subject to an exponential decay if it decreases with a rate proportional to its value. The mathematical equation of this reaction is given by

$$\frac{d}{dt} u(t) = -\lambda u, \qquad (3.3)$$

where λ is a positive parameter called decay constant. The solution of (3.3) is given by

$$u(t) = u_0 e^{-\lambda t}. \qquad (3.4)$$

We will consider two standards finite differences schemes for this ordinary differential equation: forward and backward Euler method respectively given by

$$\frac{u_{k+1} - u_k}{\Delta t} = -\lambda u_k, \qquad (3.5)$$

and

$$\frac{u_k - u_{k-1}}{\Delta t} = -\lambda u_k, \qquad (3.6)$$

where Δt represents the step-size of discretization. By solving (3.5) and (3.6), one obtains respectively

$$u_{k+1} = (1 - \lambda\Delta t)u_k \quad (3.7)$$

and

$$u_k = \frac{1}{1+\lambda\Delta t}u_{k-1}. \quad (3.8)$$

The two last equations define sequences which can be express respectively by

$$u_k = (1 - \lambda\Delta t)^k u_0 \quad (3.9)$$

and

$$u_k = \left(\frac{1}{1+\lambda\Delta t}\right)^k u_0. \quad (3.10)$$

The non-standard scheme for the ordinary differential equation (3.3) is obtained by using (3.2) and (3.4), it is given by [9],

$$\frac{u_{k+1} - u_k}{\phi(\Delta t)} = -\lambda u_k, \quad (3.11)$$

where $\phi(\Delta t) = \frac{1-e^{-\lambda\Delta t}}{-\lambda}$ is the denominator function verifying $\phi = \Delta t + O(\Delta t^2)$. While solving (3.11), we obtain

$$u_{k+1} = u_k e^{-\lambda\Delta t} \quad (3.12)$$

and furthermore

$$u_k = u_0 e^{-\lambda k\Delta t} = u_0 e^{-\lambda t_k}, \text{ where } t_k = k\Delta t. \quad (3.13)$$

This last relation proves that the scheme (3.11) is an exact scheme for the ordinary differential equation (3.3). Indeed, $u(t_k) = u_0 e^{-\lambda t_k} = u_k$. Thus the solutions of these two equations have the same behaviour, it is follows that the non-standard scheme (3.11) preserve all the properties of the solution of the problem (3.3), that is not the case of the schemes (3.5) and (3.6).

Study of the stability of the schemes.

- The sequence u_k for the relation (3.9) is a geometrical sequence of reason $q = 1 - \lambda\Delta t$, then it converges if $|q| < 1$. However for $u_0 > 0$, and for t tending towards $+\infty$, the exact solution given by (3.4) is positive and tends towards 0 and implies that the scheme (3.5) is stable and convergent if $0 < q < 1$ i.e; $0 < \Delta t < \frac{1}{\lambda}$. Thus the scheme (3.5) is stable if $\Delta t \in]0, \frac{1}{\lambda}[$ and unstable if $\Delta t \geq 1/\lambda$.
- The sequence u_k for the relation (3.10) is a geometrical sequence of reason $0 < q = \frac{1}{1+\lambda\Delta t} < 1, \forall \Delta t > 0$.

Then the scheme given by (3.6) is unconditionally stable.

- The numerical scheme given by (3.11) is unconditionally stable. Indeed, the sequence u_k given by (3.13) is geometrical with reason $q = exp(-\lambda\Delta t) < 1,\ \forall\Delta t > 0.$

Numerical results and comments about convergence

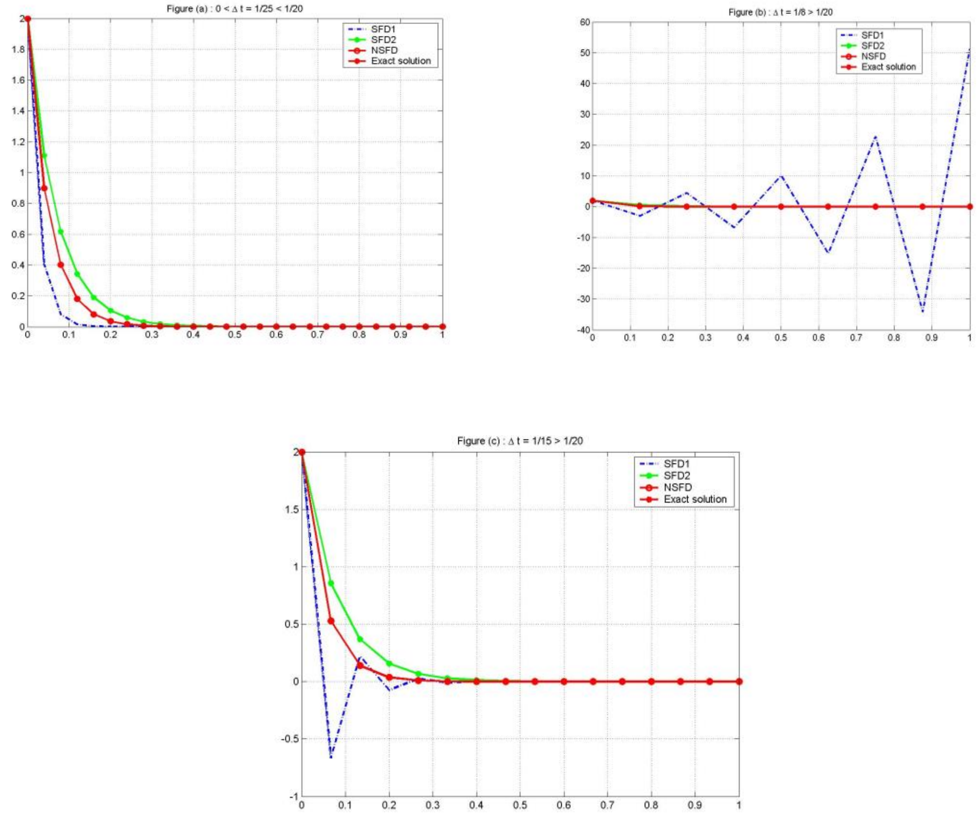


Figure 1: Exponential disintegration with λ = 20.

Note that SFD1 is the forward Euler scheme (2.8), and SFD2 is the backward Euler scheme (2.9).

- The scheme (2.8) (SFD1), is stable and converges well towards the exact solution of the problem, because the step-size verify the stability condition, i.e. $0 < \Delta t = 1/25 < 1/\lambda = 1/20$.
- The scheme (2.8) (SFD1), is unstable and does not converge, its solution oscillates with an increasing amplitude. This is due to the fact that $\Delta t = 1/8 > 1/\lambda = 1/20$.
- The scheme (2.8) (SFD1), is convergent and unstable, its solution oscillates with a decreasing amplitude. With the fact that the reason of the sequence (2.12) is lower than 1 in absolute value, there is convergence. Instability comes to the fact that the reason of the sequence is negative. Consequently, the scheme converges but not towards the solution of the problem.
- We can also remark that the solution of the non-standard scheme (NSFD) and the exact solution are merge, since the NSFD scheme is an exact scheme as shown previously.

Error analysis

Let denote by e_k the error made at the stage k.

For the relation (3.5), the error equation is,

$$\frac{e_{k+1}}{e_k} = 1 - \lambda\Delta t \qquad (3.14)$$

and for the relation (3.6), the error equation is

$$\frac{e_{k+1}}{e_k} = \frac{1}{1 + \lambda\Delta t} \qquad (3.15)$$

For the relation (3.11), the error is equation

$$\frac{e_{k+1}}{e_k} = e^{-\lambda\Delta t}. \qquad (3.16)$$

Assume that

$$\begin{cases} \Phi_{1,\lambda}(\Delta t) = |1 - \lambda\Delta t| \\ \Phi_{2,\lambda}(\Delta t) = e^{-\lambda\Delta t} \\ \Phi_{3,\lambda}(\Delta t) = \frac{1}{1+\lambda\Delta t} \end{cases}. \qquad (3.17)$$

where $\Phi_{1,\lambda}, i = 1,2,3$ is the amplification factor of the error for each of the three schemes.

Let us consider the equation $f(x) = e^{-\lambda x} + 1 - \lambda x$ define on $]\frac{1}{\lambda}, +\infty[$ where x represents Δt. $f'(x) < 0$, f is strictly decreasing for all $x \epsilon]\frac{1}{\lambda}, +\infty[$. Moreover, $f\left(\frac{1}{\lambda}\right) = e^{-1} > 0$ and $\lim_{x\to+\infty} f(x) = -\infty$, then there exists $x^0 \epsilon]\frac{1}{\lambda}, +\infty[$, unique solution of the equation $f(x) = 0$. Since $f\left(\frac{2}{\lambda}\right) = e^{-2} - 1 < 0$, hence $x^0 \epsilon]\frac{1}{\lambda}, \frac{2}{\lambda}[$. From these calculations, one obtains a comparison of the factors of amplification $\Phi_{1,\lambda}$ and $\Phi_{2,\lambda}$ as follow:

$$\begin{cases} \forall\, \Delta t \,\epsilon\,]0, \frac{1}{\lambda}],\ 0 \le \Phi_{1,\lambda}(\Delta t) \le \Phi_{2,\lambda}(\Delta t) \le \Phi_{3,\lambda}(\Delta t) \le 1 \\ \forall\, \Delta t \,\epsilon\, [\frac{1}{\lambda}, x^0[,\ 0 \le \Phi_{1,\lambda}(\Delta t) \le \Phi_{2,\lambda}(\Delta t) \le \Phi_{2,\lambda}\left(\frac{1}{\lambda}\right) \le \Phi_{3,\lambda}(x^0) \le \Phi_{3,\lambda}\left(\frac{1}{\lambda}\right) \\ \forall\, \Delta t \,\epsilon\, [x^0, \frac{2}{\lambda}[,\ \Phi_{2,\lambda}\left(\frac{2}{\lambda}\right) \le \Phi_{2,\lambda}(x^0) \le \Phi_{1,\lambda}(x^0) \le \Phi_{1,\lambda}\left(\frac{1}{\lambda}\right) \\ \forall\, \Delta t \,\epsilon\, [\frac{2}{\lambda}, +\infty[,\ 0 \le \Phi_{2,\lambda}(\Delta t) \le \Phi_{2,\lambda}\left(\frac{2}{\lambda}\right) \le \Phi_{3,\lambda}(\Delta t) \le 1 \le \Phi_{1,\lambda}(\Delta t) \end{cases}. \qquad (3.17)$$

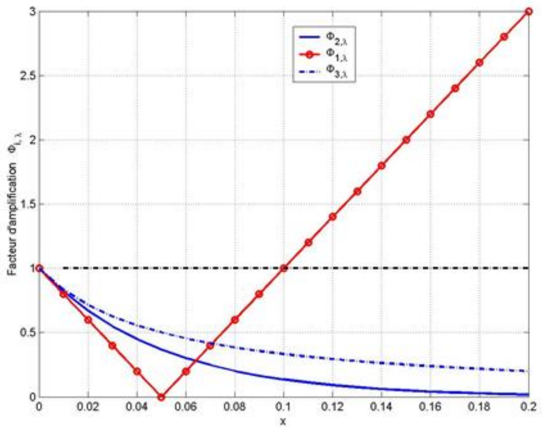


Figure 2: Amplification factor $\Phi_{1,\lambda}, i = 1,2,3$, $\lambda_0 = 20$

The figure 2 enables us to draw these conclusions:

- The non-standard scheme (3.11) is better than the schemes (3.5) and (3.6);
- The scheme (3.5) is stable and convergent for the values of $\Delta t \in]0, \frac{1}{\lambda}[$. For $\Delta t \in]\frac{1}{\lambda}, \frac{2}{\lambda}[$, one has

$0 < \Phi_{1,\lambda}(\Delta t) < 1$, the solution will oscillate with decreasing amplitudes; thus the scheme (3.5) is not stable but converges towards a solution which is not the solution of the studied problem. For $\Delta t \in [\frac{2}{\lambda}, +\infty[$, $1 < \Phi_{1,\lambda}(\Delta t) < +\infty$ the solution of the scheme (3.5) oscillates with increasing amplitudes, it has neither stability there, nor convergence;

- Although the scheme (3.6) is unconditionally stable, the Figure 2 shows that the error made by the non-standard scheme (3.11) tends quickly towards 0 than the error made by scheme (3.6).

B- Logistic equation

If one notes *u* the size of a population, *m*(*u*) the rate of mortality and *n*(*u*) the birth rate, then evolution of size of the considered population follows the ordinary differential equation

$$\frac{d}{dt}u(t) = n\big(u(t)\big)u(t) - m(u(t))u(t), \qquad (3.19)$$

Let us suppose then that there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $n(u(t)) = \lambda_1, m(u(t)) = \lambda_2 u(t)$.

The ordinary differential equation (3.19) becomes

$$\frac{d}{dt}u(t) = \lambda_1 u(t) - \lambda_2 u(t)^2. \qquad (3.20)$$

The forward Euler's scheme is given by

$$\frac{u_{k+1}-u_k}{\Delta t}=\lambda_1 u_k-\lambda_2 u_k^2, \qquad (3.21)$$

And the non-standard scheme is given by (see [9]),

$$\frac{u_{k+1}-u_k}{\frac{-1+e^{\lambda_1 \Delta t}}{\lambda_1}}=\lambda_1 u_k-\lambda_2 u_k u_{k+1}. \qquad (3.22)$$

***Theorem* 3.1.** The numerical scheme (3.22) is an exact scheme for the differential equation (3.20) and thus converges independently to the step-size $\Delta t > 0$.

Proposition 3.1. (Stability and conservation of positivity)

i)- The numerical scheme (3.21) is stable and preserves positivity if $0<\Delta t \leq \left|\frac{1}{\lambda_2 u_0-\lambda_1}\right|$.

ii)- The scheme (3.22) is unconditionally stable and if $u_k \geq 0$, then $u_{k+1} \geq 0$ for all $\Delta t \geq 0$.

Proof. For the numerical scheme (3.21), one has: $u_{k+1} = u_k(1+\lambda_1\Delta t-\lambda_2\Delta t u_k)$.

Hence, $u_{k+1} \geq 0$ if $1+\lambda_1\Delta t-\lambda_2\Delta t u_k \geq 0$. We deduce that $0<\Delta t \leq \frac{1}{\lambda_2 u_k-\lambda_1}$. Therefore, the scheme will be stable if an only if

$$0<\Delta t \leq min_{k\epsilon\mathbb{N}}\left|\frac{1}{\lambda_2 u_k-\lambda_1}\right|. \qquad (3.23)$$

If $\lambda_2 u_0-\lambda_1>0$ i.e. the death rate is higher than the birth rate, then the sequence u_k defined by (3.21) is decreasing (this means that the population decays) and we show that $\left|\frac{1}{\lambda_2 u_0-\lambda_1}\right| = \min_{k\epsilon\mathbb{N}}\left|\frac{1}{\lambda_2 u_k-\lambda_1}\right|$.

Similarly, if $\lambda_2 u_0-\lambda_1<0$ i.e. the initial birth rate is higher than the death rate, then the sequence u_k definite by (3.21) is increasing and one has $\left|\frac{1}{\lambda_2 u_0-\lambda_1}\right| = \min_{k\epsilon\mathbb{N}}\left|\frac{1}{\lambda_2 u_k-\lambda_1}\right|$. This ends the proof of i).

For the numerical scheme (3.22), since $u_k \geq 0$ and the fact that $\lambda_1 > 0$ and $\lambda_2 > 0$, one has:

$$u_{k+1}=\frac{u_k e^{\lambda_1 \Delta t}}{1+\lambda_2\phi(\Delta t)u_k} \geq 0, for\ all\ \Delta t>0\ if\ u_0 \geq 0. \blacksquare$$

Numerical results and comments about convergence

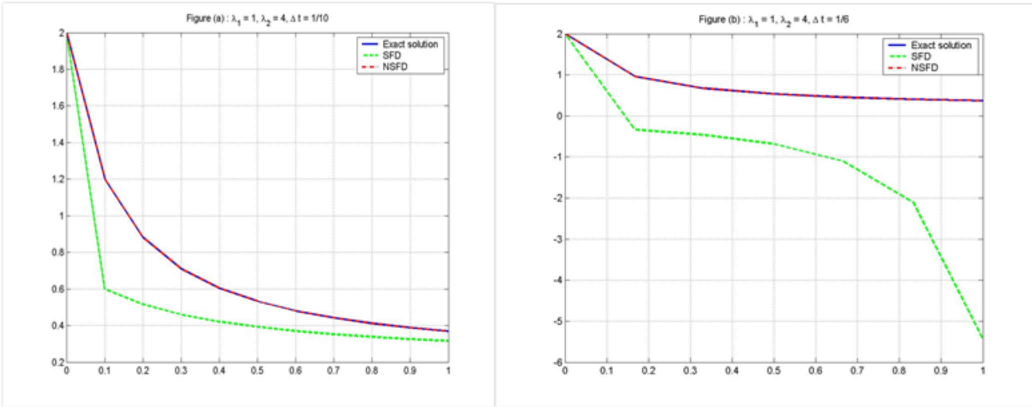


Figure 3: Logistic equation with the initial death rate higher than the birth rate.

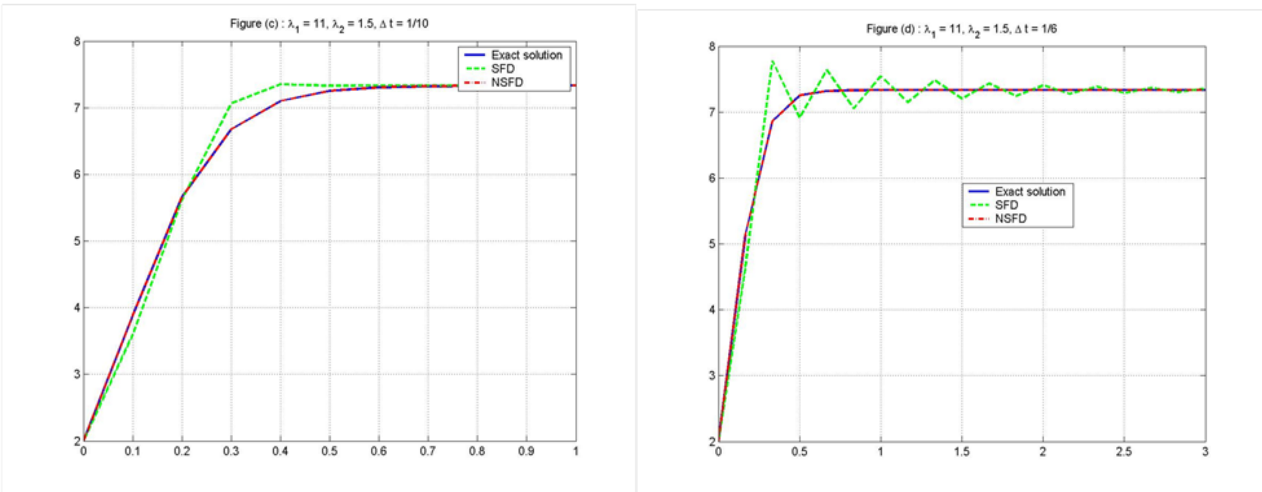


Figure 4: Logistic equation with the initial birth rate higher than the death rate.

Results of the figures 3 and 4 proves that the scheme (3.22) is very powerful compared to the scheme (3.21). Figure 3b shows that for the time step-size $\Delta t = 1/6$ which is higher than the limit value 1/7 (the case where the death rate is higher than the birth rate), the scheme (3.21) does not converge. The figure 4b shows that for the time step-size $\Delta t = 1/6$ which is higher than the limit value 1/8 (the case where the initial birth rate is higher than the death rate), the scheme (3.21) is unstable but converge because of oscillations with decreasing amplitudes. The figure 5 permit us to draw the following conclusions: the error made by the relation (3.22) is very small (around 10^{-15}) comparing with the error of the standard method given by (3.21), and for the value of Δt which does not satisfy the stability condition of the proposition 3.1, the scheme (3.21) is coarsely divergent (figure 5b), since error of this scheme growth exponentially.

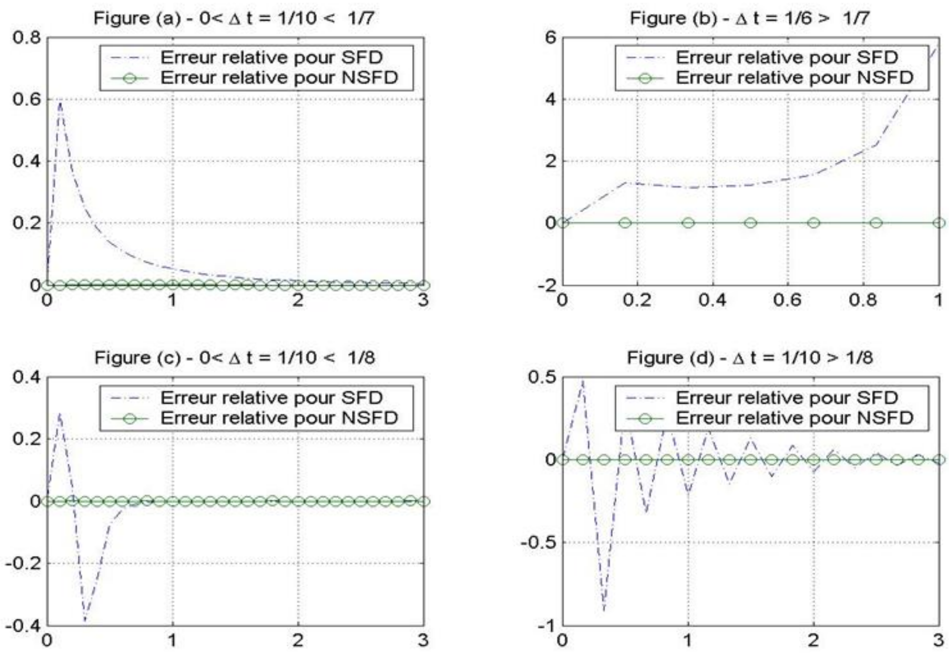


Figure 4: Relative error of the various schemes: (a) and (b) (respectively (c) and (d)) represent the curves of errors if mortality is higher (respectively lower) than the birth rate.

Second order ordinary differential equation

Let us consider the following second order ordinary differential equation $\frac{d^2}{dt^2}u(t) = \lambda \frac{d}{dt}u(t), \lambda \epsilon \mathbb{R}$.

Its general solution is given by

$$u(t) = \frac{\lambda u(t_0) - u'(t_0)}{\lambda} + \frac{1}{\lambda}u'(t_0)e^{\lambda(t-t_0)}. \qquad (3.24)$$

The standard finite differences scheme for this ordinary differential equation is given by the following expression

$$\frac{u_{k-1} - 2\, u_k + u_{k+1}}{\Delta t^2} = \lambda \frac{u_k - u_{k-1}}{\Delta t}. \qquad (3.25)$$

Proposition 3.2. The non-standard scheme for this equation is given by (see [9]),

$$\frac{u_{k-1} - 2\, u_k + u_{k+1}}{\Delta t\left(\frac{-1 + e^{\lambda_1 \Delta t}}{\lambda_1}\right)} = \lambda \frac{u_k - u_{k-1}}{\Delta t} \qquad (3.26)$$

is an exact numerical scheme for the studied differential equation.

Proof. Let us pose $A = \frac{\lambda u(t_0) - u'(t_0)}{\lambda}$ and $B = \frac{1}{\lambda}u'(t_0)e^{-\lambda t_0}$, hence the solution for (3.24) is

$$u(t) = A + Be^{\lambda t}. \qquad (3.27)$$

The relation (3.26) give us

$$u_{k+1} = u_k + e^{\lambda \Delta t}(u_k - u_{k-1}). \qquad (3.28)$$

By using the change $t \to t_k$, $u(t) \to u_k$, then $t \to t_{k-1}$, $u(t) \to u_{k-1}$ into (3.27), and by introducing the results into (3.28), we obtain

$u_{k+1} = A + Be^{\lambda \Delta t(k+1)} = u(t_{k+1})$. ■

Proposition 3.3. If $\lambda > 0$, the finite differences scheme (3.25) is stable and convergent for very small and positive Δt. If $\lambda < 0$, to ensure the convergence and the stability of the scheme (3.25), it is necessary that $0 < \Delta t < -1/\lambda$.

Proof. For scheme (3.25), we have the following $u_{k+1} - (2+\lambda\Delta t)u_k + (1+\lambda\Delta t)u_{k-1} = 0$. Its characteristic equation is given by $r^2 - (2+\lambda\Delta t)r + (1+\lambda\Delta t) = 0$, with the solutions $r_1 = 1$ *ou* $r_2 = 1+\lambda\Delta t$. This implies that the solution of the scheme (3.25) is given by

$$u_k = c_1 + c_2 r_2^k, \textit{where } c_1 = A, c_2 = Be^{\lambda t_0}.$$

As $u(t_k) \to +\infty$ when $k \to +\infty$, then for $\lambda > 0$, the numerical scheme (3.25) converge if $\Delta t > 0$ and very small. If $\lambda < 0$, then to have convergence and stability, it's sufficient that $0 < r_2 < 1$, what implies $0 < \Delta t < -1/\lambda$. ■

Numerical results

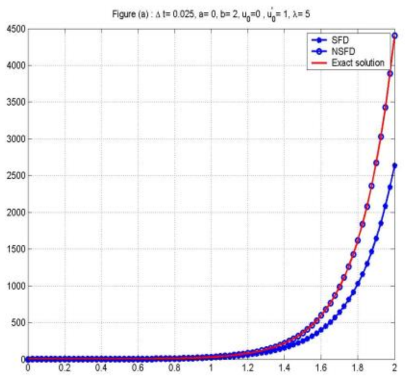


Figure 5: Equation $u''(t) = \lambda u'(t)$ with $\lambda = 5$

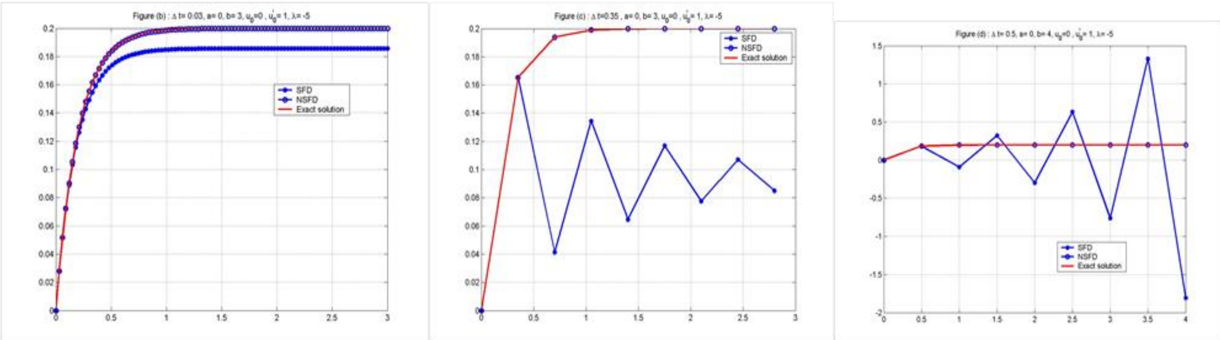


Figure 6: Equation $u''(t) = \lambda u'(t)$ with λ = -5

3.4. Numerical scheme for the advection equation in one space dimension.

Here, we initially studies some finite differences schemes for the advection equation. Then, considering the advection equation with a logistic growth reaction given by $r(u) = \lambda u(1-u)$, $\lambda \in \mathbb{R}$, we build a standard finite differences, an exact scheme, and finally we make a numerical simulation for each one of these schemes.

Let us consider the following transport problem,

$$\begin{cases} \partial_t u + c\partial_x u = 0, \; x \in]0, L[, t > 0 \\ u(x,0) = f(x), \end{cases} \qquad (3.29)$$

where $f \in C^1$ and the general solution is $u(x,t) = f(x - ct), c \in \mathbb{R}$. In the following, we set $h = \Delta x$, $k = \Delta t$ and $v = k/h$.

Lax-Friedrichs finite differences scheme for the transport problem

This scheme is centred in space and progressive in time. It uses an average centred to approximate u_m^n. The scheme is given by

$$\frac{u_m^{n+1} - \frac{1}{2}(u_{m+1}^n + u_{m-1}^n)}{k} + c\frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0\,. \qquad (3.30)$$

Proposition 3.4. The numerical scheme (3.30) is consistent if $h^2/k \to 0$ for $h, k \to 0$, and of order 1 if $k = \lambda h$, ($\lambda \in \mathbb{R}_+$, fixed) and stable under the CFL condition $k \le h/|c|$.

Proof. From (3.30), we have $u_m^{n+1} = (1/2 - cv)u_{m+1}^n + (1/2 + cv)u_{m-1}^n$. Replacing u_m^n by w^n, one obtains $w^{n+1} = \left(\frac{1}{2} - cv\right)\tau_h w^n + (\frac{1}{2} + cv)\tau_{-h} w^n$, then taking the Fourier transform of this expression, we arrive to

$$\widehat{w}^{n+1} = \left(\left(\frac{1}{2} - cv\right)e^{ikh} + \left(\frac{1}{2} + cv\right)e^{-ikh}\right)\widehat{w}^n, i.e.\ \rho(kh) = cos(kh) - i(2vcsin(kh)).$$

Hence, evaluating $|\rho|^2 \leq 1$ implies that $c^2\nu^2 - 1 \leq 0$. We obtain the following CFL condition

$$k \leq \frac{h}{|c|}. \qquad (3.31)$$

Thus, the Lax-Friedrichs scheme is stable under the CFL condition $k \leq h/|c|$. ■

Lax-Wendroff finite differences scheme for the transport problem

Let u be a smooth solution of the problem (3.29). Then, using a Taylor expansion of u following the variable t, one obtains $u_{m+1}^n = u_m^n + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + O(k^2)$.

Since $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-c\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-c\frac{\partial u}{\partial t}\right) = c^2\frac{\partial^2 u}{\partial x^2}$, then $u_{m+1}^n = u_m^n + -ck\frac{\partial u}{\partial x} + \frac{c^2k^2}{2}\frac{\partial^2 u}{\partial x^2} + O(k^2)$.

Replacing the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ respectively by their centred approximations, we infer that

$$u_{m+1}^n = u_m^n - \frac{c\nu}{2}(u_{m+1}^n - u_{m-1}^n) + \frac{c^2\nu^2}{2}(u_{m+1}^n - 2\,u_m^n + u_{m-1}^n). \qquad (3.32)$$

Proposition 3.5. The numerical scheme (3.32) is consistent, of order 2 in time and order 1 in space, stable for the CFL condition $k \leq h/|c|$.

Proof. From (3.32), we has

$$w^{n+1} = w^n - c\frac{\nu}{2}(\tau_h w^n - \tau_{-h}w^n) + \frac{c^2\nu^2}{2}(\tau_h w^n - 2w^n + \tau_{-h}w^n).$$

The Fourier transform is given by

$$\widehat{w}^{n+1} = \left[1 - i\big(\nu c sin(kh)\big) + \frac{c^2\nu^2}{2}(e^{ikh} - 2 + e^{-ikh})\right]\widehat{w}^n$$

and deduce that $\rho(kh) = 1 + c^2\nu^2(-1 + cos(kh)) - i(\nu c sin(kh))$. Once again, we evaluate $|\rho|^2 \leq 1$, and arrive to $4c^2\nu^2(\,c^2\nu^2 - 1)sin^4(\frac{kh}{2}) \leq 0$ which implies the following CFL condition

$$k \leq \frac{h}{|c|}. \qquad (3.33)$$

Hence the numerical scheme (3.32) is stable under the CFL condition $k \leq h/|c|$. ■

Standard finite differences (Euler) scheme for the transport problem

By using a forward difference in time and backward difference in space, one obtains

$$u_m^{n+1} = (1 - cv)u_m^n + cvu_{m-1}^n. \qquad (3.34)$$

The study of the stability of the scheme (3.34) by the Von-Neumann method leads to the CFL condition $|c|v \leq 1$.

Non-standard finite differences scheme for the transport problem

Proposition 3.6. The problem (3.29) admit an exact scheme given by $u_m^n + 1 = F(m - cn), si\ \Delta t = \Delta x$.

Proof. Let us assume that $\Delta t = \Delta x$, then by using (3.34), one obtains $u_m^{n+1} = (1 - c)u_m^n + cu_{m-1}^n$, avec $|c| \leq 1$. Now, $u(x_m, t_n) = f(x_m - ct_n) = f(\Delta t(m - cn))$, by replacing this in the preceding equality, one obtains

$$u_m^{n+1} = (1 - c)f\big(\Delta t(m - cn)\big) + cf\big(\Delta t(m - cn - 1)\big), with\ |c| \leq 1. \qquad (3.35)$$

Let us consider the function $F : \mathbb{R} \to \mathbb{R}$ such that $F(y) = (1 - c)f(y\Delta t) + cf(\Delta t(y - 1)), avec\ |c| \leq 1$. Thereby, the relation (3.35) lead to $u_m^{n+1} = F(m - cn)$. ■

We conclude that the non-standard finite differences scheme for the problem of partial differential equations (3.29) is given by

$$u_m^{n+1} = (1 - c)u_m^n + cu_{m-1}^n, with\ |c| \leq 1\ and\ \Delta t = \Delta x. \qquad (3.36)$$

3.5. Advection-reaction equation in one dimension space with logistic type reaction

Now let us consider the following problem of partial differential equation

$$\begin{cases} \partial_t u + c\partial_x u = \lambda u(1 - u),\ x \in]0, L[, t > 0,\ \lambda \in \mathbb{R} \\ u(x, 0) = f(x), \end{cases} \qquad (3.37)$$

where $f \in C^1$, which general solution is given by

$$u(t) = \frac{f(x-ct)}{e^{-\lambda t} + (1 - e^{-\lambda t})f(x - ct)},\ c \in \mathbb{R}. \qquad (3.38)$$

Standard finite differences (Euler) scheme for the transport problem

By using a forward difference in time and backward difference in space, one obtains

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} + c\frac{u_m^n - u_{m-1}^n}{\Delta x} = \lambda u_m^n(1 - u_m^n). \qquad (3.39)$$

Equation (3.39) is the standard finite differences scheme of the partial differential equation (3.37).

Construction of the non-standard scheme (exact scheme) for the problem (3.37)

Proposition 3.7. The exact numerical scheme semi-discrete in time for the problem of partial differential equation (3.37) is given by

$$\frac{u^{n+1}(x) - u^n(\bar{x})}{\phi} = \lambda u^n(\bar{x})(1 - u^{n+1}(x)) \quad (3.40)$$

where

$$\phi = \frac{e^{\lambda \Delta t} - 1}{\lambda} \, and \; \bar{x} = x - c\Delta t. \quad (3.41)$$

Proof. To do it, we start by showing the following equality

$$u(x, t + \Delta t) = \frac{u(x - ct, t)}{e^{-\lambda \Delta t} + (1 - e^{-\lambda \Delta t}) u(x - ct, t)}, \; c \in \mathbb{R}. \quad (3.42)$$

Then, considering the following change $t \rightarrow t_n$, $x - c\Delta t \rightarrow \bar{x}$, *and* $u(x, t_n) \rightarrow u^n(x)$ in (3.42), and doing little calculations, we obtain $u^{n+1}(x) = \left(1 - e^{\lambda \Delta t}\right) u^n(\bar{x}) u^{n+1}(x) + e^{\lambda \Delta t} u^n(\bar{x})$. Then adding $-u^n(\bar{x})$ both on the left and the right-hand side of the previous equations lead to (3.40). ■

Let us consider $\gamma = \{x_0, x_1 \ldots, x_{M^*}\}$, where $x_{m+1} - x_m = \Delta x$, a uniform grid of [0, L]. Then, the non-standard scheme for the problem (3.37) is given by

$$\frac{u^{n+1}(x_m) - u^n(\bar{x}_m)}{\phi} = \lambda u^n(\bar{x}_m)\left(1 - u^{n+1}(x_m)\right), where \; \bar{x}_m = x_m - c\Delta t. \quad (3.43)$$

3.6. Advection-reaction equation in two dimension space with logistic type reaction

Here, we will develop a non-standard scheme for the advection-reaction equation in two dimension space. It should be noted that while proceeding in the same way, one will be able to build the non-standard scheme for the transport problem in $N \geq 3$ dimension space and this without major difficulty.

We want to build a non-standard scheme for the following transport problem define on $\Omega^2 \times \mathbb{R}_+$, where $\Omega =]0, L[$:

$$\begin{cases} \partial_t u + v \cdot \nabla u = \lambda u(1 - u), \; (x, y) \in \Omega^2, t > 0, \; \lambda \in \mathbb{R} \\ u(x, y, 0) = f(x, y), \end{cases} \quad (3.44)$$

where $v(x, y, t) = \begin{pmatrix} v_x(x,y,t) \\ v_y(x,y,t) \end{pmatrix}$ is the velocity of carried particles and$\nabla u(x, y, t) = \begin{pmatrix} \partial_x u(x,y,t) \\ \partial_y u(x,y,t) \end{pmatrix}$.

We solve the problem (3.44) uses the characteristics method (see [3]). We suppose that the velocity is constant i.e. $v(x, y, t) = \binom{c}{c}$ and the solution is given by

$$u(x, y, t) = \frac{f(s)}{e^{-\lambda t} + (1 - e^{-\lambda t}) f(s)}, \; where \; s = \begin{pmatrix} x - ct \\ y - ct \end{pmatrix}. \quad (3.45)$$

Therefore, the semi-discrete time discretization of the problem (3.44) is

$$\frac{u^{n+1}(X)-u^{n}(\bar{X})}{\phi}=\lambda u^{n}(\bar{X})\big(1-u^{n+1}(X)\big),\ where\ \bar{X}=X-c\Delta t=\binom{x-ct}{y-ct} \quad (3.46)$$

Now, let us consider the uniform grid of $\Omega^2=[0,L]\times[0,L]$, $\gamma_x=\{x_0,x_1\ldots,x_{M^*}\}$ and $\gamma_y=\{y_0,y_1\ldots,y_{M^*}\}$, where $\Delta x=\Delta y=\frac{1}{M^*}$. The non-standard scheme for the transport problem in two dimension space is given by

$$\frac{u^{n+1}(X_m)-u^{n}(\bar{X}_m)}{\phi}=\lambda u^{n}(\bar{X}_m)\big(1-u^{n+1}(X_m)\big), where\ \bar{X}_m=X_m-c\Delta t=\binom{\bar{x}_m-ct}{\bar{y}_m-ct}. \quad (3.47)$$

3.7. Numerical results in one dimension space

Here, we are interested to the numerical simulation of the results obtained in the preceding part, this in order to show the effectiveness of the non-standard scheme (3.43) compared to the Lax-Friedrichs and Lax-Wendroff schemes. To do it, let us consider the following transport problem

$$\begin{cases}\partial_t u+\partial_x u = u(1-u),\ x\,\epsilon\,]0,5[, t\epsilon\,]0,1],\\ \qquad\qquad u(x,0)=f(x),\end{cases} \quad (3.48)$$

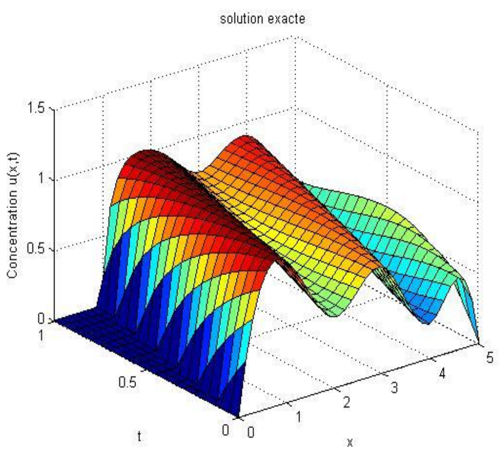
where $f(0)=0, f(1)=1, f(2)=0.5, f(3)=0.75, f(4)=0.25, f(4.5)=0.5, and\ f(5)=0.$

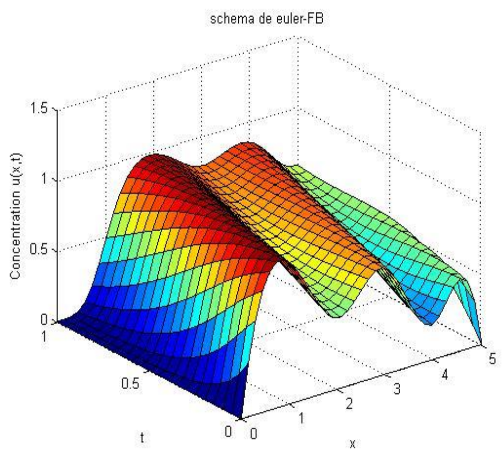
We find a polynomial function which interpolates these values by using the Lagrange or Newton interpolation. Let us recall that the non-standard scheme that we wants to simulate is,

$$\frac{u^{n+1}(x_m)-u^{n}(\bar{x}_m)}{\frac{e^{\Delta t}-1}{1}}=u^{n}(\bar{x}_m)\big(1-u^{n+1}(x_m)\big), where\ \bar{x}_m=x_m-\Delta t. \quad (3.43)$$

We consider a uniform grid of [0, 1] with the step-size $\Delta t = 0.05$, and a uniform grid of [0, 5] with the step-size $\Delta x = 0.15$. Lastly, to evaluate the term $u^n(\bar{x}_m)$, one uses the cubic spline interpolation.

Numerical results





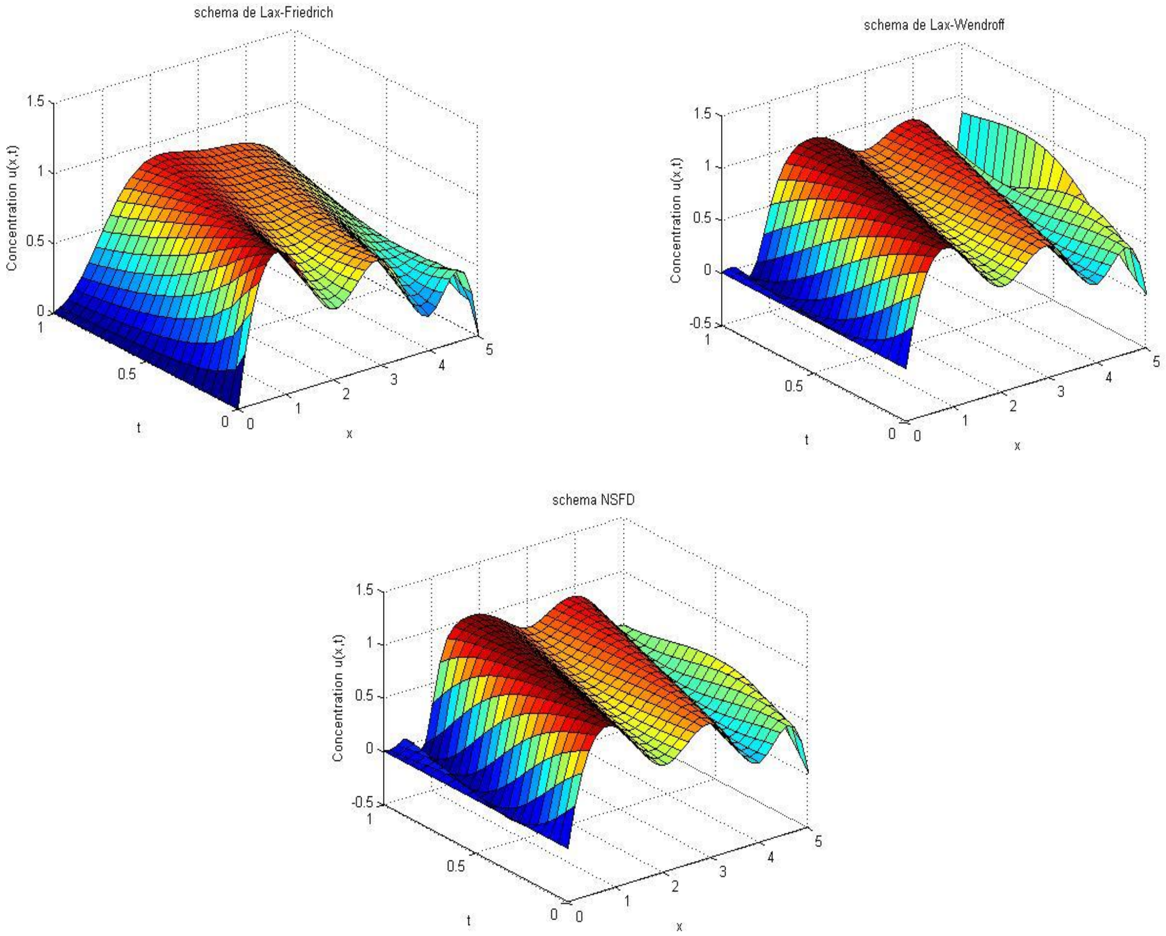


Figure 7: Surfaces of transport corresponding to the transport problem modelled by the Euler-FB, Lax-Friedrichs, Lax-Wendroff and the non-standard schemes.

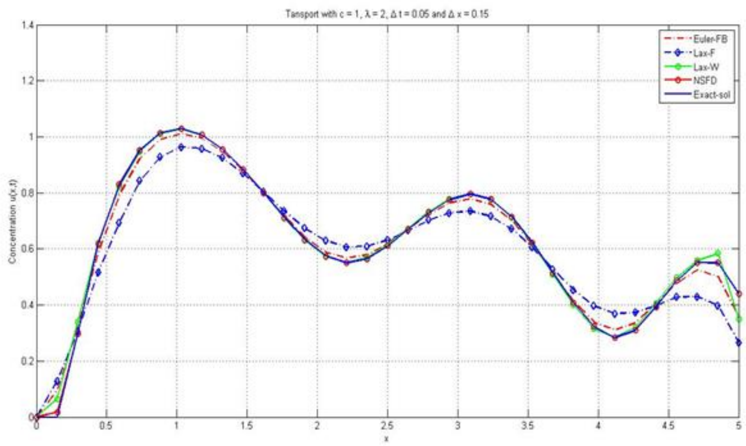


Figure 8: Solution of the transport problem (3.54), curves come from the surfaces of the preceding figure for time t = 0.2.

3.8. Conclusion of the section

In this section, we have to point out the fundamental rules of the non-standard finite differences method established by Mickens, then we made the analysis of some schemes for ordinary differential equations with or without second members, schemes for which one established stability conditions. Lastly, we made the study of four finite differences schemes for the transport equation. They are the mixed forward-Backward scheme (Euler-FB, see the numerical expression (3.39)), the Lax-Friedrichs (3.30), Lax-Wendroff (3.32) and the non-standard schemes (3.43). The figures 8 and 9 show that the non-standard scheme approaches the solution better when this one is sufficiently regular (see figure 9 in the interval [1, 5]). The oscillation observed in the zone [0, 1], comes because of errors accumulation from the interpolation as well for calculation from the exact solution as for the evaluation of the concentration $\boldsymbol{u^n(\bar{x}_m)}$ by the cubic splines. In the following section, we will use the scheme (3.43) to build the non-standard scheme for the advection-diffusion equation with logistic growth type reaction given by $\boldsymbol{r(u,x,t) = \lambda u(x,t)(1-u(x,t))}$, and we will insist too long on the evaluation of error of the method.

4. Mathematical model, and non-standard discretization in one dimension space

Here, we consider the advection-diffusion-reaction equation with the advection velocity $v(x,t)$ constant and equal to $c \in \mathbb{R}$, and the diffusion coefficient $d(x,t)$ also constant and equal to $d \in \mathbb{R}$. The advection-diffusion problem with a logistic growth reaction that we be will study is as follows,

$$\begin{cases} \partial_t u + c\partial_x u - d\partial_{xx} u = \lambda u(1-u), \ on\ \Omega \times]0,T], \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0 \end{cases} \tag{4.1}$$

where $\Omega =]0, L[, L > 0$ and $T > 0$. Later on, we will use the following notations: At time t_n, u_m^n for the approximate solution, U_m^n for the exact solution, $\zeta_m^n = U_m^n - u_m^n$ the difference between the exact and the approximate solution and $\bar{\zeta}_m^n = \bar{U}_m^n - \bar{u}^n(x_m)$, for all n.

4.1. Time semi-discretization

Let us set $\frac{Du}{Dt} = \partial_t u + c\,\partial_x u$, according to proposition 3.7, one has

$$\frac{Du}{Dt}(x,t_n) = \frac{u^{n+1}(x) - u^n(\bar{x})}{\phi}, with\ \phi = \frac{e^{\lambda\Delta t}-1}{\lambda} and\ \bar{x} = x - c\Delta t. \tag{4.2}$$

Let us express the diffusion term at time t_{n+1} by

$$\partial_{xx} u^{n+1}(x) = \frac{\partial^2 u^{n+1}(x)}{\partial_x^2}. \tag{4.3}$$

Finally, let denote the discretized logistic growth reaction by

$$R\big(u^{n+1}(x), u^n(\bar{x})\big) = \lambda u^n(\bar{x})(1 - u^{n+1}(x)). \tag{4.4}$$

where $R \in C^1([0,L] \times [0,L])$, and $R(z_1, z_2) = \lambda z_2(1 - z_1)$. By combining the relations (4.2), (4.3) and (4.4), one obtains the semi-discrete scheme in time of the equation (4.1) given by

$$\frac{u^{n+1}(x)-u^{n}(\bar{x})}{\phi}-d\partial_{xx}u^{n+1}(x)=R\big(u^{n+1}(x),u^{n}(\bar{x})\big). \quad (4.5)$$

4.2. Non-standard scheme for the advection-diffusion-reaction equation

Let us consider $\gamma = \{x_0, x_1 \ldots, x_{M^*}\}$, where $x_{m+1} - x_m = \Delta x$, a uniform grid of [0, L]. By using the centred difference in space for the diffusion term (4.3), one has

$$\partial_{xx}u^{n+1}(x) \approx \frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{\Delta x^2}. \quad (4.6)$$

Let us note δ_x^2 the finite centred difference operator in space define by

$$(\delta_x^2 u^{n+1})_m = \frac{u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}}{\Delta x^2}. \quad (4.7)$$

Then, the non-standard scheme for the partial differential equation (4.1) is

$$\frac{u_m^{n+1}-u^{n}(\bar{x}_m)}{\phi}-d(\delta_x^2 u^{n+1})_m = R\big(u_m^{n+1},u^{n}(\bar{x}_m)\big). \quad (4.8)$$

4.3. Error analysis

The following result is prove in Mickens [9]. It shows that the error of the method (4.8) is bounded.

Theorem 4.1. Let us suppose that the exact solution of the partial differential equation (4.1),$U \in C^4([0,L] \times [0,L])$ and the approximate solution u_m^{n+1} is defined by the non-standard numerical scheme (4.8), where $u^n(\bar{x}_m) = u^n(x_m - c\Delta t) = (S_3 u^n)(\bar{x}_m)$ is the interpolation of u_m^n and S_3 the cubic spline interpolation operator. Then, there exists a constant $Q > 0$ such that

$$\|U(.,t_{n+1}) - u^{n+1}(.)\|_\infty \le QS((\Delta x)^2 + \Delta t), \quad (4.9)$$

where

$$S = max\left(\max_{0\le k\le n+1}\left\|\frac{\partial^4 U^k}{\partial x^4}\right\|_\infty, \sup_{[0,L]\times J}\left|\frac{\partial^2 U}{\partial \tau^2}\right|, \sup_{[0,L]\times J}\left|\frac{\partial U}{\partial \tau}\right| \sup_{[0,L]\times[0,L]}\left|\frac{\partial R}{\partial z_2}\right|\right).$$

The theorem 4.1 is prove in [9], so we will not repeat it here. But, to achieve the proof of this theorem, Mickens use two others theorems. Here, we will slightly modify one of them (theorem 4.2) and state a new one (theorem 4.3) with his proof so that they will be adapted for our problem. We need the following notations,

$$j(m) = \{j: |\bar{x}_m - x_j| = min_{0\le k\le M^*}\{|\bar{x}_m - x_k|\}\}, \qquad (4.10)$$

and

$$\Delta x^* = |\bar{x}_m - x_{j(m)}| = min\left\{\frac{\Delta x}{2}, \widehat{K}\Delta t\right\}. \qquad (4.11)$$

Theorem 4.2. Let f be a real variable function. Suppose that $f \in C^4([0,L])$, and $\gamma = \{x_0, x_1 \ldots, x_{M^*}\}$ a uniform grid of [0,L] with the step-size Δx. Let us denote by $\boldsymbol{m} \in \mathbb{R}^{M^*+1}$ the moments vector $\boldsymbol{m_i}$ of the complete spline S_3 interpolating f on γ and finally, let $\boldsymbol{f} \in \mathbb{R}^{M^*+1}$ be the vector of exact values of the second derivative $f''(x_i)$ on the nodes of γ. Then, $\|\boldsymbol{m} - \boldsymbol{f}\|_\infty \le \frac{3}{2}\|\boldsymbol{f^4}\|_\infty(\Delta x)(\Delta x^*)$.

Theorem 4.3. Let f be a real variable function. Suppose that $f \in C^4([0,L])$, and $\gamma = \{x_0, x_1 \ldots, x_{M^*}\}$ a uniform grid of [0,L] with the step-size Δx. If S_3 is the complete cubic spline interpolating f on γ, then there exist some constants C_k such that $|f^{(k)}(\bar{x}_m) - \tilde{f}^{(k)}(\bar{x}_m)| \le C_k\|f^4\|_\infty(\Delta x)^{3-k}(\Delta x^*), k = 0,1,2,3$ where $\tilde{f} = S_3 f$ is the interpolation of f by the complete cubic spline.

Proof. (Of Theorem 4.3)

One has, $\bar{x}_m \in]x_{j(m)+1}, x_{j(m)}[$, we begin the proof for $k = 3$, until $k = 0$. For $k=3$, by the definition of the cubic spline [1], we have

$$\tilde{f}^{(3)} = \frac{\boldsymbol{m}_{j(m)+1} - \boldsymbol{m}_{j(m)}}{\Delta x}$$

then

$$|\tilde{f}^{(3)}(\bar{x}_m) - f^{(3)}(\bar{x}_m)| \le \underbrace{\left|\frac{\boldsymbol{m}_{j(m)+1} - f''(x_{j(m)+1})}{\Delta x} - \frac{\boldsymbol{m}_{j(m)} - f''(x_{j(m)})}{\Delta x}\right|}_{(I)}$$

$$+ \underbrace{\left|\frac{f''(x_{j(m)+1}) - f''(\bar{x}_m)}{\Delta x} - \frac{f''(x_{j(m)}) - f''(\bar{x}_m)}{\Delta x} - f^{(3)}(\bar{x}_m)\right|}_{(II)}.$$

Let us evaluate separately (I) and (II). According to the theorem 4.2 and the triangular inequality, one has

$$(I) \le \frac{1}{\Delta x}\left\{\frac{3}{2}\|f^4\|_\infty(\Delta x)(\Delta x_m^*) + \frac{3}{2}\|f^4\|_\infty(\Delta x)(\Delta x^*)\right\}$$

which infer that

$$(I) \le 3\|f^4\|_\infty(\Delta x^*).$$

To evaluate *(II)*, one makes the Taylor expansion of the function f'' on the neighbourhood of $\bar{x}_m$. Then, there exist $\eta_1, \eta_2 \epsilon]x_{j(m)+1}, x_{j(m)}[$, such that

$$(II) \leq \left| \frac{1}{\Delta x}\left(x_{j(m)+1} - \bar{x}_m\right) f^{(3)}(\bar{x}_m) + \frac{1}{2}\underbrace{\frac{\left(x_{j(m)+1} - \bar{x}_m\right)^2}{\Delta x}}_{(III)} \underbrace{f^{(4)}(\eta_1)}_{(IV)} \right.$$

$$\left. - \frac{1}{\Delta x}\left(x_{j(m)} - \bar{x}_m\right) f^{(3)}(\bar{x}_m) - \frac{1}{2}\underbrace{\frac{\left(x_{j(m)} - \bar{x}_m\right)^2}{\Delta x}}_{(III)} \underbrace{f^{(4)}(\eta_1)}_{(IV)} - f^{(3)}(\bar{x}_m) \right|.$$

Like $\bar{x}_m \;\epsilon]x_{j(m)+1}, x_{j(m)}[$, with $x_{j(m)+1}$ and $x_{j(m)} \epsilon \;\gamma$, then

$$(III) \leq \frac{1}{\Delta x}(\Delta x)(\Delta x^*) = (\Delta x^*).$$

In addition, $\left|f^{(4)}(\eta_1)\right| \leq \left\|f^{(4)}\right\|_\infty$ implies that $(IV) \leq \left\|f^{(4)}\right\|_\infty$. The sum of the remaining terms vanish, then we obtain $(II) \leq \left\|f^{(4)}\right\|_\infty (\Delta x^*)$.

The estimations (I) and (II) lead to the following result

$$\left|\tilde{f}^{(3)}(\bar{x}_m) - f^{(3)}(\bar{x}_m)\right| \leq 4\left\|f^{(4)}\right\|_\infty (\Delta x^*), therefor, C_3 = 4. \qquad (4.12)$$

For $k = 2$, let us choose $x_\ell \;\epsilon\; \gamma$ such as $|x_\ell - \bar{x}_m| \leq \frac{\Delta x}{2}$. By using the fundamental theorem of calculus and the triangular inequality, we infer that

$$\left|\tilde{f}''(\bar{x}_m) - f''(\bar{x}_m)\right| \leq \left|\tilde{f}''(\mathrm{x}_\ell) - f''(\mathrm{x}_\ell)\right| + \left|\int_{\mathrm{x}_\ell}^{\bar{x}_m} \left(\tilde{f}^{(3)}(t) - f^{(3)}(t)\right) dt\right|.$$

However, according to the theorem 4.2, we have $\left|\tilde{f}''(\bar{x}_m) - f''(\bar{x}_m)\right| \leq \frac{3}{2}\|f^4\|_\infty (\Delta x)(\Delta x^*)$. Hence,

$$\left|\int_{\mathrm{x}_\ell}^{\bar{x}_m} \left(\tilde{f}^{(3)}(t) - f^{(3)}(t)\right) dt\right| \leq \max_m \left|\tilde{f}^{(3)}(\mathrm{x}_\ell) - f^{(3)}(\mathrm{x}_\ell)\right| \left|\int_{\mathrm{x}_\ell}^{\bar{x}_m} dt\right|.$$

In other hand we have $\left|\int_{x_\ell}^{\bar{x}_m} dt\right| \leq \left|\int_{x_{j(m)}}^{x_{j(m)+1}} dt\right| = \Delta x$, then we obtain

$$\left|\int_{x_\ell}^{\bar{x}_m} \left(\tilde{f}^{(3)}(t) - f^{(3)}(t)\right) dt\right| \leq 4\left\|f^{(4)}\right\|_\infty \Delta x(\Delta x^*).$$

Summarizing all the estimations lead to the following

$$\left|\tilde{f}''(\bar{x}_m) - f''(\bar{x}_m)\right| \leq \frac{11}{2}\left\|f^{(4)}\right\|_\infty \Delta x(\Delta x^*), therefor, C_2 = \frac{11}{2}. \qquad (4.13)$$

For $k = 1$, one uses the fact that $\tilde{f}(x_i) = f(x_i)$ on each node x_i of γ since $\tilde{f} - f \in C^1([0,1])$. By applying the Rolle theorem to $\tilde{f} - f$ on each subinterval of the gridγ, there exists α_i in each subinterval $[x_{i+1}, x_i], i = 0,1\ldots, M^* - 1$ such as $\tilde{f}(\alpha_i) - f(\alpha_i) = 0$. Since $\tilde{f} = S_3 f$ is the complete spline interpolating *f*, one has the following bound conditions $\tilde{f}'(0) = f'(0)$ and $\tilde{f}'(1) = f'(1)$.

Like $x_m \in [0,1]$, then we choose α_ℓ sufficiently near to one zero α_i of $\tilde{f} - f$. By using the fundamental theorem of calculus and the triangular inequality like previously (for the case k = 2), one obtains

$$\left|\tilde{f}'(\bar{x}_m) - f'(\bar{x}_m)\right| \leq \left|\tilde{f}'(\alpha_\ell) - f'(\alpha_\ell)\right| + \left|\int_{x_\ell}^{\bar{x}_m} \left(\tilde{f}''(t) - f''(t)\right) dt\right|.$$

For our choice of α_ℓ, one has $\left|\tilde{f}'(\alpha_\ell) - f'(\alpha_\ell)\right| = 0$, thereafter

$$\left|\tilde{f}'(\bar{x}_m) - f'(\bar{x}_m)\right| \leq \left|\int_{x_\ell}^{\bar{x}_m} \left(\tilde{f}''(t) - f''(t)\right) dt\right| \qquad (4.14)$$

and we deduce that

$$\left|\tilde{f}'(\bar{x}_m) - f'(\bar{x}_m)\right| \leq \frac{11}{2}\left\|f^{(4)}\right\|_\infty (\Delta x)^2(\Delta x^*), therefor, C_1 = \frac{11}{2}. \qquad (4.15)$$

Lastly, for the case $k = 0$, like previously, we choose α_ℓ close to one zero of $\tilde{f} - f$ so that $\tilde{f}(\alpha_\ell) - f(\alpha_\ell) = 0$. And by using the fundamental theorem of analysis and the triangular inequality, one obtain

$$\left|\tilde{f}(\bar{x}_m) - f(\bar{x}_m)\right| \leq \left|\tilde{f}(\alpha_\ell) - f(\alpha_\ell)\right| + \left|\int_{x_\ell}^{\bar{x}_m} \left(\tilde{f}'(t) - f'(t)\right) dt\right|.$$

Then by making the same calculations as previously,

$$\left|\tilde{f}(\bar{x}_m) - f(\bar{x}_m)\right| \leq \frac{11}{4}\left\|f^{(4)}\right\|_\infty (\Delta x)^3(\Delta x^*), therefor, C_0 = \frac{11}{4}. \qquad (4.16)$$

4.4. Numerical result in one dimension of space

Let us consider the following problem

$$\begin{cases} \partial_t u + c\partial_x u - d\partial_{xx} u = \lambda u(1-u), \ on\ ]0,7[\times]0,4] \\ u(x,0) = f(x), \\ u(0,t) = -t^3 + 2t^2 + 0.7, \\ u(7,t) = 0, \end{cases} \qquad (4.17)$$

where $c = 0.14, d = 0.05, \lambda = 1$ and $f$is an unknown function from which some values resulting from an experiment are given below (see Table 2).

Table 2: Some values resulting from an experiment for the function f

x	0	1	2	3	4	4.5	5	6	7
f(x)	0.5	0.55	0.6	0.45	0.3	0.4	0.31	0.325	0

One uses the function spline of MATLAB to interpolate f in order to be able to evaluate it in any point of the segment [0,7]. Let $\gamma_x = \{x_0, x_1 \ldots, x_M\}$ be a uniform grid of [0,7], with the step $\Delta x = 0.15$ and $\gamma_t = \{t_0, t_1 \ldots, t_N\}$ a uniform grid of [0,4], with the step $\Delta t = 0.1$. With the fact that we have not an exact solution, and knowing that the non-standard method is more effective, we will simulate the non-standard scheme given by

$$\begin{cases} u_m^{n+1} = \left[\frac{4d\phi(k)+dh^2+2h^2\lambda\phi(k)\,\boldsymbol{u}^{\boldsymbol{n}}(\bar{\boldsymbol{x}}_{\boldsymbol{m}})}{2d\phi(k)}\right] u_m^{n+1} - u_{m-1}^{n+1} - \left[\frac{2h^2+2h^2\lambda\phi(k)}{2d\phi(k)}\right] u^n(\bar{x}_m), \\ u_m^0 = f(x_m), for\ all\ 0 \le m \le M, \\ u_0^n = -(t^n)^3 + 2(t^n)^2 + 0.7, for\ all\ 0 \le n \le N, \\ u_M^n = 0, for\ all\ 0 \le n \le N, \end{cases} \tag{4.18}$$

and the Euler scheme, forward in time, backward for the first derivative in x and centred at three points for the second derivative in x. This scheme is given by

$$\begin{cases} u_m^{n+1} = \left(\frac{dk}{h^2} - \frac{ck}{h}\right) u_m^{n+1} + \left(1 + k\lambda - 2\frac{dk}{h^2}\right) u_m^n + \left(\frac{dk}{h^2} + \frac{ck}{h}\right) u_{m-1}^n - k\lambda(u_m^n)^2, \\ u_m^0 = f(x_m), for\ all\ 0 \le m \le M, \\ u_0^n = -(t^n)^3 + 2(t^n)^2 + 0.7, for\ all\ 0 \le n \le N, \\ u_M^n = 0, for\ all\ 0 \le n \le N. \end{cases} \tag{4.19}$$

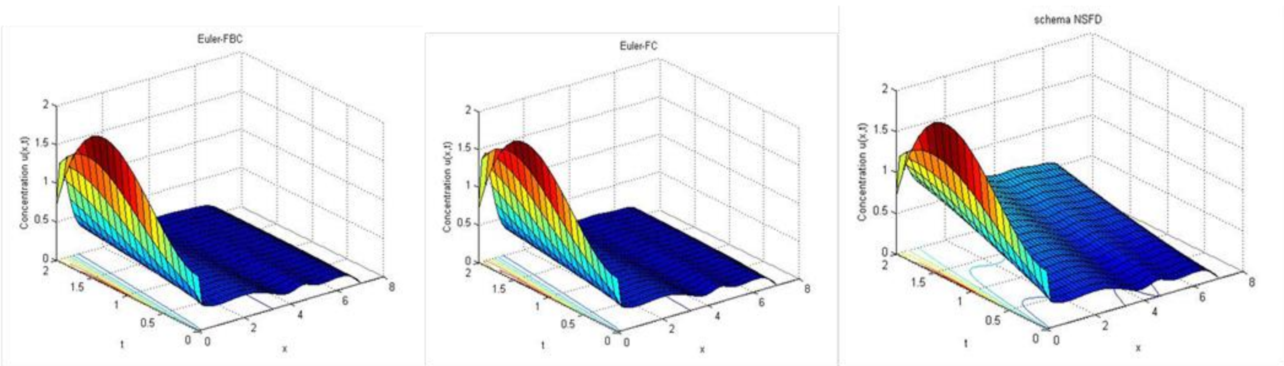


Figure 9: Surfaces of transport and diffusion corresponding to the studied problem modelled by the Euler-FBC scheme, the Euler-FC scheme and the non-standard scheme.

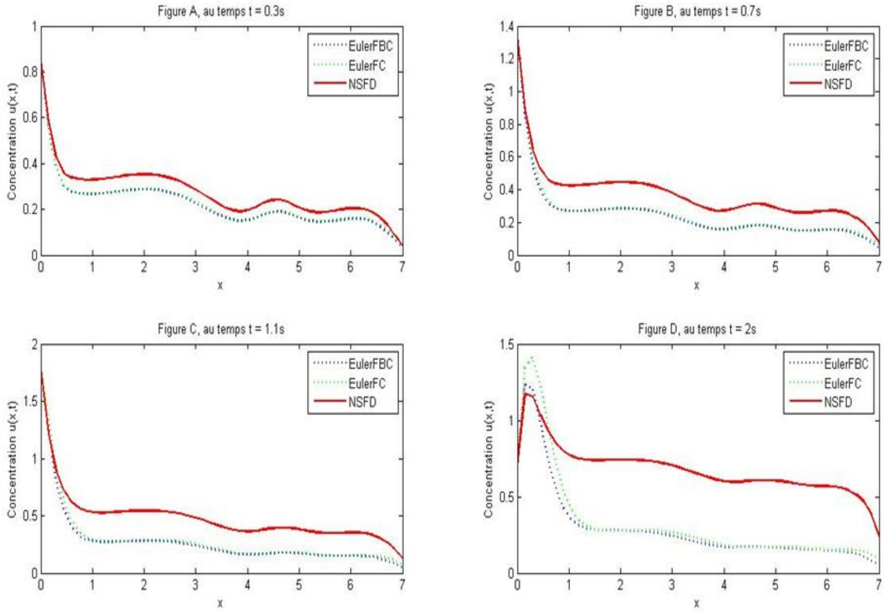


Figure 10: Solution of the advection-diffusion reaction problem (4.17), curves from the surfaces of the previous figure at time t = 0.3s, 0.7s, 1.1s and 2s.

4.5. Comment

One can notice that the surfaces obtained by the two schemes of Euler do not correctly show the diffusion phenomenon. Indeed, one notes a broad difference between the schemes of Euler and the non-standard scheme when we move away from the source. This is due to the fact that in the Euler's schemes, calculation at time $\mathbf{t^{n+1}}$ uses the data calculated at time $\mathbf{t^{n}}$, therefore there is proliferation of error, that is the reason of the large margin observed in the figure D at time t = 2s. On the other hand, the non-standard scheme uses only the data at the edge of the domain to begin calculations for each stage of the grid.

4.6. Conclusion

In this work, the purpose of we were to build a non-standard scheme for an partial differential equation: the equation of advection-diffusion-reaction in subjugated dimension one of space under an initial condition and the limit conditions of Dirichlet; where the reaction models a logistic growth, to evaluate the error of the method and finally to make a digital simulation in order to evaluate the quality of the theoretical results. To arrive there, we built a non-standard scheme for the equation of advection-reaction (transport equation) in one and two dimension of space, we also made standard schemes (the scheme of Euler (3.39), the scheme of Lax-Friedrichs (3.30) and the scheme of Lax-Wendroff (3.32)) for this equation with an aim of showing the power of the non-standard approach. Then one used the nonstandard discretization of the transport equation added to the discretization of the diffusion term to obtain a non-standard finite differences scheme (4.8) in one dimension of space for the above-mentioned problem. Passing by the use of the cubic splines interpolation, we gave an estimate of the bound of the error of the method. Since the discovery of the non-standard finite differences method (around 1990 by physicist R. E. Mickens), researchers in different fields of science have been using it due to its power and near-exact quality of results (see for instance [17,18,19,20,21,22,23,24] and references therein); but unlike numerical methods such as Euler's finite differences, the finite volume method or even the finite element method, there is no explicit theory of convergence and stability for the of non-standard finite differences method. Here, we proceed exclusively by numerical tests to assess the quality of the approximation made in comparison to the methods mentioned above. We plan in our future work, to establish an explicit mathematical theory of stability and convergence of the non-standard method.

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