TECHNISCHE UNIVERSITÄT
BERGAKADEMIE FREIBERG
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Colourings of P_5-free graphs

By the Faculty of Mathematics and Computer Science
of the Technische Universität Bergakademie Freiberg

approved

Thesis

to attain the academic degree of

doctor rerum naturalium
(Dr. rer. nat.)

submitted by **M.Sc. Maximilian Geißer**

born on the 13th of September 1992 in Karlsruhe

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Date of the award: Freiberg, April 13th, 2022

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- Prof. Dr. rer. nat. Ingo Schiermeyer
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In the selection and use of materials and in the writing of the manuscript I received support from the following persons:

- Prof. Dr. rer. nat. Ingo Schiermeyer
- Dr. rer. nat. Christoph Brause

Persons other than those above did not contribute to the writing of this thesis.

I did not seek the help of a professional doctorate-consultant. Only those persons identified as having done so received any financial payment from me for any work done for me. This thesis has not previously been published in the same or a similar form in Germany or abroad.

April 13th, 2022 M.Sc. Maximilian Geißer

Acknowledgements

This thesis is the result of my time in the Discrete Mathematics Group at TU Bergakademie Freiberg. This group creates a great atmosphere for mathematical research and development. It is always fun to discuss or create ideas with smart and kind people and see the ideas develop or not stand the test of time. Right from the beginning I was pleased about the possibility to work and learn within this group.

I want to express my gratitude to my advisor Prof. Dr. Ingo Schiermeyer for letting me choose my problems, and for joining with well-trained patience and valuable ideas. I am also thankful to Prof. Dr. Bert Randerath for the helpful remarks and for being reviewer of my thesis. During the years, I had the opportunity to work closely with Dr. Christoph Brause, who I admire for his smarts and his work ethic. I also want to thank Prof. Dr. Martin Sonntag for his support in and outside the mathematical world and for being the heart of the group.

I would like to emphasize my thanks to Michael Hanzel, Vincent Kowalsky, Dr. Simon Liebing and Matthias Wolf (alphabetical order) for listening to my thoughts and supporting me. I also had the opportunity to work with a lot of smart and kind researchers, in particular Peter Stumpf from Passau, Germany, and Dr. Jonathan Rollin from Halle, Germany and Dr. Zhiwei Guo from Xi'an, China.

Last but not least, I very much want to thank my family and my second family for all their love and support.

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1 Introduction

We use the starting section of this thesis to slowly introduce the reader to the problem which we are researching. For a shorter introduction we refer the reader to Section 1.1. In our opinion the easiest way to introduce graph theory is by letting the reader imagine a street map. It is possible to reconstruct the structure of the street map if one knows all the different crossings in the street map and the information which crossings are directly connected by a street. This underlying logical structure is called a graph, which consists of two sets, the non-empty set V for *vertices*, which correspond to the crossings, and the set $E \subseteq \{\{v, w\} \mid v, w \in V, v \neq w\}$ of so called *edges*, which correspond to the streets between the crossings. For a graph G we use $V(G)$ and $E(G)$ to reference his vertex set and edge set, respectively. Before we continue let us introduce three important, quite simple graphs. For $n \in \mathbb{N}_{>0}$, the *complete graph* of size n, also called K_n, is defined by

$$K_n = (V(K_n), E(K_n)) \coloneqq (\{v_i \mid i \in \{1, 2, \dots, n\}\}, \{\{v, w\} \mid v, w \in V(K_n), v \neq w\}).$$

For $n \in \mathbb{N}_{>2}$, we define the *path* of length n and the *cycle* of length n by

$$P_n \coloneqq (\{v_i \mid i \in \{1, 2, \dots, n\}\}, \{\{v_i, v_{i+1}\} \mid i \in \{1, \dots, n-1\}\}),$$

and

$$C_n \coloneqq (\{v_i \mid i \in \{1, 2, \dots, n\}\}, \{\{v_i, v_{i+1}\} \mid i \in \{1, \dots, n-1\}\} \cup \{\{v_1, v_n\}\}),$$

respectively. Noticing all these brackets it is a logical notation to just write $uv \in E(G)$ instead of $\{u, v\} \in E(G)$. Using a graph as the underlying mathematical structure one can look at many different problems. This area of mathematics is called graph theory. One interesting problem which arises by looking at a street map is to find the shortest path between two crossings. This problem can currently be solved quite efficiently using graph theory and these solutions are used every time a phone is asked for directions. We refer the interested reader to an article by Schrijver [61] depicting the history of this problem.

Most discrete data can be depicted in a graph. Let us present one more example of an interesting graph. Identifying each user of a given social media platform with its own vertex and connecting the vertices with an edge if and only if the corresponding people are friends on the platform creates a large friendship graph. For the platform Twitter this graph is subject of a paper by Bakhshandeh et al. [4]. In their paper they introduce, among other things, an algorithm to calculate an approximate solution to the shortest path problem. They find an average degree of separation of 3.43 between two random Twitter users, meaning that for any two users u_1 and u_2 on average there is a path of length less than four, consisting of users, which are pairwise friends, connecting u_1 and u_2. This is a surprisingly small number and therefore another instance of the so-called small-world experiment.

In this thesis we look at another graph theoretical problem namely the colouring problem. We use the following example to motivate this problem. In an atlas there is a coloured map of the worlds' countries. One notices that countries which share a border are for better readability coloured differently. The person responsible for colouring the atlas has to solve the following question. How many colours are necessary to colour the countries under the restriction that adjacent countries are coloured differently? We translate this in a graph theoretical problem as follows. The graph $G_{\text{Earth}} = (V(G_{\text{Earth}}), E(G_{\text{Earth}}))$, which contains all the relevant information to colour the atlas map, arises from the neighbourhood relation between the countries as follows. Each country of the earth is identified with its own vertex and there is an edge between two vertices if and only if the corresponding countries share a border of positive length. That is the reason why we generally say u is *adjacent* to v in G if $uv \in E(G)$. The question of finding an allowed colouring now translates into finding a $k \in \mathbb{N}_{>0}$ and a map $c : V(G_{\text{Earth}}) \to \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ for each $uv \in E(G_{\text{Earth}})$. We say a graph G is *k-colourable*, if we find such a $k \in \mathbb{N}_{>0}$ and a map c. Clearly every graph G is $|V(G)|$-colourable by colouring every vertex in its own colour. Going back to the atlas-map it is quite natural to ask for the smallest amount of colours necessary to colour the countries, since when using fewer colours there is a larger visual difference between these colours. Because of its relevance the smallest k for which a graph is k-colourable has its own name and is called the *chromatic number* of G and is denoted by $\chi(G)$. In general there is no known efficient algorithm to calculate the chromatic number of a given graph [28]. Trying all different combinations of colours leads to an exponential running time and therefore is highly impractical for larger graphs. Since in general determining the chromatic number is a difficult problem, we now only collect the maps from the atlas which fulfil the following quite natural restriction. We are interested in all maps for which each country depicted in the map is topologically connected. The graphs which arise from theses maps are so called *planar graphs*. Surprisingly four colours are enough to colour each one of these maps. This is the famous 4-colour theorem, which was proven by Appel and Haken [2] in 1977. The more general fact, that all maps with the special property are 4-colourable is clearly more useful than just knowing $\chi(G_{Earth})$.

Like in the example often times it is not just a single graph that one wants to know the chromatic number of, but rather a large collection of graphs which are of interest. So given a family of graphs the aim is to find an upper bound to the chromatic number of these graphs. Obviously there are different ways to obtain such a family of graphs. Before we explain the graph families that we are interested in we need to introduce two technical definitions. Firstly for two graphs G and H, an *isomorphism between G and H* is a bijection between $V(G)$ and $V(H)$ such that for every two vertices $u, v \in V(G)$ we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If there is an isomorphism between G and H we write G is *isomorphic* to H or $G \cong H$. The second definition we need is that of an induced subgraph. Given a non-empty set $S \subseteq V(G)$, $G[S]$ is the graph with vertex set S and edge set $E(G) \cap \{s_1 s_2 : s_1, s_2 \in S\}$. We say that H is an *induced subgraph* of G, denoted by $H \subseteq_{\text{ind}} G$, if there is some set $S \subseteq V(G)$ of vertices such that $G[S] \cong H$. So for example K_2 is an induced subgraph of every graph with an edge, but P_5 is not an induced subgraph of C_5. Now we can define the graph families that we are interested in. For a graph H we define the family $\text{For}(H)$ of graphs by

$$\text{For}(H) = \{G \mid H \text{ is not an induced subgraph of } G\}.$$

Or in other words, we are interested in graph families which occur by forbidding a certain (often small) graph H as an induced subgraph. This family of graphs is denoted by $\text{For}(H)$, short for forbidden. In general a smaller forbidden subgraph H grants a smaller family $\text{For}(H)$. For example the family $\text{For}(K_2)$ just consists of $K_1, 2K_1, 3K_1, \ldots$. One advantage of choosing the family of graphs in this way is that for every graph $G \in \text{For}(H)$ and an induced subgraph G' of G also $G' \in \text{For}(H)$. This is the so called hereditary property and for all graph families $\mathcal{G}$ which fulfil the hereditary property there is a family of graphs $\mathcal{H}$ with $\text{For}(\mathcal{H}) = \mathcal{G}$ ($\mathcal{H}$ can be chosen to be the set of all graphs not in $\mathcal{G}$ but all induced subgraphs of which are in $\mathcal{G}$).

So imagine such a family of graphs. Quite clearly the chromatic number of these graphs can be arbitrary large as long as the forbidden subgraph H is not a complete graph. This is the case, since if H is not a complete graph the complete graph of any size is a member of $\text{For}(H)$. So it is a logical idea to divide the graphs into buckets depending on the largest complete graph which they contain as an induced subgraph. Now the aim is to find an upper bound on the chromatic number for each bucket. Or in other word we try to find a function $f_H : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\chi(G) \leq f_H(\omega(G))$ for every $G \in \text{For}(H)$, where $\omega(G)$ denotes the cardinality of the largest set of pairwise adjacent vertices in G. The function f_H is called a *χ-binding function* for $\text{For}(H)$. Motivated by the Strong Perfect Graph Conjecture of Berge [5], Gyárfás [31] first introduced these functions. Often times is is quite difficult to figure out whether or not there is such a function.

Let us first imagine there is an $n \in \mathbb{N}_{>2}$ such that C_n is an induced subgraph of H.

It was first shown by Erdős [25] that in this setting there is no χ-binding function for For(H). This is the case since for every $k, \ell \in \mathbb{N}_{>0}$ there is a graph $G_{k,\ell}$ with $\chi(G_{k,\ell}) \geq k$ and which shortest cycle has length at least ℓ. So choosing ℓ as $n+1$ the infinite family $\{G_{k,\ell} \mid k \in \mathbb{N}_{>0}\}$ has clique number 2, unbounded chromatic number, and is a subset of For(H). Since this result by Erdős the study of χ-binding functions for (hereditary) graph families is one of the central problems in chromatic graph theory.

So to have any chance of finding a χ-binding function for For(H) we need that there is no $n \in \mathbb{N}_{>2}$ such that C_n is an induced subgraph of H. The easiest graphs which fulfil this condition are the paths. If the forbidden subgraph is a P_4 it was first shown by Seinsche [65] that one can even choose $f_{P_4}(\omega) = \omega$ as a binding function. Since $\chi(G) \geq \omega(G)$ for every graph G this function is the smallest non-trivial binding function. The family For(P_5) contains so many more graphs than For(P_4) that for example it is still open whether or not there is a polynomial binding function for For(P_5). So to better understand this family many researchers forbid an additional graph. A lot of results have been published in the last decades in this particular field and Chapter 2 is a collection of these results. We also refer the reader to surveys of Randerath and Schiermeyer [59], and Scott and Seymour [63] for a great overview over the years of research. Let us use this space to state that whenever we state a theorem, lemma or corollary which is not our result there is a citation and name crediting the author. If there is no name it is one of our results.

Like we saw in the example of For(P_4) it is a logical wish to find the smallest binding function. Also like mentioned above to research the family For(H) one additionally forbids a second subgraph and researches this smaller family. So after this introduction we now formally define the following often used definition of an optimal χ-binding function. Given a set $\mathcal{H}$ of graphs, let $f^\star_{\mathcal{H}}\colon \mathbb{N}_0 \to \mathbb{N}_0$ be the *optimal χ-binding function for For($\mathcal{H}$)*, that is,

$$f^\star_{\mathcal{H}}(\omega) = \max\{\chi(G) : \omega(G) = \omega, G \in \mathrm{For}(\mathcal{H})\}.$$

Finding χ-binding functions is difficult which implies that it is especially difficult to find optimal ones. For some subfamilies of P_5-free graphs we are able to determine optimal χ-binding functions through a combination of decompositions by homogeneous sets and clique-separators. Others we determine by structural analysis.

This thesis is organised as follows: We continue in this chapter with a motivation and summary of our results as well as an introduction into notation and terminology. In Chapter 2 we outline the known results in this area. Then we prove the main techniques in Chapter 3 that are used in later proofs.

In the then following chapters we discuss the different subfamilies and their χ-binding functions. We deal with the families For(P_5, *hammer*), For(P_5, *banner*), For(P_5, *dart*),

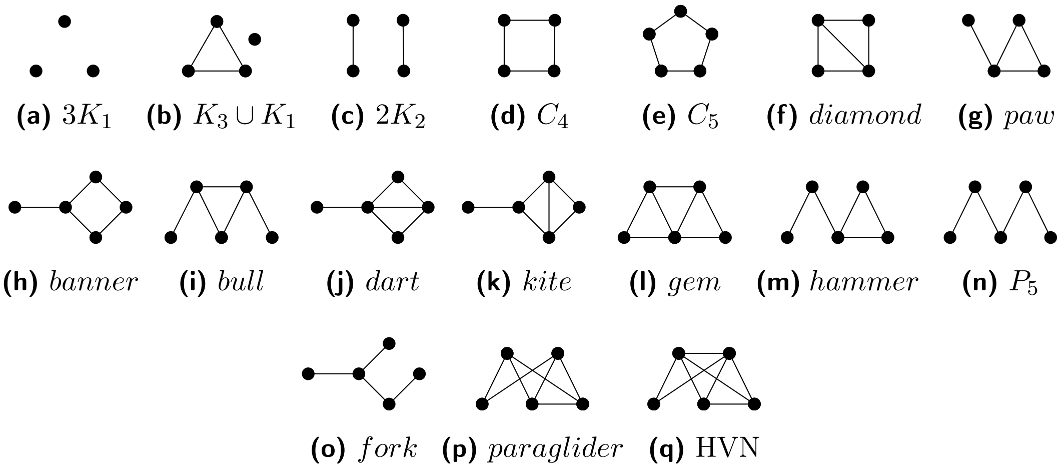


Fig. 1: Most frequently used forbidden induced subgraphs

For(P_5, *kite*) and For(P_5, HVN) in Chapter 4, Chapter 5, Chapter 6, Chapter 8 and Chapter 9, respectively. Since we find nice structural results for these families we also get result for some of their subfamilies. All these results are collected in the then following Chapter 7. There we discuss our results for For(P_5, C_4), For(P_5, *gem*), and For(P_5, *diamond*).

We lastly characterise all graphs H for which there is a constant $c(H)$, only depending on H, with $f^{\star}_{\{P_5,H\}}(\omega) \leq \omega + c(H)$, for all $\omega \in \mathbb{N}_{>0}$, in Chapter 10.

1.1 Motivation and contribution

We consider standard notation and terminology, and note that each of the considered graphs in this thesis is simple, finite and undirected unless otherwise stated. Some particular graphs are depicted in Fig. 1 and Fig. 2, and we denote a path and a cycle on n vertices by P_n and C_n, respectively. Additionally, given graphs $G, H_1, H_2, \ldots$, the graph G is $(H_1, H_2, \ldots)$-*free* if $G - S$ is non-isomorphic to H for each $S \subseteq V(G)$ and each $H \in \{H_1, H_2, \ldots\}$.

A function $L\colon V(G) \to \mathbb{N}_{>0}$ is a *(proper) colouring* if $L(u) \neq L(v)$ for each pair of adjacent vertices $u, v \in V(G)$ and, for simplicity, we say that each $k \in \{L(u) : u \in V(G)\}$ is a *colour*. The smallest number of colours for which there is a proper colouring of G is the *chromatic number* of G, denoted by $\chi(G)$. It is well known that each *clique*, which is a set of pairwise adjacent vertices, needs to be coloured by pairwise different colours in a proper colouring. Thus, the *clique number*, which is the largest cardinality of a clique in G and that is denoted by $\omega(G)$, is a lower bound on $\chi(G)$. Since the beginnings of chromatic graph theory, researchers are interested in relating these two invariants. For example, Erdős [25] showed that the difference could be arbitrarily large by proving that, for every two integers $g, k \geq 3$, there is a $(C_3, C_4, \ldots, C_g)$-free graph G with $\chi(G) \geq k$. In contrast, it attracted Berge [5] to study *perfect* graphs, which are graphs, say G, that satisfy $\chi(G - S) = \omega(G - S)$ for each $S \subseteq V(G)$. His research resulted in two famous conjectures, the Weak and the Strong Perfect Graph Conjecture. The first one, proven by Lovász [46], states that the complementary graph of a perfect graph is perfect. In contrast to the Weak Perfect Graph Conjecture, the Strong Perfect Graph Conjecture was open for a long time but is nowadays confirmed and known as the Strong Perfect Graph Theorem.

The Strong Perfect Graph Theorem (Chudnovsky et al. [20])**.** *A graph G is perfect if and only if G and $\bar{G}$ are $(C_5, C_7, \ldots)$-free.*

To generalize the notation of perfect graphs, Gyárfás [31] introduced the definition of a χ-binding function as follows. A function $f : \mathbb{N}_0 \to \mathbb{N}_0$ is a *χ-binding function* for a family of graphs $\mathcal{G}$ if and only if $\chi(G') \leq f(\omega(G'))$ holds for all induced subgraphs G' of $G \in \mathcal{G}$. If there is a χ-binding function for a graph family $\mathcal{G}$, then there is obviously a *optimal (or smallest) χ-binding function* for $\mathcal{G}$ defined by

$$f^\star(x) = \max\{\chi(G') \mid G' \text{ is an induced subgraph of } G \in \mathcal{G}, \omega(G') = x\}.$$

Gyárfás [31] also observed from the aforementioned result by Erdős [25] that the χ-binding function does not exist for the family of $(H_1, H_2, \ldots, H_k)$-free graphs whenever each of the given graphs $H_1, H_2, \ldots, H_k$ contains an induced cycle. In other words, to hope for χ-binding functions for the family of $(H_1, H_2, \ldots, H_k)$-free graphs, at least one of the graphs $H_1, H_2, \ldots, H_k$ must be a forest. Furthermore, Gyárfás [31] and, independently, Sumner [66] conjectured that there is such an upper bound on the chromatic numbers of H-free graphs whenever H is a forest.

Given a set $\mathcal{H}$ of graphs, we use the notation of $f^\star_{\mathcal{H}}$ for the optimal χ-binding function for the family of $\mathcal{H}$-free graphs, which means, since this family is hereditary $f^\star_{\mathcal{H}} \colon \mathbb{N}_0 \to \mathbb{N}_0$ is defined by

$$f^\star_{\mathcal{H}}(\omega) = \max\{\chi(G) : \omega(G) = \omega, G \text{ is } \mathcal{H}\text{-free}\}.$$

For example, the family of P_t-free graphs for $t \geq 5$ has a χ-binding function (cf. Theorem 12, [31]) although up until recently the best known upper bounds on $f^\star_{\{P_5\}}$ and $f^\star_{\{P_5, C_5\}}$ were exponential in ω [26, 22]. In 2021 Scott, Seymour, and Spirkl [64] proved a quasi-polynomial bound for P_5-free graphs. For more details about their proof we refer to Chapter 2. The right order of magnitude of $f^\star_{\{P_5\}}$ is a long-standing and still an open problem. Esperet (unpublished) even posed the difficult problem to decide whether or not every χ-bounded family admits a polynomial χ-binding function? For that reason, it is natural to ask whether there exists a polynomial χ-binding function for a χ-bounded graph family $\mathcal{G}$. To the best of our knowledge, it is also unknown whether there is a polynomial χ-binding function for the family of $(C_5, C_7, \ldots)$-free graphs (which is a short notation for the family of graphs each of which is C_{2k+5}-free for each $k \in \mathbb{N}_0$) although an exponential one exists [62]. For various graph families, χ-binding functions have been established and surveyed by Gyárfás [31], Seymour and Scott [63], and Randerath and Schiermeyer [55].

It is rather interesting that P_4-free graphs are perfect by the Strong Perfect Graph Theorem but, for supersets such as P_5-free graphs and $(C_5, C_7, \ldots)$-free graphs, the best known χ-binding functions are not even polynomial. Although it is unknown whether $f^\star_{\{P_5\}}$ and $f^\star_{\{C_5,C_7,\ldots\}}$ are polynomially or not, there is a big difference in the order of magnitude between $f^\star_{\{P_4\}}$ on one hand, and $f^\star_{\{P_5\}}$ on the other hand. For this reason we focus in this thesis on P_5-free graphs, as this family is the smallest – in terms of the forbidden induced paths – for which the right order of magnitude of $f^\star_{\{P_5\}}$ is unknown. Note that Fouquet et al. [27] show among other things that there is no linear χ-binding function for the class of P_5-free graphs. By modifying a result of [14], we obtain Lemma 42 which we prove in Chapter 3 and from which we especially deduce that the families of P_5-free graphs and of $(C_5, C_7, \ldots)$-free graphs do not have a linear χ-binding function.

Since the orders of magnitude of $f^\star_{\{P_5\}}$ and $f^\star_{\{C_5,C_7,\ldots\}}$ are unknown, it is of interest to study subfamilies of P_5-free graphs and subfamilies of $(C_5, C_7, \ldots)$-free graphs. For example, it has been proven

- $f^\star_{\{P_5,paw\}}(\omega) = \begin{Bmatrix} f^\star_{\{P_5,C_3\}}(\omega) & \text{if } \omega \leq 2, \\ \omega & \text{if } \omega > 2 \end{Bmatrix} = \begin{Bmatrix} 3 & \text{if } \omega = 2, \\ \omega & \text{if } \omega \neq 2 \end{Bmatrix}$ (cf. [48, 54] or [59]),
- $f^\star_{\{P_5,diamond\}}(\omega) \leq \omega + 1$ (cf. [54]),
- $f^\star_{\{P_5,C_4\}}(\omega), f^\star_{\{P_5,gem\}}(\omega) \leq \lceil 5\omega/4 \rceil$ (cf. [15, 19]),
- $f^\star_{\{P_5,paraglider\}}(\omega) \leq \lceil 3\omega/2 \rceil$ (cf. [36]), and
- $f^\star_{\{C_5,C_7,\ldots,bull\}}(\omega), f^\star_{\{P_5,bull\}}(\omega) \leq \binom{\omega+1}{2}$ (cf. [22]).

We refer the reader to the survey of Randerath and Schiermeyer [59] and Chapter 2 for additional results and further informations.

The research field of this thesis is the study of binding functions of (P_5, H)-free graphs for $H \in \{hammer, banner, dart, kite, \text{HVN}\}$. With our main technique which is stated in Section 3.2 we find an approach which allows us determining optimal χ-binding functions for some of these families. As particular tools, we need the terminologies of critical graphs as well as those of homogeneous sets and clique-separators. A graph G is *critical* if $\chi(G) > \chi(G - u)$ for each $u \in V(G)$. Additionally, in a connected graph G, a set S is a *homogeneous set* if $1 < |S| < |V(G)|$ and each vertex outside S is adjacent to each or none of the vertices of S, and S is a *clique-separator* if S is a clique and

$G - S$ is disconnected.

In its basic form, our approach for some subfamilies of $\text{For}(P_5)$ can be described as follows:

Whenever there is a set $\mathcal{H}$ of graphs, it is reasonable to study the chromatic number of critical $\mathcal{H}$-free graphs only for determining $f^\star_{\mathcal{H}}$ since each critical graph $G - S$ with $\chi(G - S) = \chi(G)$ and $S \subseteq V(G)$ satisfies $\omega(G - S) \leq \omega(G)$. Assuming $f^\star_{\mathcal{H}}$ to be non-decreasing and G to be not critical, we obtain by induction hypothesis

$$\chi(G) = \chi(G - S) \leq f^\star_{\mathcal{H}}(\omega(G - S)) \leq f^\star_{\mathcal{H}}(\omega(G)).$$

This observation leads to the following well-known lemma which we state here for later reference.

Lemma 1 (Folklore). *Let $\mathcal{H}$ be a set of graphs and $\mathcal{C}_{\mathcal{H}} := \{H \in \mathcal{H} \mid H \text{ is critical}\}$ and $f : \mathbb{N}_0 \to \mathbb{N}_0$ non-decreasing. If $\chi(C) \leq f(\omega(C))$ for all $C \in \mathcal{C}_{\mathcal{H}}$ then $\chi(H) \leq f(\omega(H))$ for all $H \in \mathcal{H}$.*

This simplification particularly implies that we can restrict our attention to graphs without clique-separators, which is reasoned by the fact that each graph G for which there are two graphs G_1 and G_2 such that $V(G_1) \setminus V(G_2), V(G_2) \setminus V(G_1) \neq \emptyset$, $E(G) = E(G_1) \cup E(G_2)$, and $V(G_1) \cap V(G_2)$ is a clique-separator satisfies $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ (cf. Lemma 37), and so G is not critical.

Furthermore, let us assume that M is a homogeneous set for which there is no homogeneous set containing M properly. For the neighbours of M, it does not matter how a proper colouring $L\colon V(G) \to \mathbb{N}_{>0}$ colours the vertices of M. It is only the set of colours that L assigns to the vertices in M which is of interest. From this view, it is reasonable to delete all but one vertex of M, assigning $\chi(G[M])$ as weight to the remaining vertex, and to consider set-mappings as colourings.

By refining the concepts of critical graphs and clique-separators, we are in a position to reduce the determination of optimal χ-binding functions to the study of set-mappings for graphs without clique-separators and homogeneous sets. We apply this approach and our findings, and obtain several optimal χ-binding functions. It is worth pointing out that there are just a few graph families for which optimal χ-binding functions are known. As described above, mostly one can only determine a χ-binding function, and it is often a tough and challenging problem to determine the optimal one or its order of magnitude. Our main results are collected in the following theorems. They are ordered by their occurrence in this thesis. Note that by definition of $f^\star_{\mathcal{H}}$ it is possible to state these bounds in a compact form, but for example proving $f^\star_{\{P_5, dart\}}(\omega) = f^\star_{\{3K_1\}}(\omega)$ requires roughly 30 pages.

Theorem 2. *If* $\omega \in \mathbb{N}_{>0}$*, then*

$$f^\star_{\{P_5,hammer\}}(\omega) = f^\star_{\{2K_2\}}(\omega).$$

Theorem 3. *If* $\omega \in \mathbb{N}_{>0}$*, then*

(i) $f^\star_{\{P_5,banner\}}(\omega) = f^\star_{\{3K_1\}}(\omega)$ *and*

(ii) $f^\star_{\{C_5,C_7,\ldots,banner\}}(\omega) = f^\star_{\{C_5,3K_1\}}(\omega)$.

Theorem 4. *If* $\omega \in \mathbb{N}_{>0}$*, then*

(i) $f^\star_{\{P_5,dart\}}(\omega) = f^\star_{\{3K_1\}}(\omega)$ *and*

(ii) $f^\star_{\{C_5,C_7,\ldots,dart\}}(\omega) = f^\star_{\{C_5,3K_1\}}(\omega)$.

Theorem 5. *If* $\omega \in \mathbb{N}_{>0}$*, then*

(i) $f^\star_{\{P_5,C_4\}}(\omega) = f^\star_{\{P_5,gem\}}(\omega) = \left\lceil \frac{5\omega-1}{4} \right\rceil$ *and*

(ii) $f^\star_{\{P_5,diamond\}}(\omega) = \begin{cases} 3 & \text{if } \omega = 2, \\ \omega & \text{if } \omega \neq 2. \end{cases}$

Theorem 6. *If* $\omega \in \mathbb{N}_{>0}$*, then*

$$\left\lfloor \frac{3\omega}{2} \right\rfloor \leq f^\star_{\{P_5,kite\}}(\omega) = f^\star_{\{2K_2,K_3\cup K_1,C_5\cup K_1\}}(\omega) \leq \begin{cases} \left\lfloor \frac{3\omega}{2} \right\rfloor & \text{if } \omega \leq 3, \\ 2\omega - 2 & \text{if } \omega \geq 4. \end{cases}$$

Theorem 7. *If* $\omega \in \mathbb{N}_{>0}$*, then*

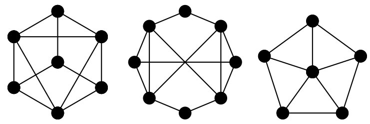
$$f^\star_{\{P_5,HVN\}}(\omega) = \begin{cases} \omega + 1 & \text{if } \omega \notin \{1,3\}, \\ \omega & \text{if } \omega = 1, \\ \omega + 2 & \text{if } \omega = 3. \end{cases}$$

Last but not least, we aim for graphs F such that

$$f^\star_{\{P_5,F\}}(\omega) \leq \omega + c(F)$$

for some constant $c(F)$ – depending on F only – and each $\omega \in \mathbb{N}_{>0}$. In particular, we prove the following characterization, where F_p denotes the complementary graph of $pK_1 \cup P_3$ for each $p \in \mathbb{N}_{\geq 0}$.

Theorem 8. *Let* F *be a graph. There is a constant* $c(F)$ *such that* $f^\star_{\{P_5,F\}}(\omega) \leq \omega + c(F)$ *for each* $\omega \in \mathbb{N}_{>0}$ *if and only if either* $F \cong P_4$ *or* F *is an induced subgraph of* F_p *for some* $p \in \mathbb{N}_{\geq 0}$.



(a) G_1 **(b)** G_2 **(c)** W_5

Fig. 2: Used graphs in the characterisation of critical graphs

By results of Kim (cf. Corollary 27, [42]) and Wagon (cf. Lemma 30, [67]),

$$f^{\star}_{\{3K_1\}}(\omega) \in \Theta\left(\frac{w^2}{\log(w)}\right) \quad \text{and} \quad f^{\star}_{\{2K_2\}}(w) \leq \binom{w+1}{2} \in \mathcal{O}(\omega^2),$$

respectively. We note that, by using a result of Gaspers and Huang [29] and an inductive proof, we reduce the upper bound on $f^{\star}_{\{2K_2\}}$ for $\omega \geq 3$ in Chapter 2. Additionally, let us note that Lemma 42 implies that the classes of $(C_5, 3K_1)$-free, $3K_1$-free and $2K_2$-free graphs do not have a linear χ-binding function.

On our way to optimal χ-binding functions for some of these families, we characterise in parallel critical graphs; all these results are collected in Theorem 9. For this purpose, a *'non-empty, $2K_1$-free'-expansion* of a graph G' is a graph G for which there are a partition of $V(G)$ into cliques $S_1, S_2, \ldots, S_{|V(G')|}$ and a bijective function $f\colon \{S_1, S_2, \ldots, S_{|V(G')|}\} \to V(G')$ such that each vertex of S_i is adjacent to each vertex of S_j if $f(S_i)$ is adjacent to $f(S_j)$ and each vertex of S_i is non-adjacent to each vertex of S_j if $f(S_i)$ is non-adjacent to $f(S_j)$ for each distinct $i, j \in [|V(G')|]$. Now our second main result reads as follows.

Theorem 9. *Let G be a critical graph.*

(i) *If G is $(P_5, banner)$-free, then G is $3K_1$-free.*

(ii) *If G is $(P_5, dart)$-free and S is a non-empty set of vertices such that each vertex in S is adjacent to each vertex of $V(G) \setminus S$ and each homogeneous set M in $G[S]$ has a vertex in $S \setminus M$ that is non-adjacent to each vertex of M, then $G - S$ is critical, and $G[S]$ is $3K_1$-free or a 'non-empty, $2K_1$-free'-expansion of G' with $G' \in \{G_1, G_2\}$.*

(iii) *If G is $(P_5, hammer)$-free, then G is $2K_2$-free.*

(iv) *If G is $(C_5, C_7, \ldots)$-free, and banner-free or dart-free, then G is $(C_5, 3K_1)$-free.*

(v) *If G is (P_5, C_4)-free, then G is a 'non-empty, $2K_1$-free'-expansion of a graph G' with $G' \in \{C_5, W_5, K_1\}$.*

(vi) *If G is (P_5, gem)-free, then G is a 'non-empty, $2K_1$-free'-expansion of a graph G' with $G' \in \{C_5, G_2, K_1\}$.*

(vii) *If G is $(P_5, diamond)$-free, then G is complete or a cycle of length 5.*

Let us shortly state some extra thoughts on Theorem 9 (ii), since it is by far the most challenging characterisation; for more information see the last page of Chapter 6. We note that an inclusion-wise minimal set $S_<$ for which each vertex is adjacent to each vertex of the possibly empty set $V(G) \setminus S_<$ meets the assumptions on the set S in Theorem 9 (ii). This observation together with Theorem 9 (ii) yields a characterisation of the critical $(P_5, dart)$-free graphs.

An interesting open conjecture by Reed [56] is that $\chi(G)$ can be bounded from above by $\lceil(\Delta(G) + \omega(G) + 1)/2\rceil$, where $\Delta(G)$ denotes the *maximum degree* of G, i.e. the largest number of vertices that have a common adjacent vertex. For example, this conjecture is proven for

- $(C_5, C_7, \ldots)$-free graphs [3],
- $3K_1$-free graphs [43, 44],
- (P_5, gem)-free graphs [19],
- graphs whose complementary graph is disconnected [53], and
- graphs G with $\chi(G) \leq \lceil 5\omega(G)/4\rceil$ [37],

and, to the best of our knowledge, it is open for $2K_2$-free graphs. By using Theorem 9, parts of its proof, and the above listed results, we obtain the following corollary:

Corollary 10. *If G is $(P_5, banner)$-free or $(P_5, dart)$-free, then*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil .$$

1.2 Notation and terminology

In this section, we introduce notation and terminology we use throughout this thesis. Whenever a notation or definition is unclear the reader can come back to this section and reread the relevant part.

Recall that we consider finite, simple, and undirected graphs if not otherwise stated. For notation and terminology not defined herein, we refer to [8]. A *graph* G consists of a non-empty vertex set $V(G)$ and an edge set $E(G)$, where each edge $e \in E(G)$ is a two elementary subset of $V(G)$. For notational simplicity, we write uv instead of $\{u, v\}$ to denote an edge of G. The *complementary graph* of G, denoted by $\bar{G}$, has vertex set $V(G)$ and edge set $\{uv : u, v \in V(G), u \neq v, uv \notin E(G)\}$. We also use the notation of *co-H* to talk about the complementary graph of the graph H, e.g. co-kite and co-domino. Additionally, given two vertices $u, v \in V(G)$ and a set $S \subseteq V(G)$, we let $N_G(u)$ denote the neighbours of u, $N_G[u] = N_G(u) \cup \{u\}$, $N_G(S)$ be the set of all vertices of $V(G) \setminus S$ that have a neighbour in S, $N_G[S] = N_G(S) \cup S$, and $\text{dist}_G(u, v)$ be the *distance* of u and v in G, which is the minimal length of a path connecting u and v in G. Note that $\text{dist}_G(u, u) = 0$ and we define $\text{dist}_G(u, S) = \min\{\text{dist}_G(u, s) \mid s \in S\}$. We also let $N^i_G(S) = \{u : \min\{\text{dist}_G(u, s) : s \in S\} = i\}$ for $i \geq 1$ and $N^0_G(S) = S$. Also for a subgraph H of G we define $N^i_G(H) = N^i_G(V(H))$ and $N_G[H] = N_G[V(H)]$. For a graph G we call a tuple $(v, w) \in V(G) \times V(G)$ a *comparable vertex pair*, if $v \neq w, vw \notin E(G)$, and $N_G(v) \subseteq N_G(w)$. A vertex $u \in V(G)$ is a *universal vertex in* G if $N_G(u) = V(G) \setminus \{u\}$. Observe that $\Delta(G) = \{|N_G(u)| : u \in V(G)\}$ is the *maximum degree* of G. Furthermore, a graph H with $V(H) = V(G)$ and $E(H) \subseteq E(G)$ is a *spanning subgraph* of G. A vertex $v \in V(G)$ is a *cutvertex of* G if $G[V(G) \setminus \{v\}]$ consists of more connected components than G.

We use $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and, for $x \in \mathbb{N}_0$, $\mathbb{N}_{>x} := \{n \in \mathbb{N}_0 \mid n > x\}$, and $\mathbb{N}_{\geq x} := \{n \in \mathbb{N}_0 \mid n \geq x\}$. So $\mathbb{N}_0$ and $\mathbb{N}_{>0}$ denote the set of non-negative integers and positive integers, respectively. For some integer $k \in \mathbb{N}_{>0}$ we use $[k] := \{x \in \mathbb{N}_{>0} \mid x \leq k\}$. The power set of set S we denote by 2^S.

Additionally, for a function f whose range is a subset of $\mathbb{N}_0$, we let

$$\text{Argmin}\{f(s) : s \in S\} = \{s : f(s) \leq f(s') \text{ for each } s' \in S\}$$

and

$$\text{Argmax}\{f(s) : s \in S\} = \{s : f(s) \geq f(s') \text{ for each } s' \in S\},$$

and say that $f \colon \mathbb{N}_0 \to \mathbb{N}_0$ is *superadditive* if $f(s_1)+f(s_2) \leq f(s_1+s_2)$ for each $s_1, s_2 \in \mathbb{N}_0$ and $f(1) \neq 0$. For two non-empty sets S, T and two functions $f_1, f_2 : S \to T$, we shortly write $f_1 \equiv t$ if $f_1(s) = t$ for each $s \in S$, and $f_1 \equiv f_2$, or $f_1 \leq f_2$, or $f_1 \geq f_2$ if $f_1(s) = f_2(s)$, or $f_1(s) \leq f_2(s)$, or $f_1(s) \geq f_2(s)$, for each $s \in S$, respectively.

Let G be a graph and $q \colon V(G) \to \mathbb{N}_0$ be a function, which we also call *vertex-weight function*. Given a non-empty set S of vertices of G, $G[S]$ is the graph with vertex set S and edge set $E(G) \cap \{s_1s_2 : s_1, s_2 \in S\}$. We say that $G[S]$ is the graph *induced* by S and S *induces* $G[S]$ in G. Given an additional graph H, an *isomorphism between* G *and* H is a bijection between the $V(G)$ and $V(H)$ such that for every two vertices $u, v \in V(G)$ we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If there is an isomorphism between G and H we write G is *isomorphic* to H or $G \cong H$. We say that H is an *induced subgraph* of G, denoted by $H \subseteq_{\text{ind}} G$, if there is some set $S \subseteq V(G)$ of vertices such that $G[S] \cong H$, we also say that S *induces* a H in G, if S induces $G[S]$ in G and $G[S] \cong H$. If $H \subseteq_{\text{ind}} G$ we reversely say G *contains* H *as an induced subgraph* or G *contains an induced* H.

For simplification purposes we often times use a fixed ordering on the vertices if we claim that S induces a H, for most graphs H. To show the fixed ordering we use $[\dots]$ instead of $\{\dots\}$. Now the list of all relevant graphs and their orderings follows. We write $[v_1, v_2, v_3, v_4, v_5]$ *induces a HVN* in G, if and only if $G[\{v_1, v_2, v_3, v_4, v_5\}]$ is isomorphic to HVN and v_2, v_3 are universal vertices in $G[\{v_1, v_2, v_3, v_4, v_5\}]$ and $v_1v_4, v_1v_5 \notin E(G)$. We write $[v_1, v_2, v_3, v_4, v_5]$ *induces a dart* in G, if and only if $G[\{v_1, v_2, v_3, v_4, v_5\}]$ is isomorphic to *dart* and v_2 is a universal vertex in $G[\{v_1, v_2, v_3, v_4, v_5\}]$ and $v_3v_4, v_4v_5 \in E(G)$. We write $[v_1, v_2, v_3, v_4, v_5]$ *induces a* P_5 in G, if and only if $v_iv_{i+1} \in E(G)$, for $i \in [4]$. We write $[v_1, v_2, \ldots, v_k]$ *induces a* C_k in G, if and only if $v_iv_{i+1} \in E(G)$, for $i \in [k-1]$. We write $[v_1, v_2, v_3, v_4]$ *induces a* $2K_2$ in G, if and only if $G[\{v_1, v_2, v_3, v_4\}]$ is isomorphic to $2K_2$ and $v_1v_2 \in E(G)$. Lastly we write $[v_1, v_2, v_3, v_4]$ *induces a* $K_1 \cup K_3$ in G, if and only if $G[\{v_1, v_2, v_3, v_4\}]$ is isomorphic to $K_1 \cup K_3$ and $v_1v_2, v_1v_3 \notin E(G)$.

An often used notation is that of $G[q]$, which denotes the graph $G[\{u : q(u) \geq 1, u \in V(G)\}]$. Assuming H to be an induced subgraph of G, we further define

$$q(S) = \sum_{s \in S} q(s) \quad \text{and} \quad q(H) = q(V(H)).$$

For simplicity in notation and terminology, we say that q instead of the restriction of q to $V(H)$ is a vertex-weight function of H.

Given two graphs G_1, G_2 with $V(G_1) \cap V(G_2) = \emptyset$ and an integer $k \geq 1$, we denote by $G_1 \cup G_2$ the *union* of G_1 and G_2, that is, $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$, and by kG_1 a graph $G_1' \cup G_2' \cup \ldots \cup G_k'$ where $G_i' \cong G_1$ and $V(G_i') \cap V(G_j') = \emptyset$ for each disjoint $i, j \in [k]$. We denote by $G_1 + G_2$ the *join* of G_1 and G_2, that is, $G_1 + G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{v_1v_2 \mid v_1 \in V(G_1), v_2 \in V(G_2)\}$.

A *family of graphs* or a *graph family* is a set containing only graphs. A *class of graphs* or a *graph class* is a family of graphs closed under isomorphism. Note that in this thesis most of the regarded graph families are also graph classes. A family of graphs where every induced subgraph of a graph is likewise a member of the family of graphs is called *hereditary*.

In this thesis, we mainly work with forbidden induced subgraphs. Thus, given two graphs G, H and a family $\mathcal{H}$ of graphs, we say that G is *H-free* if H is not an induced subgraph of G, and that G is *$\mathcal{H}$-free* if G is H-free for each $H \in \mathcal{H}$. Recall that *$(H_1, H_2, \ldots)$-free* means $\mathcal{H}$-free with $\mathcal{H} = \{H_1, H_2, \ldots\}$ and we use $\mathrm{For}(\mathcal{H})$ to denote the family of graphs consisting of all $\mathcal{H}$-free graphs. A $(C_3, C_4, C_5, \ldots)$-free graph is called a *forest*.

Let again G be a graph and $q\colon V(G) \to \mathbb{N}_0$ be a vertex-weight function. Recall that a *clique* of G is a set of vertices which are pairwise adjacent. The *q-clique number* of G, denoted by $\omega_q(G)$ is the largest integer k for which there is a clique S of G with $q(S) = k$. An independent set S of G is a set of vertices which is a clique in $\bar{G}$, that is, the vertices of S are pairwise non-adjacent in G. The *q-independence number*, denoted by $\alpha_q(G)$, equals $\omega_q(\bar{G})$. A *q-colouring* $L\colon V(G) \to 2^{\mathbb{N}_{>0}}$ is a function for which $|L(u)| = q(u)$ for each $u \in V(G)$. We note that the integers of $L(u)$ are also called *colours* of u for $u \in V(G)$, and we say that L colours the vertices of G. In view of a simple notation, we let

$$L(S) = \bigcup_{s \in S} L(s) \quad \text{and} \quad L(H) = L(V(H))$$

for each set $S \subseteq V(G)$ and each induced graph H of G. The colouring L is *proper* if each two adjacent vertices of G receive disjoint sets of integers. The graph G is *k-colourable* (with respect to q) for some integer $k \in \mathbb{N}_{>0}$ if there is some proper q-colouring L that uses at most k different integers from $\mathbb{N}_{>0}$ for the assigned sets. The smallest integer k for which G is k-colourable (with respect to q) is the *q-chromatic number* of G, denoted by $\chi_q(G)$. For the vertex-weight function q with $q(u) = 1$ for each $u \in V(G)$, we use the classical terminology of *clique number*, *independence number*, and *chromatic number* instead of q-clique number, q-independence number, and q-chromatic number, and denote these graph invariants by $\omega(G)$, $\alpha(G)$, and $\chi(G)$, respectively. Furthermore, recall that G is *perfect* if $\omega(G') = \chi(G')$ for each induced subgraph G' of G. Also for the vertex-weight function q with $q \equiv 1$ a proper q-colouring c of a graph G can simply be seen as a function $c : V(G) \to \mathbb{N}_{>0}$, with $c(u) \neq c(v)$ whenever $uv \in E(G)$. Note that in this case for a subset $S \subseteq V(G)$ we see $c(S) \subseteq \mathbb{N}_{>0}$.

Given a class $\mathcal{G}$ of graphs, we recall that a function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ is a *χ-binding function* if $\chi(G') \leq f(\omega(G'))$ for each graph $G \in \mathcal{G}$ and each induced subgraph G' of G. Since we are interested in graph classes defined by a set, say $\mathcal{H}$, of forbidden induced subgraphs, we let $f^\star_{\mathcal{H}}$ denote the *optimal χ-binding function* of the class of $\mathcal{H}$-free graphs, that is, $f^\star_{\mathcal{H}}\colon \mathbb{N}_0 \to \mathbb{N}_0$ is defined by

$$\omega \mapsto \max\{\chi(G) : \omega(G) = \omega \text{ and } G \text{ is } \mathcal{H}\text{-free}\}.$$

As we consider the maximum of a subset of $\mathbb{N}_0$, we note that $\max \emptyset = 0$. Therefore, we see that $f^\star_{\mathcal{H}}(0) = 0$ for all sets $\mathcal{H}$. Also we see that $f^\star_{\mathcal{H}}(1) = 1$ if $K_1 \notin \mathcal{H}$. Since the function $f^\star_{\{P_5\}}$ occurs often, we mostly write $f^\star_{P_5}$ instead of $f^\star_{\{P_5\}}$.

Let again G be a graph. For two disjoint sets A and B of vertices, we let $E_G[A, B]$ denote the set of all edges between A and B in G, that is $E_G[A, B] \coloneqq \{uv \in E(G) \mid u \in A, v \in B\}$. Also we say $E_G[A, B]$ is *complete* or *anticomplete* if $|E_G[A, B]| = |A| \cdot |B|$ or $|E_G[A, B]| = 0$ respectively. Note that the empty set is both complete and anticomplete to every other set. We say $E_G[S_1, S_2]$ is *mixed* if $E_G[S_1, S_2]$ is neither complete nor anticomplete.

A set M of vertices of G is a *module* if $E_G[M, N_G(M)]$ is complete. We note that a module M is a *homogeneous set* if $1 < |M| < |V(G)|$. The graph G is *prime* if there is no homogeneous set in G. A clique X of G is a *clique-separator* if the number of components of $G - X$ exceeds that of G. Let $k \geq 1$ be an integer, G_1, G_2 be two not necessarily connected induced subgraphs of G with $G = G_1 \cup G_2$ and

$V(G_1) \setminus V(G_2), V(G_2) \setminus V(G_1) \neq \emptyset$, $k \in \mathbb{N}_{>0}$, and $X_1, X_2, \ldots, X_k$ be k pairwise vertex disjoint modules in G. If

- $E_G[X_i, X_j]$ is complete in G for each distinct $i, j \in [k]$ and
- $V(G_1) \cap V(G_2) = X_1 \cup X_2 \cup \ldots \cup X_k$,

then $X_1 \cup X_2 \cup \ldots \cup X_k$ is a *clique-separator of modules* in G.

Let $q, q' \colon V(G) \to \mathbb{N}_0$ be two vertex-weight functions of a graph G. We write $q' \lhd_\chi^G q$ if $\chi_{q'}(G) = \chi_q(G)$, $q'(G) < q(G)$, and $q'(u) \leq q(u)$ for each $u \in V(G)$. Additionally, q is $\lhd_\chi^G$*-minimal* if there is no vertex-weight function $q' \colon V(G) \to \mathbb{N}_0$ with $q' \lhd_\chi^G q$ and $q \not\equiv 0$. We note that a graph G is *critical* if $q \colon V(G) \to [1]$ is $\lhd_\chi^G$-minimal. Or simpler a graph G is vertex-critical or short critical if $\chi(G - v) < \chi(G)$ for every $v \in V(G)$.

Let G be a graph and P be a property that a graph can have. A P*-expansion* of a vertex u in G is a graph that can be obtained from G by replacing u by a graph G' that has property P and making each vertex of G' adjacent to each neighbour of u. In this thesis, given a vertex ordering $\prec$, we associate $\prec$ with a bijective function $f_\prec \colon V(G) \to [|V(G)|]$ which is defined by the equivalence that $u \prec v$ if and only if $f_\prec(u) < f_\prec(v)$. A P*-expansion* of G is a graph G' for which there is a vertex ordering $\prec$ of G and a finite series $\{G_i\}_{i=1}^{|V(G)|+1}$ of graphs such that

- $G = G_1$ and $G' = G_{|V(G)|+1}$, and
- G_{i+1} is a P-expansion of $f_\prec^{-1}(i)$ in G_i for each $i \in [|V(G)|]$.

If $q \colon V(G) \to \mathbb{N}_0$ is a vertex-weight function, then a q*-expansion* of G is a 'complete graph'-expansion of G in which each vertex $u \in V(G)$ is replaced by a clique of size $q(u)$. We note that, for a *'non-empty,* $2K_1$*-free'*-expansion G' of G, there is a vertex-weight function $q \colon V(G) \to \mathbb{N}_{>0}$ such that G' is a q-expansion of G. Furthermore, a *buoy* and a *connected buoy* are a 'non-empty vertex set'-expansion and a 'connected'-expansion of a cycle of length 5, respectively. A *maximal connected buoy* C in G is an induced connected buoy in G for which there is no other induced connected buoy (distinct from C) in G having C as an induced subgraph.

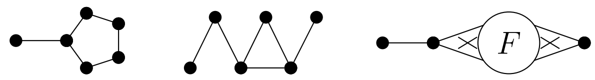
Let C be a cycle of length 5 and $q \colon V(G) \to \mathbb{N}_0$ be a vertex-weight function. If $L \colon V(G) \to 2^{\mathbb{N}_{>0}}$ is a proper q-colouring of G and $c_1, c_2 \in V(C)$ are two vertices, then

$$L^{(1)}(c_1) = \{k : k \in L(c_1), k \notin L(c) \text{ for each } c \in V(C) \setminus \{c_1\}\}$$

and

$$L^{(2)}(c_1, c_2) = \{k : k \in L(c_1) \cap L(c_2), k \notin L(c) \text{ for each } c \in V(C) \setminus \{c_1, c_2\}\}.$$

In Fig. 1 and Fig. 3, the most frequently used (forbidden) induced subgraphs of this thesis are depicted. As usual, C_n, K_n, and P_n denote a cycle, a complete graph, and



(a) A_5 **(b)** $T_{0,1,2}$ **(c)** $Q[F]$

Fig. 3: Some additional frequently used forbidden induced subgraphs

a path of order n, respectively, and $K_{n,m}$ denotes a complete bipartite graph whose partite sets have sizes n and m. Additionally, if $P\colon u_1u_2u_3u_4$ is a path on 4 vertices and F is an arbitrary graph that is vertex disjoint from P, then $Q[F]$ is the 'equals F'-expansion of u_3 in P.

When calculating χ-binding function it is good practice to state a family of graphs which grants a lower bound. For this situation the following notation is useful. For disjoint graphs $H_1, \dots H_5$ we define the graph $C_5[H_1, H_2, \dots, H_5]$ to be the graph with vertex set $\bigcup_{i=1}^{5} V(H_i)$ and edge set

$$\bigcup_{i=1}^{5} E(H_i) \cup \bigcup_{i=1}^{4} \{uv \mid u \in V(H_i), v \in V(H_{i+1})\} \cup \{uv \mid u \in V(H_5), v \in V(H_1)\}.$$

Given a graph G and a vertex-weight function $q\colon V(G) \to \mathbb{N}_0$, let $\mathcal{C}_5(G)$ be the set of all induced cycles of length 5 in G and

$$\mathcal{C}_5^\star(G, q) = \text{Argmax}\{\chi_q(C) : C \in \mathcal{C}_5(G)\}.$$

We often write $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ to shortly state, that $C \in \mathcal{C}_5(G)$ and the vertices of C are labelled by $c_1, \dots, c_5$ with $c_ic_{i+1} \in E(G)$ for $1 \leq i \leq 4$. Additionally, recall that G is $(C_5, C_7, \dots)$*-free* if G is C_{2k+5}-free for each $k \in \mathbb{N}_0$.

We note that index calculations are always considered with respect to the modulo operation. For example, all index calculations are considered modulo 5 whenever we consider a $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ or a buoy $C\colon C_1C_2C_3C_4C_5C_1$.

In what follows, we may assume that C is a cycle of length 5. An *orientation* of C is an assignment of a direction to each edge. As the obtained graph is a directed graph, we note that there are exactly two orientations of C that are directed cycles. In view of simplicity, whenever we work with such a cycle C, we implicitly fix one orientation that leads to a directed cycle $\vec{C}$. Furthermore, for each vertex $c \in V(C)$, we write c^- and c^+ for the vertices of C such that $(c^-, c), (c, c^+) \in E(\vec{C})$. In view of simplicity, we write c^{-2} and c^{+2} for $(c^-)^-$ and $(c^+)^+$, respectively.

For the remainder of the thesis let the set $\mathcal{G}^\star$ be defined as follows. It consists of all connected graphs G such that, taken an arbitrary cycle $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$, we have that $V(G) - N_G[V(C)]$ is an independent set and that there is some integer $i \in [5]$ such that $E_G[\{\{c_i, c_{i+2}, c_{i+3}\}, N_G(V(C))\}]$ is complete and $E_G[\{\{c_{i+1}, c_{i+4}\}, N_G(V(C))\}]$ is anticomplete.

A graph G is a *matched co-bipartite* graph if G is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| + 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching. A graph G is called *complete multipartite* if there is an $n \in \mathbb{N}_{>0}$ and $a_1, a_2, \ldots, a_n \in \mathbb{N}_{>0}$ such that $\bar{G} \cong K_{a_1} \cup K_{a_2} \cup \ldots K_{a_n}$.

Let us use this space to define the *Ramsey number* $R(m, n)$, for $m, n \in \mathbb{N}_{>0}$. The number $R(m, n)$ is the minimum number of vertices such that all graphs of order $R(m, n)$ contain an independent set of order m or a clique of order n.

2 P_5-free universe

A classical result by Erdős [25] in the field of chromatic graph theory shows that the difference between chromatic and clique numbers of a graph can be arbitrarily large even for graphs of large girth.

Theorem 11 (Erdős [25])**.** *For any positive integers $k, \ell \geq 3$, there exists a graph G with girth $g(G) \geq \ell$ and chromatic number $\chi(G) \geq k$*

However, on the positive side, in terms of forbidden induced subgraphs it is possible to characterize graphs G whose each induced subgraph has equal clique and chromatic number (cf. Strong Perfect Graph Theorem). Recall that such a graph G is called perfect. A large collection of 120 graph classes, which are all perfect, has been surveyed by Hougardy [35]. Naturally, the behaviour of the chromatic number of non-perfect graphs is of wide interest.

A concept relating the chromatic and clique numbers of a graph and surrounding the Strong Perfect Graph Conjecture is that of χ-binding functions for graph classes. Recall the definition introduced by Gyárfás [31]. Given a class $\mathcal{G}$ of graphs, a function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ is a *χ-binding function* for $\mathcal{G}$ if $\chi(G - S) \leq f(\omega(G - S))$ for each $G \in \mathcal{G}$ and each $S \subsetneq V(G)$. The function $f^\star\colon \mathbb{N}_0 \to \mathbb{N}_0$ with

$$\omega \mapsto \max\{\chi(G - S) : G \in \mathcal{G}, S \subseteq V(G), \omega(G - S) = \omega\}$$

is the *optimal χ-binding function* of $\mathcal{G}$.

By using Theorem 11 one can show that in general there is no χ-binding function for a family $\mathcal{G}$ of graphs. Another wellknown family to illustrate that fact is based on a construction from Mycielski [47]. In general the Mycielski construction grants a way to construct a graph $\mu(G)$ with the following properties, if given a graph G with $\omega(G) \geq 2$. Firstly $\chi(\mu(G)) = \chi(G) + 1$ but also $\omega(\mu(G)) = \omega(G)$. The Mycielski-graph $\mu(G)$ of a graph G is defined as follows. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and V_1 be a copy of $V(G)$ named $\{v_1^1, v_2^1, \ldots, v_n^1\}$, and u be a single vertex. Then the $V(\mu(G)) = V(G) \cup V_1 \cup \{u\}$ and

$$E(\mu(G)) = E(G) \cup \{v_i v_j^1 : v_i v_j \in E(G)\} \cup \{v_j^1 u : \forall j \in [n]\}.$$

We define the graph family $\mathcal{M}$ by $\mathcal{M} := \{\mu^k(K_2) \mid k \in \mathbb{N}_{\geq 0}\}$. Hence, we find $\omega(G) = 2$ for all $G \in \mathcal{M}$ and $\chi(\mu^{i-2}(K_2)) = i$, for $i \in \mathbb{N}_{\geq 2}$. Thus, the family $\mathcal{M}$ has no χ-binding function.

However, for some restricted classes of graphs such binding functions exist. Recall that for brevity, given some graphs $H_1, H_2, \ldots$, we let $f^\star_{\{H_1,H_2,\ldots\}}$ denote the optimal χ-binding function of the class of $(H_1, H_2, \ldots)$-free graphs. In this chapter we collect and discuss known results in the area of χ-binding functions for subfamilies of P_5-free graphs.

Let us first talk about the biggest family and superfamily of all later talked about families: The family of P_5-free graphs. The first result is a bound by Gyarfas [31]:

Theorem 12 (Gyárfás [31])**.** *For $n \in \mathbb{N}_{>1}$ and $\omega \in \mathbb{N}_{>0}$*

$$\frac{R(\lceil n/2\rceil, \omega+1)-1}{\lceil n/2\rceil - 1} \leq f^\star_{P_n}(\omega) \leq (n-1)^{\omega-1}.$$

The lower bound follows from the observation that an induced P_n in a graph G contains an independent set of size $\lceil n/2\rceil$ as follows. Let G be a graph with $|V(G)| = R(\lceil n/2\rceil, \omega+1)-1$ with neither an independent set of size $\lceil n/2\rceil$ nor a clique of size $\omega+1$. Thus, $\alpha(G) = \lceil n/2\rceil - 1$ and G is especially P_n-free, and $\omega(G) = \omega$. Therefore, $\chi(G) \geq |V(G)|/\alpha(G) = R(\lceil n/2\rceil, \omega+1) - 1/(\lceil n/2\rceil - 1)$, where the first inequality is true for every graph by definition of χ and α. This proves the lower bound. He also mentions that the truth is probably close to the lower bound, and that the lower bound is exact for $n = 4$ by a previously proven result from Seinsche [65].

Proving this upper bound is already nontrivial. The proof by Gyárfás is inductively over $\omega(G)$. In the induction step t to $t+1$ he supposes for the sake of contradiction that there is a graph G with $\omega(G) = t+1$ and $\chi(G) > (n-1)^t$. In this graph he finds an induced P_n by defining nesting vertex-sets $V_1, V_2, \ldots, V_n$ with $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_n$ with special properties.

The first improvement to the upper bound uses online colourings. Let us not dive too deep into online colourings, but the idea is, that the graph which we want to colour is not completely known in the beginning but instead is presented vertex by vertex. In this online setting Kierstead et al. [41] prove the following.

Theorem 13 (Kierstead et al. [41])**.** *There exists an on-line algorithm A such that $\chi_A(G) \leq (4^{w(G)} - 1)/3$, for every P_5-free graph G.*

Gravier et al. [30] improve on this bound. In their paper they especially prove the following corollary. Note that their result is more general but we omit the more general result here and state what is relevant for our purpose.

Corollary 14 (Gravier et al. [30])**.** *For $\omega \in \mathbb{N}_{>0}, n \in \mathbb{N}_{>2}, f^\star_{P_n}(\omega) \leq (n-2)^{\omega-1}$.*

The next improvement to this bound is by Esperet et al. [26] from 2013. By proving that (P_5, K_4)-free graphs are 5-colourable, they improve the bound of Gravier et al.

for $\omega = 3$ and $n = 5$. They also state the graph $C_5[K_1, C_5, K_1, C_5, K_1]$, as defined in Section 1.2, to prove the following equality.

Theorem 15 (Esperet et al. [26])**.**

$$f^{\star}_{P_5}(3) = 5.$$

Combining their new bound and the proof from Gravier et al. [30] implies for $\omega \in \mathbb{N}_{>3}$ that

$$f^{\star}_{P_5}(\omega) \leq 5 \cdot 3^{\omega-3}.$$

Thus, $f^{\star}_{P_5}(\omega) \leq 3^{\omega-c}$, where $c = 3 - \log 5/\log 3 \approx 1.535$.

In August 2021 Scott, Seymour and Spirkl [64] published a paper in which they prove, for $\omega \in \mathbb{N}_{\geq 4}$,

$$f^{\star}_{P_5}(\omega) \leq \omega^{\log_2(\omega)}.$$

This is the currently best known general bound for $f^{\star}_{P_5}$. Bounds of this form are called quasi-polynomial. Note that the previously stated bound is only smaller for $\omega = 4$. The proof to this statement is quite short and analytical. The first claim in their paper which they use multiple times proves an upper bound for $\chi(G \setminus X)$ for every cutset X. Note that for $k \in [3]$ the exact values of $f^{\star}_{P_5}(k)$ are known. They define the function $f : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ by $f(1) = 1$, $f(2) = 3$, $f(3) = 5$ and $f(k) = k^{\log_2(k)}$ for $k \in \mathbb{N}_{\geq 4}$ and show that $f(w-1) + (w+2) \cdot f(\lfloor w/2 \rfloor) \leq f(w)$ for $w \geq 5$ and in an additional claim they prove that a function fulfilling the just stated inequality and some other simple properties is a binding function for every P_5-free graph. Note that their result can be improved if one is able to find a function also fulfilling the stated inequality which is smaller than f.

Let us now talk about the Strong Perfect Graph Theorem (SPGT) and its tight relation with the research of χ-binding functions for subfamilies of P_5-free graphs. The proof of the SPGT is one of the biggest achievements in the last decades of graph theory. It is one of the most challenging now proven conjectures in graph theory. During more than four decades numerous attempts by different researchers were made to solve it. The final concluding paper consists of over 100 pages and contains multiple ideas.

The Strong Perfect Graph Theorem (Chudnovsky et al. [20])**.** *A graph G is perfect if and only if G and $\bar{G}$ are $(C_5, C_7, \ldots)$-free.*

This question was introducted by Berge [5] and is therefore known as Berge's conjecture. The SPGT is useful in the research of χ-binding functions for subfamilies of P_5-free graphs. This is the case, since to find a χ-binding function one only has to look at the non-perfect graphs and the SPGT gives structural support for these. Since P_5-free graphs are especially $(C_7, C_9, \ldots)$-free and since $C_5 \cong \bar{C}_5$ it can be assumed by SPGT, that G contains an induced odd antihole. Many researches use this result to make a structural analysis of the existing odd antihole and its neighbourhood.

On a side note let us shortly talk about P_4-free graphs. P_4-free graphs are perfect, but there is also the more general Observation by Randerath and Schiermeyer [55] which states that for any subgraph $T \subseteq_{\text{ind}} P_4$ the family $\text{For}(T)$ is perfect.

Observation 16 (Randerath and Schiermeyer [55]). *Let $\mathcal{G}$ be a χ-bounded family of graphs defined in terms of only one forbidden induced subgraph T. Then T is acyclic. Furthermore, if $T \subseteq_{ind} P_4$ then $\mathcal{G}$ has the (smallest) χ-binding function $f_T^\star(\omega) = \omega$, or otherwise there exists no linear χ-binding function f for $\mathcal{G}$.*

Note that $f_{K_2}^\star(2) = 0$ but this is a trivial result. For that reason Gyárfás [31] in his introductory paper already assumes that $f(\omega) \geq \omega$ for every χ-binding function f and every $\omega \in \mathbb{N}_{>0}$. The same is true for [55] and this is the reason why the word "smallest" in Observation 16 is in brackets.

2.1 (P_5, H)-free graphs

For the remainder of the chapter we at least forbid one additional graph, called H. For the χ-binding function it is important, whether or not $\alpha(H) \geq 3$. Since in the case $\alpha(H) \geq 3$ the best possible χ-binding function for the family of (P_5, H)-free graphs is $f_{3K_1}^\star \in \Theta(\omega^2/\log(\omega))$ as we discuss in Section 2.2.1.

2.1.1 $\alpha(H) = 2$

Fouquet et al [27] prove for $\omega \in \mathbb{N}_{>0}$, $k \in \{j \in \mathbb{N}_{>0} \mid \exists i \in \mathbb{N}_0 : j = 2^i\}$

$$f_{\{P_5,\text{house}\}}^\star(\omega) \leq \binom{\omega+1}{2}$$

and

$$k^{\log_2(5/2)} \approx k^{1.322} \leq f_{\{P_5,\text{house}\}}^\star(k).$$

Since the lower bound is discussed as a small side note in Section 4.1 of their paper, we discuss how to achieve the bound by their recursive definition. They recursively construct a family of $(P_5, \bar{P}_5)$-free graphs whose chromatic number does increase non linearly in the clique number. They start with $G_0 \cong K_1$ and G_{k+1} is the 'be G_k'-expansion of C_5 for $k \in \mathbb{N}_0$. They note that $\omega(G_{k+1}) = 2 \cdot \omega(G_k)$ and prove that $\chi(G_{k+1}) = \left\lceil \frac{5\chi(G_k)}{2} \right\rceil$ for $k \in \mathbb{N}_0$, which also follows from the more general result we state in Corollary 46. Thus, the searched binding function $f := f_{P_5,\text{house}}^\star$ has the following two properties

$$f(1) = 1 \text{ and for } k \in \mathbb{N}_{>0} : f(2^k) \geq \left\lceil \frac{5}{2} f(2^{k-1}) \right\rceil .$$

Therefore,

$$f(2^k) \geq \frac{5}{2} f(2^{k-1}) \geq \frac{5}{2}(\frac{5}{2} f(2^{k-2})) \geq \cdots \geq (\frac{5}{2})^k \cdot f(2^{k-k}) \geq (\frac{5}{2})^k.$$

Note that using the exact bound and not omitting the ceiling function does grant the same asymptotic lower bound, since $\sum_{i=0}^{k-1} (\frac{5}{2})^i \leq (\frac{5}{2})^k$. By substituting 2^k by x one gets $f(x) \geq x^{\log_2(5/2)} \approx x^{1.322}$. Also let us add, that the graph G_2 is currently the graph which grants the biggest known lower bound for $f^\star_{P_5}(4)$. Note that this graph family is also *bull*-free.

We next want to talk about the upper bound. They extend a result by Blázsik [6], for $\mathrm{For}(C_4, 2K_2)$ to $\mathrm{For}(P_5, \bar{P}_5)$. Note that by SPGT $(C_5, P_5, \bar{P}_5)$-free graphs are perfect. So for a $(P_5, \bar{P}_5)$-free graph G their idea is as follows. They choose a minimal subset T of $V(G)$, such that every $C_5 \in \mathcal{C}_5(G)$ contains a vertex which belongs to T. A subset fulfilling these properties is called a *minimal transversal T of the C_5's.* So the main result of their paper is the following theorem.

Theorem 17 (Fouquet et al [27]). *Every minimal transversal T of the C_5's of a $(P_5, \bar{P}_5)$-free graph G is such that $\omega(T) \leq \omega(G) - 1$.*

This shows that every $(P_5, \bar{P}_5)$-free graph can be partitioned into two sets, called T and $V(G) \setminus T$ such that $\omega(G[T]) \leq \omega(G) - 1$ and $\chi(G[V(G) \setminus T]) = \omega(G)$. Inductively they now prove the quadratic upper bound.

We use a result by Brandstädt and Mosca [9] about prime $(P_5, kite)$-free graphs. Interestingly they are interested in these prime graphs for a different reason. Instead of trying to χ-bound this family they are looking for a polynomial algorithm to determine the maximum weight independent set. In this algorithmic problem the aim is to find the largest independent set in a given graph. Generally this problem is $\mathcal{NP}$-complete even for K_3-free graphs [50]. They prove that for the family $(P_5, kite)$-free graphs this problem is polynomially solvable. Since we make use of the following lemma, we shortly want to talk about its proof.

Lemma 18 (Brandstädt and Mosca [9]). *If a prime $(P_5, kite)$-free graph contains an induced $2K_2$ then it is a matched co-bipartite graph.*

In the proof of this lemma they use the following result by Hoàng and Reed [34]. The graphs A and domino are depicted in Figure 4.

Lemma 19 (Hoàng and Reed [34]). *If a prime graph contains an induced $2K_2$ then it contains an induced P_5 or $\bar{A}$ or co-domino.*

(a) A **(b)** domino

Fig. 4: Induced subgraphs used in the paper of Brandstädt and Mosca [9]

Making use of the structural result of Hoàng and Reed and noting that P_5 and $\bar{A}$ are not $(P_5, kite)$-free they assume that the researched graph contains an induced co-domino. Now consequently researching the structure of the neighbourhood of the co-domino they prove the statement.

Next we look at a paper by Brause et al. [14]. The main focus of this paper is to prove χ-binding functions for some subclasses of $2K_2$-free graphs. But in the last section of their paper they consider $(P_5, hammer)$-free graphs and show that for $\omega \in \mathbb{N}_{>0}$

$$f^{\star}_{\{P_5,hammer\}}(\omega) \leq \binom{\omega+1}{2}.$$

They discuss no lower bound. Note that we prove something stronger in Chapter 4 by proving that $f^{\star}_{2K2} = f^{\star}_{P_5,hammer}$. Many mathematicians research the family of $2K_2$-free graphs and that is why we talk about this family and what is known about $f^{\star}_{2K_2}$ in the upcoming Section 2.2.2. Just note that this new result currently only slightly improves the bound, because not much is known about the general bound for $f^{\star}_{2K_2}$.

One of the first researched families is the family of (P_5, paw)-free graphs. Note that this family is also important for our research of (P_5, HVN)-free graphs, since $\mathrm{HVN} = K_1 + paw$. Let us first state the known results:

$$f^{\star}_{\{P_5,paw\}}(\omega) = \begin{Bmatrix} f^{\star}_{\{P_5,C_3\}}(\omega) & \text{if } \omega \leq 2, \\ \omega & \text{if } \omega > 2 \end{Bmatrix} = \begin{Bmatrix} 3 & \text{if } \omega = 2, \\ \omega & \text{if } \omega \neq 2 \end{Bmatrix} \text{ (cf. [48, 54] or [59]).}$$

Forbidding *paw* is a huge restriction and these graphs are completely characterised by Olariu [48].

Theorem 20 (Olariu [48])**.** *G is a paw-free graph if and only if each component of G is K_3-free or complete multipartite.*

Since complete multipartite graphs are perfect, the graphs which are relevant to research to achieve this bound are K_3-free graphs. We note that Randerath [54] characterises all non-bipartite (P_5, K_3)-free graphs which grants the bound.

Like we mention in the introductory chapter the following result is proven

$$f^{\star}_{\{P_5,gem\}}(\omega) \leq \lceil 5\omega/4 \rceil \text{ (cf. [19]).}$$

This bound is best possible for ω even. Clearly for example for $\omega = 1$ it is not best possible. In that paper they use a theorem from [38], which concretely works with the function $f : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ defined by $f(w) = \lceil 5\omega/4 \rceil$, so it seems difficult to use the papers result to get the best possible bound. We show in Chapter 7 as a conclusion of other results that $f^{\star}_{\{P_5,gem\}}(\omega) = \lceil (5\omega - 1)/4 \rceil$, which is the best possible bound.

In a paper by Huang and Karthick [36] they prove

$$f^{\star}_{\{P_5,paraglider\}}(5) = 8 \text{ and } \lceil 3\omega/2 \rceil - 1 \leq f^{\star}_{\{P_5,paraglider\}}(\omega) \leq \lceil 3\omega/2 \rceil,$$

for $\omega \in \mathbb{N}_{>2} \setminus \{5\}$. To get this strong result they do lots of structural analysis of the neighbourhood of a given C_5. After that they first assume that additionally to the C_5 there is a vertex which is adjacent to three non-consecutive vertices of the C_5. By proving that in this case the resulting graphs are off nice structure they assume from now on that no C_5 has such a vertex in its neighbourhood. This idea of assuming the graph contains a certain induced subgraph and analysing the structure they do for two more graphs. In these steps they obtain graph classes with certain structural properties and in the last section they colour the graphs from these classes. For the case ω equals to 5 they find two $(P_5, paraglider)$-free graphs namely the complementary graph of the Clebsch graph $\bar{\mathcal{C}}$ and a subgraph of $\bar{\mathcal{C}}$ with $\omega(\bar{\mathcal{C}}) = 5$ and $\chi(\bar{\mathcal{C}}) = 8 = \lceil 3 \cdot 5/2 \rceil$.

In a paper by Hoàng and McDiarmid [33] they introduce the notation of 2-divisibility. A graph G is said to be 2-*divisible* if for all (nonempty) induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $\omega(A) < \omega(H)$ and $\omega(B) < \omega(H)$. In a recent paper by Chudnovsky and Sivaraman [22] they prove by a short, inductive proof that for every 2-divisible graph G $\chi(G) \leq 2^{\omega(G)-1}$. By now proving that every (P_5, C_5)-free graph is 2-divisible they conclude

$$f^{\star}_{\{P_5,C_5\}}(\omega) \leq 2^{\omega-1}.$$

This bound is probably far from optimal, but it is the currently best known bound at least for $\omega \leq 19$. For $\omega \geq 20$ the quasi-polynomial bound for P_5-free graphs by Scott et al. [64] is smaller. In their paper they do not mention a lower bound for this function. The graph C_5 is one of the few graphs H where the family of (P_5, H)-free graphs has no known polynomial χ-binding function. Just additionally forbidding C_5 seems to be quite a small restriction. Also according to the Strong Perfect Graph Theorem one still has to consider the cases that G contains an induced $\bar{C}_{2k+1}$ for every $k \geq 3$.

2.1.2 $\alpha(H) \geq 3$

Schiermeyer [57] considers the graph $K_1 + (K_1 \cup P_4)$, which is obtained from a *gem* by adding a pendant edge to its vertex of degree 4. Therefore, it is called *gem*$^+$ and sometimes *parachute*. The following bound is sufficient to show Reed's Conjecture for this family as long as $\omega(G)$ is not too large.

Theorem 21 (Schiermeyer [57]). *Let G be a (P_5, gem^+)-free graph. Then $\chi(G) \leq \omega^2(G)$.*

This proof is a short and elegant proof by induction on $\omega(G)$. It uses the fact that P_5-free graphs contain a dominating clique or an induced dominating P_3. By subdividing the neighbourhood of the dominating subgraph into perfect subgraphs this bound is achieved. By proving a χ-binding function for the large family of (P_5, gem^+)-free graphs they prove a χ-binding function for all subfamilies. Subfamilies are for example the $(P_5, dart)$-free graphs and $(P_5, claw)$-free graphs. This result does not grant an optimal bound for the family of $(P_5, dart)$-free graphs as we show in Chapter 6 and no lower bound is stated.

Karthick et al. [39] are interested in the Weighted Vertex Colouring (WVC) problem and whether or not in can be solved in polynomial time. The WVC problem is explained as follows: given a graph G and a weight function $q : V(G) \to \mathbb{N}_0$, calculate $\chi_q(G)$. They for example research the family of $(P_5, dart)$-free graphs. Note that they do not give bounds on the weighted chromatic number $\chi_q(G)$ but instead figure out how fast one can calculate this number. To answer this question it is also necessary to study the structure of the prime graphs. This is the reason their result is stated here even though they do not research χ-binding functions in [39]. They prove:

Theorem 22 (Karthick et al. [39]). *Let G be a prime $(P_5, dart)$-free graph that contains an induced C_5. Then either $|V(G)| \leq 18$ or G is $3K_1$-free.*

They use Theorem 22 together with the also proven fact, that the WVC problem is polynomial solvable for the family of $(P_5, dart, C_5)$-free graphs to prove their claim. This suffices to show that for this family the WVC problem can be solved in polynomial time, since for finite graphs and $3K_1$-free graphs it is known. To get the explicit bound for $f^\star_{\{P_5,dart\}}$ one has to research the structure of all prime graphs according to our Lemma 41. This is exactly what we do in Chapter 6.

Brause et al. [10] figure out a polynomial χ-binding function for the family of $(P_5, K_{2,t})$-free graph. Concretely they prove for $k \in \mathbb{N}_{>1}$, $\omega \in \mathbb{N}_{>0}$

$$f^\star_{\{P_5,K_{2,t}\}}(\omega) \leq c_t \cdot \omega^t \text{ for a constant } c_t.$$

Note that this result is quite general and includes for $t = 2$ the family of (P_5, C_4)-free graphs. Therefore, it is not surprising, that the bound for $t = 2$ is not optimal.

Hoàng [32] introduces the notation of perfect divisibility. A graph G is said to be *perfectly divisible* if for all induced subgraphs H of G, $V(H)$ can be partitioned into two sets A, B such that $H[A]$ is perfect and $\omega(H[B]) < \omega(H)$. In the previously stated paper by Chudnovsky and Sivaraman [22] they also prove inductively that the chromatic number $\chi(G)$ is upper bounded by $\binom{\omega(G)+1}{2}$, for every perfectly divisible graph G. By now proving that every $(P_5, bull)$-free and every $(bull, C_5, C_7, \dots)$-free graph is perfectly divisible they conclude

$$f^\star_{\{P_5,bull\}}(\omega), f^\star_{\{C_5,C_7,\dots,bull\}}(\omega) \leq \binom{\omega+1}{2},$$

for $\omega \in \mathbb{N}_{>0}$. This bound is the currently best known bound. There is no talk about a lower bound to these functions.

For integers $n_1 \geq n_2 \geq \dots \geq n_p \geq 2$, the *generalized windmill graph* $W(n_1, n_2, \dots, n_p)$ is defined by $W(n_1, n_2, \dots, n_p) := K_1 + (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_p})$. Schiermeyer [58] proves a polynomial χ-binding function for the class of $(P_5, W(n_1, n_2, \dots, n_p))$-free graphs. For $p \geq 2$ and a constant $c(n_1, \dots, n_p)$, which only depends on the integers

$$f^\star_{\{P_5,W(n_1,n_2,\dots,n_p)\}}(\omega) \leq c(n_1, \dots, n_p) \cdot \omega^{1+\sum_{i=1}^{p-1} n_i}.$$

It is clearly really difficult to find an optimal χ-binding function for this large graph family. So the first aim of this paper is not to find an optimal χ-binding function but instead it is to find a polynomial χ-binding function for a large graph family. They prove more general results which they then apply to get the bound for this graph family. Note that in the following theorem we summarize results from Schiermeyer [58]. These results help to get an estimation for the χ-binding function of a larger graph class if the forbidden subgraph can be build under certain construction rules by smaller graphs. These bounds can be used generally to get a first approximation for the magnitude of a χ-binding function.

Theorem 23 (Schiermeyer [58])**.** *Let $n_1 \in \mathbb{N}_{>1}$, H be a graph such that there is a constant $c \in \mathbb{R}_{>0}$ with $f^\star_H(\omega) \leq c \cdot \omega^t$ for some $t \in \mathbb{N}_{>0}$ and every $\omega \in \mathbb{N}_{>0}$. Then there are constants $c(H), c(n_1, H), \tilde{c}(H) \in \mathbb{R}_{>0}$ such that*

$$\begin{aligned} f^\star_{K_2 \cup H}(\omega) &\leq c(H) \cdot \omega^{2+t}, \\ f^\star_{\{P_k, K_{n_1} \cup H\}}(\omega) &\leq c(n_1, H) \cdot \omega^{n_1+t}, \text{ and} \\ f^\star_{\{P_5, K_1 + H\}}(\omega) &\leq \tilde{c}(H) \cdot \omega^{t+1}. \end{aligned}$$

Note that they save a factor of ω^{n_p} in their windmill bound by using a generalization of the following result.

Theorem 24 (Schiermeyer [58])**.** *Let $n_1, n_2 \in \mathbb{N}_{>1}$ with $n_1 \geq n_2$. Then*

$$f^\star_{\{P_k, K_{n_1} \cup K_{n_2}\}} \leq c(n_1) \cdot \omega^{n_1},$$

for a constant $c(n_1)$.

2.2 $3K_1$ and $2K_2$

If one wants to colour the family of (P_5, H)-free graphs it is sometimes sufficient to colour the family of $(2K_2, H)$ or $(3K_1, H)$-free graphs. Clearly both $2K_2$ and $3K_1$ are subgraphs of P_5, but for certain graphs H the (P_5, H)-free graphs with high chromatic number, relative to their clique number, are all even $3K_1$ or $2K_2$-free. That is why it is necessary to talk about the known χ-binding functions of these families.

2.2.1 $3K_1$-free universe

In this section we want to talk about the family of $3K_1$-free graph. Its chromatic number is highly related to the Ramsey number $R(3, k)$. For that reason we want to state some known results regarding this specific Ramsey number.

Theorem 25 (Ajtai et al. [1]). $R(3, k) \in \mathcal{O}(k^2/\log k)$

Fifteen years later Kim proves the following theorem, which is considered to be a landslide result in this area.

Theorem 26 (Kim [42]). $R(3, k) \in \Theta(k^2/\log k)$

In the following lemma we introduce a concrete upper and a concrete lower bound of $f^\star_{\{3K_1\}}(\omega)$ only depending on ω and $R(3, \omega + 1)$, for every $\omega \in \mathbb{N}_{>0}$. To achieve that we use an upper bound on the chromatic number by Schiermeyer [60]. We do not know of an article stating these bounds. For that reason, we shortly prove them.

Corollary 27. *For $\omega \in \mathbb{N}_{>0}$,*

$$\left\lceil \frac{R(3, \omega + 1) - 1}{2} \right\rceil \leq f^\star_{\{3K_1\}}(\omega) \leq \left\lfloor \frac{R(3, \omega + 1) - 2 + \omega}{2} \right\rfloor$$

and thus by Theorem 26

$$f^\star_{3K_1}(\omega) \in \Theta(\omega^2/\log \omega).$$

Proof. Note that $\chi(G'') \cdot \alpha(G'') \geq |V(G'')|$ for every graph G'', which follows directly from the fact that each colour class is an independent set. Let $w \in \mathbb{N}_{>0}$ be fixed and $R := R(3, w + 1) - 1$. There is a $3K_1$-free graph G', with $\omega(G') = w$ and $|V(G')| = R$. Since $\alpha(G') \leq 2$, we obtain $\chi(G') \geq R/\alpha(G') \geq R/2$. Thus, $\chi(G') \geq \lceil R/2 \rceil$, since $\chi(G')$ is an integer, which proves the lower bound.

Schiermeyer [60] proves that $\chi(G) \leq (|V(G)| + \omega(G) + 1 - \alpha(G))/2$ for each connected graph G. We shortly prove that this bound is also true for a disconnected graph G. Let $k \in \mathbb{N}_{>1}$ and $V_1, V_2, \ldots, V_k \subseteq V(G)$ be such that $G[V_i]$ induces a connected component of G, for $i \in [k]$, $\bigcup_{i\in[k]} V_i = V(G)$, and $\chi(G) = \chi(G[V_1])$. We shortly write G_1 instead of $G[V_1]$. Thus,

$$\begin{aligned}\chi(G) = \chi(G_1) &\leq \frac{V(G_1) + \omega(G_1) + 1 - \alpha(G_1)}{2}\\ &\leq \frac{|V(G_1)| + \omega(G) + 1 - \alpha(G_1) - \sum_{i=2}^{k} |V(G[V_i])| + \sum_{i=2}^{k} |V(G[V_i])|}{2}\\ &\leq \frac{|V(G)| + \omega(G) + 1 - \alpha(G)}{2},\end{aligned}$$

by the bound by Schiermeyer [60], $\omega(G_1) \leq \omega(G)$, and $\alpha(G_1)+\sum_{i=2}^{k} |V(G[V_i])| \geq \alpha(G)$.

Let G be an arbitrary $3K_1$-free graph. By the definition of the Ramsey number $R(3, \omega(G)+1)$, we know $|V(G)| \leq R(3, \omega(G)+1)-1$. If $\alpha(G) = 1$, then $\chi(G) = \omega(G)$. Otherwise, $\alpha(G) = 2$ and, thus,

$$\chi(G) \leq \frac{|V(G)| + \omega(G) + 1 - \alpha(G)}{2} \leq \frac{R(3, \omega(G) + 1) - 2 + \omega(G)}{2}.$$

Since $R(3, \omega(G) + 1) - 2 \geq \omega(G)$, for each $\omega(G) \in \mathbb{N}_{>0}$, we find $\chi(G) \leq (R(3, \omega(G) + 1) - 2 + \omega(G))/2$ in both cases, which completes the proof by the arbitrariness of G and since $\chi(G)$ is an integer. □

So the asymptotic growth of the function $f^{\star}_{3K_1}(\omega)$ is completely solved, but the optimal binding function is still widely open. For that reason for example Choudum et al. [16] study some subfamilies of For$(3K_1)$ and prove bounds. In the introductory section of their paper quite some subfamilies of For$(3K_1)$ and their χ-binding functions are stated. We also refer to an article by Pedersen [49] for a χ-binding function for the class of $(3K_1, K_1 \cup K_4)$-free graphs.

2.2.2 $2K_2$-free universe

In this section we talk about the known results regarding the function $f^{\star}_{\{2K_2\}}$. Let us first state the following useful structural result for $2K_2$-free graphs which Chung et al. [24] prove.

Lemma 28 (Chung et al. [24]). *If G is a connected $2K_2$-free graph with $\omega(G) \geq 3$, then there is a clique of size $\omega(G)$ that is dominating in G.*

The lower bound to $f^{\star}_{\{2K_2\}}$ is a result by Gyárfás [31] using a theorem proven by Chung [23] five years prior.

Theorem 29 (Gyárfás [31]). *There exists an $\epsilon > 0$ s.t. for each $\omega \in \mathbb{N}_{>0}$,*

$$\frac{\omega^{1+\epsilon}}{3} \leq f^{\star}_{\{2K_2\}}(\omega).$$

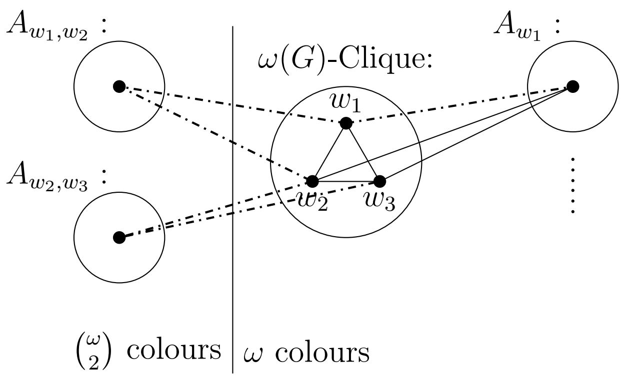


Fig. 5: Illustration to Wagon's proof

The following result from 1978 is asymptotically still the best known general upper bound for this family. For that reason we want to talk about its nice proof in a bit more detail.

Theorem 30 (Wagon [67])**.** *For* $\omega \in \mathbb{N}_{>0}$*,*

$$f^{\star}_{\{2K_2\}}(\omega) \leq \binom{\omega+1}{2}.$$

In Figure 5 the main idea of the proof is visualized. Let G be a $2K_2$-free graph. Starting with a clique W of size $\omega(G)$ labelled with $w_1, w_2, \ldots, w_{\omega(G)}$ in any $2K_2$-free graph one can partition the remaining vertices in the following sets. For $i \in [\omega(G)]$ the set A_{w_i} is defined as the vertices $x \in V(G) \setminus W$ with $N_G(x) \cap W = W \setminus \{w_i\}$. Note that the set $\{w_i\} \cup A_{w_i}$ is an independent set, for $i \in [\omega(G)]$, since otherwise there is a clique of size $\omega(G)+1$ in G which is a contradiction. For $i, j \in [\omega(G)]$ with $i \neq j$ the set A_{w_i,w_j} is defined as the set of vertices $x \in V(G)$ with $N_G(x) \cap W \subseteq W \setminus \{w_i, w_j\}$. The set A_{w_i,w_j} is also an independent set, for $i, j \in [\omega(G)]$ with $i \neq j$, otherwise G contains a $2K_2$ as an induced subgraph, again a contradiction. Let $M = \{(i,j) \in [\omega(G)] \times [\omega(G] \mid i < j\}$ and $A_2 = \bigcup_{(i,j)\in M} A_{w_i,w_j}$, then $V(G) = A_2 \cup W \cup \bigcup_{i\in[\omega(G)]} A_{w_i}$ and

$$\chi(A_2) = \chi(\bigcup_{(i,j)\in M} A_{w_i,w_j}) \leq \sum_{(i,j)\in M} \chi(A_{w_i,w_j}) \leq \sum_{(i,j)\in M} 1 = \binom{\omega(G)}{2}.$$

Since $\{w_i\} \cup A_{w_i}$ is an independent set, we find $\chi(G - A_2) \leq \omega(G)$, which proves Wagon's bound as follows:

$$\chi(G) \leq \chi(G[V(G) \setminus A_2]) + \chi(G[A_2]) \leq \binom{\omega(G)}{2} + \omega(G) = \binom{\omega(G)+1}{2}.$$

But for $\omega(G) = 3$ Wagon's bound is already not best possible. Erdős first conjectured in 1985, that $f^{\star}_{\{2K_2\}}(3) = 4$, where the Wagon bound is 6. This Conjecture was proven by Nagy and Szentmiklóssy but the proof was never officially published. So the first official paper proving that result is from 2018 by Gasper and Huang [29]. The tightness of the bound is achieved by the graph W_5, the wheel on 6 vertices.

Theorem 31 (Gasper and Huang [29])**.** $f^\star_{\{2K_2\}}(3) = 4$

Let us restate that in this thesis we for example prove in Chapter 4 that $f^\star_{\{P_5,hammer\}} = f^\star_{\{2K_2\}}$. Thus, every improvement to $f^\star_{\{2K_2\}}$ is an improvement to $f^\star_{\{P_5,hammer\}}$. For this reason we use the stated result by Gasper and Huang [29] to make an improvement on the general bound by Wagon.

Corollary 32 ([11])**.** *For* $\omega \in \mathbb{N}_{>0}$*,*

$$f^\star_{\{2K_2\}}(\omega) \leq \binom{\omega+1}{2} - 2\left\lfloor \frac{\omega}{3} \right\rfloor.$$

Proof. We prove this by induction on ω. For $\omega \leq 3$ this states $f^\star_{\{2K_2\}}(1) \leq \binom{1+1}{2} - 0 = 1$, $f^\star_{\{2K_2\}}(2) \leq \binom{2+1}{2} - 0 = 3$, and $f^\star_{\{2K_2\}}(3) \leq \binom{3+1}{2} - 2 = 4$, where the first two inequalities are true by Theorem 30 and the last inequality is true by Theorem 31. So we assume there is an $\omega_0 \in \mathbb{N}_{\geq 3}$ such that $f^\star_{\{2K_2\}}(\omega) \leq \binom{\omega+1}{2} - 2\left\lfloor \frac{\omega}{3} \right\rfloor$ for each $\omega \in [\omega_0]$.

So let G be a $2K_2$-free graph with $\omega(G) = \omega_0 + 1$. By the result by Chung et al. [24] there is a dominating clique W of size $\omega(G)$ in G. Fix $v_1, v_2, v_3 \in W$. Now we define the sets M and D as

$$\begin{aligned} M &:= \{v \in V(G) \setminus W \mid E_G[\{v\}, W \setminus \{v_1, v_2, v_3\}] \text{ is complete}\} \text{ and} \\ D &:= \{v \in V(G) \setminus W \mid E_G[\{v\}, \{v_1, v_2, v_3\}] \text{ is complete}\}. \end{aligned}$$

For each vertex v in $V(G) \setminus (W \cup D \cup M)$ there is a $i \in [3]$ and a $j \in [\omega(G)] \setminus [3]$ with $vv_i, vv_j \notin E(G)$. So we define $I = [3] \times [\omega(G)] \setminus [3]$ and for $(i,j) \in I$ we define

$$X_{(i,j)} := \{v \in V(G) \setminus (W \cup D \cup M) \mid vv_i, vv_j \notin E(G)\}.$$

Note that for $(i,j) \in I$ the set $X_{(i,j)}$ is an independent set, since G is $2K_2$-free. We obtain $V(G) = \{v_1, v_2, v_3\} \cup M \cup \{v_4, \ldots, v_{\omega(G)}\} \cup D \cup \bigcup_{(i,j)\in I} X_{i,j}$. Note that $\omega(G[\{v_1, v_2, v_3\} \cup M]) = 3$ and $\omega(G[\{v_4, \ldots, v_{\omega(G)}\} \cup D]) = \omega(G) - 3$, since the largest clique in G has size $\omega(G)$. Thus, by induction hypotheses, we get $\chi(G[\{v_1, v_2, v_3\} \cup M]) \leq 4$ and $\chi(G[\{v_4, \ldots, v_{\omega(G)}\} \cup D]) \leq \binom{\omega(G)-3+1}{2} - 2\left\lfloor \frac{\omega(G)-3}{3} \right\rfloor$.

Let for the following calculation $\omega = \omega(G)$:

$$\begin{aligned} \chi(G) &\leq \chi(G[\{v_1, v_2, v_3\} \cup M]) + \chi(G[\{v_4, \ldots, v_\omega\} \cup D]) + \chi(G[\bigcup_{(i,j)\in I} X_{i,j}]) \\ &\leq 4 + \binom{\omega - 3 + 1}{2} - 2\left\lfloor \frac{\omega - 3}{3} \right\rfloor + \sum_{(i,j)\in I} 1 \end{aligned}$$

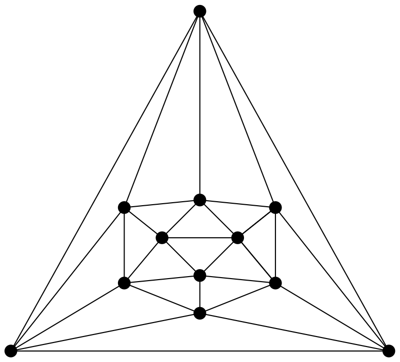


Fig. 6: Icosahedron I

$$
\begin{aligned}
&= 4 + \binom{\omega-2}{2} - 2(\left\lfloor \frac{\omega}{3} \right\rfloor - 1) + 3(\omega - 3) \\
&= \binom{\omega-2}{2} + 3\omega - 3 - 2\left\lfloor \frac{\omega}{3} \right\rfloor = \frac{(\omega-2)(\omega-3)+6\omega-6}{2} - 2\left\lfloor \frac{\omega}{3} \right\rfloor \\
&= \frac{\omega^2 - 5\omega + 6 + 6\omega - 6}{2} - 2\left\lfloor \frac{\omega}{3} \right\rfloor = \binom{\omega+1}{2} - 2\left\lfloor \frac{\omega}{3} \right\rfloor .
\end{aligned}
$$

So every finite $2K_2$-free graph G is by induction $(\binom{\omega(G)+1}{2} - 2\left\lfloor \frac{\omega(G)}{3} \right\rfloor)$-colourable. □

In a recent paper by Chudnovsky et al. [18] they study the class of $(fork, C_4)$-free graphs. A valid question is why this family is discussed here. This family is relevant for our research since the complementary graph of such a graph is $(2K_2, kite)$-free and so this seems like a fitting place. Their main work in their paper can be divided into three big parts. They first prove a structure theorem for $(fork, C_4)$-free graphs. From this theorem, which we use in Chapter 8, they deduce the following corollary.

Corollary 33 (Chudnovsky et al. [18])**.** *Let G be a connected $(fork, C_4)$-free graph. Then G is $K_{1,3}$-free or G has a universal vertex or G has a clique separator.*

This corollary is used to show that to $\left\lceil \frac{3\omega(G)}{2} \right\rceil$-colour a $(fork, C_4)$-free graph G it is sufficient to $\left\lceil \frac{3\omega(G')}{2} \right\rceil$-colour every $(K_{1,3}, C_4)$-free graph G'. In the second structure theorem they characterise the structure of said graphs. Relevant graphs for this characterisation are the icosahedron (cf. Figure 6) and the so called crown. Note that the icosahedron I is completely triangulated and therefore C_4-free and $K_{1,3}$-free, since for every $v \in V(I)$ we have $G[N_I[v]] \cong W_5$. In their last section they colour the relevant graphs. This bound is not known to be optimal but again by using the clique-expansion of the icosahedron I they show that for $\omega \in 3\mathbb{N}_{>0}$:

$$\frac{4\omega}{3} \leq f^{\star}_{\{fork, C_4\}}(\omega) \leq \left\lceil \frac{3\omega}{2} \right\rceil .$$

Note that quite a few subfamilies of $2K_2$-free graphs have been studied. For the interested reader we refer to the previously stated paper by Brause et al. [14], the paper by Karthick and Mishra [40] and Prashant and Gokulnath [52].

3 Techniques

To achieve our aim of determining χ-binding functions different techniques are used. In this Chapter we collect the techniques which are used multiple times in proofs of this thesis. In Section 3.1 we start with techniques which are generally applicable. Where the techniques of Section 3.2 are applicable for the large family of $Q[P_4]$-free graphs. In the following Section 3.3 we talk about some results for χ-binding functions. Finally the last quite technical Section 3.4 is later used to colour certain graphs which contain an induced, weighted C_5.

3.1 General techniques

In chromatic graph theory, the private neighbourhood reduction is an important tool. There is a similar reduction technique for vertex-weight functions of graphs, which is implicitly defined in the next lemma. Note that for the unweighted version, Lemma 34 describes the private neighbourhood reduction. In particular, for $q\colon V(G) \to [1]$, we have $\chi(G) = \chi(G - u_1)$ if there are two non-adjacent vertices $u_1, u_2 \in V(G)$ with $N_G(u_1) \subseteq N_G(u_2)$. Thus, the following lemma implies that a critical graph does not contain a comparable vertex pair.

Lemma 34 ([12])**.** *If $q\colon V(G) \to \mathbb{N}_0$ is a $\lhd_\chi^G$-minimal vertex-weight function and $S \subseteq V(G)$, $u \in V(G) \setminus S$ with $E_G[\{u\}, S]$ is anticomplete, and $q(u) > 0$ and $N_G(u) \subseteq N_G(s)$ for each $s \in S$, then $q(u) > \chi_q(G[S])$.*

Proof. For the sake of a contradiction, let us suppose $q(u) \leq \chi_q(G[S])$. Additionally, let $q'\colon V(G) \to \mathbb{N}_0$ be a vertex-weight function with

$$v \mapsto \begin{cases} 0 & \text{if } v = u, \\ q(v) & \text{if } v \neq u. \end{cases}$$

Note that $q' \not\equiv 0$ and for a proper q'-colouring $L_{q'}\colon V(G) \to 2^{[\chi_{q'}(G)]}$ of G, one can find a set L_u such that $L_u \subseteq L_{q'}(S)$ and $|L_u| = q(u) \leq \chi_q(G[S]) \leq |L_{q'}(S)|$. Hence, from the proper q-colouring $L_q\colon V(G) \to 2^{\mathbb{N}_{>0}}$ with

$$v \mapsto \begin{cases} L_u & \text{if } v = u, \\ L_{q'}(v) & \text{if } v \neq u, \end{cases}$$

it follows $\chi_q(G) \leq \chi_{q'}(G)$. Thus, $\chi_q(G) = \chi_{q'}(G)$, which contradicts our assumption that q is $\lhd_\chi^G$-minimal. Hence, $q(u) > \chi_q(G[S])$. □

Since we often create weighted graphs the following lemma is used multiple times. It is a central result in Lovász' [67] proof of the Weak Perfect Graph Theorem.

Lemma 35 (Lovász [46])**.** *If G is a perfect graph, then each 'perfect'-expansion of G is perfect.*

We continue by an observation concerning the chromatic and clique numbers of q-expansions of a graph.

Observation 36 ([13])**.** *If G is a graph, $q\colon V(G) \to \mathbb{N}_0$ is a vertex-weight function, and G' is a q-expansion of G, then*

$$\chi(G') = \chi_q(G) \quad \textit{and} \quad \omega(G') = \omega_q(G).$$

Note that Observation 36 together with Lemma 35 implies $\chi_q(G) = \omega_q(G)$ for each perfect graph G and each vertex-weight function $q\colon V(G) \to \mathbb{N}_0$.

We concentrate next on our combination of homogeneous sets and clique-separators, namely the so-called clique-separators of modules. Note that each clique-separator is a clique-separator of modules. Having this observation in mind, the following lemma generalises the fact that critical graphs do not contain clique-separators since it implies that $G[q]$, for some $\lhd_\chi^G$-minimal vertex-weight function $q\colon V(G) \to \mathbb{N}_0$, does not contain a clique-separator of modules.

Lemma 37 ([13])**.** *If G, G_1, G_2 are three graphs with $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2)$ is a clique-separator of modules in G, and $q\colon V(G) \to \mathbb{N}_0$ is a vertex-weight function, then*

$$\chi_q(G) = \max\{\chi_q(G_1), \chi_q(G_2)\} \quad \textit{and} \quad \omega_q(G) = \max\{\omega_q(G_1), \omega_q(G_2)\}.$$

Proof. Let $k \in \mathbb{N}_{>0}$ and $X, X_1, X_2 \dots, X_k \subseteq V(G)$ be sets such that $X = X_1 \cup X_2 \cup \dots \cup X_k = V(G_1) \cap V(G_2)$ and $X_1, X_2, \dots, X_k$ are the modules of X. Furthermore, for each $n \in [2]$, let $L_n\colon V(G_n) \to 2^{[\chi_q(G_n)]}$ be a q-colouring which minimises $|L_n(X)|$. Since $E_G[X_i, X_j]$ is complete for each distinct $i, j \in [k]$ with $i < j$, by renaming colours if necessary, we may assume $L_n(X_1) = [\chi_q(G_n[X_1])]$ and

$$L_n(X_j) = [\chi_q(G_n[X_1 \cup X_2 \cup \dots \cup X_j])] \setminus [\chi_q(G_n[X_1 \cup X_2 \cup \dots \cup X_{j-1}])]$$

for each $j \in [k] \setminus \{1\}$, that is, L_n colours the vertices of X_1 with subsets of $[\chi_q(G_n[X_1])]$, the vertices of X_2 with sets that contain only colours of $\{\chi_q(G_n[X_1])+1, \chi_q(G_n[X_1])+2, \ldots, \chi_q(G_n[X_1 \cup X_2])\}$, ..., and the vertices of X_k with sets that contain only colours of $\{\chi_q(G_n[X_1 \cup X_2 \cup \ldots \cup X_{k-1}])+1, \chi_q(G_n[X_1 \cup X_2 \cup \ldots \cup X_{k-1}])+2, \ldots, \chi_q(G_n(X))\}$.

We show next that we may assume that the two proper q-colourings L_1 and L_2 coincide on X_j with a proper q-colouring of $G[X_j]$ for each $j \in [k]$. If $L_{X_j} \colon X_j \to 2^{L_n(X_j)}$ is a proper q-colouring of $G[X_j]$, then $L'_n \colon V(G_n) \to 2^{[\chi_q(G_n)]}$ with

$$v \mapsto \begin{cases} L_n(v) & \text{if } v \in V(G_n) \setminus X_j, \\ L_{X_j}(v) & \text{if } v \in X_j \end{cases}$$

is a proper q-colouring of G_n since X_j a module. Thus, by our choice of L_n, L_n uses $\chi_q(G[X_j])$ colours for the vertices of X_j, and so,

$$L_n(X) = \bigcup_{j=1}^{k} L(X_j) = [\chi_q(G[X_1]) + \chi_q(G[X_2]) + \ldots + \chi_q(G[X_k])] = [\chi_q(G[X])].$$

Hence, we may assume $L_1(v) = L_2(v)$ for each $v \in X$. Thus,

$$\chi_q(G) \leq \max\{\chi_q(G_1), \chi_q(G_2)\} \leq \chi_q(G),$$

since G_1 and G_2 are induced subgraphs of G. Finally, $\omega_q(G) = \max\{\omega_q(G_1), \omega_q(G_2)\}$ since $E_G[V(G_1) \setminus X, V(G_2) \setminus X]$ is anticomplete, which completes our proof. □

3.2 Techniques for $Q[P_4]$-free graphs

Note that $Q[P_4]$ contains an induced *banner*, *dart*, *gem* and *kite*. We wish to establish some results for $Q[P_4]$-free graphs but begin by considering modules of $Q[F]$-free graphs, where F is arbitrary and not necessarily related to P_4.

Lemma 38 ([13])**.** *If F is a graph and G is a $Q[F]$-free graph, then, for each module M in G, $G[M]$ is F-free or $N_G(M)$ is a clique-separator of modules or $N_G^2(M) = \emptyset$.*

Proof. If $M = V(G)$, then $N_G^2(M) = \emptyset$, and so let us assume that M is a module in G such that $|M| < |V(G)|$, and $S \subseteq M$ with $G[S] \cong F$, and $N_G^2(M) \neq \emptyset$. We continue by showing that $N_G(M)$ is a clique-separator of modules. Let $X_1, X_2, \ldots, X_\ell$ be the sets of vertices which induce the components of $\bar{G}[N_G(M)]$. Since $E_G[M \cup X_i, X_j]$ is complete for each distinct $i, j \in [\ell]$ and $N_G^2(M) \neq \emptyset$, we may suppose, for the sake of a contradiction, that there is some $k \in [\ell]$ and a vertex $w \in N_G^2(M)$ for which $X_k \cap N_G(w) \neq \emptyset$ and $X_k \setminus N_G(w) \neq \emptyset$. Hence, by the connectivity of $\bar{G}[X_k]$, we may assume that $x_1 \in X_k \cap N_G(w)$ and $x_2 \in X_k \setminus N_G(w)$ are non-adjacent. Thus,

$S \cup \{w, x_1, x_2\}$ induces a $Q[F]$, which contradicts our assumption that G is $Q[F]$-free. Thus, X_k is a module, and $N_G(M)$ is a clique-separator of modules, which completes our proof. □

Let us focus on $Q[P_4]$-free graphs next. It is rather interesting that every vertex-weight function for $Q[P_4]$-free graphs can be nicely decomposed.

Lemma 39 ([13])**.** *Let G be a $Q[P_4]$-free graph. If $q\colon V(G) \to \mathbb{N}_0$ is a vertex-weight function with $q \not\equiv 0$, then there exist an integer $k \in \mathbb{N}_{>0}$, k pairwise disjoint non-empty sets $M_1, M_2, \ldots, M_k \subseteq V(G[q])$, and k $\lhd_\chi^G$-minimal vertex-weight functions $q_1, q_2, \ldots, q_k\colon V(G) \to \mathbb{N}_0$ such that $V(G[q_i]) \subseteq M_i$, $\chi_q(G[M_i]) = \chi_{q_i}(G)$, $\omega_q(G[M_i]) \geq \omega_{q_i}(G)$, and $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of $G[q_i]$ which is a prime graph without clique-separators of modules for each $i \in [k]$, $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$, and*

$$\chi_q(G) = \sum_{i=1}^{k} \chi_q(G[M_i]).$$

Furthermore, $\omega_q(G[M_i]) = \omega_{q_i}(G)$ for each $i \in [k]$ if q is $\lhd_\chi^G$-minimal.

Proof. For simplicity, if (G, q) is a pair for which G is a $Q[P_4]$-free graph and $q\colon V(G) \to \mathbb{N}_0$ is a vertex-weight function with $q \not\equiv 0$, and both satisfy the statement of the lemma, then we say that (G, q) is *decomposable*. For the sake of a contradiction, let us suppose that (G, q) is a minimal counterexample to our lemma, that is, $q \not\equiv 0$ and (G, q) is not decomposable but each pair (G', q') with either G' is an induced subgraph of G with $G' \neq G$ and $q' \not\equiv 0$, or $G' = G$ and $|V(G[q'])| < |V(G[q])|$ and $q' \not\equiv 0$, or $G' = G$ and $|V(G[q'])| = |V(G[q])|$ and $q' \lhd_\chi^G q$ is decomposable.

If there is a vertex $u \in V(G)$ with $q(u) = 0$, then, since (G, q) is a minimal counterexample and $G[q]$ is an induced subgraph of G with $G[q] \neq G$, we have that $(G[q], q)$ is decomposable, which also implies that (G, q) is decomposable. The latter contradiction on our supposition on (G, q) implies $G = G[q]$.

We show next that q is $\lhd_\chi^G$-minimal by supposing, for the sake of a contradiction, the contrary. Since $q \not\equiv 0$, there is a $q'\colon V(G) \to \mathbb{N}_0$ which is $\lhd_\chi^G$-minimal with $q' \lhd_\chi^G q$. Then (G, q') is decomposable into pairwise disjoint non-empty sets $M'_1, M'_2, \ldots, M'_k$ and vertex-weight functions $q'_1, q'_2, \ldots, q'_k\colon V(G[q']) \to \mathbb{N}_0$ since q is a minimal counterexample. Clearly, $V(G[q']) \subseteq V(G[q])$. Since $q' \lhd_\chi^G q$, we have $\chi_{q'}(G) = \chi_q(G)$. Additionally, $\chi_q(G[M'_i]) \geq \chi_{q'}(G[M'_i])$ and $\omega_q(G[M'_i]) \geq \omega_{q'}(G[M'_i])$ for each $i \in [k]$. In view of the desired result, it remains to prove $\chi_q(G[M'_i]) \leq \chi_{q'}(G[M'_i])$ for each $i \in [k]$. Since $E_G[M'_i, M'_j]$ is complete for each distinct $i, j \in [k]$, we have

$$\chi_q(G[M'_i]) + \sum_{j \in [k] \setminus \{i\}} \chi_q(G[M'_j]) \leq \chi_q(G) = \chi_{q'}(G)$$

$$= \sum_{i=1}^{k} \chi_{q'}(G[M'_i]) \leq \chi_{q'}(G[M'_i]) + \sum_{j\in[k]\setminus\{i\}} \chi_q(G[M'_j]),$$

and so $\chi_q(G[M'_i]) = \chi_{q'}(G[M'_i])$ for each $i \in [k]$. Thus, (G, q) is decomposable into the modules $M'_1, M'_2, \ldots, M'_k$ and the vertex-weight functions $q'_1, q'_2, \ldots, q'_k : V(G[q]) \to \mathbb{N}_0$, which contradicts our supposition on (G, q). Therefore, we have that q is $\lhd^G_\chi$-minimal, and so $G = G[q]$ is connected.

Let M_1 be an inclusion-wise minimal module in G for which $N^2_G(M_1) = \emptyset$. Note that possibly $M_1 = V(G)$. Since q is $\lhd^G_\chi$-minimal, $G[M_1]$ is connected. Let M be a module in $G[M_1]$ with $N^2_{G[M_1]}(M) = \emptyset$. Hence, $N_G(M) = (M_1 \setminus M) \cup (V(G) \setminus M_1) = V(G) \setminus M$, which implies $M = M_1$ by the minimality of $|M|$.

We may assume first that $M_1 \neq V(G)$. Thus, by the definition of M_1, $E_G[M_1, V(G) \setminus M_1]$ is complete. For $S \in \{M_1, V(G) \setminus M_1\}$, let $q^S : S \to \mathbb{N}_0$ be defined by

$$u \mapsto \begin{cases} q(u) & \text{if } u \in S, \\ 0 & \text{if } u \notin S. \end{cases}$$

Note that $q^{M_1}, q^{V(G)\setminus M_1} \not\equiv 0$, since $M_1, V(G) \setminus M_1 \neq \emptyset$, $\chi_q(G[M_1]) = \chi_{q^{M_1}}(G)$ and $\chi_q(G - M_1) = \chi_{q^{V(G)\setminus M_1}}(G)$, and so

$$\chi_q(G) = \chi_{q^{M_1}}(G) + \chi_{q^{V(G)\setminus M_1}}(G).$$

Thus, since q is $\lhd^G_\chi$-minimal, q^{M_1} and $q^{V(G)\setminus M_1}$ are $\lhd^G_\chi$-minimal. Hence, since we know that $|V(G[q^{M_1}])|, |V(G[q^{V(G)\setminus M_1}])| < |V(G[q])|$ and (G, q) is a minimal counterexample, we have that $(G[M_1], q^{M_1})$ and $(G[V(G) \setminus M_1], q^{V(G)\setminus M_1})$ are decomposable into pairwise disjoint non-empty sets $M'_1, M'_2, \ldots, M'_{k_1}$ and $M'_{k_1+1}, M'_{k_1+2}, \ldots, M'_{k_1+k_2}$ as well as $\lhd^G_\chi$-minimal vertex-weight functions $q'_1, q'_2, \ldots, q'_{k_1} : V(G) \to \mathbb{N}_0$ and $q'_{k_1+1}, q'_{k_1+2}, \ldots, q'_{k_1+k_2} : V(G) \to \mathbb{N}_0$, respectively. Hence, the function q is decomposable into the modules $M'_1, M'_2, \ldots, M'_{k_1+k_2}$ and the vertex weight functions $q'_1, q'_2, \ldots, q'_{k_1+k_2} : V(G) \to \mathbb{N}_0$. Additionally, since q^S is $\lhd^G_\chi$-minimal, we have $\omega_q(G[M'_i]) = \omega_{q^S}(G[M'_i]) = \omega_{q'_i}(G)$ for each $i \in [k_1 + k_2]$ and, depending on i, some $S \in \{M_1, V(G) \setminus M_1\}$.

It remains to assume $M_1 = V(G)$. Recall that q is $\lhd^G_\chi$-minimal, and so G has no clique-separator of modules by Lemma 37. Furthermore, G is connected. If G is also prime, then we see that (G, q) is decomposable by choosing $k = 1$ and $q_1 \equiv q$; a contradiction. Thus, there is a homogeneous set in G. Let us recall that for every homogeneous set H in $G = G[M_1]$, by the choice of M_1, $N^2_G(H) \neq \emptyset$. By Lemma 38, we see that $G[H]$ is P_4-free, for every homogeneous set H. Let M_2, M_3 be two homogeneous sets in G with $M_2 \cap M_3 \neq \emptyset$. For the sake of a contradiction, let us suppose that $M_2 \cup M_3$ is not a homogeneous set in G. Hence, $M_2 \setminus M_3, M_3 \setminus M_2 \neq \emptyset$, and we let $m_2 \in M_2 \setminus M_3$, $m_3 \in M_3 \setminus M_2$, and $m_4 \in M_2 \cap M_3$ be arbitrary vertices. Since M_2 and M_3 are modules, we have

$$N_G(m_2) \setminus (M_2 \cup M_3) = N_G(m_4) \setminus (M_2 \cup M_3) = N_G(m_3) \setminus (M_2 \cup M_3),$$

and so $V(G) = M_2 \cup M_3$ since $M_2 \cup M_3$ is not a homogeneous set in G. Clearly, $M_2 \cap M_3$ is a module in G. Since G has no clique-separators of modules, $M_2 \cap M_3$ is not a clique-separator of modules, and so a vertex of $M_2 \setminus M_3$ is adjacent to a vertex of $M_3 \setminus M_2$. Hence, by the fact that M_2 and M_3 are modules, we have that each vertex of $M_2 \cap M_3$ is adjacent to each vertex of $V(G) \setminus (M_2 \cap M_3)$, and so $N_G^2(M_2 \cap M_3) = \emptyset$, which contradicts the choice of M_1. Hence, there is some integer $k \in \mathbb{N}_{>0}$ and k pairwise disjoint homogeneous sets $M'_1, M'_2, \ldots, M'_k$ of G with $M \subseteq M'_i$ for each homogeneous set M in G and, depending on M, some $i \in [k]$. Recall that $G[M'_i]$ is P_4-free for each $i \in [k]$. The Strong Perfect Graph Theorem implies that $G[M'_i]$ is perfect, and so $\chi_q(G[M'_i]) = \omega_q(G[M'_i])$ by Lemma 35 and Observation 36 for each $i \in [k]$. Since q is $\lhd_\chi^G$-minimal, we obtain that M'_i is a clique, and we let u'_i be a vertex of M'_i for each $i \in [k]$. Hence, let $q_1 \colon V(G) \to \mathbb{N}_0$ be a vertex-weight function with

$$u \mapsto \begin{cases} q(M'_i) & \text{if } u = u'_i \text{ for some } i \in [k], \\ 0 & \text{if } u \in M'_i \setminus \{u'_i\} \text{ for some } i \in [k], \\ q(u) & \text{if } u \notin \bigcup_{i=1}^{k} M'_i. \end{cases}$$

Clearly, $G[M_1]$ is a 'non-empty, $2K_1$-free'-expansion of $G[q_1]$. It is further easily seen $\chi_q(G) = \chi_q(G[M_1]) = \chi_{q_1}(G)$, $\omega_q(G) = \omega_q(G[M_1]) = \omega_{q_1}(G)$, and that q_1 is $\lhd_\chi^G$-minimal. Since $G - ((M'_1 \cup M'_2 \cup \ldots \cup M'_k) \setminus \{u'_1, u'_2, \ldots, u'_k\})$ is prime, $G[q_1]$ is prime as well. For the sake of a contradiction, let us suppose that X is a clique-separator of modules in $G[q_1]$. Since $G[q_1]$ is prime, every module of X is of size 1. Let

$$X(x) = \begin{cases} \{x\} & \text{if } x \notin \{u'_1, u'_2, \ldots, u'_k\}, \\ M'_i & \text{if } x = u_i \text{ for some } i \in [k]. \end{cases}$$

Since $M'_1, M'_2, \ldots, M'_k$ are pairwise disjoint modules which are cliques and for which $u'_i \in M'_i$ for each $i \in [k]$, $\bigcup_{x \in X} X(x)$ is a clique-separator of modules in G, which is a contradiction to the fact that, by Lemma 37, such a set cannot exist. Hence, (G, q) is decomposable into the module $V(G)$ and the vertex-weight function q_1, and our proof is complete. □

We first note that Lemma 39 evokes a nice characterisation of critical $Q[P_4]$-free graphs.

Corollary 40 ([13])**.** *If G is a critical $Q[P_4]$-free graph, then there is some integer $k \in \mathbb{N}_{>0}$ such that $V(G)$ can be partitioned into sets $M_1, M_2, \ldots, M_k$ such that $E_G[M_i, M_j]$ is complete for distinct $i, j \in [k]$, and $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of a prime graph without clique-separator of modules for each $i \in [k]$.*

Proof. Note that the vertex-weight function $q\colon V(G) \to [1]$ is $\lhd_\chi^G$-minimal since G is critical. By Lemma 39, there exist an integer $k \in \mathbb{N}_{>0}$, k pairwise disjoint non-empty sets $M_1, M_2, \ldots, M_k \subseteq V(G)$, and k $\lhd_\chi^G$-minimal vertex-weight functions $q_1, q_2, \ldots,$ $q_k\colon V(G) \to \mathbb{N}_0$ such that $V(G[q_i]) \subseteq M_i$ and $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of $G[q_i]$ which is a prime graph without clique-separators of modules for each $i \in [k]$, $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$, and

$$\chi(G) = \sum_{i=1}^{k} \chi(G[M_i]).$$

Since G is critical, we conclude from the latter equality that $M_1, M_2, \ldots, M_k$ is indeed a partition of $V(G)$, which completes the proof. □

Corollary 40 is important for the proof of Theorem 9. However, by Lemma 39, we are now in a position to formulate our central lemma which reasons to study proper q-colourings of prime graphs without clique-separators of modules whenever we are interested in χ-binding functions for subclasses of $Q[P_4]$-free graphs.

Lemma 41 ([13])**.** *Let G be a $Q[P_4]$-free graph, $q\colon V(G) \to \mathbb{N}_0$ be a vertex-weight function, and $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ be a superadditive function. If $\chi_{q'}(G) \leq f(\omega_{q'}(G))$ for each $\lhd_\chi^G$-minimal vertex-weight function $q'\colon V(G) \to \mathbb{N}_0$ for which $G[q']$ is prime and has no clique-separator of modules, then*

$$\chi_q(G) \leq f(\omega_q(G)).$$

Proof. If $q \equiv 0$, then $\chi_q(G) = 0 = f(0) = f(\omega_q(G))$, since f is superadditive. Thus, we may assume $q \not\equiv 0$. By Lemma 39, there is an integer $k \in \mathbb{N}_{>0}$ and there are k $\lhd_\chi^G$-minimal vertex-weight functions $q_1, q_2, \ldots, q_k\colon V(G) \to \mathbb{N}_0$ such that

$$\chi_q(G) = \sum_{i=1}^{k} \chi_{q_i}(G) \quad \text{and} \quad \omega_q(G) \geq \sum_{i=1}^{k} \omega_q(G[M_i]) \geq \sum_{i=1}^{k} \omega_{q_i}(G).$$

Furthermore, $G[q_i]$ is a prime graph without clique-separators of modules, and so $\chi_{q_i}(G) \leq f(\omega_{q_i}(G))$ for each $i \in [k]$. The superadditivity of f implies

$$\chi_q(G) = \sum_{i=1}^{k} \chi_{q_i}(G) \leq \sum_{i=1}^{k} f(\omega_{q_i}(G)) \leq f\left(\sum_{i=1}^{k} \omega_{q_i}(G)\right) \leq f(w_q(G)),$$

which completes our proof. □

3.3 Binding functions

In this section we establish four lemmas concerning the general structure of χ-binding functions for certain graph classes.

Lemma 42 ([13]). *If $\mathcal{H}$ is a set of graphs and h is an integer such that each $H \in \mathcal{H}$ satisfies that its complementary graph $\bar{H}$ contains an induced cycle of length at most h, then the class of $\mathcal{H}$-free graphs has no linear χ-binding function.*

Proof. We may assume that $f^{\star}_{\mathcal{H}}$ exists. Thus, $\mathcal{H} \neq \emptyset$ and we get $h \geq 3$, since the precondition is fulfilled. By a result of Bollobás [7], for each two integers $g, \Delta \geq 3$, there is a $(C_3, C_4, \ldots, C_g)$-free graph $G_{g,\Delta}$ with $\Delta(G_{g,\Delta}) = \Delta$ and

$$\frac{\alpha(G_{g,\Delta})}{|V(G_{g,\Delta})|} < \frac{2\log(\Delta)}{\Delta}.$$

Hence, there is a series $\{\bar{G}_{h,i}\}_{i=3}^{\infty}$ such that, for each $i \geq 3$, $\bar{G}_{h,i}$ is a graph whose complementary graph is $G_{h,i}$. We show next that $\bar{G}_{h,i}$ is $\mathcal{H}$-free, for $i \geq 3$. Suppose not, then there is a $H \in \mathcal{H}$ with $H \subseteq_{\text{ind}} \bar{G}_{h,i}$. By the definition of $\mathcal{H}$, there is a $k \in [h] \setminus [2]$ with $C_k \subseteq_{\text{ind}} \bar{H}$. Thus, $C_k \subseteq_{\text{ind}} \bar{H} \subseteq_{\text{ind}} G_{h,i}$, which is a contradiction to the fact that the graph $G_{h,i}$ is $(C_3, C_4, \ldots, C_h)$-free by definition. Since $G_{h,i}$ is C_3-free, it follows $\alpha(\bar{G}_{h,i}) = \omega(G_{h,i}) \leq 2$. Furthermore,

$$\omega(\bar{G}_{h,i}) < \frac{2\log(i)}{i} \cdot |V(\bar{G}_{h,i})|,$$

and so

$$\frac{i}{4 \cdot \log(i)} \cdot \omega(\bar{G}_{h,i}) < \frac{|V(\bar{G}_{h,i})|}{2} \leq \frac{|V(\bar{G}_{h,i})|}{\alpha(\bar{G}_{h,i})} \leq \chi(\bar{G}_{h,i}).$$

Note that $i/(4 \cdot \log(i))$ tends to $+\infty$ as i tends to $+\infty$. Thus, there is no linear χ-binding function for the class of $\mathcal{H}$-free graphs. □

Lemma 41 has obviously huge impact on studying χ-binding function. However, in view of its application, we need that some optimal χ-binding functions, $f^{\star}_{\{3K_1\}}, f^{\star}_{\{C_5,3K_1\}}$, and $f^{\star}_{\{2K_2\}}$ in particular, are superadditive.

Lemma 43 ([13]). *If $\mathcal{H}$ is a set of graphs such that each $H \in \mathcal{H}$ does not contain a complete bipartite spanning subgraph, then $f^{\star}_{\mathcal{H}}$ is superadditive or the class of $\mathcal{H}$-free graphs has no χ-binding function.*

Proof. We may assume that $f^{\star}_{\mathcal{H}}$ exists. Note that $f^{\star}_{\mathcal{H}}(1) = 1 \neq 0$, since $K_1 \notin \mathcal{H}$. Let $w_1, w_2 \geq 1$ be two integers, G_1 be a $\mathcal{H}$-free graph with $\omega(G_1) = w_1$ and $\chi(G_1) = f^{\star}_{\mathcal{H}}(w_1)$, and G_2 be a $\mathcal{H}$-free graph with $\omega(G_2) = w_2$ and $\chi(G_2) = f^{\star}_{\mathcal{H}}(w_2)$ that is vertex disjoint from G_1. Note that G_1 and G_2 exist since $K_{w_i} \in \text{For}(\mathcal{H})$ for $i \in [2]$ by the definition of $\mathcal{H}$.

Let G be the graph obtained from G_1 and G_2 by adding all edges between the vertices of G_1 and the vertices of G_2. We prove first that G is $\mathcal{H}$-free. For the sake of a contradiction, let us suppose that there is some $H \in \mathcal{H}$ for which G contains a set S

of vertices inducing H. Since both G_1 and G_2 are H-free, $s_1 = |S \cap V(G_1)| > 0$ and $s_2 = |S \cap V(G_2)| > 0$. Therefore, the graph $G[S]$ has a spanning subgraph that is a isomorphic to K_{s_1,s_2}. But now $G[S] \cong H$ gives a contradiction to our assumption that H does not have a spanning subgraph which is a complete bipartite graph. Hence, G is $\mathcal{H}$-free.

Clearly, $\omega(G) = w_1 + w_2$ and $\chi(G) = \chi(G_1) + \chi(G_2) = f^\star_{\mathcal{H}}(w_1) + f^\star_{\mathcal{H}}(w_2)$, and so

$$f^\star_{\mathcal{H}}(w_1 + w_2) \geq \chi(G) = f^\star_{\mathcal{H}}(w_1) + f^\star_{\mathcal{H}}(w_2),$$

which completes our proof. □

Weakening the precondition of the previous lemma the following lemma grants no longer a superadditive χ-binding function but a sufficient condition for a graph family $\mathcal{H}$ such that $f^\star_{\mathcal{H}}$ is at least strictly increasing.

Lemma 44. *If $\mathcal{H}$ is a set of graphs such that each $H \in \mathcal{H}$ does not contain a universal vertex, then $f^\star_{\mathcal{H}}$ is strictly increasing or the class of $\mathcal{H}$-free graphs has no χ-binding function.*

Proof. We may assume that $f^\star_{\mathcal{H}}$ exists. We claim that $f^\star_{\mathcal{H}}(k) < f^\star_{\mathcal{H}}(k+1)$, for every $k \in \mathbb{N}_{>0}$. This claim we prove by induction on k as follows. Clearly $f^\star_{\mathcal{H}}(1) = 1 < 2 \leq f^\star_{\mathcal{H}}(2)$, since $K_1, K_2 \in \text{For}(\mathcal{H})$. So let $k \in \mathbb{N}_{>1}$ such that $f^\star_{\mathcal{H}}(k') < f^\star_{\mathcal{H}}(k'+1)$ for all $k' \in \mathbb{N}_{>0}$ with $k' < k$. Since $f^\star_{\mathcal{H}}(k) \neq 0$, there is a $\mathcal{H}$-free graph G with $\chi(G) = f^\star_{\mathcal{H}}(k)$ and $\omega(G) = k$. We define the graph G' as $G' \coloneqq G + \{v_1\}$ for $v_1 \notin V(G)$. For the sake of a contradiction, let us suppose that there is some $H \in \mathcal{H}$ for which G' contains a set S of vertices inducing H. Since $H \in \mathcal{H}$, the graph H does not contain a universal vertex so $v_1 \notin S$. Thus, $H \cong G'[S] = G[S]$ which is a contradiction to the fact, that G is $\mathcal{H}$-free. Thus, the graph G' is an $\mathcal{H}$-free graph with $\omega(G') = k+1$. Therefore,

$$f^\star_{\mathcal{H}}(k) < \chi(G) + 1 = \chi(G') \leq f^\star_{\mathcal{H}}(k+1),$$

which completes our proof. □

For example $f^\star_{P_5,kite}$ is strictly increasing according to Chapter 8 even though *dart* contains a universal vertex. Thus, the reverse of Lemma 44 is not true. On the other hand, we introduce in the following lemma another sufficient condition for a graph family $\mathcal{H}$ such that the optimal χ-binding function $f^\star_{\mathcal{H}}$ is non-decreasing.

Lemma 45. *If $\mathcal{H}$ is a set of graphs such that for all $H \in \mathcal{H}$ every connected component of H is non-isomorphic to a complete graph, then $f^\star_{\mathcal{H}}$ is non-decreasing or the class of $\mathcal{H}$-free graphs has no χ-binding function.*

Proof. We may assume that $f^\star_{\mathcal{H}}$ exists. We claim that $f^\star_{\mathcal{H}}(k) \leq f^\star_{\mathcal{H}}(k+1)$, for every $k \in \mathbb{N}_{>0}$. This claim we prove by induction on k as follows. Clearly $f^\star_{\mathcal{H}}(1) = 1 \leq 2 \leq f^\star_{\mathcal{H}}(2)$, since $K_1, K_2 \in \mathrm{For}(\mathcal{H})$. So let $k \in \mathbb{N}_{>1}$ such that $f^\star_{\mathcal{H}}(k') \leq f^\star_{\mathcal{H}}(k'+1)$ for all $k' \in \mathbb{N}_{>0}$ with $k' < k$. Since $f^\star_{\mathcal{H}}(k) \neq 0$, there is a $\mathcal{H}$-free graph G with $\chi(G) = f^\star_{\mathcal{H}}(k)$ and $\omega(G) = k$. We define the graph G' as $G' \coloneqq G \cup K_{k+1}$. For the sake of a contradiction, let us suppose that there is some $H \in \mathcal{H}$ for which G' contains a set S of vertices inducing H. Since $H \in \mathcal{H}$, the graph H does not contain a connected component which is isomorphic to a complete graph. Therefore, $S \cap V(G) = S \cap V(G')$. Thus, $H \cong G'[S] = G[S]$ which is a contradiction to the fact, that G is $\mathcal{H}$-free. Thus, the graph G' is $\mathcal{H}$-free graph with $\omega(G') = k+1$ and therefore

$$f^\star_{\mathcal{H}}(k) = \chi(G) \leq \chi(G') \leq f^\star_{\mathcal{H}}(k+1),$$

which completes our proof. □

3.4 Techniques to colour graphs with weighted cycles

We use the results of this section in our later proofs to colour certain graphs which contain induced cycles of length 5. In particular, we frequently deal with cycles $C \cong C_5$ and vertex-weight functions $q\colon V(C) \to \mathbb{N}_0$. This section is quite technical and we use the results of it to colour the special graphs G_1, G_2, G_3, G_4 which occur in Chapter 6. We also use these results multiple times in other Chapters and, for that reason, we state them here.

Following Narayanan and Shende [45], who proved

$$\chi(G) = \max\left\{\omega(G), \left\lceil \frac{|V(G)|}{\alpha(G)} \right\rceil\right\}$$

for each 'non-empty, $2K_1$-free'-expansion G of a cycle of length at least 4, we can determine the q-chromatic number of a C_5 by Observation 36.

Corollary 46 ([13])**.** *Let $\omega \in \mathbb{N}_{>0}$. If C is a cycle of length 5 and $q\colon V(C) \to \mathbb{N}_0$ is a vertex-weight function such that $\omega_q(C) = \omega$, then*

$$\chi_q(C) = \max\left\{\omega_q(C), \left\lceil \frac{q(C)}{2} \right\rceil\right\} \leq \left\lceil \frac{5\omega_q(C) - 1}{4} \right\rceil,$$

and this bound is tight.

Proof. In view of Observation 36, it remains to show

$$\left\lceil \frac{q(C)}{2} \right\rceil \leq \left\lceil \frac{5\omega_q(C) - 1}{4} \right\rceil$$

and that this bound is tight. Renaming vertices if necessary, let us assume that $C\colon c_1c_2c_3c_4c_5c_1$ is defined such that $\omega_q(C) = q(\{c_1, c_2\})$ and $q(c_1) \geq q(c_2)$. Thus, $q(\{c_3, c_4\}) \leq \omega_q(C)$ and $q(c_5) \leq \lfloor \omega_q(C)/2 \rfloor$. Furthermore, for $n, m \in \mathbb{N}_0$ with $\omega_q(C) = 4n + m$ and $m < 4$, we have

$$\left\lceil \frac{q(C)}{2} \right\rceil \leq \omega_q(C) + \left\lceil \frac{\left\lfloor \frac{\omega_q(C)}{2} \right\rfloor}{2} \right\rceil = \omega_q(C) + \left\{ \begin{matrix} n & \text{if } m \leq 1 \\ n+1 & \text{if } m \geq 2 \end{matrix} \right\}$$
$$= \omega_q(C) + \left\lceil \frac{\omega_q(C) - 1}{4} \right\rceil = \left\lceil \frac{5\omega_q(C) - 1}{4} \right\rceil.$$

From this chain of inequalities it follows that the bound is tight if $q(c_1) = q(c_3) = \lceil \omega/2 \rceil$ and $q(c_2) = q(c_4) = q(c_5) = \lfloor \omega/2 \rfloor$, which completes our proof. □

Corollary 46 is important for our later considerations. However for some subclasses, we also need the following stronger result. This result roughly states that if and only if the largest weighted clique in a C_5 is not too big, we can colour the weighted C_5 by using all colours twice except for some extra colours which we use on one special vertex.

Corollary 47 ([12]). *Let $C\colon c_1c_2c_3c_4c_5c_1$ be a cycle of length 5, $q, q'\colon V(C) \to \mathbb{N}_0$ be two vertex-weight functions, and $k \in \mathbb{N}_0$ be an integer such that $q(C) - k \equiv 0 \bmod 2$, $q(c_3) \geq k$, and q' is defined by*

$$c_i \mapsto \begin{cases} q(c_i) - k & \text{if } i = 3, \\ q(c_i) & \text{if } i \neq 3. \end{cases}$$

There is some proper q-colouring $L\colon V(C) \to 2^{\mathbb{N}_{>0}}$ with $|L^{(1)}(c_3)| = k$ and

$$L(C) = L^{(1)}(c_3) \cup \left(\bigcup_{i=1}^{5} L^{(2)}(c_i, c_{i+2}) \right)$$

if and only if

$$\omega_{q'}(C) \leq \frac{q'(C)}{2}.$$

Proof. Let $L'\colon V(C) \to 2^{\mathbb{N}_{>0}}$ be a proper q'-colouring, $L'(C) = [\ell]$, and $L\colon V(G) \to 2^{\mathbb{N}_{>0}}$ be a proper q-colouring with

$$c \mapsto \begin{cases} L'(c_i) \cup \{\ell+1, \ldots, \ell+k\} & \text{if } i = 3, \\ L'(c_i) & \text{if } i \neq 3. \end{cases}$$

If $L(C) = L^{(1)}(c_3) \cup \left(\bigcup_{i=1}^{5} L^{(2)}(c_i, c_{i+2}) \right)$ and $|L^{(1)}(c_3)| = k$, then

$$\omega_{q'}(C) \leq \chi_{q'}(C) \leq \ell = \left| \bigcup_{i=1}^{5} L^{(2)}(c_i, c_{i+2}) \right| = \frac{q(C) - k}{2} = \frac{q'(C)}{2}.$$

If $\omega_{q'}(C) \leq q'(C)/2$, then $\chi_{q'}(C) = q'(C)/2$ by Corollary 46 and since $k \leq q(c_3)$. Thus, assuming $\ell = \chi_{q'}(C)$, we have

$$L'(C) = \bigcup_{i=1}^{5} (L')^{(2)}(c_i, c_{i+2}) = \bigcup_{i=1}^{5} L^{(2)}(c_i, c_{i+2})$$

and $\{\chi_{q'}(C)+1, \ldots, \chi_{q'}(C)+k\} = L^{(1)}(c_3)$, which completes our proof. □

In some of our proofs we use a minimal counterexample approach to properly q-colour graphs. The next preliminary lemma helps us to gain some structural results for all weighted graphs containing an induced C_5. It is necessary to determine the weighted chromatic number of the special graphs G_1, G_2, G_3, G_4 (cf. Chapter 8) but is more generally applicable and therefore stated here. Before we prove this lemma let us shortly show one of its uses. If all these assumptions are fulfilled by some smartly chosen I and $f_{q'}$, we often find that (ii) holds which grants that $\omega_q(G) = \omega_q(G - I)$. Thus, there is at least one $\omega_q(G)$-Clique in G which consists of vertices of $V(G) \setminus I$ only. We choose different independent sets and, thus, obtain quite some structure for the researched graphs.

Lemma 48 ([12]). *Let G be a graph, I be a non-empty independent set in G, q, $q' \colon V(G) \to \mathbb{N}_0$ be two vertex-weight functions such that $q'(u) = q(u) - 1$ if $u \in I$ and $q'(u) = q(u)$ if $u \notin I$, $C \in \mathcal{C}_5^\star(G, q)$ and $C' \in \mathcal{C}_5^\star(G, q')$ be two cycles, and $f_q, f_{q'} \in \mathbb{N}_0$ be two integers such that $\chi_{q'}(G) \geq f_{q'}$. If*

$$\chi_q(G) > \max\{\omega_q(G), \chi_q(C), f_q\} \quad \textit{and} \quad \chi_{q'}(G) \leq \max\{\omega_{q'}(G), \chi_{q'}(C'), f_{q'}\},$$

then at least one of following three statements holds:

(i) $1 \leq \max\{\omega_q(G), \chi_q(C), f_q\} \leq f_{q'}$,

(ii) $f_{q'} < \max\{\omega_{q'}(G), \chi_{q'}(C')\}$, $\max\{\chi_q(C), f_q\} \leq \omega_q(G)$, *and* $\omega_q(G) = \omega_q(G - I)$,

(iii) $f_{q'} < \max\{\omega_{q'}(G), \chi_{q'}(C')\}$, $\max\{\omega_q(G), f_q\} \leq \chi_q(C)$, $|V(C') \cap I| \leq 1$, *and*

$$\chi_q(G) - 1 = \chi_q(C) = \chi_{q'}(C') = \left\lceil \frac{q'(C')}{2} \right\rceil = \left\lceil \frac{q(C')}{2} \right\rceil.$$

Proof. Clearly, we have $\chi_{q'}(G) \geq \{\omega_{q'}(G), \chi_{q'}(C'), f_{q'}\}$, and so

$$\chi_{q'}(G) = \omega_{q'}(G) \quad \text{or} \quad \chi_{q'}(G) = \chi_{q'}(C'), \quad \text{or} \quad \chi_{q'}(G) = f_{q'}.$$

Additionally, we note $\chi_q(G) \leq \chi_{q'}(G) + 1$ since I is an independent set. Since $q(u) \geq 1$ for each $u \in I$, we have $\omega_q(G) \geq q(u) \geq 1$.

If $\chi_{q'}(G) = f_{q'}$, then $\max\{\omega_q(G), \chi_q(C), f_q\} \leq f_{q'}$ since $\chi_q(G) \leq \chi_{q'}(G) + 1$. Hence, we may assume

$$\max\{\omega_{q'}(G), \chi_{q'}(C')\} = \chi_{q'}(G) > f_{q'}$$

for the rest of our proof.

If $\chi_{q'}(G) = \omega_{q'}(G)$, then we obtain $\omega_q(G) = \omega_{q'}(G)$ from

$$\omega_q(G) + 1 \leq \chi_q(G) \leq \chi_{q'}(G) + 1 = \omega_{q'}(G) + 1 \leq \omega_q(G) + 1.$$

Thus, each clique S with $q'(S) = \omega_{q'}(G)$ does not intersect I, and so

$$\omega_q(G) = \omega_{q'}(G) = \omega_{q'}(G - I) \leq \omega_q(G - I) \leq \omega_q(G).$$

Since $\chi_q(G) = \omega_q(G) + 1$, we additionally have $\max\{\chi_q(C), f_q\} \leq \omega_q(G)$ by our assumption $\chi_q(G) > \max\{\omega_q(G), \chi_q(C), f_q\}$.

If $\chi_{q'}(G) > \omega_{q'}(G)$ and $\chi_{q'}(G) = \chi_{q'}(C')$, then

$$\chi_{q'}(C') = \left\lceil \frac{q'(C')}{2} \right\rceil$$

by Corollary 46. Furthermore,

$$\chi_q(C) + 1 \leq \chi_q(G) \leq \chi_{q'}(G) + 1 = \chi_{q'}(C') + 1 \leq \chi_q(C') + 1 \leq \chi_q(C) + 1,$$

which implies

$$\chi_q(C) = \chi_{q'}(C') = \left\lceil \frac{q'(C')}{2} \right\rceil \leq \left\lceil \frac{q(C')}{2} \right\rceil \leq \chi_q(C') \leq \chi_q(C),$$

and so $|I \cap V(C')| \leq 1$. Since $\chi_q(G) = \chi_q(C) + 1$, we additionally have $\max\{\omega_q(G), f_q\} \leq \chi_q(C)$ by our assumption $\chi_q(G) > \max\{\omega_q(G), \chi_q(C), f_q\}$. □

4 $(P_5, hammer)$-free graphs

In this chapter, we prove $f^\star_{\{P_5,hammer\}} = f^\star_{\{2K_2\}}$ which is Theorem 2 and that each critical $(P_5, hammer)$-free graph is $2K_2$-free which is Theorem 9 (iii).

Since each $2K_2$-free graph is especially $(P_5, hammer)$-free we know that

$$f^\star_{\{P_5,hammer\}}(\omega) \geq f^\star_{\{2K_2\}}(\omega), \text{ for } \omega \in \mathbb{N}_{>0}.$$

Note that, by Lemma 43, $f^\star_{\{P_5,hammer\}}$ is superadditive and thus non-decreasing. By Lemma 1 it now suffices to show that each critical $(P_5, hammer)$-free graph is $2K_2$-free to prove the desired results. So we show exactly that. We note that there are $(P_5, hammer)$-free graphs that are not $Q[P_4]$-free, for example the graph $Q[P_4]$ itself. Hence, we cannot make use of Corollary 40 but Lemma 37 is still applicable.

For the sake of a contradiction, let us suppose that G is a critical $(P_5, hammer)$-free graph that contains an induced $2K_2$. We clearly can assume that G is connected and that $q\colon V(G) \to [1]$ is $\lhd^G_\chi$-minimal. For two vertices $u, v \in V(G)$, we define the set $X_{u,v}$ by $X_{u,v} \coloneqq N_G(u) \cap N_G(v)$.

Let u_1u_2 be an arbitrary edge of G such that $|E(G - N_G[\{u_1, u_2\}])| \geq 1$. If $v \in N_G(\{u_1, u_2\})$, $w \in N_G(v) \cap N^2_G(\{u_1, u_2\})$, and $x \in N_G(w) \setminus N_G[\{u_1, u_2, v\}]$, then, renaming vertices if necessary, we assume $u_1v \in E(G)$. Thus, $[x, w, v, u_1, u_2]$ induces a P_5 if $u_2v \notin E(G)$ and a *hammer* if $u_2v \in E(G)$, which is a contradiction to the fact that G is $(P_5, hammer)$-free. Hence, $N^i_G(\{u_1, u_2\}) = \emptyset$ for $i \geq 3$, and each vertex subset of $N^2_G(\{u_1, u_2\})$ inducing a component of $G[N^2_G(\{u_1, u_2\})]$ is a module. Since $|E(G - N_G[\{u_1, u_2\}])| \geq 1$, there is some set W of vertices which induces a component of $G[N^2_G(\{u_1, u_2\})]$ with at least one edge, say w_1w_2.

Let us first show that deleting $X_{u_1,u_2} \cap X_{w_1,w_2}$ disconnects the graph. Suppose not and let $P : p_1, p_2, \cdots p_k$ be the shortest path in $G' = G-(X_{u_1,u_2} \cap X_{w_1,w_2})$ starting in $\{u_1, u_2\}$ and ending in $\{w_1, w_2\}$. By otherwise renaming the vertices we assume without loss of generality that $p_1 = u_1$ and $p_k = w_1$. Note that $3 \leq k \leq 4$, since $w_1, w_2 \in N^2_G(\{u_1, u_2\})$ and G' is P_5-free. Since P is the shortest path we know that $u_ip_j \notin E(G')$ for $i \in [2], j \in \{3, 4\} \cap [k]$ and $w_ip_j \notin E(G)$ for $i \in [2], j \in \{k - 3, k - 2\} \cap [k]$. If $k = 4$, $P \cup \{u_2\}$ induces a *hammer*, if $u_2p_2 \in E(G)$, and $P \cup \{u_2\}$ induces a P_5, if $u_2p_2 \notin E(G)$. So $k = 3$ and since $p_2 \notin X_{u_1,u_2} \cap X_{w_1,w_2}$ we know $p_2u_2 \notin E(G)$ or $p_2w_2 \notin E(G)$. Again by Symmetry we assume $p_2w_2 \notin E(G)$. But now $P \cup \{w_2\} \cup \{u_2\}$ induces a *hammer*, if $u_2p_2 \in E(G)$, and $P \cup \{w_2\} \cup \{u_2\}$ induces a P_5, if $u_2p_2 \notin E(G)$. So u_1 and w_1 are not in the same component in $G - (X_{u_1,u_2} \cap X_{w_1,w_2})$ and deleting $X_{u_1,u_2} \cap X_{w_1,w_2}$ disconnects the graph.

Let $X_1, X_2, \dots, X_\ell$ be the sets of vertices which induce the components of $\bar{G}[X_{u_1,u_2} \cap X_{w_1,w_2}]$, and $i \in [\ell]$. We are going to show that X_i is a module. For the sake of a contradiction, let us suppose that there is a vertex $y \in V(G) \setminus (X_{u_1,u_2} \cap X_{w_1,w_2})$ with $X_i \cap N_G(y) \neq \emptyset$ and $X_i \setminus N_G(y) \neq \emptyset$. Since $\bar{G}[X_i]$ is connected, we may assume that $x_1 \in X_i \cap N_G(y)$ and $x_2 \in X_i \setminus N_G(y)$ are non-adjacent. Let Y be the set of vertices which induces the component of $G - (X_{u_1,u_2} \cap X_{w_1,w_2})$ that contains y. If $|Y| = 1$, then $N_G(y) \subseteq N_G(u_1)$, which contradicts Lemma 34 since $q\colon V(G) \to [1]$ is $\lhd_\chi^G$-minimal. Thus, $|Y| \geq 2$ and there is a vertex $y' \in Y \cap N_G(y)$. Since u_1 and w_1 are not in the same component in $G - (X_{u_1,u_2} \cap X_{w_1,w_2})$, we have $u_1, u_2 \notin Y$ or $w_1, w_2 \notin Y$. Renaming vertices if necessary, we may assume $u_1, u_2 \notin Y$. Since Y induces a component of $G - N_G[\{u_1, u_2\}]$, it is a module. Thus, $x_1y' \in E(G)$ but $x_2y' \notin E(G)$, and $[x_2, u_1, x_1, y, y']$ induces a *hammer*; a contradiction. Hence, y does not exist, and X_i is a module. Let M_1 be set of vertices which are in the connected component of u_1 in $G - (X_{u_1,u_2} \cap X_{w_1,w_2})$ and $Z_1 = M_1 \cup (X_{u_1,u_2} \cap X_{w_1,w_2})$ and $Z_2 = V(G) \setminus M_1$ Clearly, $Z_1 \cap Z_2 = X_{u_1,u_2} \cap X_{w_1,w_2}$ and $G = G[Z_1] \cup G[Z_2]$. Thus, $X_{u_1,u_2} \cap X_{w_1,w_2}$ is a clique-separator of the modules $X_1, X_2, \dots, X_\ell$, and we have

$$\chi(G) = \max\{\chi(G[Z_1]), \chi(G[Z_2])\}$$

by Lemma 37. Since $u_1, u_2 \in Z_1$ and $w_1, w_2 \in Z_2$, we have that G is not critical, which contradicts our assumption on G. Thus, every critical $(P_5, hammer)$-free graph is $2K_2$-free, which completes our proof.

5 $(P_5, banner)$-free graphs

This chapter is devoted to a proof of the statements of Theorem 3, Theorem 9 and Corollary 10 concerning *banner*-free graphs. So we prove $f^\star_{\{P_5,banner\}}(\omega) = f^\star_{\{3K_1\}}(\omega)$, $f^\star_{\{C_5,C_7,\ldots,banner\}}(\omega) = f^\star_{\{C_5,3K_1\}}(\omega)$, for $\omega \in \mathbb{N}_{>0}$, which together form Theorem 3, that each critical $(P_5, banner)$-free is $3K_1$-free, which is Theorem 9 (i), and that each critical $(banner, C_5, C_7, \ldots)$-free graph is $(C_5, 3K_1)$-free, which is Theorem 9 (iv). Lastly we show one part of Corollary 10. That is, if G is $(P_5, banner)$-free, then

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil .$$

We note that instead of verifying $f^\star_{\mathcal{H}}(\omega) \leq f(\omega)$ for the corresponding χ-binding function $f\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$, we show the slightly stronger statement

$$\chi_q(G) \leq f(\omega_q(G))$$

for each $\mathcal{H}$-free graph G and each vertex-weight function $q\colon V(G) \to \mathbb{N}_0$.

We note that each graph of $\{banner, C_7, C_9, \ldots, P_5\}$ contains at least one induced $3K_1$. Consequently, for each $\omega \in \mathbb{N}_{>0}$, we have

$$f^\star_{\{C_5,C_7,\ldots,banner\}}(\omega) \geq f^\star_{\{C_5,3K_1\}}(\omega) \quad \text{and} \quad f^\star_{\{P_5,banner\}}(\omega) \geq f^\star_{\{3K_1\}}(\omega).$$

Since neither C_5 nor $3K_1$ contains a spanning subgraph that is complete bipartite and

$$f^\star_{\{C_5,3K_1\}}(\omega) \leq f^\star_{\{3K_1\}}(\omega) \in \Theta(\omega^2/\log(\omega)),$$

it follows that $f^\star_{\{3K_1\}}$ and $f^\star_{\{C_5,3K_1\}}$ are superadditive by Lemma 43, where the order of magnitude of the function is subject of Corollary 27. Additionally, each *banner*-free graph is $Q[P_4]$-free. Thus, given a graph G, which is $(C_5, C_7, \ldots, banner)$-free or $(P_5, banner)$-free, by Lemma 41, we can focus on studying the q-chromatic number of G for $\lhd^G_\chi$-minimal vertex-weight functions $q\colon V(G) \to \mathbb{N}_0$ for which $G[q]$ is prime and has no clique-separator of modules.

For prime *banner*-free graphs, the following two results are known.

Theorem 49 (Hoáng [32])**.** *If G is a prime $(C_5, C_7, \ldots, banner)$-free graph of independence number at least 3, then G is perfect.*

Theorem 50 (Karthick, Maffray, and Pastor [39]). *If G is a prime $(P_5, banner)$-free graph of independence number at least 3, then G is perfect.*

If G is $(C_5, C_7, \dots, banner)$-free or $(P_5, banner)$-free, and $q\colon V(G) \to \mathbb{N}_0$ is a $\lhd_\chi^G$-minimal vertex-weight function for which $G[q]$ is prime and has no clique-separator of modules, then $G[q]$ is perfect or $3K_1$-free by Theorem 49 and Theorem 50. Additionally, a q-expansion G' of $G[q]$ is perfect by Lemma 35 or $3K_1$-free by construction, respectively. We obtain, by Observation 36,

$$\chi_q(G) = \chi_q(G[q]) = \chi(G') \le f^\star_{\{3K_1\}}(\omega(G')) = f^\star_{\{3K_1\}}(\omega_q(G[q])) = f^\star_{\{3K_1\}}(\omega_q(G)).$$

Hence,

$$f^\star_{\{P_5, banner\}} = f^\star_{\{3K_1\}} \quad \text{and} \quad f^\star_{\{C_5, C_7, \dots, banner\}} = f^\star_{\{C_5, 3K_1\}}.$$

Using the previously stated theorems we next prove that every critical $(P_5, banner)$-free graph and every critical $(C_5, C_7, \dots, banner)$-free graph is $3K_1$-free. Let G be a critical $(P_5, banner)$-free graph or a critical $(C_5, C_7, \dots, banner)$-free graph. By Corollary 40, the vertex set of G can be partitioned into $k \ge 1$ sets $M_1, M_2, \dots, M_k$ such that $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of a prime graph G_i^p for each $i \in [k]$ and $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$. By Theorem 49 and Theorem 50, G_i^p is either $3K_1$-free or perfect for each $i \in [k]$. Thus, $G[M_i]$ is $3K_1$-free or, by Lemma 35, $G[M_i]$ is perfect. In the latter case, $G[M_i]$ is complete since G is critical. Thus, in both cases, $G[M_i]$ is $3K_1$-free. Since $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$, G is $3K_1$-free as well.

Let us lastly prove that Corollary 10 (Reed's Conjecture) is true for $(P_5, banner)$-free graphs. Let G be a $(P_5, banner)$-free graph and G' be a critical graph with $V(G') \subseteq V(G)$ and $\chi(G') = \chi(G)$. By Theorem 9(i) we know that G' is $3K_1$-free. Since Reed's conjecture is proven for $3K_1$-free graphs [43, 44] we get

$$\chi(G) = \chi(G') \le \left\lceil \frac{\Delta(G') + \omega(G') + 1}{2} \right\rceil \le \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil,$$

which proves one part of Corollary 10.

6 $(P_5, dart)$-free graphs

This chapter is devoted to a proof of the statements of Theorem 4, Theorem 9 and Corollary 10 concerning *dart*-free graphs. So we prove $f^\star_{\{P_5,dart\}}(\omega) = f^\star_{\{3K_1\}}(\omega)$, $f^\star_{\{C_5,C_7,\ldots,dart\}}(\omega) = f^\star_{\{C_5,3K_1\}}(\omega)$, which together form Theorem 4, and that each critical $(dart, C_5, C_7, \ldots)$-free graph is $(C_5, 3K_1)$-free, which is one part of Theorem 9 (iv). This chapter can conceptually also be found in [12].

We also fully characterise all critical $(P_5, dart)$-free graphs according to Theorem 9 (ii). There we state that for each critical $(P_5, dart)$-free graph G and S a non-empty set of vertices such that each vertex in S is adjacent to each vertex of $V(G) \setminus S$ and each homogeneous set M in $G[S]$ has a vertex in $S \setminus M$ that is non-adjacent to each vertex of M, then $G - S$ is critical, and $G[S]$ is $3K_1$-free or a 'non-empty, $2K_1$-free'-expansion of G' with $G' \in \{G_1, G_2\}$.

We also show one part of Corollary 10. That is, that if G is $(P_5, dart)$-free, then

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil .$$

Assuming Theorem 9 to be proven we firstly prove Corollary 10. Note that to prove Reed's Conjecture for all $(P_5, dart)$-free graphs it clearly suffices to prove it for all critical graphs; of those we know the structure. Theorem 9 and the fact that Reed's conjecture is proven for $3K_1$-free graphs [43, 44], graphs whose complementary graphs are disconnected [53], and graphs G with $\chi(G) \leq \lceil 5\omega(G)/4 \rceil$ [37] imply that is suffices to show the latter inequality for each 'non-empty, $2K_1$-free'-expansion of G_1 and of G_2 in order to prove Corollary 10.

So it remains to show the statements of Theorem 4 and Theorem 9 that particularly contain a proof of the inequality $\chi_q(G_i) \leq \lceil (5\omega_q(G_i) - 1)/4 \rceil$ for each $i \in [2]$ and each vertex-weight function $q\colon V(G_i) \to \mathbb{N}_0$. We note that instead of for example verifying $f^\star_{\mathcal{H}}(\omega) \leq f(\omega)$ for the corresponding χ-binding function $f\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$, we show the slightly stronger statement

$$\chi_q(G) \leq f(\omega_q(G))$$

for each $\mathcal{H}$-free graph G and each vertex-weight function $q\colon V(G) \to \mathbb{N}_0$.

At the beginning, let us mention that we start our proof similarly to that of *banner*-free graphs. Namely, each graph of $\{dart, C_7, C_9, \ldots, P_5\}$ contains at least one induced

$3K_1$, and so

$$f^\star_{\{C_5,C_7,\ldots,dart\}}(\omega) \geq f^\star_{\{C_5,3K_1\}}(\omega) \quad \text{and} \quad f^\star_{\{P_5,dart\}}(\omega) \geq f^\star_{\{3K_1\}}(\omega)$$

for each $\omega \geq 1$.

Let us note that $f^\star_{\{3K_1\}}$ and $f^\star_{\{C_5,3K_1\}}$ are superadditive by Lemma 43 since neither C_5 nor $3K_1$ contains a spanning subgraph that is complete bipartite and $f^\star_{\{C_5,3K_1\}}(\omega) \leq f^\star_{\{3K_1\}}(\omega) \in \Theta(\omega^2/\log(\omega))$. Additionally, each *dart*-free graph is $Q[P_4]$-free. Thus, given a graph G, which is ($C_5, C_7, \ldots, dart$)-free or ($P_5, dart$)-free, by Lemma 41, we can focus on studying the q-chromatic number of G for $\lhd^G_\chi$-minimal vertex-weight functions $q\colon V(G) \to \mathbb{N}_0$ for which $G[q]$ is prime and has no clique-separator of modules.

To finally get our two optimal χ-binding functions $f^\star_{\{P_5,dart\}}$ and $f^\star_{\{C_5,C_7,\ldots,dart\}}$, we need to divide our proof into smaller parts. First of all, we show that $\bar{G}$ is $(C_7, C_9, \ldots)$-free whenever G is a prime *dart*-free graph of independence number at least 3 that is P_5-free or C_5-free.

Lemma 51. *If G is a prime dart-free graph with independence number at least 3, which is C_5- or P_5-free, then the complementary graph $\bar{G}$ is $(C_7, C_9, \ldots)$-free.*

Proof. For the sake of a contradiction, let us suppose that G is a prime *dart*-free graph of independence number at least 3 which is C_5- or P_5-free, and for which C_{2k+1} is an induced subgraph of $\bar{G}$ for some integer $k \geq 3$, say $C\colon c_1c_2\ldots c_{2k+1}c_1 \in \mathcal{C}_{2k+1}(\bar{G})$. Clearly, G is connected, since G is prime and $|V(G)| \geq 3$. Let M be the set of vertices of $V(G) \setminus V(C)$ such that $E_G[\{m\}, V(C)]$ is mixed if and only if $m \in M$, and D be the vertices of $N_G(V(C))$ such that $E_G[\{d\}, V(C)]$ is complete if and only if $d \in D$.

Let $m \in M$ be an arbitrary vertex. If there is some $i \in [2k+1]$ such that $c_im, c_{i+1}m \notin E(G)$, then, renaming vertices if necessary, we may assume that $c_{i+2}m \in E(G)$. Since $[c_{i+1}, c, m, c_{i+2}, c_i]$ does not induce a *dart* for each $c \in \{c_{i+4}, c_{i+5}\}$, we have $c_{i+4}m, c_{i+5}m \notin E(G)$. But now, $[m, c_{i+2}, c_{i+4}, c_i, c_{i+5}]$ induces a *dart*; a contradiction. Thus, $c_im \in E(G)$ or $c_{i+1}m \in E(G)$ for each $i \in [2k+1]$. Since $2k+1$ is odd, there is some $t(m) \in [2k+1]$ such that $c_{t(m)}m \notin E(G)$ but $c_{t(m)-1}m, c_{t(m)+1}m, c_{t(m)+2}m \in E(G)$.

If $u \in N_G(V(C))$ and $v \in V(G) \setminus N_G[V(C)]$ are two adjacent vertices, then we see that $[v, u, c_3, c_1, c_4]$ if $u \in D$ and $[v, u, c_{t(u)+1}, c_{t(u)-1}, c_{t(u)+2}]$ if $u \in M$ induces a *dart*; a contradiction. Hence, $E_G[N_G[V(C)], V(G) \setminus N_G[V(C)]]$ is anticomplete, and the connectivity of G implies $V(G) = N_G[V(C)]$.

Let I be an independent set of size 3 in G such that

$$\sum_{a \in I} \mathrm{dist}_{\bar{G}}(a, V(C))$$

is minimal.

Since $G[V(C)]$ is $3K_1$-free, and $c_i u \in E(G)$ or $c_{i+1}u \in E(G)$ for each $u \in D \cup M$ and each $i \in [2k+1]$, we have $|I \cap V(C)| \leq 1$. Let a_1, a_2 be two vertices of $I \setminus V(C)$. We assume first that there is some vertex $c_j \in V(C) \setminus (N_G(a_1) \cup N_G(a_2))$ for some $j \in [2k+1]$. Hence, $a_i c_{j'} \in E(G)$ for each $i \in [2]$ and each $j' \in \{j-1, j+1\}$. For each $c \in \{c_{j+2}, c_{j+3}\}$, since $[c_j, c, a_1, c_{j-1}, a_2]$ does not induce a *dart*, we have $a_1 c \notin E(G)$ or $a_2 c \notin E(G)$. Furthermore, recall that $a_i c_{j+2} \in E(G)$ or $a_i c_{j+3} \in E(G)$ for each $i \in [2]$. Thus, renaming vertices if necessary, we may assume $a_1 c_{j+2}, a_2 c_{j+3} \in E(G)$ and $a_1 c_{j+3}, a_2 c_{j+2} \notin E(G)$. Hence, $[a_1, c_{j+2}, c_j, c_{j+3}, c_{j+1}]$ induces a C_5 and $[a_1, c_{j+2}, c_j, c_{j+3}, a_2]$ induces a P_5, which is a contradiction to the fact that G is C_5- or P_5-free. Hence, $I \cap V(C) = \emptyset$, and $V(C) \setminus (N_G(a_1) \cup N_G(a_2)) = \emptyset$ for each distinct $a_1, a_2 \in I$.

Let $I = \{a_1, a_2, a_3\}$ and, renaming vertices if necessary, let us assume

$$\mathrm{dist}_{\bar{G}}(a_1, V(C)) \leq \mathrm{dist}_{\bar{G}}(a_2, V(C)), \mathrm{dist}_{\bar{G}}(a_3, V(C)).$$

We consider first the case where $a_1 \in M$. Recall that by definition $a_1 c_{t(a_1)} \notin E(G)$ but $a_1 c_{t(a_1)-1}, a_1 c_{t(a_1)+1}, a_1 c_{t(a_1)+2} \in E(G)$ and $a_2 c_{t(a_1)}, a_3 c_{t(a_1)} \in E(G)$. Since $[a_1, c_{t(a_1)+2}, a_2, c_{t(a_1)}, a_3]$ does not induce a *dart*, we have $a_2 c_{t(a_1)+2} \notin E(G)$ or $a_3 c_{t(a_1)+2} \notin E(G)$. Thus, the fact $V(C) \setminus (N_G(a_2) \cup N_G(a_3)) = \emptyset$ implies that either $a_2 c_{t(a_1)+2} \notin E(G)$ or $a_3 c_{t(a_1)+2} \notin E(G)$. Renaming vertices if necessary, we may assume the latter case, and so $a_3 c_{t(a_1)+1} \in E(G)$. Since $[a_3, c_{t(a_1)-1}, a_1, c_{t(a_1)+2}, a_2]$ does not induce a *dart*, we have some $i \in [3]$ such that $a_i c_{t(a_1)-1} \notin E(G)$. Clearly, $i \neq 1$ and, since the set $V(C) \setminus (N_G(a_2) \cup N_G(a_3))$ is empty, the integer i is uniquely determined. Thus, $[a_1, c_{t(a_1)+1}, a_3, c_{t(a_1)}, c_{t(a_1)+2}]$ induces a C_5 and $[a_1, c_{t(a_1)-1}, a_{5-i}, c_{t(a_1)}, a_i]$ induces a P_5, which contradicts our assumption that G is C_5- or P_5-free. Thus, $a_1 \in D$ and, since $2 \leq \mathrm{dist}_{\bar{G}}(a_1, V(C)) \leq \mathrm{dist}_{\bar{G}}(a_2, V(C)), \mathrm{dist}_{\bar{G}}(a_3, V(C))$, we have $I \subseteq D$.

Let $u \in V(G) \setminus (N_G[a_1] \cup \{a_2, a_3\})$. Since $a_1 \in D$ and $V(G) = N_G[V(C)]$, it follows $u \notin V(C)$ and there is some $j \in [2k+1]$ such that $c_j \in N_G(u)$, respectively. Furthermore, $[a_1, c_j, a_2, u, a_3]$ does not induce a *dart*, and so $a_2 u \notin E(G)$ or $a_3 u \notin E(G)$. Renaming vertices if necessary, we may assume the latter case. By the choice of I, we have $\mathrm{dist}_{\bar{G}}(V(C), a_2) \leq \mathrm{dist}_{\bar{G}}(V(C), u)$. Thus, $\mathrm{dist}_{\bar{G}}(V(C), a_1) \leq \mathrm{dist}_{\bar{G}}(V(C), a_2) \leq \mathrm{dist}_{\bar{G}}(V(C), v)$ for each $v \in V(G) \setminus N_G[a_1]$. In particular, it follows $\mathrm{dist}_{\bar{G}}(V(C), a_1) = \infty$. Let D' with $I \subseteq D' \subseteq D$ be the set of vertices inducing a component of $\bar{G}$. Since $\mathrm{dist}_{\bar{G}}(V(C), a_1) = \infty$, we have that $E_G[D', V(G) \setminus D']$ is complete, and so D' is a homogeneous set, which contradicts the fact that G is prime. Thus, our proof is complete. $\square$

Using the Lemma 51 it follows from the Strong Perfect Graph Theorem that every prime $(C_5, C_7, \ldots, dart)$-free graph is perfect or $3K_1$-free. Similarly as we argue in Chapter 5

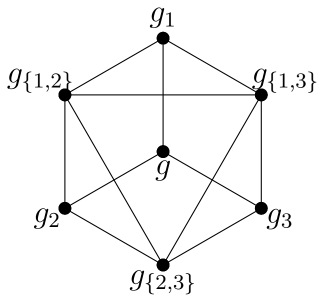


Fig. 7: G_1

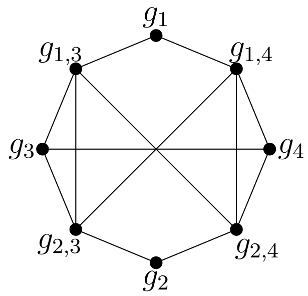


Fig. 8: G_2

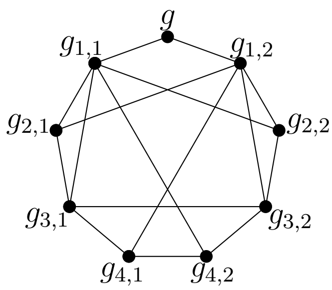


Fig. 9: G_3

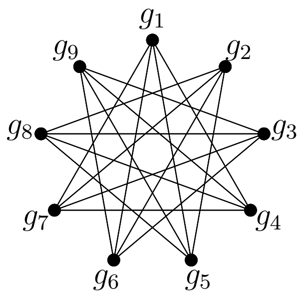


Fig. 10: G_4

for $(C_5, C_7, \ldots, banner)$-free graphs, we prove next that each critical $(C_5, C_7, \ldots, dart)$-free graph is $3K_1$-free by applying Corollary 40. Let G be a critical $(C_5, C_7, \ldots, dart)$-free graph. By Corollary 40, the vertex set of G can be partitioned into $k \geq 1$ sets $M_1, M_2, \ldots, M_k$ such that $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of a prime graph G_i^p for each $i \in [k]$ and $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$. By Lemma 51 G_i^p is $3K_1$-free or perfect for each $i \in [k]$. Thus, $G[M_i]$ is $3K_1$-free or, by Lemma 35 and Observation 36, $G[M_i]$ is perfect. In the latter case, $G[M_i]$ is complete since G is critical. Thus, in both cases, $G[M_i]$ is $3K_1$-free. Since $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$, G is $3K_1$-free as well.

Since we characterized all critical graphs we know, by Lemma 1, that $f^\star_{\{C_5, C_7, \ldots, dart\}} = f^\star_{\{C_5, 3K_1\}}$.

In contrast to prime $(P_5, banner)$-free graphs which are perfect by Theorem 50 if the independence number is at least 3, there exist prime $(P_5, dart)$-free graphs which are not perfect although their independence number is at least 3, for example G_1, G_2, G_3, and G_4, depicted in Figs. 7-10. We note that, by a result of Karthick, Maffray, and Pastor [39], each such graph contains at most 18 vertices. However, in order to apply Lemma 41, we need a full characterisation of these graphs.

Lemma 52. *If G is a prime $(P_5, dart)$-free graph of independence number at least 3, then either G is W_5-free and $\bar{G}$ is A_5-free, or $G \cong G_1$.*

Proof. Let G be a prime $(P_5, dart)$-free graph of independence number at least 3 with

$G \ncong G_1$. Since G is prime, we immediately obtain that G is connected. We show first that G contains an induced cycle of length 5. Clearly, G is $(C_7, C_9, \ldots)$-free, and from Lemma 51 we deduce that $\bar{G}$ is $(C_7, C_9, \ldots)$-free as well. Hence, G contains an induced cycle of length 5 or it is perfect by the Strong Perfect Graph Theorem. But in the latter case, G is W_5-free and $\bar{G}$ is A_5-free.

For some $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$, let $M(C)$ be the set of vertices of $V(G) \setminus V(C)$ such that $E_G[\{m\}, V(C)]$ is mixed if and only if $m \in M(C)$, and let $D(C)$ be the set of vertices of $N_G(V(C))$ such that $E_G[\{d\}, V(C)]$ is complete if and only if $d \in D(C)$. Furthermore, for some vertex $u \in N_G(V(C))$, let $i_C(u) \in [5]$ and $j_C(u), k_C(u) \in \mathbb{N}_0$ be such that

(i) $c_{i_C(u)}u, c_{i_C(u)+1}u, \ldots, c_{i_C(u)+j_C(u)}u \in E(G)$ and $c_{i_C(u)+j_C(u)+1}u \notin E(G)$,

(ii) $c_{i_C(u)-1}u, c_{i_C(u)-2}u, \ldots, c_{i_C(u)-k_C(u)}u \notin E(G)$ and $c_{i_C(u)-(k_C(u)+1)}u \in E(G)$,

(iii) with respect to (i) and (ii), $j_C(u)$ is minimum, and

(iv) with respect to (i), (ii), and (iii), $k_C(u)$ is maximum.

Since $[m, c_{i_C(m)}, c_{i_C(m)-1}, c_{i_C(m)-2}, c_{i_C(m)-3}]$ does not induce a P_5, we have $k_C(m) \leq 2$ for each $m \in M(C)$. For each $i \in [5]$, let

$$\begin{aligned} A_i(C) &= \{a : N_G(a) \cap V(C) = \{c_i, c_{i+2}\}\} \quad \text{and} \\ B_i(C) &= \{b : N_G(b) \cap V(C) = \{c_i, c_{i+2}, c_{i+3}\}\}. \end{aligned}$$

Clearly, $A_i(C) \cup B_i(C) \subseteq M(C)$ and $i_C(u) = i$ if $u \in A_i(C) \cup B_i(C)$ for each $i \in [5]$. With

$$\begin{aligned} X_{\geq 2}(C) &= \{x : x \in N_G(V(C)), j_C(x) \geq 2\} \quad \text{and} \\ X_{\geq 3}(C) &= \{x : x \in N_G(V(C)), j_C(x) \geq 3\}, \end{aligned}$$

we obtain

$$M(C) = \left(\bigcup_{i=1}^{5} A_i(C) \cup B_i(C) \right) \cup (X_{\geq 2}(C) \setminus D(C))$$

by the fact that $k_C(m) \leq 2$ for each $m \in M(C)$. Obviously, $E_G[N_G^2(V(C)), X_{\geq 2}(C)]$ is anticomplete since $[w, x, c_{i_C(u)}, c_{i_C(u)+1}, c_{i_C(u)+2}]$ does not induce a *dart* for each $w \in N_G^2(V(C))$ and each $x \in N_G(w) \cap X_{\geq 2}(C)$. Consequently, $V(G) = D(C) \cup M(C) \cup V(C)$ if $M(C) \setminus X_{\geq 2}(C) = \emptyset$. Furthermore, let

$$A(C, x) = \begin{cases} A_{i_C(x)-2}(C) \setminus N_G(x) & \text{if } x \in X_{\geq 3}(C) \setminus D(C), \\ \emptyset & \text{if } x \in D(C), \end{cases}$$

and

$$B(C, x) = \begin{cases} B_{i_C(x)-1}(C) \setminus N_G(x) & \text{if } x \in X_{\geq 3}(C) \setminus D(C), \\ \emptyset & \text{if } x \in D(C) \end{cases}$$

for each $x \in X_{\geq 3}(C)$.

We continue by proving four claims from which we finally deduce our desired result.

Claim 52.1. *If $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ with $X_{\geq 3}(C) \neq \emptyset$, then $M(C) \setminus X_{\geq 2}(C) = \bigcap_{x \in X_{\geq 3}(C)} B(C, x)$ and $|M(C) \setminus X_{\geq 2}(C)| \leq 1$.*

Proof. For the sake of simplicity, we divide the proof of this claim into three parts and prove step-by-step for each $C \in \mathcal{C}_5(G)$:

(i) $M(C) \setminus X_{\geq 2}(C) = \bigcap_{x \in X_{\geq 3}(C)} (A(C, x) \cup B(C, x))$,

(ii) $M(C) \setminus X_{\geq 2}(C) = \bigcap_{x \in X_{\geq 3}(C)} B(C, x)$, and

(iii) $|M(C) \setminus X_{\geq 2}(C)| \leq 1$.

Note that $\bigcap_{x \in X_{\geq 3}(C)} (A(C, x) \cup B(C, x)) \subseteq M(C) \setminus X_{\geq 2}(C)$, and (i) implies (ii) and (iii) if $D(C) \neq \emptyset$. Hence, for (ii) and (iii), we may assume $D(C) = \emptyset$.

For the sake of a contradiction, let us suppose that (i) is false. Let $m \in M(C) \setminus X_{\geq 2}(C)$ and $x \in X_{\geq 3}(C)$ be two arbitrary vertices. Note that $j_C(m) = 0$ and $c_{i_C(m)-1}m$, $c_{i_C(m)+1}m \notin E(G)$. Furthermore, the maximality of $k_C(m)$ implies $c_{i_C(m)+2}m \in E(G)$. If $x \in D(C)$, then, redefining $i_C(x)$ if necessary, we may assume $i_C(m) = i_C(x)$. Hence, $[m, c_{i_C(x)}, c_{i_C(x)-1}, x, c_{i_C(x)+1}]$ if $mx \notin E(G)$ and $[c_{i_C(x)-1}, x, m, c_{i_C(x)+2}, c_{i_C(x)+1}]$ if $mx \in E(G)$ induces a *dart*; a contradiction. Thus, $x \notin D(C)$, and so $j_C(x) = 3$ and $k_C(x) = 1$. If $i_C(m) = i_C(x)$, then

- $[c_{i_C(x)-1}, c_{i_C(x)}, c_{i_C(x)+1}, x, m]$ if $mx \in E(G)$,
- $[m, c_{i_C(x)+2}, c_{i_C(x)+1}, x, c_{i_C(x)+3}]$ if $c_{i_C(x)+3}m, mx \notin E(G)$, and
- $[c_{i_C(x)-1}, c_{i_C(x)+3}, m, c_{i_C(x)+2}, x]$ if $c_{i_C(x)+3}m \in E(G)$ but $mx \notin E(G)$

induces a *dart*; a contradiction. Hence, $i_C(m) \neq i_C(x)$. If $i_C(m) = i_C(x) + 1$, then $[c_{i_C(x)}, x, m, c_{i_C(x)+3}, c_{i_C(x)+2}]$ if $mx \in E(G)$ and $[m, c_{i_C(x)+1}, c_{i_C(x)}, x, c_{i_C(x)+2}]$ if $mx \notin E(G)$ induces a *dart*; a contradiction. Hence, $i_C(m) \neq i_C(x) + 1$. If $i_C(m) = i_C(x) + 2$, then

- $[c_{i_C(x)+3}, x, c_{i_C(x)+1}, c_{i_C(x)}, m]$ if $c_{i_C(x)}m, mx \in E(G)$,
- $[c_{i_C(x)}, x, m, c_{i_C(x)+2}, c_{i_C(x)+3}]$ if $c_{i_C(x)}m \notin E(G)$ but $mx \in E(G)$, and
- $[m, c_{i_C(x)+2}, c_{i_C(x)+1}, x, c_{i_C(x)+3}]$ if $mx \notin E(G)$

induces a *dart*; a contradiction. Hence, $i_C(m) \neq i_C(x) + 2$. If $i_C(m) = i_C(x) + 3$, then $[c_{i_C(x)-1}, c_{i_C(x)+3}, c_{i_C(x)+2}, x, m]$ if $mx \in E(G)$ and $[c_{i_C(x)-1}, c_{i_C(x)}, m, c_{i_C(x)+1}, x]$ if $c_{i_C(x)+1}m \in E(G)$ but $mx \notin E(G)$ induces a *dart*; a contradiction. Hence, $c_{i_C(x)+1}m$, $mx \notin E(G)$, and so $m \in A(C, x)$. If $i_C(m) = i_C(x) + 4$, then

- $[c_{i_C(x)+3}, x, c_{i_C(x)}, c_{i_C(x)+1}, m]$ if $mx \in E(G)$ and

- $[m, c_{i_C(x)+1}, c_{i_C(x)}, x, c_{i_C(x)+2}]$ if $c_{i_C(x)+2}m, mx \notin E(G)$

induces a *dart*; a contradiction. Hence, $c_{i_C(x)+2}m \in E(G)$ but $mx \notin E(G)$, and so $m \in B(C, x)$, which completes our proof for (i).

For (ii), let us assume that $x \in X_{\geq 3}(C) \setminus D(C)$ is an arbitrary vertex, and, for the sake of a contradiction, let us suppose that $A(C, x) \neq \emptyset$. Let S be the set of vertices of G such that $c_{i_C(x)}s, c_{i_C(x)+3}s \in E(G)$ but $c_{i_C(x)+1}s, c_{i_C(x)+2}s, sx \notin E(G)$ if and only if $s \in S$. Note that $A(C, x) \cup \{c_{i_C(x)+4}\} \subseteq S$ and $\bar{G}[A(C, x) \cup \{c_{i_C(x)+4}\}]$ is connected. Hence, let A be the set of vertices that induces the component of $\bar{G}[S]$ which contains all vertices of $A(C, x) \cup \{c_{i_C(x)+4}\}$. We note that, for each $a \in A$, $C_a\colon c_{i_C(x)}c_{i_C(x)+1}c_{i_C(x)+2}c_{i_C(x)+3}ac_{i_C(x)}$ is an induced C_5 in G and $N_G(x) \cap V(C_a) = N_G(x) \cap V(C)$. Since A is not a homogeneous set in G, there is some vertex $u \in V(G) \setminus A$ that has a neighbour, say a_1, and a non-neighbour, say a_2, in A. Since $\bar{G}[A]$ is connected, we can assume $a_1a_2 \notin E(G)$. Clearly, $u \notin A \cup V(C) \cup \{x\}$. Note that $a_2 \in A(C_{a_1}, x)$. Thus, by (i), $u \notin X_{\geq 3}(C_{a_1})$. If $u \in X_{\geq 2}(C_{a_1}) \setminus X_{\geq 3}(C_{a_1})$, then $|N_G(u) \cap V(C_{a_2})| = 2$ since $a_1u \in E(G)$ but $a_2u \notin E(G)$. Thus, by (i), $u \in A(C_{a_2}, x)$, and so $u \in S$. To be more precise, since $a_2u \notin E(G)$, we have $u \in A$ by the choice of A, which is a contradiction to the fact $u \in V(G) \setminus A$. Consequently, by (i), it remains to consider the case where $u \in A(C_{a_1}, x) \cup B(C_{a_1}, x)$, and so, since $a_1u \in E(G)$, we have $u \in B(C_{a_1}, x)$. Hence, $k_{C_{a_2}}(u) = 3$, which contradicts the fact that $k_{C_{a_2}}(v) \leq 2$ for each $v \in N_G(V(C_{a_2}))$ as shown above. Consequently, $A(C, x) = \emptyset$, which proves (ii).

We finally prove (iii) and assume that there exists some vertex $x \in X_{\geq 3}(C) \setminus D(C)$. For the sake of a contradiction, let us suppose $|M(C) \setminus X_{\geq 2}(C)| > 1$. Recall that $E_G[N_G^2(V(C)), X_{\geq 2}(C)]$ is anticomplete. Therefore, since $[x, c_{i_C(x)+3}, c_{i_C(x)+4}, b, w]$ does not induce a P_5 for each $b \in B(C, x)$ and each $w \in N_G^2(V(C))$, we additionally have $E_G[B(C, x), N_G^2(V(C))]$ is anticomplete. Thus, the connectivity of G and (ii) imply

$$V(G) = V(C) \cup D(C) \cup M(C) = V(C) \cup \left(\bigcap_{x \in X_{\geq 3}(C)} B(C, x) \right) \cup X_{\geq 2}(C).$$

Since $B(C, x) \subseteq M(C) \setminus X_{\geq 2}(C)$, (ii) implies $B(C, x) = M(C) \setminus X_{\geq 2}(C)$. Additionally, since $B(C, x)$ is not a homogeneous set, there are vertices $b_1, b_2 \in B(C, x)$ and $u \in V(G) \setminus B(C, x)$ such that u is adjacent to b_1 but not to b_2. Hence, $u \in X_{\geq 2}(C)$ by (ii). By (ii) and the fact $b_1u \in E(G)$, it follows $u \notin X_{\geq 3}(C)$. Thus, $u \in X_{\geq 2}(C) \setminus X_{\geq 3}(C)$. In particular, $j_C(u) = k_C(u) = 2$. Furthermore, $[c_{i_C(x)}, c_{i_C(x)+1}, b_1, c_{i_C(x)+2}, b_2]$ does not induce a *dart*, and so $b_1b_2 \in E(G)$. Since $[u, b_1, c_{i_C(x)+1}, b_2, c_{i_C(x)+4}]$ and $[u, b_1, c_{i_C(x)+2}, b_2, c_{i_C(x)+4}]$ do not induce a *dart* in G, we have $c_{i_C(x)+1}u, c_{i_C(x)+2}u \in E(G)$ or $c_{i_C(x)+4}u \in E(G)$. Let us consider first the case where $c_{i_C(x)+1}u, c_{i_C(x)+2}u \in E(G)$. Since $k_C(u) = 2$, it follows $c_{i_C(x)}u \notin E(G)$ or $c_{i_C(x)+3}u \notin E(G)$. Renaming vertices if necessary, we may assume $c_{i_C(x)}u \notin E(G)$. Thus, $[c_{i_C(x)}, c_{i_C(x)+1}, u, b_1, b_2]$ induces a

$dart$; a contradiction. Thus, let us consider the second case where $c_{i_C(x)+4}u \in E(G)$. But now,

- $[c_{i_C(x)}, c_{i_C(x)-1}, u, b_1, b_2]$ if $c_{i_C(x)}u \notin E(G)$,
- $[c_{i_C(x)+3}, c_{i_C(x)+4}, u, b_1, b_2]$ if $c_{i_C(x)+3}u \notin E(G)$, and
- $[b_2, c_{i_C(x)+4}, c_{i_C(x)+3}, u, c_{i_C(x)}]$ if $c_{i_C(x)}u, c_{i_C(x)+3}u \in E(G)$

induces a $dart$; a contradiction. Hence, $|M(C) \setminus X_{\geq 2}(C)| \leq 1$, (iii) follows. and our proof is complete. $(\square)$

Claim 52.2. *If* $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ *with* $X_{\geq 3}(C) \neq \emptyset$ *and* $M(C) \setminus X_{\geq 2}(C) \neq \emptyset$, *then* $|X_{\geq 3}(C)| = 1$.

Proof. Let $x \in X_{\geq 3}(C)$. For the sake of a contradiction and by Claim 52.1, let us suppose that there is a vertex $b \in B(C, x)$ but $|X_{\geq 3}(C)| \geq 2$. For each $x_1, x_2 \in X_{\geq 3}(C)$, we have $N_G(x_1) \cap V(C) = N_G(x_2) \cap V(C) \neq V(C)$ by Claim 52.1, and $x_1x_2 \in E(G)$ by the fact that $[c_{i_C(x_1)-1}, c_{i_C(x_1)}, x_1, c_{i_C(x_1)+1}, x_2]$ does not induce a $dart$. Since $X_{\geq 3}(C)$ is not a homogeneous set, there is some vertex $u \in V(G) \setminus X_{\geq 3}(C)$ that is, renaming vertices if necessary, adjacent to x_1 but non-adjacent to x_2. Recall that $E_G[N^2_G(V(C)), X_{\geq 2}(C)]$ is anticomplete, and so $u \in N_G(V(C))$. Hence, by Claim 52.1, $u \in X_{\geq 2}(C) \setminus X_{\geq 3}(C)$, and so $j_C(u) = k_C(u) = 2$. Furthermore, $[u, x_1, c_{i_C(x_1)}, x_2, c_{i_C(x_1)+3}]$ does not induce a $dart$, which means $c_{i_C(x_1)}u \in E(G)$ or $c_{i_C(x_1)+3}u \in E(G)$. Renaming vertices if necessary, we may assume $c_{i_C(x_1)}u \in E(G)$. Since $[c_{i_C(x_1)-1}, c_{i_C(x_1)}, u, x_1, x_2]$ does not induce a $dart$, it follows $c_{i_C(x_1)-1}u \in E(G)$. From $j_C(u) = k_C(u) = 2$, we obtain further that either $c_{i_C(x_1)+1}u \in E(G)$ or $c_{i_C(x_1)+3}u \in E(G)$. If $c_{i_C(x_1)+1}u \in E(G)$, then $[u, c_{i_C(x_1)+1}, b, c_{i_C(x_1)+2}, x_2]$ induces a $dart$ if $bu \notin E(G)$ and $[c_{i_C(x_1)+3}, c_{i_C(x_1)+2}, b, u, c_{i_C(x_1)}]$ induces a P_5 if $bu \in E(G)$, which contradicts our assumption that G is $(P_5, dart)$-free. Hence, $c_{i_C(x_1)+1}u \notin E(G)$ and $c_{i_C(x_1)+3}u \in E(G)$. But now, $[b, c_{i_C(x_1)+4}, c_{i_C(x_1)+3}, u, c_{i_C(x_1)}]$ if $bu \notin E(G)$ and $[b, u, c_{i_C(x_1)}, x_1, c_{i_C(x_1)+3}]$ if $bu \in E(G)$ induces a $dart$; the final contradiction. It implies $|X_{\geq 3}(C)| \leq 1$, which completes our proof. $(\square)$

Claim 52.3. *If* $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$, *then* $M(C) \setminus X_{\geq 2}(C) \neq \emptyset$.

Proof. For the sake of a contradiction, let us suppose $M(C) \setminus X_{\geq 2}(C) = \emptyset$. Note that since $E_G[N^2_G(V(C)), X_{\geq 2}(C)]$ is anticomplete, and, by the connectivity of G, it follows $V(G) = V(C) \cup X_{\geq 2}(C)$. In view of Lemma 52, let $\{a_1, a_2, a_3\}$ be a set of three pairwise non-adjacent vertices, such that

(i) $\sum_{i=1}^{3} \operatorname{dist}_{\bar{G}}(a_i, V(C))$ is minimum, and

(ii) with respect to (i), $\operatorname{dist}_{\bar{G}}(a_1, V(C)) \leq \operatorname{dist}_{\bar{G}}(a_2, V(C)) \leq \operatorname{dist}_{\bar{G}}(a_3, V(C))$.

Since $M(C) \setminus X_{\geq 2}(C) = \emptyset$, and so $j_C(x) \geq 2$ for each $x \in N_G(V(C))$, we have $|\{a_1, a_2, a_3\} \cap V(C)| \leq 1$, and so $\mathrm{dist}_{\bar{G}}(a_2, V(C)) \geq 1$.

If $\mathrm{dist}_{\bar{G}}(a_1, V(C)) = \infty$, then the set, say S, of vertices inducing the component of $\bar{G}$ that contains a_1, a_2, a_3 satisfies that $E_G[S, V(G) \setminus S]$ is complete, and so, S is a homogeneous set in G, which is contradiction to our assumption that G is prime. Thus, $\mathrm{dist}_{\bar{G}}(a_1, V(C)) < \infty$. Hence, let $P\colon p_1p_2 \dots p_\ell$, $\ell \geq 1$, be a shortest path connecting a_1 and a vertex of C in $\bar{G}$, where $a_1 = p_1$ and $p_\ell \in V(C)$. Renaming vertices if necessary, we may assume $p_\ell = c_1$. By the minimality of $\sum_{i=1}^{3} \mathrm{dist}_{\bar{G}}(a_i, V(C))$, we further have $a_2p_2, a_3p_2 \in E(G)$.

If $\ell = 1$, then $a_1 = c_1$. Since $a_2, a_3 \notin V(C)$ and $j_C(a_2), j_C(a_3) \geq 2$, it follows $a_2c_3, a_2c_4, a_3c_3, a_3c_4 \in E(G)$. Furthermore, $[c_1, c_2, a_2, c_3, a_3]$ does not induce a *dart*, and so $a_2c_2 \notin E(G)$ or $a_3c_2 \notin E(G)$. Similarly, $a_2c_5 \notin E(G)$ or $a_3c_5 \notin E(G)$. However, $j_C(a_2) = j_C(a_3) = 2$, and so, renaming vertices if necessary, we may assume $a_2c_2 \in E(G)$ and $a_3c_5 \in E(G)$. Thus, $[a_2, c_2, c_1, c_5, a_3]$ induces a P_5, which is a contradiction to our assumption that G is P_5-free. Hence, $\ell \geq 2$.

If $\ell \geq 3$, then $E_G[\{a_1, a_2, a_3\}, V(C)]$ is complete. Since $V(G) = V(C) \cup X_{\geq 2}(C)$, there is some $i \in [5]$ such that $p_2c_i \in E(G)$, and so $[a_1, c_i, a_2, p_2, a_3]$ induces a *dart*, which is a contradiction to our assumption that G is *dart*-free. Thus, $\ell = 2$.

Since $\ell = 2$, we have $a_1 \notin D(C)$. Hence, $a_1c_{i_C(a_1)-1} \notin E(G)$ but $a_1c_{i_C(a_1)}$, $a_1c_{i_C(a_1)+1}$, $a_1c_{i_C(a_1)+2} \in E(G)$. Recall that further $a_2c_{i_C(a_1)-1}, a_3c_{i_C(a_1)-1} \in E(G)$ by the minimality of $\sum_{i=1}^{3} \mathrm{dist}_{\bar{G}}(a_i, V(C))$. The set $[a_1, c_{i_C(a_1)}, a_2, c_{i_C(a_1)-1}, a_3]$ does not induce a *dart*, and so there is some $i \in \{2, 3\}$ such that $a_ic_{i_C(a_1)} \notin E(G)$. Again, by the minimality of $\sum_{i=1}^{3} \mathrm{dist}_{\bar{G}}(a_i, V(C))$, we have $a_{5-i}c_{i_C(a_1)} \in E(G)$. Similarly, since $[a_i, c_{i_C(a_1)+1}, a_1, c_{i_C(a_1)}, a_{5-i}]$ does not induce a *dart* and $a_1c_{i_C(a_1)+1} \in E(G)$, we have $a_2c_{i_C(a_1)+1} \notin E(G)$ or $a_3c_{i_C(a_1)+1} \notin E(G)$. Hence, there is some $j \in \{2, 3\}$ such that $a_jc_{i_C(a_1)+1} \notin E(G)$. Again, by the minimality of $\sum_{i=1}^{3} \mathrm{dist}_{\bar{G}}(a_i, V(C))$, it follows $a_{5-j}c_{i_C(a_1)} \in E(G)$. But now, $[a_1, c_{i_C(a_1)+1}, a_{5-j}, c_{i_C(a_1)-1}, a_j]$ induces a P_5, which is a contradiction to our assumption that G is P_5-free. The last contradiction completes our proof. (□)

Claim 52.4. *G is G_1-free.*

Proof. For the sake of a contradiction, let us suppose that S induces a G_1 and the vertices of S are denoted as in Fig. 7. Furthermore, let $T \subseteq V(G)$ be the set of vertices such that $(N_G(t) \cap S) \setminus \{g\} = \{g_1, g_2, g_3\}$ for each $t \in T$. Note that $g \in T$ and $(S \setminus \{g\}) \cup \{t\}$ induces a G_1 for each $t \in T$. If $V(G) = S \cup T$, then $T = \{g\}$ since G is prime, and we conclude $G \cong G_1$, which is a contradiction to our assumption that $G \ncong G_1$. Hence, by the connectivity of G, we may assume that there is some vertex $u \in N_G(S \cup T)$. Renaming vertices if necessary, we may assume $u \in N_G(S)$.

For each i, j, k with $\{i, j, k\} = [3]$, $C_{\{i,j\}}\colon g g_i g_{\{i,k\}} g_{\{j,k\}} g_j g \in \mathcal{C}_5(G)$, $g_k \in M(C_{\{i,j\}}) \setminus X_{\geq 2}(C_{\{i,j\}})$, and $g_{\{i,j\}} \in X_{\geq 3}(C_{\{i,j\}})$. From Claim 52.1 and Claim 52.2, we deduce

$$N_G(V(C_{\{i,j\}})) = M(C_{\{i,j\}}) \cup D(C_{\{i,j\}}) = \{g_k, g_{\{i,j\}}\} \cup (X_{\geq 2}(C_{\{i,j\}}) \setminus X_{\geq 3}(C_{\{i,j\}})).$$

Thus, u satisfies $j_{C_{\{i,j\}}}(u) = k_{C_{\{i,j\}}}(u) = 2$.

We assume first $gu \notin E(G)$. It follows $g_{\{i,k\}}u, g_{\{j,k\}}u \in E(G)$, and either $g_i u \in E(G)$ or $g_j u \in E(G)$ for each $\{i, j, k\} = [3]$, where the latter observation cannot be satisfied for all three triples $\{1, 2\}, \{1, 3\}$, and $\{2, 3\}$. Thus, $gu \in E(G)$.

If there are integers i, j, k with $\{i, j, k\} = [3]$ such that $g_i u, g_j u \in E(G)$, then $g_{\{i,k\}}u$, $g_{\{j,k\}}u \notin E(G)$ since $j_{C_{\{i,j\}}}(u) = k_{C_{\{i,j\}}}(u) = 2$. Thus, either $g_k u \in E(G)$ or $g_{i,j}u \in E(G)$. Since $u \notin T$, we have $g_k u \notin E(G)$ and $g_{i,j}u \in E(G)$, and so $[g_k, g_{i,k}, g_i, u, g_j]$ induces a P_5; a contradiction.

Finally, we consider the case that there is some $i \in [3]$ such that $g_i u \in E(G)$ but $g_j u \notin E(G)$ for each $j \in [3] \setminus \{i\}$. But now, $u \notin X_{\geq 2}(C_{[3]\setminus\{i\}})$, which is a contradiction to the above observations. Hence, $N_G(S \cup T) = \emptyset$, which completes our proof. (□)

For each $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$, we have $M(C) \setminus X_{\geq 2}(C) \neq \emptyset$ by Claim 52.3. Furthermore, Claim 52.1 implies that $V(C) \cup \{m, x\}$ induces a G_1 in G if $m \in M(C) \setminus X_{\geq 2}(C)$ and $x \in X_{\geq 3}(C)$. Thus, since G is G_1-free by Claim 52.4, $X_{\geq 3}(C) = \emptyset$, and so G is W_5-free and $\bar{G}$ is A_5-free, which completes our proof. □

By Lemma 51 and Lemma 52, it remains to study the q-chromatic number of G_1 and of prime $(P_5, dart, W_5)$-free graphs of independence number at least 3 whose complementary graphs are $(A_5, C_7, C_9, \ldots)$-free. We study graphs of this type by proving the next slightly stronger result. Note that the complementary graph of a *dart*-free graph is $T_{0,1,2}$-free. We show this stronger result because in this form this lemma is also applicable for (P_5, gem)-free graphs as we show in Chapter 7. For the definition of $\mathcal{G}^\star$ we refer to page 23.

Lemma 53. *If G is a prime (P_5, W_5)-free graph for which $\bar{G}$ is $(A_5, C_7, C_9, \ldots, T_{0,1,2})$-free, then G is perfect or $G \in \mathcal{G}^\star$ or $G \cong G'$ with*

$$G' \in \{C_5, G_2, G_3, G_3 - g, G_3 - g_{4,1}, G_3 - \{g, g_{4,1}\}, G_3 - \{g_{2,2}, g_{4,1}\}, G_3 - \{g, g_{2,2}, g_{4,1}\}, G_4\}.$$

Proof. For some maximal connected buoy $C\colon C_1C_2C_3C_4C_5C_1$ in G and each $i \in [5]$, let

$$A_i(C) = \{a : N_G(a) \cap V(C) = C_i \cup C_{i+2}\} \quad \text{and}$$
$$B_i(C) = \{b : N_G(b) \cap V(C) = C_i \cup C_{i+2} \cup C_{i+3}\}.$$

Furthermore, let

$$\mathcal{C}_5^\circ(G) = \text{Argmax}\{|B_1(C) \cup B_2(C) \cup \ldots \cup B_5(C)| : C \in \mathcal{C}_5(G)\}.$$

We introduce first five claims from which we finally deduce our desired result.

Claim 53.1. *If $C\colon C_1C_2C_3C_4C_5C_1$ is a maximal connected buoy, then $N_G(V(C)) = \bigcup_{i\in[5]} A_i(C) \cup B_i(C)$, and $A_{j-1}(C) \cup C_j$ is independent for each $j \in [5]$.*

Proof. Let $v \in N_G(V(C))$ be an arbitrary vertex. Since G is W_5-free and $\bar{G}$ is A_5-free, there are two integers $i_1, i_2 \in [5]$ such that $E_G[\{v\}, C_{i_1} \cup C_{i_2}]$ is anticomplete.

For the sake of a contradiction, let us suppose that, for each two integers $j_1, j_2 \in [5]$ with $j_2 = j_1 + 2$, one of the two sets $E_G[\{v\}, C_{j_1}]$, $E_G[\{v\}, C_{j_2}]$ is not complete. Since $v \in N_G(C)$, there are some $k \in [5]$ and a vertex $c_k \in C_k \cap N_G(v)$. Since, for every triple $(c_{k+1}, c_{k+2}, c_{k+3}) \in C_{k+1} \times C_{k+2} \times C_{k+3}$, $[v, c_k, c_{k+1}, c_{k+2}, c_{k+3}]$ does not induce a P_5, there is some $\ell \in \{k+1, k+2, k+3\}$ such that $E_G[\{v\}, C_\ell]$ is complete. Let $c_\ell \in C_\ell$. By our supposition, there are some $c_{\ell+2} \in C_{\ell+2} \setminus N_G(v)$ and $c_{\ell+3} \in C_{\ell+3} \setminus N_G(v)$. Since $[v, c_\ell, c_{\ell-1}, c_{\ell+3}, c_{\ell+2}]$ for each $c_{\ell-1} \in C_{\ell-1}$ and $[v, c_\ell, c_{\ell+1}, c_{\ell+2}, c_{\ell+3}]$ for each $c_{\ell+1} \in C_{\ell+1}$ do not induce copies of P_5, we have that $E_G[\{v\}, C_{\ell-1} \cup C_{\ell+1}]$ is complete, which contradicts our supposition. Thus, there are two integers $j_1, j_2 \in [5]$ such that $j_2 = j_1+2$ and $E_G[\{v\}, C_{j_1} \cup C_{j_2}]$ is complete.

If $i_2 = i_1 + 1$, then $j_1 = i_2 + 1$ and $j_2 = i_1 - 1$, and, by the maximality of C, we have that $E_G[\{v\}, C_{j-2}]$ is anticomplete, and so $v \in A_{j_1}(C)$. Thus, renaming vertices if necessary, we may assume $i_2 = i_1 + 2$, $j_1 = i_1 + 1$, and $j_2 = i_2 + 1$. For two adjacent vertices $c_{i_1-1}, c'_{i_1-1} \in C_{i_1-1}$ with $c_{i_1-1} \in N_G(v)$, we have $c'_{i_1-1} \in N_G(v)$ since $[c'_{i_1-1}, c_{i_1-1}, v, c_{j_1}, c_{i_2}]$ for some $c_{j_1} \in C_{j_1}$ and some $c_{i_2} \in C_{i_2}$ does not induce a P_5. By the connectedness of C_{i_1-1}, this observation implies $v \in A_{j_1}(C) \cup B_{j_1}(C)$. Furthermore, by the arbitrariness of v, it follows $N_G(V(C)) = \bigcup_{i\in[5]} A_i(C) \cup B_i(C)$. Thus, C_j is a module for each $j \in [5]$, and so $|C_j| = 1$ since G is prime. In particular, each connected buoy $C'\colon C'_1C'_2C'_3C'_4C'_5C'_1$ is indeed an induced cycle, and so $A_{j-1}(C) \cup C_j$ is an independent set for each $j \in [5]$. (□)

Claim 53.2. *If $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$, then*

(i) *$N^3_G(V(C)) = \emptyset$ and*

(ii) *$N^2_G(V(C))$ is an independent set.*

Proof. Let W be a set of vertices inducing a component in $G - N_G[V(C)]$ and let us suppose, for the sake of a contradiction, that $w_1 \in N^2_G(V(C)) \cap W$ and $w_2 \in [N^2_G(V(C)) \cup N^3_G(V(C))] \cap W$ are two arbitrarily chosen adjacent vertices. By Claim 53.1, there is some vertex in $\bigcup_{i=1}^5 (A_i(C) \cup B_i(C))$ that is adjacent to w_1. Let v be an arbitrary neighbour of w_1 in $N_G(V(C))$. Renaming vertices if necessary, we may assume $v \in A_i(C) \cup B_i(C)$. Since $[c_{i-1}, c_i, v, w_1, w_2]$ does not induce a P_5, we have $vw_2 \in E(G)$, and so $w_2 \in N^2_G(V(C))$. Thus, since v is arbitrarily chosen, $N_G(w_1) \cap N_G(V(C)) \subseteq$

$N_G(w_2) \cap N_G(V(C))$, and so, by the arbitrariness of w_1 and w_2, W is a homogeneous set in G, which contradicts the fact that G is prime. Hence, (i) and (ii) follow. (□)

Claim 53.3. *If $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$, then,*

(i) *for each $j \in [5]$ and each $a_j \in A_j(C)$, $E_G[\{a_j\}, A_{j-1}(C) \cup A_{j+1}(C) \cup B_{j+1}(C)]$ is complete and $E_G[\{a_j\}, A_j(C) \cup B_j(C) \cup B_{j+2}(C) \cup N^2_G(V(C)) \setminus \{a_j\}]$ is anticomplete.*

(ii) *for each $j \in [5]$, each set $B_j(C)$ is a module in $G[N_G[V(C)]]$,*

(iii) *there is some integer $p(C) \in [5]$ such that*

  (a) *$B_{p(C)+1}(C) \cup B_{p(C)+3}(C) \cup B_{p(C)+4}(C) = \emptyset$,*

  (b) *$E_G[B_{p(C)}(C), B_{p(C)+2}(C)]$ is anticomplete,*

  (c) *$E_G[B_{p(C)}(C) \cup B_{p(C)+2}(C), N^2_G(V(C))]$ is complete if none of the three sets $B_{p(C)}(C), B_{p(C)+2}(C)$, and $N^2_G(V(C))$ is empty,*

  (d) *$|B_{p(C)}(C) \cup B_{p(C)+2}(C) \cup N^2_G(V(C))| = 3$ or at least one of the three sets $B_{p(C)}(C)$, $B_{p(C)+2}(C)$, $N^2_G(V(C))$ is empty, and*

(iv) *$\bigcup_{i=1}^5 A_i(C) = \emptyset$ or $N^2_G(V(C)) = \emptyset$.*

Proof. Let us assume $a_j \in A_j(C)$. Note that Claim 53.1 implies that $A_j(C)$ is independent. By considering the cycle $C' : a_jc_{j+2}c_{j+3}c_{j+4}c_ja_j \in \mathcal{C}_5(G)$, the same claim implies

$$A_{j-1}(C) \cup A_{j+1}(C) \cup B_{j+1}(C) \subseteq N_G(a_j) \quad \text{and} \quad N_G(a_j) \cap [B_j(C) \cup B_{j+2}(C)] = \emptyset.$$

Furthermore, since $[w, a_j, c_{j+2}, c_{j+3}, c_{j+4}]$ does not induce a P_5 for some $w \in N^2_G(V(C))$, we have that $E_G[\{a_j\}, N^2_G(V(C))]$ is anticomplete. Thus, (i) follows.

Recall that $N^i_G(V(C)) = \emptyset$ for each $i \geq 3$ by Claim 53.2. Hence, $V(G) \setminus N_G[V(C)] = N^2_G(V(C))$. For simplicity, whenever there is some $i \in [5]$ and a vertex b_i, we let $b_i \in B_i(C)$. Since neither $[c_{i+2}, b_i, c_i, c_{i-1}, b_{i+1}]$ induces a P_5 in G if $b_ib_{i+1} \notin E(G)$ nor $\{c_{i+3}, c_{i+2}, b_i, b_{i+1}, c_{i-1}, c_i\}$ induces a $T_{0,1,2}$ in $\bar{G}$, we have that $B_i(C) = \emptyset$ or $B_{i+1}(C) = \emptyset$ for each $i \in [5]$. Thus, (a) is proven. Furthermore, we conclude (b) from the fact that the set $\{b_{p(C)+2}, c_{p(C)-1}, c_{p(C)}, b_{p(C)}, c_{p(C)+2}, c_{p(C)+1}\}$ does not induce a $T_{0,1,2}$ in $\bar{G}$.

Let us assume that $w \in N^2_G(V(C))$ is an arbitrarily chosen vertex, and there are two vertices $b_{p(C)} \in B_{p(C)}(C)$ and $b_{p(C)+2} \in B_{p(C)+2}(C)$. Since $w \in N^2_G(V(C))$, (i) implies that there is a vertex $b \in B_{p(C)}(C) \cup B_{p(C)+2}(C)$ which is adjacent to w. Renaming vertices if necessary, we may assume $b \in \{b_{p(C)}, b_{p(C)+2}\}$. Since neither $[w, b_{p(C)}, c_{p(C)-2}, c_{p(C)-1}, b_{p(C)+2}]$ nor $[w, b_{p(C)+2}, c_{p(C)-1}, c_{p(C)-2}, b_{p(C)}]$ induces a P_5, we have $b_{p(C)}w, b_{p(C)+2}w \in E(G)$. Thus, by considering the cycle $C'\colon wb_{p(C)}c_{p(C)-2}c_{p(C)-1}$ $b_{p(C)+2}w \in \mathcal{C}_5(G)$, Claim 53.1 and (b) imply $B_{p(C)}(C) \cup B_{p(C)+2}(C) \subseteq N_G(w)$, and so

(c) follows. Recall that, by (i), $E_G[\{a_j\}, N_G^2(V(C))]$ is anticomplete. Hence, $N_G^2(V(C))$ is a module, and (d) follows if (ii) holds since G is prime.

For the sake of a contradiction, let us suppose that $B_j(C)$ is not a module in graph $G[N_G[V(C)]]$. Thus, there are two vertices $b_j, b'_j \in B_j(C)$ and a vertex $v \in N_G(V(C)) \setminus B_j(C)$ such that $b_j v \in E(G)$ but $b'_j v \notin E(G)$. By Claim 53.1, there is some $i \in [5]$ such that $v \in A_i(C) \cup B_i(C)$. By (a), (b), and the fact $v \notin B_j(C)$, we have $v \notin B_i(C)$, and so $v \in A_i(C)$. Furthermore, (i) implies $i = j+1$ or $i = j+2$. By the symmetry of the cycle, we may assume $i = j+1$. But now, $\{c_{j+3}, b'_j, c_{j+2}, b_j, v, c_{j+1}\}$ if $b_j b'_j \notin E(G)$ and $\{b_j, v, c_{j+3}, b'_j, c_j, c_{j+4}\}$ if $b_j b'_j \in E(G)$ induces a $T_{0,1,2}$ in $\bar{G}$, which is a contradiction to our assumption that $\bar{G}$ is $T_{0,1,2}$-free. Thus, (ii) as well as (d) follow.

We finally show (iv). Let $w \in N_G^2(V(C))$. By (i), by Claim 53.1, and by renaming vertices if necessary, we may assume $b_1 \in B_1(C)$ is adjacent to w. Note that (i) implies that $E_G[A_5(C), B_1(C)]$ is complete and $E_G[A_1(C) \cup A_4(C), B_1(C)]$ is anticomplete. Furthermore, $E_G[A_i(C), N_G^2(V(C))]$ is anticomplete by (i) for each $i \in [5]$. Since neither $[w, b_1, c_1, c_2, a_2]$ nor $[w, b_1, c_1, c_5, a_3]$ induces a P_5 for each $a_2 \in A_2(C)$ and each $a_3 \in A_3(C)$, we have that $E_G[A_2(C) \cup A_3(C), \{b_1\}]$ is complete. Thus, $E_G[A_i(C) \cup \{c_{i+1}\}, \{b_1\}]$ is either complete or anticomplete for each $i \in [5]$. For the sake of a contradiction, let us suppose that there is some $i \in [5]$ such that $A_i(C) \neq \emptyset$. The fact that $A_i(C) \cup \{c_{i+1}\}$ is not a homogeneous set implies that there are vertices $a_i, c'_i \in A_i(C) \cup \{c_{i+1}\}$ and $v \notin A_i(C) \cup \{c_{i+1}\}$ such that $a_i v \notin E(G)$ but $c'_i v \in E(G)$. We let $C'\colon c'_i c_{i+2} c_{i+3} c_{i+4} c_i c'_i \in \mathcal{C}_5(G)$. For the sake of simplicity, let us rename the vertices of C such that $C'\colon c'_1 c'_2 c'_3 c'_4 c'_5 c'_1 \in \mathcal{C}_5(G)$ and $c_{i+2} = c'_{i+1}$. Note that by the fact that $E_G[A_i(C) \cup \{c_{i+1}\}, \{b_1\}]$ is either complete or anticomplete, we have $b \in \bigcup_{j=1}^5 B_j(C')$. Furthermore, since $c'_i w \notin E(G)$ by (i), it follows $w \in N_G^2(V(C'))$. By Claim 53.1,

$$v \in \bigcup_{j=1}^{5} (A_j(C') \cup B_j(C')).$$

Since $a_i v \notin E(G)$ and $a_i \in A_{i-1}(C')$, (i) and (iii) imply $v \in B_{p(C')}(C') \cup B_{p(C')+2}(C')$. Let $j \in \{p(C'), p(C')+2\}$ such that $v \in B_j(C')$. If $vw \in E(G)$, then, similarly as for b_1 and C, we have that $E_G[A_{i-1}(C') \cup \{c'_i\}, \{v\}]$ is either complete or anticomplete, which contradicts the fact that $a_i \in A_{i-1}(C')$ and $a_i v \notin E(G)$ while $c'_i v \in E(G)$. If $vw \notin E(G)$, then $b_1 \neq v$, and so, by (c), it follows $b_1, v \in B_j(C')$. Since $B_j(C')$ is a module in $G[N_G[V(C')]]$ by (ii), we have $b_1 c'_i \in E(G)$ but $a_i b_1 \notin E(G)$, which contradicts the fact that $E_G[A_i(C) \cup \{c_{i+1}\}, \{b_1\}]$ is either complete or anticomplete. Thus, $\bigcup_{i=1}^5 A_i(C) = \emptyset$ and (iv) follows. (□)

Claim 53.4. *If $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5^\circ(G)$ and $\bigcup_{i=1}^5 A_i(C) \neq \emptyset$, then*

(i) *there are two vertices $b_1 \in B_{i+1}(C)$ and $b_2 \in B_{i+3}(C)$ such that $\{a_i, a_{i+2}, b_1, b_2\} \cup V(C)$ induces a G_4 if there exist an integer $i \in [5]$ and two adjacent vertices*

$a_i \in A_i(C)$ and $a_{i+2} \in A_{i+2}(C)$,

(ii) *for each $i \in [5]$ and each $a \in A_i(C)$, there is some $b \in B_{i+3}(C) \cup B_{i+4}(C)$ that is non-adjacent to a,*

(iii) *for each $i \in \{p(C), p(C)+2\}$ and each $b \in B_i(C)$, there is at most one $a \in A_{i+1}(C) \cup A_{i+2}(C)$ that is non-adjacent to b, and*

(iv) *for each $i \in \{p(C), p(C)+2\}$, $|A_{i+1}(C) \cup A_{i+2}(C)| \leq |B_i(C)| \leq 1$.*

Proof. Before we start, let us note that $N_G^2(V(C)) = \emptyset$ by Claim 53.3 (iv), and so $|B_i(C)| \leq 1$ for each $i \in [5]$ by Claim 53.3 (ii) and since G is prime.

We focus first on verifying (i). Note that $C'\colon a_i c_{i+2} c_{i+3} c_{i+4} c_i a_i \in \mathcal{C}_5(G)$ but $a_{i+2} \in A_{i+2}(C) \cap (B_{p(C')}(C') \cup B_{p(C')+2}(C'))$. Since $C \in \mathcal{C}_5^\circ(G)$, there is some $b_1 \in (B_{p(C)}(C) \cup B_{p(C)+2}(C)) \setminus (B_{p(C')}(C') \cup B_{p(C')+2}(C'))$. By Claim 53.1,

$$b_1 \in \bigcup_{j=1}^{5} A_j(C').$$

Thus, $a_i b_1 \notin E(G)$ but $b_1 c_{i+1} \in E(G)$. If $b_1 \in B_{i+4}(C)$, then $a_{i+2} b_1 \notin E(G)$ by Claim 53.3 (i), and $[b_1, c_{i+1}, c_i, a_i, a_{i+2}]$ induces a P_5. From this contradiction to our assumption on G, we conclude $b_1 \notin B_{i+4}(C)$. Since $b_1 \in \bigcup_{j=1}^5 A_j(C')$ by Claim 53.1, we have $b_1 \in B_{i+3}(C)$. Furthermore, $a_{i+2} b_1 \in E(G)$ by Claim 53.3 (i). Similarly, considering $C''\colon a_{i+2} c_{i+4} c_i c_{i+1} c_{i+2} a_{i+2}$ instead of C', we obtain that there is some $b_2 \in B_{i+1}(C)$ with $a_i b_2 \in E(G)$ but $a_{i+2} b_2 \notin E(G)$. By Claim 53.3 (b), $b_1 b_2 \notin E(G)$, and so $G[V(C) \cup \{a_i, a_{i+2}, b_1, b_2\}] \cong G_4$, which implies (i).

We continue by proving (ii). For the sake of a contradiction, let us suppose that there is some $i \in [5]$ and some vertex $a \in A_i(C)$ such that each $b \in B_{i+3} \cup B_{i+4}$ is adjacent to a. Since G is prime, $\{a, c_{i+1}\}$ is not a homogeneous set. Thus, there is some vertex $v \in V(G)$ such that either $av \in E(G)$ and $c_{i+1} v \notin E(G)$ or $av \notin E(G)$ and $c_{i+1} v \in E(G)$. Clearly, $v \in N_G(V(C))$ and $C'\colon c_i a c_{i+2} c_{i+3} c_{i+4} c_i \in \mathcal{C}_5(G)$. Thus, from Claim 53.1 and Claim 53.3 we deduce

$$v \in \begin{cases} [A_{i-2}(C) \cup A_{i+2}(C)] \cap [B_{p(C')}(C') \cup B_{p(C')+2}(C')] & \text{if } av \in E(G), c_{i+1} v \notin E(G), \\ \left[\bigcup_{i=1}^5 A_i(C')\right] \cap [B_{i+3}(C) \cup B_{i+4}(C)] & \text{if } av \notin E(G), c_{i+1} v \in E(G). \end{cases}$$

By our assumption on a, we conclude $v \notin B_{i+3}(C) \cup B_{i+4}(C)$, which means $av \in E(G)$ and $c_{i+1} v \notin E(G)$. Hence, (i) implies that there is some $b \in B_{i+3}(C) \cup B_{i+4}(C)$ that is non-adjacent to a. This conclusion is a contradiction to our supposition on a. Thus, (ii) follows.

We focus next on a proof for (iii) and let $b \in B_i(C)$. For the sake of a contradiction, let us suppose that there are two integers $j, k \in \{i+1, i+2\}$, which are not necessarily distinct, and two vertices $a_1 \in A_j(C)$ and $a_2 \in A_k(C)$ that are non-adjacent to b. If

$j \neq k$, then, renaming vertices if necessary, we may assume $j = i+1$ and $k = i+2$. By Claim 53.3 (i), $a_1a_2 \in E(G)$, and so $[c_i, b, c_k, a_2, a_1]$ induces a P_5; a contradiction. Hence, $j = k$ and, renaming vertices if necessary, we by symmetry may assume $j = k = i+1$. Since G is prime, $\{a_1, a_2\}$ is not a homogeneous set. Thus, renaming vertices if necessary, there is some vertex $v \in V(G)$ such that $a_1v \in E(G)$ but $a_2v \notin E(G)$. Clearly, $v \in N_G(V(C)) \setminus \{a_1, a_2, b\}$. Considering the two cycles $C' : a_1c_{i+3}c_{i+4}c_ic_{i+1}a_1$ and $C'' : a_2c_{i+3}c_{i+4}c_ic_{i+1}a_2$, Claim 53.1 implies

$$v \in [B_{p(C')}(C') \cup B_{p(C')+2}(C')] \cap \left[\bigcup_{i=1}^{5} A_i(C'')\right].$$

In particular, either $c_{i+1}v \in E(G)$ or $c_{i+3}v \in E(G)$. Note that further either $N_G(v) \cap V(C) = N_G(v) \cap V(C'')$ or $N_G(v) \cap V(C) = (N_G(v) \cap V(C'')) \cup \{c_{i+2}\}$. If $c_{i+1}v \in E(G)$, then $c_{i+3}v \notin E(G)$. Hence, $c_{i+4}v \in E(G)$. By Claim 53.3 (iii) (a), $c_{i+2}v \notin E(G)$. However, $bv \in E(G)$ by Claim 53.3 (i). Note that $a_1 \in A_{i+1}(C)$ and $v \in A_{i+4}(C)$ are adjacent. By (i), there is some $b' \in B_{i+2}(C)$ such that $b'v \notin E(G)$. By Claim 53.3 (i) and (iii), $a_1b', a_2b' \in E(G)$ but $bb' \notin E(G)$. Recall that $a_1a_2 \notin E(G)$ since $A_{i+1}(C)$ is independent by Claim 53.1. Thus, $[a_2, b', a_1, v, b]$ induces a P_5; a contradiction. Hence, $c_{i+1}v \notin E(G)$ but $c_{i+3}v \in E(G)$, and so $c_iv \in E(G)$. If $c_{i+2}v \in E(G)$, then $b, v \in B_i(C)$, which contradicts the fact that $|B_i(C)| \leq 1$. Thus, $c_{i+2}v \notin E(G)$ and $v \in A_{i+3}(C)$. By Claim 53.3 (i), $bv \notin E(G)$. Hence, $[c_{i+2}, b, c_i, v, a_1]$ induces a P_5; a contradiction. This final contradiction completes our proof for (iii).

Let us finally consider (iv) and let us assume $A_{i+1}(C) \cup A_{i+2}(C) \neq \emptyset$. By (ii) and the fact that $B_{i-1}(C) \cup B_{i+1}(C) = \emptyset$, it follows that, for each $a \in A_{i+1}(C) \cup A_{i+2}(C)$, there is a vertex in $B_i(C)$ that is non-adjacent to a. Since $|B_i(C)| \leq 1$, the vertex $b \in B_i(C)$ is non-adjacent to all vertices of $A_{i+1}(C) \cup A_{i+2}(C)$. By (iii), it follows $|A_{i+1}(C) \cup A_{i+2}(C)| \leq 1$, and thus (iv) follows. (□)

Claim 53.5. *If $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ and $N_G^2(V(C)) \neq \emptyset$, then*

(i) *$G \cong G_2$ if none of the three sets $B_{p(C)}(C), B_{p(C)+2}(C)$, and $N_G^2(V(C))$ is empty, and*

(ii) *for each $C' \in \mathcal{C}_5(G)$, $N_G^2(V(C')) \neq \emptyset$.*

Proof. We focus on a short proof for (i) first. By Claim 53.1, $N_G(V(C)) = \bigcup_{i=1}^{5}(A_i(C) \cup B_i(C))$. Furthermore, from Claim 53.3 (iii) (a) and (iv) as well as from the fact $N_G^2(V(C)) \neq \emptyset$, we obtain $N_G(V(C)) = B_{p(C)}(C) \cup B_{p(C)+2}(C)$. By Claim 53.3 (iii) (d), $|B_{p(C)}(C)| = |B_{p(C)+2}(C)| = |N_G^2(V(C))| = 1$. Additionally, $V(G) = N_G[V(C)] \cup N_G^2(V(C))$ by Claim 53.2 and the result follows from Claim 53.3 (iii) (b) and (c).

Let us consider (ii). Clearly, by (i) and the fact that (ii) holds for G if $G \cong G_2$, we may assume either $B_{p(C)}(C) = \emptyset$ or $B_{p(C)+2}(C) = \emptyset$. Renaming vertices if necessary, we may assume the latter case. Furthermore, we only need to consider some arbitrary $C' \in \mathcal{C}_5(G)\backslash\{C\}$. Note that Claim 53.1 and Claim 53.3 (iii) and (iv) imply $N_G(V(C)) = B_{p(C)}(C)$. Thus, $E_G[B_{p(C)}(C), \{c_{p(C)}, c_{p(C)+2}, c_{p(C)+3}, \}]$ is complete. Since $N_G^2(V(C))$ is independent, we have $V(C) \cap V(C') = \emptyset$. Consequently,

$$\mathrm{dist}_G(c_{p(C)-1}, V(C')), \mathrm{dist}_G(c_{p(C)+1}, V(C')) \geq 2,$$

which completes our proof for (ii). (□)

Now, the proof of the lemma can be completed as follows:

Let us assume that G is not perfect. Since G is P_5-free and $\bar{G}$ is $(C_7, C_9, \ldots)$-free, the Strong Perfect Graph Theorem implies $\mathcal{C}_5(G) \neq \emptyset$.

Let $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ be an arbitrary cycle. Recall that, by Claim 53.1 and Claim 53.2 (i),

$$V(G) = V(C) \cup \left(\bigcup_{i=1}^{5}(A_i(C) \cup B_i(C))\right) \cup N_G^2(V(C)).$$

From Claim 53.3 (iii) (a), we have that there is some integer $p(C) \in [5]$ such that $B_{p(C)+1}(C) \cup B_{p(C)+3}(C) \cup B_{p(C)+4}(C) = \emptyset$.

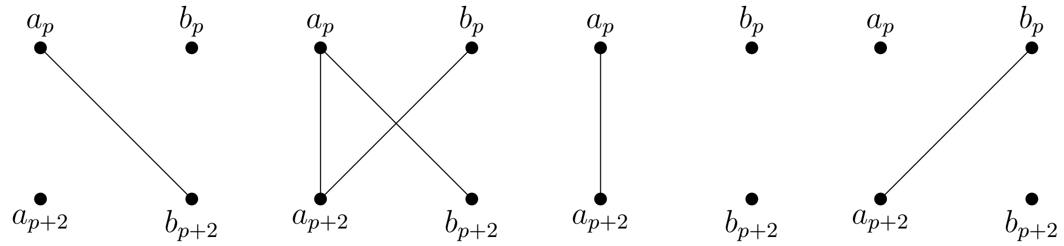
If none of the three sets $B_{p(C)}(C), B_{p(C)+2}(C), N_G^2(V(C))$ is empty, then $G \cong G_2$ by Claim 53.5 (i).

If $B_{p(C)+2}(C) = \emptyset$ but $N_G^2(V(C)) \neq \emptyset$, then $\bigcup_{i=1}^5 A_i(C) = \emptyset$ by Claim 53.3 (iv), and so $V(G) = V(C) \cup B_{p(C)}(C) \cup N_G^2(V(C))$. Additionally, $E_G[\{c_{p(C)+1}, c_{p(C)+4}\}, N_G(V(C))]$ is anticomplete, $E_G[\{c_{p(C)}, c_{p(C)+2}, c_{p(C)+3}\}, N_G(V(C))]$ is complete, and $N_G^2(V(C))$ is independent, by Claim 53.2 (ii). By Claim 53.5 (ii), it follows $N_G^2(V(C')) \neq \emptyset$ for each $C'\colon c'_1c'_2c'_3c'_4c'_5c'_1 \in \mathcal{C}_5(G)$. Arguing in the exact same way for C' as we did for C we obtain that $V(G) - N_G[V(C')]$ is independent and that there is some integer $i \in [5]$ such that $E_G[\{c'_i, c'_{i+2}, c'_{i+3}\}, N_G(V(C'))]$ is complete and $E_G[\{c'_{i+1}, c'_{i+4}\}, N_G(V(C'))]$ is anticomplete, since in this case $G \ncong G_2$ and $N_G^2(V(C')) \neq \emptyset$. Hence, $G \in \mathcal{G}^\star$. Analogously, $G \in \mathcal{G}^\star$ if $B_{p(C)}(C) = \emptyset$ but $N_G^2(V(C)) \neq \emptyset$. Thus, we may consider the case where $N_G^2(V(C)) = \emptyset$.

Let us assume for the rest of our proof that we additionally have $C \in \mathcal{C}_5^\circ(G)$. By Claim 53.3 (ii) and the fact that G is prime, $|B_i(C)| \leq 1$ for each $i \in \{p(C), p(C)+2\}$. Furthermore, by Claim 53.4 (iv),

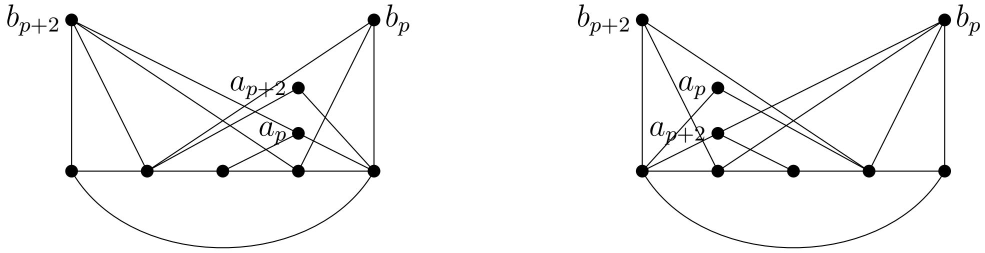
$$|A_{p(C)+1}(C) \cup A_{p(C)+2}(C)| \leq |B_{p(C)}(C)| \leq 1 \quad \text{and}$$
$$|A_{p(C)+3}(C) \cup A_{p(C)+4}(C)| \leq |B_{p(C)+2}(C)| \leq 1.$$

Moreover, Claim 53.4 (ii) implies $A_{p(C)}(C) = \emptyset$. Thus, $|V(G)| \leq 9$.



(a) $a_p \in A_{p(C)+1}(C)$, $a_{p+2} \in A_{p(C)+3}(C)$ **(b)** $a_p \in A_{p(C)+1}(C)$, $a_{p+2} \in A_{p(C)+4}(C)$ **(c)** $a_p \in A_{p(C)+2}(C)$, $a_{p+2} \in A_{p(C)+3}(C)$ **(d)** $a_p \in A_{p(C)+2}(C)$, $a_{p+2} \in A_{p(C)+4}(C)$

Fig. 11: Illustration of the adjacences in $\{b_p, b_{p+2}, a_p, a_{p+2}\}$



(a) Case $A_{p(C)+2}(C) \cup A_{p(C)+4}(C) = \emptyset$ **(b)** Case $A_{p(C)+1}(C) \cup A_{p(C)+3}(C) = \emptyset$

Fig. 12: Illustration of the symmetry between cases $A_{p(C)+1}(C) \cup A_{p(C)+3}(C) = \emptyset$ and $A_{p(C)+2}(C) \cup A_{p(C)+4}(C) = \emptyset$

If there is a vertex $a_p \in A_{p(C)+1}(C) \cup A_{p(C)+2}(C)$, then there is also a vertex $b_p \in B_{p(C)}(C)$ with $a_pb_p \notin E(G)$ by Claim 53.4 (ii). Furthermore, by Claim 53.3 (i), $E_G[\{a_p\}, B_{p(C)+2}(C)]$ is complete if $a_p \in A_{p(C)+1}(C)$ and anticomplete otherwise. Similarly, if there is a vertex $a_{p+2} \in A_{p(C)+3}(C) \cup A_{p(C)+4}(C)$, then there is also a vertex $b_{p+2} \in B_{p(C)+2}(C)$ with $a_{p+2}b_{p+2} \notin E(G)$, and $E_G[\{a_{p+2}\}, B_{p(C)}(C)]$ is complete if $a_{p+2} \in A_{p(C)+4}(C)$ and anticomplete otherwise. Recall that $E_G[B_{p(C)}(C), B_{p(C)+2}(C)]$ is anticomplete by Claim 53.3 (iii) (b). So note that the adjacencies on the set $\{a_p, b_p, b_{p+2}\}$ and on the set $\{a_{p+2}, b_p, b_{p+2}\}$ are forced regardless of the existence of a_{p+2} and a_p, respectively. It is left to argue whether or not $a_pa_{p+2} \in E(G)$ in those four cases. A complete illustration can be seen in Figure 11. If $a_p \in A_{p(C)+1}(C)$ and $a_{p+2} \in A_{p(C)+4}(C)$, then $a_pa_{p+2} \in E(G)$ since $[b_p, a_{p+2}, c_{p(C)+1}, a_p, b_{p+2}]$ does not induce a P_5, and so $G \cong G_4$ by Claim 53.4 (i). If $a_p \in A_{p(C)+2}(C)$ and $a_{p+2} \in A_{p(C)+3}(C)$, then $a_pa_{p+2} \in E(G)$ by Claim 53.3 (i), and so

$$\{c_{p(C)+1}, c_{p(C)}, b_{p+2}, c_{p(C)+4}, a_p, a_{p+2}, c_{p(C)+3}, b_p, c_{p(C)+2}\}$$

induces a G_3, note that we counter-clockwise order the vertices as in Figure 9 starting at g. Hence, $A_{p(C)+1}(C) \cup A_{p(C)+3}(C) = \emptyset$ or $A_{p(C)+2}(C) \cup A_{p(C)+4}(C) = \emptyset$. Using the symmetry of the cycle, which is illustrated in Figure 12, and renaming vertices if necessary, we may assume the latter case. If the vertices $a_p \in A_{p(C)+1}(C)$ and

$a_{p+2} \in A_{p(C)+3}(C)$ exist, then $a_p a_{p+2} \notin E(G)$ since otherwise Claim 53.4 (i) implies the existence of a vertex $b \in B_{p(C)+4}(C)$, which is not possible by Claim 53.3 (iii) (a). Thus, $\{a_{p+2}, c_{p(C)}, c_{p(C)+4}, b_{p+2}, a_p, c_{p(C)+1}, \ c_{p(C)+2}, b_p, c_{p(C)+3}\}$ induces a G_3 if a_p and a_{p+2} exist, and so we may assume that a_p or a_{p+2} does not exist. Hence,

- $G \cong G_4 - g_1 \cong G_3 - g$ or $G \cong G_3 - g_{4,1}$ or $G \cong G_3 - \{g_{2,2}, g_{4,1}\}$ if a_{p+2} and b_{p+2} exist,
- $G \cong G_3 - g$ or $G \cong G_3 - \{g, g_{4,1}\}$ or $G \cong G_3 - \{g, g_{2,2}, g_{4,1}\}$ if a_{p+2} does not but b_{p+2} exists,
- $G \cong G_3 - \{g_{2,2}, g_{4,1}\}$ or $G \cong G_3 - \{g, g_{2,2}, g_{4,1}\}$ if $V(G) \neq V(C)$, and neither a_{p+2} nor b_{p+2} exists, and
- $G \cong C_5$ if $V(G) = V(C)$.

The last observation completes our proof. □

By Lemma 51, Lemma 52, and Lemma 53, all prime $(P_5, dart)$-free graphs of independence number at least 3 are characterised. We continue by colouring these graphs.

Lemma 54. *If $G \in \mathcal{G}^\star$ is a $(P_5, Q[P_4])$-free graph such that $\bar{G}$ is $(C_7, C_9, \ldots)$-free, and $q\colon V(G) \to \mathbb{N}_0$ is a vertex-weight function, then*

$$\chi_q(G) = \max\{\omega_q(G), \max\{\chi_q(C) : C \in \mathcal{C}_5(G)\}\}.$$

Proof. Clearly,

$$\chi_q(G) \geq \max\{\omega_q(G), \max\{\chi_q(C) : C \in \mathcal{C}_5(G)\}\}.$$

For the sake of a contradiction, let us suppose that q is a minimal counterexample, that is,

$$\chi_q(G) > \max\{\omega_q(G), \max\{\chi_q(C) : C \in \mathcal{C}_5(G)\}\} \quad \text{and}$$
$$\chi_{q'}(G) \leq \max\{\omega_{q'}(G), \max\{\chi_{q'}(C) : C \in \mathcal{C}_5(G)\}\}$$

for each vertex-weight function $q'\colon V(G) \to \mathbb{N}_0$ with $q'(G) < q(G)$. We clearly may assume that q is $\lhd_\chi^G$-minimal.

If $G[q]$ is C_5-free, then it is perfect by the Strong Perfect Graph Theorem, and so

$$\chi_q(G) = \chi_q(G[q]) = \omega_q(G[q]) = \omega_q(G)$$

by Lemma 35 and Observation 36. Hence, we may assume $\mathcal{C}_5^\star(G[q], q) \neq \emptyset$.

Let $C\colon c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5^\star(G[q], q)$ and, in view of an application of Lemma 48, $C' \in \mathcal{C}_5(G)$ with $V(C) \neq V(C')$. Renaming vertices if necessary, we may assume that

$E_G[\{c_1, c_3, c_4\}, N_G(V(C))]$ is complete. Thus, $|V(C') \cap V(C)| \leq 3$. As an immediate consequence, we obtain $|V(C') \cap V(C)| \leq 1$ from the latter fact since C' is (C_3, C_4)-free. In particular, it follows $|V(C') \setminus N_G[V(C)]| \leq 2$ and $|V(C') \cap N_G(V(C))| \geq 2$ since $V(G) \setminus N_G[V(C)]$ is independent. Since $E_G[\{c_1, c_3, c_4\} \cap V(C'), N_G(V(C))]$ is complete, we have $|V(C') \cap V(C)| = 0$ or that $N_G(V(C)) \cap V(C')$ is independent. However, the latter case cannot occur since $V(C') \setminus N_G[V(C)]$ is independent as well. Thus, $V(C') \cap V(C) = \emptyset$. Since $\{c_2, c_1, p_1, p_2, p_3, p_4, c_4\}$ does not induce a copy of $Q[P_4]$ for each four vertices $p_1, p_2, p_3, p_4 \in N_G(V(C))$, $G[N_G(V(C))]$ is P_4-free. Hence, $|V(C') \setminus N_G[V(C)]| = 2$ and $|V(C') \cap N_G(V(C))| = 3$. As an interesting conclusion, we have $|V(C') \cap I| \geq 2$ for each $C' \in \mathcal{C}_5(G[q])$ if $V(G[q]) \setminus N_G[V(C)] \subseteq I$ and $|I \cap V(C)| \geq 2$.

Let $I_1 = \{c_1, c_4\} \cup (V(G[q]) \setminus N_G[V(C)])$, $I_2 = \{c_2, c_4\} \cup (V(G[q]) \setminus N_G[V(C)])$, $f_{q'} = 0$, and $f_q = \omega_q(G[q])$. By applying Lemma 48 on $G[q]$, we conclude $\chi_q(C) \leq \omega_q(G[q]) = \omega_q(G)$,

$$\begin{aligned}\omega_q(G) &= \omega_q(G - I_1) = q(\{c_2, c_3\}), \quad \text{and} \\ \omega_q(G) &= \omega_q(G - I_2) = \max\{q(\{c_1, c_5\}), q(\{c_1\} \cup S)\}\end{aligned}$$

for some clique S in $G[N_G(V(C))]$. However, since $q(c_5) \geq 1$, Lemma 34 implies $q(c_5) > \chi_q(G[S]) = \omega_q(G[S]) = q(S)$. Thus, $\omega_q(G) = q(\{c_1, c_5\})$, and so

$$2\omega_q(G) < q(\{c_1, c_5\}) + q(\{c_2, c_3\}) + q(c_4) = q(C) \leq 2\chi_q(C) \leq 2\omega_q(G).$$

This contradiction proves our lemma. □

Lemma 55. *If $q\colon V(G_4) \to \mathbb{N}_0$ is a vertex-weight function, then*

$$\chi_q(G_4) = \max\{\omega_q(G_4), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_4)\}\}.$$

Proof. Clearly, $\chi_q(G_4) \geq \max\{\omega_q(G_4), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_4)\}\}$. For the sake of a contradiction, let us suppose that q is a minimal counterexample, that is,

$$\chi_q(G_4) > \max\{\omega_q(G_4), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_4)\}\}$$

and

$$\chi_{q'}(G_4) \leq \max\left\{\omega_{q'}(G_4), \max\{\chi_{q'}(C) : C \in \mathcal{C}_5(G_4)\}\right\}$$

for each vertex-weight function $q'\colon V(G_4) \to \mathbb{N}_0$ with $q'(G_4) < q(G_4)$.

Let $C \in \mathcal{C}_5(G_4)$. By the pigeonhole principle, there is an integer $i \in [9]$ such that $g_{i+4}, g_{i+5} \in V(C)$. Clearly, both vertices have distance 2 in G_4, $N_{G_4}(g_{i+4}) \cap N_{G_4}(g_{i+5}) = \{g_i, g_{i+1}, g_{i+8}\}$, $N_{G_4}(g_{i+4}) \setminus N_{G_4}(g_{i+5}) = \{g_{i+7}\}$, and $N_{G_4}(g_{i+5}) \setminus N_G(g_{i+4}) = \{g_{i+2}\}$. Since $g_{i+1}g_{i+7}, g_{i+2}g_{i+8} \in E(G)$, we have $C = C_{g_i}\colon g_ig_{i+4}g_{i+7}g_{i+2}g_{i+5}g_i$. Hence,

$$\mathcal{C}_5(G_4) = \{C_{g_i} : i \in [9]\}.$$

Note that G_4 and $\bar{G}_4$ are $(C_7, C_9, \ldots)$-free, and so $\chi_q(G_4) = \chi_q(G_4[q]) = \omega_q(G_4[q]) = \omega_q(G_4)$ by the Strong Perfect Graph Theorem, Lemma 35, and Observation 36 if $\mathcal{C}_5(G_4[q]) = \emptyset$. From this contradiction to our supposition on q, we have $\mathcal{C}_5(G_4[q]) \neq \emptyset$, and so $q(g_i) > 0$ or $q(g_{i+1}) > 0$ for each $i \in [9]$. Since 9 is odd, there is some integer $i \in [9]$ such that $q(g_{i+4}), q(g_{i+5}) > 0$. However, for the sake of a contradiction, let us suppose that, for each $j \in [9]$, there is some $k \in \{j, j+1, j+2\}$ such that $q(g_k) = 0$. Hence, $q(g_{i+3}) = q(g_{i+6}) = 0$, and so $q(g_{i+2}), q(g_{i+7}) > 0$. Since $q(g_{i+3}) = q(g_{i+6}) = 0$ and $\mathcal{C}_5(G_4[q]) \neq \emptyset$, we have $\mathcal{C}_5(G_4[q]) = \{C_{g_i}\}$, and so $q(g_{i+1}) = q(g_{i+8}) = 0$. Thus, $G_4[q] \cong C_5$ which contradicts our supposition on q. Hence, there is some integer $j \in [9]$ such that $q(g_{j-1}), q(g_j), q(g_{j+1}) > 0$.

Let $I = \{g_{j-1}, g_j, g_{j+1}\}$ and $q' \colon V(G_4) \to \mathbb{N}_0$ be a vertex-weight function with

$$u \mapsto \begin{cases} q(u) - 1 & \text{if } u \in I, \\ q(u) & \text{if } u \notin I. \end{cases}$$

By applying Lemma 48 on G_4 with $f_q = \omega_q(G_4)$ and $f_{q'} = 0$, we obtain

$$\chi_q(C) \leq \omega_q(G_4) = \omega_q(G_4 - I) = \max\{q(\{g_{j+2}, g_{j+6}\}), q(\{g_{j+2}, g_{j+7}\}), q(\{g_{j+3}, g_{j+7}\})\}$$

or

$$\omega_q(G_4) \leq \chi_q(C) = \left\lceil \frac{q'(C')}{2} \right\rceil = \left\lceil \frac{q(C')}{2} \right\rceil$$

for each $C \in \mathcal{C}_5^\star(G_4, q)$ and each $C' \in \mathcal{C}_5^\star(G_4, q')$.

We consider first the latter case. Since $|V(C') \cap I| \geq 1$, we have that $q(C')$ is even, and so

$$\omega_q(C') \leq \omega_q(G_4) \leq \frac{q(C')}{2} = \chi_q(C') \leq \chi_q(C)$$

by Corollary 46. For $C_{g_i} \in \operatorname{Argmax}\{q(C'') : C'' \in \mathcal{C}_5(G_4)\}$ with some $i \in [9]$, it follows $C_{g_i} \in \mathcal{C}_5^\star(G_4, q)$. Renaming cycles if necessary, we may assume $C = C_{g_i}$. Hence, $\lfloor q(C)/2 \rfloor \geq \omega_q(G_4)$. Let $k \in \{0, 1\}$ be such that $q(C) \equiv k \bmod 2$. If $q(g_i) < k$, then $q(g_i) = 0$ and $k = 1$. Hence,

$$\left\lfloor \frac{q(C)}{2} \right\rfloor = \frac{q(C) - 1}{2} = \frac{q(\{g_{i+2}, g_{i+4}, g_{i+5}, g_{i+7}\}) - 1}{2} \leq \frac{2\omega_q(C) - 1}{2} < \omega_q(G_4),$$

which is a contradiction. Thus, we have $q(g_i) \geq k$, and we let $q'' \colon V(C) \to \mathbb{N}_0$ be a vertex-weight function with

$$u \mapsto \begin{cases} q(u) - k & \text{if } u = g_i, \\ q(u) & \text{if } u \neq g_i. \end{cases}$$

For simplicity, let $C \colon c_1c_2c_3c_4c_5c_1$ where $c_3 = g_i$ and $c_4 = g_{i+4}$. Hence,

$$\frac{q''(C)}{2} = \left\lfloor \frac{q(C)}{2} \right\rfloor \geq \omega_q(G_4) \geq \omega_q(C) \geq \omega_{q''}(C).$$

By Corollary 47, there is some proper q-colouring $L_C\colon V(C) \to 2^{\mathbb{N}_{>0}}$ such that

$$|L_C^{(1)}(g_i)| = k \quad \text{and} \quad L_C(C) = L_C^{(1)}(g_i) \cup \left(\bigcup_{i'=1}^{5} L_C^{(2)}(c_{i'}, c_{i'+2}) \right).$$

Note that, since $q(C) \geq 2\omega_q(G_4)$,

$$q(C) = |L_C^{(1)}(g_i)| + 2 \cdot \sum_{i'=1}^{5} |L_C^{(2)}(c_{i'}, c_{i'+2})| \geq 2\omega_q(G_4).$$

Using $|L_C^{(1)}(g_i)| \leq 1$, this even implies $\omega_q(G_4) \leq \sum_{i'=1}^{5} |L_C^{(2)}(c_{i'}, c_{i'+2})|$. The maximality of $q(C)$ additionally grants

$$q(g_{i+3}) \leq q(g_{i+4}) = |L_C^{(2)}(g_{i+4}, g_{i+2}) \cup L_C^{(2)}(g_{i+5}, g_{i+4})|$$

and

$$q(g_{i+6}) \leq q(g_{i+5}) = |L_C^{(2)}(g_{i+5}, g_{i+4}) \cup L_C^{(2)}(g_{i+7}, g_{i+5})|.$$

The sets $\{g_i, g_{i+3}, g_{i+6}\}$, $\{g_{i+1}, g_{i+4}, g_{i+7}\}$ and $\{g_{i+2}, g_{i+5}, g_{i+8}\}$ are cliques and $\omega_q(G_4) \leq \sum_{i'=1}^{5} |L_C^{(2)}(c_{i'}, c_{i'+2})|$, therefore

$$q(\{g_{i+3}, g_{i+6}\}) \leq |L_C^{(2)}(g_{i+4}, g_{i+2}) \cup L_C^{(2)}(g_{i+5}, g_{i+4}) \cup L_C^{(2)}(g_{i+7}, g_{i+5})|,$$

$$q(g_{i+1}) \leq |L_C^{(2)}(g_{i+2}, g_i)| \quad \text{and} \quad q(g_{i+8}) \leq |L_C^{(2)}(g_i, g_{i+7})|.$$

For each $i' \in \{i+1, i+3, i+6, i+8\}$, let $L_{g_{i'}}^a \subseteq L_C^{(2)}(g_{i'+1}, g_{i'-1})$ such that

$$|L_{g_{i'}}^a| = \min\{q(g_j), |L_C^{(2)}(g_{i'+1}, g_{i'-1})|\}.$$

Furthermore, let $L_{g_{i+3}}^b, L_{g_{i+6}}^b \subseteq L_C^{(2)}(g_{i+5}, g_{i+4})$ be two disjoint sets such that

$$q(g_{i+3}) = |L_{g_{i+3}}^a| + |L_{g_{i+3}}^b| \quad \text{and} \quad q(g_{i+6}) = |L_{g_{i+6}}^a| + |L_{g_{i+6}}^b|,$$

which is possible by the previous restrictions on $q(g_{i+3})$, $q(g_{i+6})$, and $q(\{g_{i+3}, g_{i+6}\})$. Finally, let $L_{g_{i+1}}^b = L_{g_{i+8}}^b = \emptyset$. Thus, $L\colon V(G_4) \to 2^{\mathbb{N}_{>0}}$ with

$$u \mapsto \begin{cases} L_C(u) & \text{if } u \in V(C), \\ L_u^a \cup L_u^b & \text{if } u \notin V(C) \end{cases}$$

is a proper q-colouring of G_4, and so $\chi_q(G_4) \leq \chi_q(C)$, which is a contradiction to our supposition on q. Hence,

$$\chi_q(C) \leq \omega_q(G_4) = \max\{q(\{g_{j+2}, g_{j+6}\}), q(\{g_{j+2}, g_{j+7}\}), q(\{g_{j+3}, g_{j+7}\})\}.$$

Renaming vertices if necessary, we may assume $\omega_q(G_4) = q(\{g_3, g_8\})$. Note that

$$q(C_{g_i}) \leq 2\chi_q(C_{g_i}) \leq 2\chi_q(C) \leq 2\omega_q(G_4)$$

by Corollary 46 and the fact that $C \in \mathcal{C}_5^\star(G_4, q)$ for each $i \in [9]$. Let $L_{g_3}, L_{g_8} \subseteq [\omega_q(G_4)]$ be disjoint sets such that $|L_{g_3}| = q(g_3)$ and $|L_{g_8}| = q(g_8)$. Clearly, $L_{g_3} \cup L_{g_8} = [\omega_q(G_4)]$. Since $q(\{g_2, g_5, g_8\}), q(\{g_3, g_6, g_9\}) \leq \omega_q(G_4) = q(\{g_3, g_8\})$, there are pairwise disjoint sets $L_{g_2}, L_{g_5} \subseteq L_{g_3}$ and $L_{g_6}, L_{g_9} \subseteq L_{g_8}$ such that $|L_{g_2}| + |L_{g_5}| \leq |L_{g_3}|$, $|L_{g_6}| + |L_{g_9}| \leq |L_{g_8}|$, and $|L_u| = q(u)$ for each $u \in \{g_2, g_5, g_6, g_9\}$. Since $q(\{g_4, g_8\}), q(\{g_3, g_7\}) \leq \omega_q(G_4) = q(\{g_3, g_8\})$, we have $q(g_4) \leq q(g_3) = |L_{g_3}|$ and $q(g_7) \leq q(g_8) = |L_{g_8}|$. Hence, let $L_{g_4} \subseteq L_{g_3}$ and $L_{g_7} \subseteq L_{g_8}$ be such that $L_{g_4} \subseteq L_{g_5}$ or $L_{g_5} \subseteq L_{g_4}$, $L_{g_7} \subseteq L_{g_6}$ or $L_{g_6} \subseteq L_{g_7}$, and $|L_{g_4}| = q(g_4)$ and $|L_{g_7}| = q(g_7)$. Since $q(\{g_1, g_4, g_7\}) \leq \omega_q(G_4)$ and $q(C_{g_1}), q(C_{g_3}), q(C_{g_8}) \leq 2\omega_q(G_4)$ but $\omega_q(G_4) = q(\{g_3, g_8\})$, we have

$$q(g_1) \leq \min\{\omega_q(G_4) - |L_{g_4}| - |L_{g_7}|, \omega_q(G_4) - |L_{g_5}| - |L_{g_6}|,$$
$$\omega_q(G_4) - |L_{g_5}| - |L_{g_7}|, \omega_q(G_4) - |L_{g_4}| - |L_{g_6}|\}.$$

Thus, for $L_{g_1} \subseteq [\omega_q(G_4)] \setminus ((L_{g_4} \cup L_{g_5}) \cup (L_{g_6} \cup L_{g_7}))$ with $|L_{g_1}| = q(g_1)$, it follows that $L\colon V(G_4) \to 2^{\mathbb{N}_{>0}}$ with $u \mapsto L_u$ is a proper q-colouring of G_4, and so $\chi_q(G_4) \leq \omega_q(G_4)$. However, the last observation contradicts the fact that q is a minimal counterexample. Thus, our proof is complete. □

Lemma 56. *If $q\colon V(G_3) \to \mathbb{N}_0$ is a vertex-weight function, then*

$$\chi_q(G_3) = \max\{\omega_q(G_3), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_3)\}\}.$$

Proof. For some arbitrary vertex weight-function $q'\colon V(G_3) \to \mathbb{N}_0$, let

$$R_{q'}(G_3) = \max\{\omega_{q'}(G_3), \max\{\chi_{q'}(C) : C \in \mathcal{C}_5(G_3)\}\}.$$

Note that

$$R_{q'}(G_3) = \max\left\{\omega_{q'}(G_3), \max\left\{\left\lceil\frac{q'(C)}{2}\right\rceil : C \in \mathcal{C}_5(G_3)\right\}\right\}$$

by Corollary 46.

Clearly, $\chi_q(G_3) \geq R_q(G_3)$ and it remains to prove $\chi_q(G_3) \leq R_q(G_3)$. For the sake of a contradiction, let us suppose that q is a minimal counterexample, that is, $\chi_q(G_3) > R_q(G_3)$ but $\chi_{q'}(G_3) \leq R_{q'}(G_3)$ for each vertex-weight function $q'\colon V(G_3) \to \mathbb{N}_0$ with $q'(G_3) < q(G_3)$. Note that q is $\lhd_\chi^{G_3}$-minimal.

Since $G_3 - g \cong G_4 - g_1$, it follows $\chi_q(G_3 - g) = R_q(G_3 - g)$ by Lemma 55. Hence, we may assume $q(g) \geq 1$. By Lemma 34, $q(g_{2,1}), q(g_{2,2}) < q(g)$. If $q(g_{3,1}) = q(g_{3,2}) = 0$, then $\{g, g_{2,1}, g_{2,2}\}$ is a module in $G_3[q]$, and the $\lhd_\chi^{G_3}$-minimality of q implies that $q(g_{2,1}) = q(g_{2,2}) = 0$, and so $\chi_q(G_3) = \chi_G(C) = R_q(G_3)$ for $C\colon g g_{1,1} g_{4,2} g_{4,1} g_{1,2} g$ by Corollary 46, which contradicts our supposition that q is a minimal counterexample. Hence, renaming vertices if necessary, we may assume $q(g_{3,2}) > 0$.

Recall that G_3 is P_5-free. Furthermore, G_3 has four vertices of degree at least 4, and so $\bar{G}_3$ is $(C_7, C_9, \ldots)$-free. Additionally, we note that $G_3 - g_{1,1}$ and $G_3 - g_{1,2}$ are C_5-free, and so both graphs are perfect by the Strong Perfect Graph Theorem. Lemma 35 and Observation 36 imply $\chi_q(G_3 - g_{1,i}) = \omega_q(G_3 - g_{1,i})$ for each $i \in [2]$. By our supposition on G_3, we conclude $q(g_{1,1}), q(g_{1,2}) \geq 1$. Additionally, we let $C \in \mathcal{C}_5^\star(G_3[q], q)$.

Let $I_1 = \{g_{1,1}, g_{1,2}\}$. Since $G_3 - g_{1,1}$ and $G_3 - g_{1,2}$ are C_5-free, $|V(C') \cap I_1| \geq 2$ for each $C' \in \mathcal{C}_5(G_3[q])$. By applying Lemma 48 on $G_3[q]$ with $f_{q'} = 0$ and $f_q = \omega_q(G_3[q])$, we obtain

$$\begin{aligned}\omega_q(G_3) &= \omega_q(G_3[q]) = \omega_q(G_3[q] - I_1)\\ &= \max\{q(\{g_{3,1}, g_{3,2}\}), q(\{g_{3,1}, g_{4,1}\}), q(\{g_{3,2}, g_{4,2}\}), q(\{g_{4,1}, g_{4,2}\})\}.\end{aligned}$$

For the sake of simplicity, let $u \in \{g_{3,1}, g_{4,2}\}$ and $v \in \{g_{3,2}, g_{4,1}\}$ such that $\omega_q(G_3) = q(\{u, v\})$. Since $vg_{1,2} \in E(G)$ and $ug_{1,1} \in E(G)$, $q(u) \geq q(g_{1,2}) > 0$ and $q(v) \geq q(g_{1,1}) > 0$.

Let $I_2 = \{g_{1,1}, g_{3,2}, g_{4,1}\} \cap V(G_3[q])$. By the above observations, we have $q_{1,1}, g_{3,2} \in I_2$ but $q(g_{4,1}) = 0$ or $q(g_{4,1}) \geq 1$. Since $G_3 - g_{1,1}$ and $G_3 - \{g_{3,2}, g_{4,1}\}$ are C_5-free, $g_{1,1} \in V(C')$ and $|V(C') \cap \{g_{3,2}, g_{4,1}\}| \geq 1$ for each $C' \in \mathcal{C}_5(G_3)$, respectively. Thus, $|V(C') \cap I_2| \geq 2$ for each $C' \in \mathcal{C}_5(G_3[q])$, and, by applying Lemma 48 on $G_3[q]$ with $f_{q'} = 0$ and $f_q = \omega_q(G_3[q])$, we conclude $\chi_q(C) \leq \omega_q(G_3[q]) = \omega_q(G_3)$ and

$$\omega_q(G_3) = \omega_q(G_3[q]) = \omega_q(G_3[q] - I_2) = \max\{q(\{g, g_{1,2}\}), q(\{g_{1,2}, g_{2,1}\})\}$$

no matter whether $q(g_{4,1}) = 0$ or $q(g_{4,1}) \geq 1$. Since $q(g) > q(g_{2,1})$, we have $\omega_q(G_3) = q(\{g, g_{1,2}\})$. With $C''\colon gg_{1,1}uvg_{1,2}g \in \mathcal{C}_5(G_3[q])$ we obtain

$$q(C'') \geq 2\omega_q(G_3) + q(g_{1,1}) > 2\omega_q(G_3) \geq 2\chi_q(C) \geq 2\chi_q(C'') \geq 2\left\lceil \frac{q(C'')}{2} \right\rceil \geq q(C''),$$

which is a contradiction. Hence, our proof is complete. □

Lemma 57. *If $q\colon V(G_2) \to \mathbb{N}_0$ is a vertex-weight function, then*

$$\chi_q(G_2) = \max\left\{\omega_q(G_2), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_2)\}, \left\lceil \frac{q(G_2)}{3} \right\rceil\right\} \leq \left\lceil \frac{5\omega_q(G_2) - 1}{4} \right\rceil.$$

Proof. We start our proof by showing the second inequality first. For each $i \in [2]$ and each $j \in \{3, 4\}$, the sets $\{g_i, g_{i,j}\}$, $\{g_{1,j}, g_{2,j}, g_j\}$, and $\{g_3, g_4\}$ are cliques in G_1. Therefore,

$$2q(G_2) = q(\{g_3, g_4\}) + \sum_{j \in \{3,4\}} \left(q(\{g_1, g_{1,j}\}) + q(\{g_2, g_{2,j}\} + q(\{g_{1,j}, g_{2,j}, g_j\})\right) \leq 7\omega_q(G_2)$$

and so, for $n, m \in \mathbb{N}_0$ with $\omega_q(G_2) = 6n + m$ and $m < 6$,

$$\left\lceil \frac{q(G_2)}{3} \right\rceil \leq \left\lceil \frac{\left\lfloor \frac{7\omega_q(G_2)}{2} \right\rfloor}{3} \right\rceil = \omega_q(G_2) + \left\lceil \frac{\left\lfloor \frac{\omega_q(G_2)}{2} \right\rfloor}{3} \right\rceil = \omega_q(G_2) + \begin{cases} n & \text{if } m \leq 1, \\ n+1 & \text{if } m \geq 2 \end{cases}$$

$$= \left\lceil \frac{7\omega_q(G_2)-1}{6} \right\rceil \leq \left\lceil \frac{5\omega_q(G_2)-1}{4} \right\rceil.$$

Now, Corollary 46 completes the proof of the second inequality.

Clearly, $\chi_q(G_2) \geq \max\{\omega_q(G_2), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_2)\}\}$ and

$$\chi_q(G_2) = \chi_q(G_2[q]) \geq \lceil q(G_2[q])/3 \rceil = \lceil q(G_2)/3 \rceil$$

since $\alpha(G_2[q]) \leq 3$. It remains to prove

$$\chi_q(G_2) \leq \max\left\{\omega_q(G_2), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_2)\}, \left\lceil \frac{q(G_2)}{3} \right\rceil \right\}.$$

We continue by supposing, for the sake of a contradiction, that q is a minimal counterexample, that is,

$$\chi_q(G_2) > \max\left\{\omega_q(G_2), \max\{\chi_q(C) : C \in \mathcal{C}_5(G_2)\}, \left\lceil \frac{q(G_2)}{3} \right\rceil \right\}$$

but

$$\chi_{q'}(G_2) = \max\left\{\omega_{q'}(G_2), \max\{\chi_{q'}(C) : C \in \mathcal{C}_5(G_2)\}, \left\lceil \frac{q'(G_2)}{3} \right\rceil \right\}$$

for each vertex-weight function $q' \colon V(G_2) \to \mathbb{N}_0$ with $q'(G_2) < q(G_2)$. Hence, we may assume that q is $\lhd_\chi^{G_2}$-minimal.

Observe that $\mathcal{C}_5(G_2) = \{C_{g_1} \colon g_3 g_{1,3} g_1 g_{1,4} g_4 g_3, C_{g_2} \colon g_3 g_{2,3} g_2 g_{2,4} g_4 g_3\}$. Note that $G_2 - g_{i,j} \in \mathcal{G}^\star$ and $G_2 - g_i \cong G_4 - \{g_4, g_7\}$ for each $i \in [2]$ and $j \in \{3,4\}$. Hence, by Lemma 54 and Lemma 55, we may assume $q(g_i) \geq 1$ and $q(g_{i,j}) \geq 1$ for each $i \in [2]$ and $j \in \{3,4\}$. Furthermore, G_2 and $\bar{G}_2$ are $(C_7, C_9, \ldots)$-free, and $G_2 - g_j$ is C_5-free for each $j \in \{3,4\}$. Hence, by the Strong Perfect Graph Theorem, Lemma 35, Observation 36, and our supposition on G_2, we may assume $G_2[q] = G_2$. In particular, since $q(g_i) \geq 1$, Lemma 34 implies

$$q(g_i) > \chi_q(G[\{g_{3-i,3}, g_{3-i,4}\}]) = \max\{q(g_{3-i,3}), q(g_{3-i,4})\},$$

and so $q(\{g_{i,3}, g_{3-i,4}\}) < \omega_q(G_2)$ for each $i \in [2]$.

For each $i \in [2]$ and $j \in \{3,4\}$, note that $I_j = \{q_1, q_2, q_j\}$ and $I_{i,j} = \{q_i, q_j, q_{3-i,7-j}\}$ are independent sets in G_2. Additionally $|I_j \cap V(C_g)| = |I_{i,j} \cap V(C_g)| = 2$ for each $g \in \{g_1, g_2\}$. By applying Lemma 48 on G_2 for each of the six independent sets with

$$f_q = \left\lceil \frac{q(G_2)}{3} \right\rceil \quad \text{and} \quad f_{q'} = f_q - 1 \left(= \left\lceil \frac{q(G_2)}{3} \right\rceil - 1 = \left\lceil \frac{q'(G_2)}{3} \right\rceil \right),$$

and since $q(\{g_{1,3}, g_{2,4}\}), q(\{g_{2,3}, g_{1,4}\}) < \omega_q(G_2)$, we obtain $f_q \leq \omega_q(G_2)$ as well as

$$\omega_q(G_2) = \omega_q(G_2 - I_j) = q(\{q_3, g_{1,3}, g_{2,3}\}) = q(\{q_4, g_{1,4}, g_{2,4}\})$$

and

$$\omega_q(G_2) = \omega_q(G_2 - I_{i,j}) = q(\{q_1, g_{1,3}\}) = q(\{q_1, g_{1,4}\}) = q(\{q_2, g_{2,3}\}) = q(\{q_2, g_{2,4}\})$$

for each $i \in [2]$ and each $j \in \{3, 4\}$. Hence, there are some integers $a, b, c \in \mathbb{N}_{>0}$ such that

$$q(g_{1,3}) = q(g_{1,4}) = a, q(g_{2,3}) = q(g_{2,4}) = b, q(g_3) = q(g_4) = c, q(g_1) = b+c, q(g_2) = a+c,$$

and so

$$a+b+c+1 \leq a+b+c+\left\lceil \frac{c}{3} \right\rceil \leq \left\lceil \frac{3a+3b+4c}{3} \right\rceil = \left\lceil \frac{q(G_2)}{3} \right\rceil = f_q \leq \omega_q(G_2) = a+b+c.$$

This final contradiction completes our proof. □

Lemma 58. *If $q\colon V(G_1) \to \mathbb{N}_0$ is a vertex-weight function, then*

$$\chi_q(G_1) = \max\left\{\omega_q(G_1), \left\lceil \frac{q(G_1) - \min\{q(g_i) : i \in [3]\}}{2} \right\rceil, \left\lceil \frac{q(G_1) + q(\{g_{\{1,2\}}, g_{\{1,3\}}, g_{\{2,3\}}\})}{3} \right\rceil\right\}$$
$$\leq \left\lceil \frac{5\omega_q(G_1) - 1}{4} \right\rceil.$$

Proof. For simplicity, we let $S = \{g_{\{1,2\}}, g_{\{1,3\}}, g_{\{2,3\}}\}$, $T = \{g_1, g_2, g_3\}$,

$$R_{q'}(G_1) = \max\left\{\left\lceil \frac{q'(G_1) - \min\{q'(g_i) : i \in [3]\}}{2} \right\rceil, \left\lceil \frac{q'(G_1) + q'(S)}{3} \right\rceil\right\}$$

for each vertex-weight function $q'\colon V(G_1) \to \mathbb{N}_0$, and $f_1, f_2\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ be two functions with

$$w \mapsto w \quad \text{and} \quad w \mapsto \left\lceil \frac{5w-1}{4} \right\rceil,$$

respectively. Note that $f_2(w) \geq f_1(w) = w$ for each $w \in \mathbb{N}_{>0}$. Additionally, renaming vertices if necessary, we may assume $q(g_1) \leq q(g_2) \leq q(g_3)$.

Let $L\colon V(G_1) \to 2^{[\chi_q(G_1)]}$ be a proper q-colouring of G_1. Note that S is a clique in G_1, and so $|L(S)| = q(S)$. Additionally, each colour of $L(S)$ can be used at most twice by L. Hence, since $\alpha(G_1) = 3$, we have

$$\chi_q(G_1) \geq |L(S)| + |L(G_1) \setminus L(S)| \geq q(S) + \left\lceil \frac{q(G_1) - 2q(S)}{3} \right\rceil = \left\lceil \frac{q(G_1) + q(S)}{3} \right\rceil.$$

Furthermore, $\alpha(G_1 - g_i) = 2$, which implies

$$\chi_q(G_1) \geq \chi_q(G_1 - g_i) \geq \left\lceil \frac{q(G_1) - q(g_i)}{2} \right\rceil$$

for each $i \in [3]$. Thus, $\chi_q(G_1) \geq \max\{\omega_q(G_1), R_q(G_1)\}$ and, for the rest of our proof, it suffices to show

$$\chi_q(G_1) \leq \max\{f_\ell(\omega_q(G_1)), R_q(G_1) + 1 - \ell\},$$

for each $\ell \in [2]$. For the sake of a contradiction, let us suppose that (q, ℓ) is a minimal counterexample, that is,

$$\chi_q(G_1) > \max\{f_\ell(\omega_q(G_1)), R_q(G_1) + 1 - \ell\}$$

but

$$\chi_{q'}(G_1) \leq \max\{f_{\ell'}(\omega_{q'}(G_1)), R_{q'}(G_1) + 1 - \ell'\}$$

for each $\ell' \in [2]$ if the vertex-weight function $q' \colon V(G_1) \to \mathbb{N}_0$ satisfies $q'(G_1) < q(G_1)$, and for each $\ell' \in [\ell - 1]$ if $q \equiv q'$. Recall $f_\ell(\omega_q(G_1)) \geq \omega_q(G_1)$, and so $\chi_q(G_1) > \omega_q(G_1)$.

We first argue that $q(u) > 0$ for each $u \in V(G_1) \setminus \{g_1\}$. Observe that $G_1 - g, \bar{G}_1 - g, G_1 - \{g_1, g_2\}, \bar{G}_1 - \{g_1, g_2\}$, are $(C_5, C_7, \ldots)$-free. Thus, $G_1 - g$ and $G_1 - \{g_1, g_2\}$ are perfect by the Strong Perfect Graph Theorem. Since $\chi_q(G_1) > \omega_q(G_1)$, we have that $G_1[q]$ is not perfect by Lemma 35 and Observation 36, and so $q(g) > 0$ and $q(g_3) \geq q(g_2) > 0$. If $q(g_{[3]\setminus\{i\}}) = 0$ for some $i \in [3]$, then $G_1 - g_{[3]\setminus\{i\}} \cong G_4 - \{g_2, g_4, g_7\}$ and the combination of Corollary 46 and Lemma 55 implies

$$\begin{aligned} &\max\{f_\ell(\omega_q(G_1)), R_q(G_1) + 1 - \ell\} < \chi_q(G_1) \\ = \max&\left\{\omega_q(G_1), \left\lceil \frac{q(G_1) - q(\{g_i, g_{[3]\setminus\{i\}}\})}{2} \right\rceil\right\} \leq \max\{f_\ell(\omega_q(G_1)), R_q(G_1)\}. \end{aligned}$$

Hence, $\ell = 2$. However, again by Corollary 46 and Lemma 55, $f_2(\omega_q(G_1)) \geq \chi_q(G_1)$. From this contradiction to our supposition on (q, ℓ), we obtain that $q(g_{\{1,2\}}), q(g_{\{1,3\}}), q(g_{\{2,3\}}) > 0$. Hence, $u = g_1$ if $u \in V(G_1)$ is a vertex with $q(u) = 0$. Additionally, $\omega_q(G_1) \geq 3$.

For each $i \in [3]$, we fix $j(i), k(i) \in [3]$ such that $\{i, j(i), k(i)\} = [3]$ and let $q_i \colon V(G_1) \to \mathbb{N}_0$ be the vertex-weight function with

$$u \mapsto \begin{cases} q(u) - 1 & \text{if } u \in \{g, g_{\{j(i),k(i)\}}\}, \\ q(u) & \text{if } u \notin \{g, g_{\{j(i),k(i)\}}\}. \end{cases}$$

It follows $q_i(G_1) < q(G_1)$, $R_{q_i}(G_1) = R_q(G_1) - 1$, $\omega_{q_i}(G_1) \leq \omega_q(G_1)$, and so

$$\begin{aligned} R_q(G_1) + 1 - \ell \leq \max\{f_\ell(\omega_q(G_1)), R_q(G_1) + 1 - \ell\} &< \chi_q(G_1) \leq \chi_{q_i}(G_1) + 1 \\ &\leq \max\{f_\ell(\omega_{q_i}(G_1)) + 1, R_{q_i}(G_1) + 2 - \ell\} \\ &= \max\{f_\ell(\omega_{q_i}(G_1)) + 1, R_q(G_1) + 1 - \ell\} \\ &= f_\ell(\omega_{q_i}(G_1)) + 1 \leq f_\ell(\omega_q(G_1)) + 1 \leq \chi_q(G_1) \end{aligned}$$

by the minimality of (q, ℓ) and since $\{g, g_{\{j(i),k(i)\}}\}$ is an independent set in G_1. Hence, $R_q(G_1) + 1 - \ell \leq f_\ell(\omega_q(G_1))$. Since $f_\ell(\omega_q(G_1) - 1) < f_\ell(\omega_q(G_1))$, it follows further

$$\omega_q(G_1) = \omega_{q_i}(G_1) = \omega_q(G_1 - \{g, g_{\{j(i),k(i)\}}\}) = q(\{g_i, g_{\{i,j(i)\}}, g_{\{i,k(i)\}}\}),$$

for each $i \in [3]$. Note that this especially implies that $q(g_{\{2,3\}}) \leq q(g_1)$, since S is a clique. Consequently,

$$\begin{aligned} R_q(G_1) &\geq \left\lceil \frac{q(G_1) + q(S)}{3} \right\rceil = \left\lceil \frac{\left(\sum_{i=1}^{3} q(\{g_i, g_{\{i,j(i)\}}, g_{\{i,k(i)\}}\})\right) + q(g)}{3} \right\rceil \\ &= \omega_q(G_1) + \left\lceil \frac{q(g)}{3} \right\rceil \geq \omega_q(G_1) + 1. \end{aligned}$$

Thus, since $R_q(G_1) + 1 - \ell \leq f_\ell(\omega_q(G_1))$, it follows $\ell = 2$. In particular, we have

$$\begin{aligned} \max\{\omega_q(G_1) + 1, R_q(G_1)\} \leq f_2(\omega_q(G_1)) + 1 \leq \chi_q(G_1) \\ \leq \max\{\omega_q(G_1), R_q(G_1)\} = R_q(G_1) \end{aligned}$$

by the minimality of (q, ℓ), which implies $\chi_q(G_1) = R_q(G_1) = f_2(\omega_q(G_1)) + 1$.

Since $q(\{g, g_i\}) \leq \omega_q(G_1)$ for each $i \in [3]$, we have

$$3q(g) \leq 3\omega_q(G_1) - q(T) = \left(\sum_{i=1}^{3} q(\{g_i, g_{\{i,j(i)\}}, g_{\{i,k(i)\}}\}) \right) - q(T) = 2q(S) \leq 2\omega_q(G_1).$$

Hence, $q(g) \leq 3$ if $3 \leq \omega_q(G_1) \leq 5$, $q(g) \leq 5$ if $6 \leq \omega_q(G_1) \leq 8$, and $q(g)/3 \leq (\omega_q(G_1) - 1)/4$ if $\omega_q(G_1) \geq 9$, which implies

$$\begin{aligned} \left\lceil \frac{q(G_1) + q(S)}{3} \right\rceil + 1 = \omega_q(G_1) + \left\lceil \frac{q(g)}{3} \right\rceil + 1 \leq \omega_q(G_1) + \left\lceil \frac{\omega_q(G_1) - 1}{4} \right\rceil + 1 \\ = f_2(\omega_q(G_1)) + 1 = R_q(G_1) = \left\lceil \frac{q(G_1) - q(g_1)}{2} \right\rceil. \end{aligned}$$

Thus, since $q(G_1) - q(g_1) - q(g) + q(g_{\{2,3\}}) = 2\omega_q(G)$, it follows $q(g) > q(g_{2,3})$. Let $q'\colon V(G_1) \to \mathbb{N}_0$ be a vertex-weight function defined by

$$u \mapsto \begin{cases} 0 & \text{if } u \in \{g_1, g_{\{2,3\}}\}, \\ q(g) - q(g_{\{2,3\}}) & \text{if } u = g, \\ q(u) & \text{if } u \notin \{g, g_1, g_{\{2,3\}}\}. \end{cases}$$

Clearly, $G_1[q'] \cong C_5$ and $\omega_q(G_1) \geq \omega_{q'}(G_1) + q(g_{\{2,3\}})$, since $q(g_{\{2,3\}}) \leq q(g_1)$. By Corollary 46 and the fact that $\{g, g_{\{2,3\}}\}$ is an independent set in G_1,

$$\begin{aligned} f_2(\omega_q(G_1)) + 1 = R_q(G_1) = \left\lceil \frac{q(G_1) - q(g_1)}{2} \right\rceil \leq \chi_q(G_1 - g_1) \\ \leq \chi_{q'}(G_1 - g_1) + q(g_{\{2,3\}}) \leq \left\lceil \frac{5\omega_{q'}(G_1) - 1}{4} \right\rceil + q(g_{\{2,3\}}) \leq f_2(\omega_q(G_1)), \end{aligned}$$

which is a contradiction. Thus, (q, ℓ) is not a minimal counterexample and our proof is complete. □

We are finally in a position to show $\chi_q(G) \leq f^\star_{\{3K_1\}}(\omega_q(G))$ for each $(P_5, dart)$-free graph G and each vertex weight function $q\colon V(G) \to \mathbb{N}_0$. Recall and observe that $f^\star_{\{3K_1\}}$ is superadditive and that it remains to prove

$$\chi_q(G) \leq f^\star_{\{3K_1\}}(\omega_q(G))$$

for each vertex weight function $q\colon V(G) \to \mathbb{N}_{>0}$ of a prime $(P_5, dart)$-free graph G by Lemma 41. The latter inequality follows immediately if G is $3K_1$-free. Hence, we may assume $\alpha(G) \geq 3$. By Lemma 51, we obtain that $\bar{G}$ is $(C_7, C_9, \ldots)$-free. Additionally, Lemma 52 implies that either G is W_5-free and $\bar{G}$ is A_5-free, or $G \cong G_1$. By Lemma 53, and since G is $(C_7, C_9, \ldots)$-free and $\bar{G}$ is $T_{0,1,2}$-free, we further have that G is perfect or $G_1^p \cong G'$ for

$$\begin{aligned} G' \in \{&C_5, G_1, G_2, G_3, G_4,\\ &G_3 - g, G_3 - g_{4,1}, G_3 - \{g, g_{4,1}\}, G_3 - \{g_{2,2}, g_{4,1}\}, G_3 - \{g, g_{2,2}, g_{4,1}\}\} \cup \mathcal{G}^\star. \end{aligned}$$

If G is perfect, then $\chi_q(G) = \omega_q(G)$ by Lemma 35 and Observation 36, and, if $G \cong G'$ for some induced subgraph G' of $G'' \in \{G_1, G_2, G_3, G_4\} \cup \mathcal{G}^\star$, then Corollary 46, Lemma 54, Lemma 55, Lemma 56, Lemma 57, and Lemma 58 imply

$$\chi_q(G) \leq \left\lceil \frac{5\omega_q(G) - 1}{4} \right\rceil.$$

However, the q'-expansion of C_5 is $3K_1$-free for each vertex-weight function $q'\colon V(C_5) \to \mathbb{N}_0$, and so Observation 36 and Corollary 46 imply

$$\left\lceil \frac{5\omega_q(G) - 1}{4} \right\rceil \leq f^\star_{\{3K_1\}}(\omega_q(G)).$$

Hence, $\chi_q(G) \leq f^\star_{\{3K_1\}}(\omega_q(G))$ for each $(P_5, dart)$-free graph G, which particularly implies $f^\star_{\{P_5, dart\}} = f^\star_{\{3K_1\}}$.

Let G be a critical $(P_5, dart)$-free graph, and S be a non-empty set of vertices such that $E_G[S, V(G) \setminus S]$ is complete and each homogeneous set M in $G[S]$ satisfies $N^2_{G[S]}(M) \neq \emptyset$.

Let us firstly argue that such a set S exists. Starting with $S_0 = V(G)$, we either notice that S_0 fulfils the second property as well or there is a homogeneous set H_0 in $G[S_0]$ with $N^2_{G[S_0]}(H_0) = \emptyset$. Now defining $S_1 = H_0$ we see that $E_G[S_1, V(G) \setminus S_1]$ is complete and we either notice that S_1 fulfils the second property as well or there is a homogeneous set H_1 in $G[S_1]$ with $N^2_{G[S_1]}(H_1) = \emptyset$. So we get a strictly decreasing sequence $S_0 \supsetneq S_1 \supsetneq \ldots$ of vertex sets and since $|V(G)|$ is finite there exists a set S with $|S| \geq \min\{2, |V(G)|\}$ fulfilling both properties.

Clearly, $G[S]$ and $G - S$ are critical. By Corollary 40, S can be partitioned into modules $M_1, M_2, \ldots, M_k$ such that $E_G[M_i, M_j]$ is complete for distinct $i, j \in [k]$, and $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of a prime graph G_i^p without clique-separator of modules for each $i \in [k]$.

We consider first the case that $S = M_1$. Hence, there is a vertex-weight function $q_S \colon V(G_1^p) \to \mathbb{N}_{>0}$ such that $G[S]$ is the q_S-expansion of the prime graph G_1^p. From Lemma 51, Lemma 52, and Lemma 53, we obtain that G_1^p is $3K_1$-free or G_1^p is perfect or $G_1^p \cong G'$ for

$$\begin{aligned} G' \in \{&C_5, G_1, G_2, G_3, G_4, \\ &G_3 - g, G_3 - g_{4,1}, G_3 - \{g, g_{4,1}\}, G_3 - \{g_{2,2}, g_{4,1}\}, G_3 - \{g, g_{2,2}, g_{4,1}\}\} \cup \mathcal{G}^\star. \end{aligned}$$

Note that q_S is $\lhd_\chi^{G_1^p}$-minimal since $G[S]$ is critical. Thus, Lemma 54, Lemma 55, and Lemma 56 imply that G_1^p is $3K_1$-free or G_1^p is perfect or $G_1^p \cong G'$ for some $G' \in \{C_5, K_1, K_2, G_1, G_2\}$. If G_1^p is perfect, then $G[S]$ is perfect by Lemma 35, and so $G[S]$ is a complete graph and especially $3K_1$-free since $G[S]$ is critical. If $G_1^p \cong G'$ for some $G' \in \{C_5, K_1, K_2\}$ or in general if G_1^p is $3K_1$-free, then $G[S]$ is $3K_1$-free, which gives the desired result.

Hence, we may assume $S \setminus M_1 \neq \emptyset$. Clearly, M_1 and $S \setminus M_1$ are modules in $G[S]$, and $E_G[M_1, S \setminus M_1]$ is complete, by the partition of S. We obtain $N^2_{G[S]}(M_1) = \emptyset$ and $N^2_{G[S]}(S \setminus M_1) = \emptyset$, which implies $|M_1| = |S \setminus M_1| = 1$ by the definition of S. Thus, $|V(G[S])| = 2$ and $G[S]$ is $3K_1$-free, which completes our proof for the critical $(P_5, dart)$-free graphs.

7 Consequences for other graph classes

In this chapter we obtain χ-binding functions for (P_5, gem)- and $(P_5, diamond)$-free graphs by applying Lemma 39 and the structural results we obtain in Chapter 6 concerning $(P_5, dart)$-free graphs. Similarly we obtain a χ-binding function for (P_5, C_4)-free graphs from the structural results of Chapter 5, where we talk about $(P_5, banner)$-free graphs.

Note that *diamond* is an induced subgraph of *dart* and C_4 is an induced subgraph of *banner*, so every *banner*-free graph is especially C_4-free. The same is not true for *gem*-free graphs, but in Lemma 53 we look at graphs which are especially *gem*-free. So we apply this lemma in the respective section. Hence, one can say that we obtain our results on C_4-free graphs, *gem*-free graphs, and *diamond*-free graphs as by-products of the previous results about $(P_5, banner)$- and $(P_5, dart)$-free graphs.

7.1 (P_5, C_4)-free graphs

In this section we prove that $f^\star_{\{P_5,C_4\}}(\omega) = \left\lceil \frac{5\omega-1}{4} \right\rceil$, for $\omega \in \mathbb{N}_{>0}$, which is one part of Theorem 5(i) and that every critical (P_5, C_4)-free graph G is complete or a 'non-empty, $2K_1$-free'-expansion of a graph G' with $G' \in \{C_5, W_5\}$, which is Theorem 9(v).

Let G be a critical (P_5, C_4)-free. We first show Theorem 9(v) and use it to prove $f^\star_{\{P_5,C_4\}}(\omega) = \left\lceil \frac{5\omega-1}{4} \right\rceil$, for $\omega \in \mathbb{N}_{>0}$.

By Corollary 40, there is some integer $k \in \mathbb{N}_{>0}$ such that $V(G)$ can be partitioned into sets $M_1, M_2, \ldots, M_k$ such that $E_G[M_i, M_j]$ is complete for distinct $i, j \in [k]$, and $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of a prime graph without clique-separator of modules for each $i \in [k]$. Let us assume $\alpha(G[M_1]) \geq \alpha(G[M_i])$ for each $i \in [k]$. If $\alpha(G[M_1]) = 1$, then $G = G[M_1 \cup M_2 \cup \ldots \cup M_k]$ is a complete graph. In view of the desired result, it remains to assume $\alpha(G[M_1]) \geq 2$. Since G is C_4-free, we have that $V(G) \setminus M_1$ is a clique in G or $V(G) \setminus M_1 = \emptyset$. In the first case $G - M_1$ is complete and a 'non-empty, $2K_1$-free'-expansion of $G[u]$ for some $u \in V(G) \setminus M_1$. We note that since $\alpha(G[M_1]) \geq 2$ and G is critical, we have $\chi(G[M_1]) > \omega(G[M_1])$. Thus, $G[M_1]$ is not perfect. Let G_p be the prime graph such that $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of G_p. By Lemma 35, the graph G_p is not perfect. Additionally, G_p is a prime $(P_5, C_4, banner)$-free graph, and so G_p is $3K_1$-free by Theorem 50. Hence, $\bar{G_p}$ is $(2K_2, C_3)$-free and non-perfect, by the Strong Perfect Graph Theorem. By the Strong Perfect Graph Theorem, the graph $\bar{G_p}$ even contains an induced C_5 and therefore is non-bipartite. Randerath's [54] characterisation of non-bipartite (P_5, C_3)-free graphs imply that the prime ones are copies of C_5. Thus, we get $\bar{G_p} \cong C_5$, which implies $G_p \cong C_5$ and so $G[M_1]$ is a 'non-empty, $2K_1$-free'-expansion of a C_5. Thus, combining all the cases there exists a function $q' \colon V(G') \to \mathbb{N}_{>0}$ such that G is a q'-expansion of $G' \in \{C_5, W_5, K_1\}$, which completes our claim.

Now onto the χ-binding function. By Lemma 1 and Observation 36 to prove the upper bound it now suffices to show that

$$\chi_{q'}(G') \leq \left\lceil \frac{5\omega_{q'}(G') - 1}{4} \right\rceil,$$

for each $G' \in \{C_5, W_5, K_1\}$, since the given function is non-decreasing. Which is trivial for $G' = K_1$. By Corollary 46 this is true for $G' = C_5$. Additionally, we denote the universal vertex in $V(W_5)$ by u. Hence, we have

$$\begin{aligned} \chi_{q'}(W_5) &\leq \chi_{q'}(W_5 - u) + \chi_{q'}(W_5[\{u\}]) \\ &\leq \left\lceil \frac{5\omega_{q'}(W_5 - u) - 1}{4} \right\rceil + \omega_{q'}(W_5[\{u\}]) \leq \left\lceil \frac{5\omega_{q'}(W_5) - 1}{4} \right\rceil \end{aligned}$$

by Corollary 46 and since $W_5 \cong C_5 + K_1$.

Lastly every q-expansion of C_5 with $q \colon V(C_5) \to \mathbb{N}_0$ with $q \not\equiv 0$ is (P_5, C_4)-free. By Observation 36 and Corollary 46, we have, for $\omega \in \mathbb{N}_{>0}$,

$$f^{\star}_{\{P_5, C_4\}}(\omega) \geq \left\lceil \frac{5\omega - 1}{4} \right\rceil,$$

which completes our proof.

7.2 (P_5, gem)-free graphs

In this section we prove that $f^{\star}_{\{P_5, gem\}}(\omega) = \left\lceil \frac{5\omega - 1}{4} \right\rceil$, for $\omega \in \mathbb{N}_{>0}$, which is one part of Theorem 5(i) and that every critical (P_5, gem)-free graph G is a 'non-empty, $2K_1$-free'-expansion of a graph G' with $G' \in \{K_1, C_5, G_2\}$, which is Theorem 9(vi). It is further interesting to note that we obtain the structural result for the prime (P_5, gem)-free graphs from our characterisation of $(P_5, dart)$-free graphs.

Firstly every q-expansion of C_5 with $q \colon V(C_5) \to \mathbb{N}_0$ with $q \not\equiv 0$ is (P_5, gem)-free. By Observation 36 and Corollary 46, we have, for $\omega \in \mathbb{N}_{>0}$,

$$f^{\star}_{\{P_5, gem\}}(\omega) \geq \left\lceil \frac{5\omega - 1}{4} \right\rceil.$$

Concerning (P_5, gem)-free graphs, we may assume that G is (P_5, gem)-free and that $q\colon V(G) \to \mathbb{N}_0$ is $\lhd_\chi^G$-minimal. Note that $\chi_q(G) = \chi_q(G[q]), \omega_q(G) = \omega_q(G[q]), G[q]$ is (P_5, gem)-free, and so we may assume $G = G[q]$. We show that G is complete or a 'non-empty, $2K_1$-free'-expansion of a graph G' with $G' \in \{C_5, G_2\}$. By Lemma 39, there exist an integer $k \in \mathbb{N}_{>0}$, k pairwise disjoint non-empty sets $M_1, M_2, \ldots, M_k \subseteq V(G[q])$, and k $\lhd_\chi^G$-minimal vertex-weight functions $q_1, q_2, \ldots, q_k\colon V(G) \to \mathbb{N}_0$ such that $V(G[q_i]) \subseteq M_i$, $\chi_q(G[M_i]) = \chi_{q_i}(G)$, $\omega_q(G[M_i]) = \omega_{q_i}(G)$, and $G[M_i]$ is a 'non-empty, $2K_1$-free'-expansion of $G[q_i]$ which is a prime graph without clique-separators of modules for each $i \in [k]$, $E_G[M_i, M_j]$ is complete for each distinct $i, j \in [k]$, and

$$\chi_q(G) = \sum_{i=1}^{k} \chi_q(G[M_i]).$$

Note that $V(G) = \bigcup_{i=1}^{k} M_i$, since q is $\lhd_\chi^G$-minimal. We first show that if $k \geq 2$, then G is complete. In this case we have that $G - M_i$ is P_4-free for each $i \in [k]$, since G is *gem*-free. By the Strong Perfect Graph Theorem, Lemma 35, Observation 36, and the fact q is $\lhd_\chi^{G-M_i}$-minimal, we have that $G - M_i$ is complete, for $i \in [k]$. Hence, G is complete if $k \geq 2$. Thus, we may assume $k = 1$ and G is not complete. Clearly, G is (P_5, W_5)-free and $\bar{G}$ is $(A_5, C_7, C_9, \ldots, T_{0,1,2})$-free. Hence, $G[q_1]$ is perfect or $G[q_1] \cong G'$ with

$$\begin{aligned} G' \in \{&C_5, G_2, G_3, G_4,\\ &G_3 - g, G_3 - g_{4,1}, G_3 - \{g, g_{4,1}\}, G_3 - \{g_{2,2}, g_{4,1}\}, G_3 - \{g, g_{2,2}, g_{4,1}\}\} \cup \mathcal{G}^\star \end{aligned}$$

by Lemma 53. Since q_1 is $\lhd_\chi^G$-minimal, we obtain $G' \in \{C_5, G_2\}$ similarly as for *dart*-free graphs, by Lemma 54, Lemma 55, and Lemma 56. If we collect both cases, we find that G is a 'non-empty, $2K_1$-free'-expansion of a graph G' with $G' \in \{K_1, C_5, G_2\}$. Thus, we obtain the desired characterisation of Theorem 9. Additionally, returning to our $\lhd_\chi^G$-minimal vertex-weight function q, for each vertex-weight function $q^>\colon V(G) \to \mathbb{N}$ with $q \lhd_\chi^G q^>$, we have

$$\chi_{q^>}(G) = \chi_q(G) \leq \left\lceil \frac{5\omega_q(G) - 1}{4} \right\rceil \leq \left\lceil \frac{5\omega_{q^>}(G) - 1}{4} \right\rceil$$

by Corollary 46 and Lemma 57, which completes our proof for this part of Theorem 5(i).

7.3 $(P_5, diamond)$-free graphs

We note that Theorem 5 (ii), which is

$$f^\star_{\{P_5, diamond\}}(\omega) = \begin{cases} 3 & \text{if } \omega = 2, \\ \omega & \text{if } \omega \neq 2, \end{cases} \quad \text{for } \omega \in \mathbb{N}_{>0},$$

and Theorem 9 (vii), which characterizes the critical graphs, can be obtained from Theorem 9 (vi), proven in Section 7.2, as follows.

Let G, G' be two $(P_5, diamond)$-free graphs that are not necessarily distinct but for which $\chi(G) = \chi(G')$, $\omega(G) \geq \omega(G')$, and G' is critical. Clearly, G' is *gem*-free, and so G' is complete or a 'non-empty, $2K_1$-free'-expansion of a graph $G'' \in \{C_5, G_2\}$ by Theorem 9 (vi). In the latter case, since G' is not G''-free but *diamond*-free, we have $G' \cong G''$. By Lemma 57, we see that $\chi(G_2) = \omega(G_2) = 3$, and so G' is complete or $G' \cong C_5$, which proves Theorem 9 (vii). Additionally,

$$\chi(G) = \chi(G') \leq \left.\begin{cases} 3 & \text{if } \omega(G') = 2, \\ \omega(G') & \text{if } \omega(G') \neq 2 \end{cases}\right\} \leq \begin{cases} 3 & \text{if } \omega(G) = 2, \\ \omega(G) & \text{if } \omega(G) \neq 2. \end{cases}$$

From the fact that C_5 and K_n are $(P_5, diamond)$-free for each $n \geq 1$, we obtain Theorem 5(ii).

8 $(P_5, kite)$-free graphs

In this chapter we look at the family of $(P_5, kite)$-free graphs. This chapter can conceptually also be found in [12]. Instead of finding a binding function for this graph class directly we argue that $f^\star_{\{K_1 \cup K_3, K_1 \cup C_5, 2K_2\}} = f^\star_{\{P_5, kite\}}$ and prove a linear bound for $f^\star_{\{K_1 \cup K_3, K_1 \cup C_5, 2K_2\}}$ in Theorem 63. To show that we prove Lemma 62 by using a combination of known results and new ideas. Let us state the known results first.

Lemma 59 (Brandstädt and Mosca [9]). *If G is a prime $(P_5, kite)$-free graph, then G is a matched co-bipartite graph or $2K_2$-free.*

By Wagon [67] and followup research by Gaspers and Huang [29] we know the following corollary.

Corollary 60 (Wagon [67], Gaspers et al. [29]). *If G is $(2K_2, K_4)$-free, then $\chi(G) \leq \left\lfloor \frac{3\omega(G)}{2} \right\rfloor$.*

There is also a recent paper by Chudnosky et al. [18] in which they research the family of (co-$kite$, C_4)-free graphs. Another name commonly given to the graph co-$kite$ is fork. To understand this lemma we additionally need to define when we call a graph candled. A graph H is called a *candelabrum* (with base Z) if its vertices can be partitioned into non trivial disjoint sets Y, Z such that Y is an independent set, Z is a clique, and Y and Z are matched. One can add a candelabrum to a graph G via the following procedure: Let H be a candelabrum with base Z. Take the disjoint union of G and H, then add edges to make Z complete to $V(G)$. We refer to this construction procedure as *candling* the graph G. We say that a graph G is *candled* if it can be constructed by candling some induced subgraph $G_0 \subseteq G$.

Lemma 61 (Chudnovsky et al. [18][17]). *If G is a (co-kite, C_4)-free graph, then*

(i) *G is not connected or*

(ii) *G contains a universal vertex or*

(iii) *G contains a homogeneous clique or*

(iv) *G is candled or*

(v) *$\bar{G}$ is candled or*

(vi) *G is $K_{1,3}$-free.*

Using the above stated results we are able to show the following lemma.

Lemma 62. *Let $f\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ be such that*

(i) *$f(w) \geq \lfloor 3w/2 \rfloor$ for each $w \in \mathbb{N}_{>0}$,*

(ii) *$f(w_1) + f(w_2) \leq f(w_1 + w_2)$ for each $w_1, w_2 \in \mathbb{N}_{>0}$ and*

(iii) *$\chi(G) \leq f(\omega(G))$ for each connected, prime, $(K_1 \cup K_3, K_1 \cup C_5, 2K_2)$-free graph G whose complementary graph is a connected graph.*

If G is a $(P_5, kite)$-free graph, then $\chi(G) \leq f(\omega(G))$.

Proof. For the sake of a contradiction, let us suppose that G is a $(P_5, kite)$-free graph with $\chi(G) > f(\omega(G))$. We may assume that G is a counterexample of minimum order, that is, $\chi(G') \leq f(\omega(G'))$ for each $(P_5, kite)$-free graph G' with $|V(G')| < |V(G)|$. It is easily seen that f is strictly increasing by (i) and (ii). Thus, the graph G is connected, critical, and not perfect. Furthermore, Lemma 37 implies that G has no clique separator of modules.

We prove next that G is $2K_2$-free. Let $M \subseteq V(G)$ be a module in G such that $V(G) \setminus M \neq \emptyset$. Since G is critical and, thus, does not contain a clique separator of modules, Lemma 34 and Lemma 38 imply $|M| = 1$ or $N_G^2(M) = \emptyset$. It follows $E_G[M, V(G) \setminus M]$ is complete in the latter case, and so we obtain

$$\begin{aligned} f(\omega(G)) < \chi(G) = \chi(G[M]) + \chi(G - M) &\leq f(\omega(G[M])) + f(\omega(G - M)) \\ &\leq f(\omega(G[M]) + \omega(G - M)) = f(\omega(G)) \end{aligned}$$

from the facts that G is a counterexample of minimum order and that $f(w_1) + f(w_2) \leq f(w_1 + w_2)$ for each $w_1, w_2 \in \mathbb{N}_{>0}$. By this contradiction, we obtain that each module M is either of size 1 or of size $|V(G)|$. In other words, G is prime. Observe that in contrast to G each induced subgraph, say G', of a matched co-bipartite graph is $\omega(G')$-colourable. Hence, each matched co-bipartite graph is perfect and, thus, since G is not perfect, G is $2K_2$-free by Lemma 59.

We proceed by showing that G is $(K_1 \cup K_3)$-free. Note that $\bar{G}$ is (co-$kite, C_4$)-free. Since G is connected, $\bar{G}$ has no universal vertex. Furthermore, since G is prime, $\bar{G}$ is prime as well. Thus, $\bar{G}$ has no homogeneous set and is connected. By Lemma 37 and the fact that G is critical, neither G nor $\bar{G}$ are candled. Lemma 61 implies that $\bar{G}$ is $K_{1,3}$-free, and thus G is $(K_1 \cup K_3)$-free.

Our next goal is to prove that G is $(K_1 \cup C_5)$-free. Let us assume that C is an arbitrary induced 5-cycle in G and C is oriented, meaning that $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ and, recall Section 1.2, for $c \in V(C)$ we denote by c^+ and c^- the neighbours of c in

$V(C)$, depending on the orientation. Additionally, let A be the set of all vertices of $G - V(C)$ that have a neighbour and a non-neighbour in C, B be the set of vertices of $G - V(C)$ that are adjacent to all vertices of C, and D be the set of vertices that have no neighbour in C. For the sake of a contradiction, let us suppose that $D \neq \emptyset$. Since G is $(2K_2, K_1 \cup K_3)$-free, every vertex $a \in A$ satisfies that either $N_G(a) \cap V(C)$ or $V(C) \setminus N_G(a)$ is an independent set of size 2. Let A_2 be the set of vertices of A that have two neighbours in C and A_3 be the set of vertices that have three neighbours in C. Since G is $2K_2$-free, the set D is an independent set in G and $E_G[A_2, D] = \emptyset$. Furthermore, $E_G[A_3 \cup B, D]$ is complete since G is $(K_1 \cup K_3)$-free. Since G is prime, it follows $|D| = 1$.

For the sake of contradiction let us suppose that there is some vertex $a \in A_2$, and $c \in V(C)$ is such that $c^-, c^+ \in N_G(a)$. By Lemma 34, there is some $u \in N_G(a)$ that is not a neighbour of c. Thus, $u \notin D$. Since $[a, u, c^{-2}, c^{+2}]$ does not induce a $2K_2$, u is adjacent to c^{-2} or c^{+2}. By symmetry, we may assume that $c^{-2}u \in E(G)$. Since $[c^{-2}, u, c^+, c]$ does not induce a $2K_2$, it follows that $c^+u \in E(G)$. Since $[d, c^+, a, u]$ does not induce a $K_1 \cup K_3$, it follows $du \in E(G)$. Furthermore, $c^{+2}u \notin E(G)$ but $c^-u \in E(G)$ since $u \in A_3$, and so $[c^{+2}, a, c^-, u]$ induces a $K_1 \cup K_3$, which contradicts the fact that G is $(K_1 \cup K_3)$-free. Hence, $A_2 = \emptyset$ and $A = A_3$.

Observe that B is a module in $G - A_3$ and $G - (A_3 \cup B)$ is disconnected. By Lemma 37 and the fact that G is a counterexample of minimal order, we obtain $A_3 \neq \emptyset$. For each $a \in A_3$, let $B_a = B \setminus N_G(a)$. Since there is a vertex $c \in V(C) \setminus N_G(a)$, and every vertex of B is adjacent to every vertex of $V(C)$, and G is $(K_1 \cup K_3)$-free, it follows that $\{a\} \cup B_a$ is an independent set in G. Let, for each $c \in V(C)$, $A_{3,c}$ be the set of vertices of A_3 that are adjacent to c^{-2}, c, and c^{+2}. Clearly, $A_3 = \bigcup_{c \in V(C)} A_{3,c}$. Since $[c^+, a_1, a_2, d]$ does not induce a $K_1 \cup K_3$ for each $a_1, a_2 \in A_{3,c} \cup A_{3,c^{+2}}$, it follows that $A_{3,c} \cup A_{3,c^{+2}}$ is an independent set in G. Furthermore, for each $c \in V(C)$, we have $B_{a_1} = B_{a_2}$ if $a_1 \in A_{3,c}$ and $a_2 \in A_{3,c^{+2}}$ since neither $[a_1, a_2, b, c^-]$ nor $[a_2, a_1, b, c^{-2}]$ induces a $K_1 \cup K_3$ for each $b \in B_{a_1} \cup B_{a_2}$. Let $c \in V(C)$ be chosen such that $A_{3,c} \neq \emptyset$ and, subject to this condition, $|A_{3,c^{+2}}|$ is maximum. Since $A_3 \neq \emptyset$, we have $A_{3,c} \neq \emptyset$. If $A_{3,c^{+2}} = \emptyset$, then $A_{3,c^{-2}} = \emptyset$, and $A_{3,c^-} = \emptyset$ or $A_{3,c^+} = \emptyset$. By symmetry, we may assume $A_{3,c^-} = \emptyset$, and so $\{c, c^{+2}\} \cup A_{3,c^+}$, $\{c^-\} \cup A_{3,c}$, $\{c^{-2}, c^+, d\}$ is a partition of $V(G - B)$ into three independent sets. Thus,

$$\begin{aligned}\chi(G) &\leq \chi(G[B]) + \chi(G - B) \leq f(\omega(G[B])) + 3 \\ &\leq f(\omega(G[B])) + f(2) \leq f(\omega(G[B]) + 2) \leq f(\omega(G)).\end{aligned}$$

From this contradiction on our supposition on G, we obtain $A_{3,c^{+2}} \neq \emptyset$. Recall that $B_{a_1} = B_{a_2}$ for each $a_1 \in A_{3,c}$, each $a_2 \in A_{3,c^{+2}}$. Thus, $B_{a_1} = B_{a_2}$ for each two vertices $a_1, a_2 \in A_{3,c} \cup A_{3,c^{+2}}$. Observe that $\{c^{-2}, c\} \cup A_{3,c^-}$, $\{c^-, c^+, d\}$, $\{c^{+2}\} \cup A_{3,c^{-2}} \cup A_{3,c^+}$, $A_{3,c} \cup A_{3,c^{+2}} \cup B_{a_1}$ is a partition of $V(G - (B \setminus B_{a_1}))$ into four independent sets, and so $\chi(G-(B\setminus B_{a_1}))\leq 4$. For $a_1\in A_{3,c}$ and $a_2\in A_{3,c^{+2}}$, it follows

$$\begin{aligned}f(\omega(G))<\chi(G)\leq\ &\chi(G[B\setminus B_{a_1}])+4\leq f(\omega(G[B\setminus B_{a_1}]))+4\\&\leq f(\omega(G[B\setminus B_{a_1}]))+f(3)\leq f(\omega(G[B\setminus B_{a_1}])+3)\end{aligned}$$

by the facts that G is a counterexample of minimal order, that $f(w)\geq\lfloor 3w/2\rfloor$ for each $w\in\mathbb{N}_0$, and that $f(w_1)+f(w_2)\leq f(w_1+w_2)$ for each $w_1,w_2\in\mathbb{N}_0$. Therefore, $\omega(G)\leq\omega(G[B\setminus B_{a_1}])+2$ since f is non-decreasing. Hence,

$$\omega(G)\leq\omega(G[B\setminus B_{a_1}])+2\leq\omega(G[B])+2\leq\omega(G),$$

and so $\omega(G[B\setminus B_{a_1}])=\omega(G[B])=\omega(G)-2$. On the other hand, for some clique W of size $\omega(G[B])$ in $G[B\setminus B_{a_1}]$, we have that $W\cup\{a_1,c^{-2},c^{+2}\}$ is a clique in G and therefore $\omega(G)\geq\omega(G[B])+3$. This contradiction implies that $D=\emptyset$, and that G is $(K_1\cup C_5)$-free by the arbitrariness of C.

Recall that G is connected, prime, $(K_1\cup K_3, K_1\cup C_5, 2K_2)$-free graph and $\bar{G}$ is connected. Thus, $\chi(G)\leq f(\omega(G))$. From this final contradiction to our supposition, we obtain $\chi(G)\leq f(\omega(G))$. □

Theorem 63.

$$f^\star_{\{P_5,kite\}}\equiv f^\star_{\{2K_2,kite\}}\equiv f^\star_{\{2K_2,K_1\cup K_3\}}\equiv f^\star_{\{2K_2,K_1\cup K_3,K_1\cup C_5\}}$$

and for $\omega\in\mathbb{N}_{>0}$

$$\left\lfloor\frac{3\omega}{2}\right\rfloor\leq f^\star_{\{2K_2,K_1\cup K_3,K_1\cup C_5\}}(\omega)\leq\begin{cases}\lfloor\frac{3\omega}{2}\rfloor & \text{if } \omega\leq 3,\\ 2\omega-2 & \text{if } \omega\geq 4.\end{cases}$$

Proof. Since $f^\star_{\{P_5,kite\}}\leq f^\star_{P_5}$ and by Theorem 12, we know that the class of $(P_5, kite)$-free graphs has a χ-binding function. Note that

$$f^\star_{\{P_5,kite\}}\geq f^\star_{\{2K_2,kite\}}\geq f^\star_{\{2K_2,K_1\cup K_3\}}\geq f^\star_{\{2K_2,K_1\cup K_3,K_1\cup C_5\}},$$

since in each equality either another forbidden subgraph gets added or the forbidden graph H is replaced by an induced subgraph of H.

Since each graph of $\{2K_2, K_1\cup K_3, K_1\cup C_5\}$ does not contain a complete bipartite spanning subgraph, we conclude that $f^\star_{\{2K_2,K_1\cup K_3,K_1\cup C_5\}}$ is superadditive, by Lemma 43. Thus, this functions fulfils condition (ii) of Lemma 62

We show next that $f^\star_{\{2K_2,K_1\cup K_3,K_1\cup C_5\}}$ also fulfils the condition (i) of Lemma 62. We construct the family $\{G_\omega\mid\omega\in\mathbb{N}_{>0}\}$ of $(2K_2, K_1\cup K_3, K_1\cup C_5)$-free graphs. Let $G_q\cong K_1$ and for $\omega\in 2\mathbb{N}_{>0}$ we define G_ω as the complete join of $\omega/2$ distinct C_5's. Also for $\omega\in\mathbb{N}_{>2}\setminus(2\mathbb{N}_{>0})$ we define $G_\omega=G_{\omega-1}+K_1$. Note that $\omega(G_\omega)=\omega$ and

$\chi(G_\omega) = \lfloor 3w(G_\omega)/2 \rfloor$ for $\omega \in \mathbb{N}_{>0}$. Additionally, each graph of the family $\{G_\omega \mid \omega \in \mathbb{N}_{>0}\}$ is $(2K_2, K_1 \cup K_3, K_1 \cup C_5)$-free as follows. The complementary graph $\bar{G}_\omega$ consists of a disjoint union of C_5's with at most one isolated vertex, which is clearly a $(C_4, K_{1,3}, W_5)$-free graph. So $f^\star_{\{2K_2, K_1\cup K_3, K_1 \cup C_5\}}(w) \geq \lfloor 3w/2 \rfloor$ for each $w \in \mathbb{N}_{>0}$.

The function $f^\star_{\{2K_2, K_1\cup K_3, K_1\cup C_5\}}$ also fulfils condition (iii) of Lemma 62 by definition. Therefore, Lemma 62 finally implies that $f^\star_{\{P_5, kite\}} \leq f^\star_{\{2K_2, K_1\cup K_3, K_1\cup C_5\}}$, which proves the first statement of the theorem.

We prove the second statement by induction on $\omega(G)$. For this it suffices to prove $\chi(G) \leq 2\omega(G) - 2$ for graphs G that are $(K_1 \cup K_3, K_1 \cup C_5, 2K_2)$-free and that have clique number at least 3, by Lemma 60. For $\omega = 3$ we get $\lceil 3\omega/2 \rceil = 2\omega - 2$, which is the induction base. So let G be a graph with $\omega(G) = k \geq 4$ and $\{w_1, w_2, \ldots, w_k\}$ be a clique of size $\omega(G)$ in G. We define $S \subseteq V(G)$ as the non-neighbours of w_1. Since the graph is $(K_1 \cup K_3, K_1 \cup C_5, 2K_2)$-free $G[S]$ does not contain an odd cycle as an induced subgraph. Thus, since a graph with no odd cycles is bipartite, we know that $\chi(G[S \cup \{w_1\}]) \leq 2$. Note that $\omega(G - (S \cup \{w_1\})) = \omega(G) - 1$, since $E_G[\{w_1\}, V(G - (S \cup \{w_1\}))]$ is complete and $\{w_2, \ldots, w_k\} \subseteq V(G - (S \cup \{w_1\}))$. By induction hypothesis we now conclude

$$\chi(G) \leq \chi(G[S \cup \{w_1\}]) + \chi(G - (S \cup \{w_1\})) \leq 2 + 2(\omega(G) - 1) - 2 = 2\omega(G) - 2.$$

This inequality chain completes the proof of the theorem. □

9 (P_5, HVN)-free graphs

In this section we discuss the optimal χ-binding function for (P_5, HVN)-free graphs (cf. Theorem 7). Let us repeat Theorem 7 which states

$$f^{\star}_{\{P_5,\mathrm{HVN}\}}(\omega) = \begin{cases} \omega+1 & \text{if } \omega \notin \{1,3\}, \\ \omega & \text{if } \omega = 1, \\ \omega+2 & \text{if } \omega = 3, \end{cases}$$

for $\omega \in \mathbb{N}_{>0}$. To prove this theorem we need Lemma 64 and Lemma 65, which we prove in Section 9.2 and Section 9.3 respectively. Recall that a critical graph does not contain a comparable vertex pair and does not contain a cutvertex, which follows from Lemma 34 and Lemma 37 respectively. Assuming Lemma 64 and Lemma 65 to be already proven, we prove the theorem in the remainder of this section.

Lemma 64. *If G is a critical (P_5, HVN, C_5)-free graph then G is perfect or $G \cong \bar{C}_7$.*

Lemma 65. *If G is a critical (P_5, HVN)-free graph with $\omega(G) \geq 4$ which contains an induced C_5, then $\chi(G) \leq \omega(G) + 1$.*

We first argue that for $\omega \leq 3$, the theorem is known. Every graph G with $\omega(G) \leq 3$ is clearly HVN-free, (P_5, K_2)-free graphs are 1-colourable, (P_5, K_3)-free graphs are 3-colourable [66], and (P_5, K_4)-free graphs are 5-colourable [26]. Also according to the respective papers these bounds are best possible.

So we fix for the remainder of this paragraph $\omega \geq 4$. The following construction shows $f^{\star}_{\{P_5,\mathrm{HVN}\}}(\omega) \geq \omega + 1$. We define the graph G_ω by $C_5[K_1, K_{\omega-1}, K_1, K_{\omega-1}, K_1]$. Note that $\omega(G_\omega) = \omega$ and G_ω is (P_5, HVN)-free, so it remains to show that $\chi(G_\omega) = \omega + 1$. Let C be a C_5 with vertex-weight function q fulfilling $\omega_q(C) = \omega$ and $q(C) = 2 \cdot \omega + 1$. Note that the chromatic number of a weighted C_5 only depends on the size of the largest clique and the sum of the weights, thus, by Corollary 46,

$$\chi(G_\omega) = \chi_q(C) = \max\left\{\omega_q(C), \left\lceil \frac{q(C)}{2} \right\rceil\right\} = \omega + 1.$$

Thus, it remains to show that $f^{\star}_{\{P_5,\mathrm{HVN}\}}(\omega) \leq \omega + 1$. Let G be an arbitrary (HVN, P_5)-free graph with $\omega(G) = \omega$. Let G' be a critical subgraph of G with $\chi(G) = \chi(G')$. If

G' is C_5-free, we find, by Lemma 64,

$$\chi(G) = \chi(G') = \begin{cases} \omega(G') + 1, & \text{if } G' \cong \bar{C}_7 \\ \omega(G'), & \text{else.} \end{cases}$$

Thus, $\chi(G) \leq \omega + 1$, since $\omega(G') \leq \omega(G) = \omega$. If on the other hand the graph G' contains an induced C_5, we distinguish two cases. If $\omega(G') \geq 4$, then $\chi(G) = \chi(G') \leq \omega(G') + 1 \leq \omega + 1$, by Lemma 65. Otherwise $\omega(G') \leq 3$ and we find $\chi(G) = \chi(G') \leq 5 \leq \omega + 1$, since (P_5, K_4)-free graphs are 5-colourable [26].

Thus, it remains to prove Lemma 64 and Lemma 65.

9.1 Results for (P_5, paw)-free graphs

Before we prove Lemma 64 and Lemma 65 we first need to better understand the family of (P_5, paw)-free graphs. Note that $paw + K_1 \cong \mathrm{HVN}$, so these families are closely related. In this section we use known results to talk about the critical (P_5, K_3)-free graphs and the critical (P_5, paw)-free graphs. From that we deduce $f^{\star}_{\{P_5, paw\}}$ and introduce a special colouring.

Lemma 66 (Sumner [66])**.** *The critical (P_5, K_3)-free graphs are K_1, K_2 and C_5.*

Proof. Let G be a critical (P_5, K_3)-free graph. Clearly G is connected. If G is perfect, G is isomorphic to K_2 or K_1. If G is not perfect, then G contains an induced C_5, by the SPGT, because it is (P_5, K_3)-free. Let $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$. For the sake of contradiction we suppose there is a $x \in N_G(C)$. Then there is an $i \in [5]$ with $N_G(x) \cap V(C) = \{c_i, c_{i+2}\}$, since G is (P_5, K_3)-free. We know that (x, c_{i+1}) is not a comparable vertex pair, by Lemma 34, so there is a $y \in V(G)$ with $yx \in E(G)$ and $yc_{i+1} \notin E(G)$. Since G is P_5-free, $y \in N_G(C)$. Thus, there is a $j \in [5]$ with $N_G(y) \cap V(C) = \{c_j, c_{j+2}\}$. We see that $j \notin \{i, i+2\}$, otherwise $\{x, y, c_j\}$ induces a K_3, and $j \neq i+3$, otherwise $\{x, y, c_i\}$ induces a K_3. But now $yc_{i+1} \in E(G)$; a contradiction. Thus, our supposition is false and $G \cong C_5$. □

Lemma 67 (Olariu [48])**.** *The critical (P_5, paw)-free graphs are the complete graphs and C_5.*

Proof. Let G be a critical (P_5, paw)-free graph. Clearly G is connected. According to Olariu (cf. Theorem 20, [48]), G is a complete multipartite graph or K_3-free. In the second case G is K_1, K_2, or C_5 according to Lemma 66. In the first case G is perfect and, since critical, a complete graph. □

Corollary 68. *For* $\omega \in \mathbb{N}_{>0}$,

$$f^{\star}_{\{P_5,paw\}}(\omega) = \begin{cases} \omega & \text{if } \omega \neq 2, \\ \omega + 1 & \text{if } \omega = 2. \end{cases}$$

Proof. Since C_5 and K_ω are (P_5, paw)-free graphs, for $\omega \in \mathbb{N}_{>0}$, the stated bound is a lower bound of $f^{\star}_{\{P_5,paw\}}$. To prove the reverse direction let G be an arbitrary (P_5, paw)-free graph and G' a critical induced subgraph of G with $\chi(G) = \chi(G')$. By Lemma 67 $G' \cong K_{\omega(G')}$ or $G' \cong C_5$. In the first case we see $\chi(G) = \chi(G') = \omega(G') \leq \omega(G) \leq \chi(G)$. In the latter case we find $2 = \omega(G') \leq \omega(G) \leq \chi(G) = \chi(G') = 3$. Thus, $\omega(G) = 2$ and $\chi(G) = \omega(G) + 1$ or $\omega(G) = 3 = \chi(G)$. Thus, the proof is complete. □

In the later proofs we not only need that a (P_5, paw)-free graph G has small chromatic number, but also that it can be $\chi(G)$-coloured even if some vertices are already precoloured.

Lemma 69. *If G is a (P_5, paw)-free graph and I_1, I_2 are vertex-disjoint independent sets of G, then there is a colouring $c_{I_1,I_2} : V(G) \to [\max\{\chi(G), 3\}]$ with $|c_{I_1,I_2}(I_1 \cup I_2)| \leq 1$, if $I_2 \neq \emptyset$ and $|c_{I_1,I_2}(I_1 \cup I_2)| \leq 2$ else.*

Proof. Note that it suffices to show this result for a connected graph G, since proving it for every connected graph and applying the result to each component of a disconnected graph grants the result by renaming colours. By Olariu (cf. Theorem 20, [48]), the graph G is complete multipartite or K_3-free. If G is complete multipartite the optimal $\omega(G)$-colouring of G fulfils both bounds. If G is K_3-free and $\chi(G) \leq 2$ the result is true by simply colouring I_1 with an additional colour, if $I_2 = \emptyset$, or by optimally colouring the graph which implies $|c_{I_1,I_2}(I_1 \cup I_2)| \leq 2$ in the other case. The last remaining case is that G is K_3-free and $\chi(G) \geq 3$. In this case we see that G is non-perfect. Therefore, the graph G contains an induced C_5, since G is (P_5, K_3)-free and by the Strong Perfect Graph Theorem. Since G is K_3-free, Randerath [54] proves that G is isomorphic to $C_5[k_1 \cdot K_1, k_2 \cdot K_2, \ldots, k_5 \cdot K_5]$, for some $k_1, k_2, \ldots, k_5 \in \mathbb{N}_{>0}$. Let us denote the vertices in the independent sets of this C_5 in order by V_1, V_2, V_3, V_4, V_5 respectively. By otherwise renaming the vertices we may assume that $V_5 \cap (I_1 \cup I_2) = \emptyset$. Also we assume $I_1 \subseteq V_1 \cup V_3$. We define the colouring c_{I_1,I_2} which colours the vertices of $V_1 \cup V_3$ with 1, the vertices of $V_2 \cup V_4$ with 2 and the vertices of V_5 with 3. This proves the lemma. □

9.2 Proof of Lemma 64

If G is not perfect then G contains an induced $\bar{C}_7$, by the Strong Perfect Graph Theorem, since HVN $\subseteq_{\text{ind}} \bar{C}_{2p+1}$, for $p \geq 4$, and $P_5 \subseteq_{\text{ind}} C_{2p+1}$, for $p \geq 3$. Let $V(\bar{C}_7) = C$

and the vertices of the $\bar{C}_7$ be labelled by $c_1, \dots, c_7$ with $c_ic_{i+1} \notin E(G)$ for $1 \leq i \leq 7$, where all additions on the cycle are considered modulo 7.

(C1): For every $w \in N_G(C)$, there exists an $i \in [7]$ with $wc_i, wc_{i+1} \in E(G)$: Suppose not, then there is a $w \in N_G(C)$, such that for all $i \in [7]$ $wc_i \notin E(G)$ or $wc_{i+1} \notin E(G)$. Now there exists a $j \in [7]$ with $wc_j \in E(G), wc_{j-1}, wc_{j+1}, wc_{j+2} \notin E(G)$, since 7 is odd. But $[w, c_j, c_{j+2}, c_{j-1}, c_{j+1}]$ induces a P_5; a contradiction.

(C2): If $w \in N_G(C)$ with $wc_i, wc_{i+1}, wc_{i+2} \in E(G)$ then $wc_{i+4}, wc_{i+5} \notin E(G)$: Otherwise we see that $[c_{i+1}, w, c_{i+4}, c_{i+2}, c_i]$ or $[c_{i+1}, w, c_{i+5}, c_{i+2}, c_i]$ induces a HVN; a contradiction.

We define

$$W_i^3 := \{w \in N_G(C) \mid C \cap N_G(w) = \{v_i, v_{i+1}, v_{i+2}\}\},$$
$$W_i^4 := \{w \in N_G(C) \mid C \cap N_G(w) = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}\},$$
$$W^3 := \bigcup_{i \in [7]} W_i^3,$$
$$W^4 := \bigcup_{i \in [7]} W_i^4.$$

(C3): $N_G(C) = W^3 \cup W^4$: If $w \in N_G(C)$, then there is an $i \in [7]$ with $wc_i, wc_{i+1} \in E(G)$, by (C1). If $wc_{i-1}, wc_{i+2} \notin E(G)$, $[w, c_i, c_{i+2}, c_{i-1}, c_{i+1}]$ induces a C_5; a contradiction. Thus, $wc_{i-1} \in E(G)$ or $wc_{i+2} \in E(G)$. By symmetry of the cycle we assume the latter. By (C2), $wc_{i+4}, wc_{i+5} \notin E(G)$. If $wc_{i+3}, wc_{i-1} \in E(G)$, then using (C2) with $wc_{i-1}, wc_i, wc_{i+1} \in E(G)$ we get the contradiction $wc_{i+3} \notin E(G)$. Thus, $w \in W^3 \cup W^4$.

(C4): $N_G^2(C) = \emptyset$: For the sake of contradiction we suppose $N_G^2(C) \neq \emptyset$. Let $n_2 \in N_G^2(C)$ then, by (C3), there is a $w \in W^3 \cup W^4$ with $wn_2 \in E(G)$. There is an $i \in [7]$ with $w \in W_i^3 \cup W_i^4$. Now $[n_2, w, c_i, c_{i+4}, c_{i+6}]$ induces a P_5; a contradiction.

(C5): G is K_4-free: Suppose not, then there is an induced K_4 in G, which we call K, with $n_C(K) := |V(K) \cap C|$. Clearly $n_C(K) \leq 3$. We next look at the remaining cases one by one. By (C3), $n_C(K) < 3$. Suppose $n_C(K) = 2$, then there is an $i \in [7]$ with $V(K) \cap C = \{c_i, c_{i+2}\}$ or $V(K) \cap C = \{c_i, c_{i+3}\}$. Again by (C3), $V(K) \cup \{c_{i+5}\}$ induces a HVN; a contradiction.

Suppose $n_C(K) = 1$ and $V(K) \cap N_G(C) = \{x, y, z\}$. For $i \in [7]$ we define $n_i := |E_G[\{c_i\}, \{x, y, z\}]|$ and by otherwise renaming the vertices in C let $n_3 = 3$. We know that $n_5, n_1 \leq 1$, since otherwise there is a induced K_4 in G, called K', with $n_C(K') \geq 2$; a contradiction to the previous case. Clearly $n_5, n_1 \neq 1$, since otherwise $V(K) \cup \{n_1\}$ or $V(K) \cup \{n_5\}$ induces a HVN. So $n_5 = n_1 = 0$. Since $|E_G[\{x, y, z\}, C]| \geq 9$, we find $n_2 = 3, n_4 = 3$, by (C3). Thus, there is an induced K_4 in G, called K', with $n_C(K') \geq 2$; a contradiction to the previous case.

Suppose last $n_C(K) = 0$. Thus, $V(K) \cap N_G(C) = 4$, by (C4). This implies, by (C3), $|E_G[V(K), C]| \geq 12$, so by the pigeonhole principle there is an $i \in [5]$ with $n_i \geq 2$. Clearly $n_i > 2$, since the graph is HVN-free. So there is an induced K_4, called K', with $n_C(K') = 1$; the final contradiction to a previous case.

Chudnosky et al. [21] prove, that (K_4, C_5, C_7, C_9, ...)-free graphs are 4-colourable. Since $\chi(\bar{C}_7) = 4$, $\bar{C}_7 \subseteq_{\text{ind}} G$, and G is critical, we conclude $G \cong \bar{C}_7$.

9.3 Proof of Lemma 65

For the remainder of the section we may suppose for the sake of contradiction that the graph G is counterexample of minimum order to this lemma. So G is a connected, critical (P_5, HVN)-free graph which contains an induced C_5 and $\chi(G) \geq \omega(G) + 2 \geq 6$. Since $\omega(G) \geq 4$, we find $G[N_G(v)]$ is a (P_5, paw)-free graph and thus $\chi(G[N_G(v)]) \leq \omega(G) - 1$, by Corollary 68, for each $v \in V(G)$.

Let $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$. We define, depending on C, the following sets:

$$\begin{aligned}
A_i(C) &:= \{w \in N_G(C) \mid V(C) \cap N_G(w) = \{c_i, c_{i+2}\}\}, \text{ for } i \in [5],\\
B_i(C) &:= \{w \in N_G(C) \mid V(C) \cap N_G(w) = \{c_i, c_{i+1}, c_{i+2}\}\}, \text{ for } i \in [5],\\
Y_i(C) &:= \{w \in N_G(C) \mid V(C) \cap N_G(w) = \{c_i, c_{i+2}, c_{i+3}\}\}, \text{ for } i \in [5],\\
H_i(C) &:= \{w \in N_G(C) \mid V(C) \cap N_G(w) = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}\}, \text{ for } i \in [5],\\
D(C) &:= \{w \in N_G(C) \mid V(C) \cap N_G(w) = V(C)\},\\
A(C) &:= \bigcup_{i \in [5]} A_i(C),\\
B(C) &:= \bigcup_{i \in [5]} B_i(C),\\
Y(C) &:= \bigcup_{i \in [5]} Y_i(C),\\
H(C) &:= \bigcup_{i \in [5]} H_i(C).
\end{aligned}$$

Since G is P_5-free, $N_G(C) = A(C) \cup B(C) \cup Y(C) \cup H(C) \cup D(C)$. Note that, we often omit the C in these notations. For each $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ we define

$$n_B(C) := |\{i \in [5] \mid B_i(C) \neq \emptyset\}|.$$

Also we define $n_B^{\max} := \max\{n_B(C) \mid C \in \mathcal{C}_5(G)\}$, which only depends on the minimal counterexample G.

The remainder of the proof is now organized as follows. In the following Claim 69.1 we analyse the structure of the neighbourhood of any given C_5 in G. The different results are labelled for later reference. (S4) for example directly implies that $n_B(C) \leq 2$ for any $C \in \mathcal{C}_5(G)$. Thus, $n_B^{\max} \leq 2$. Using the structure from Claim 69.1 we show in the then following three claims, by means of a complete case distinction, that the minimal counterexample G does not exist. Note that the last subclaim in each of these three claims is a clear contradiction to something previously assumed and the claims cover all possible cases. So all that is left to do is to prove the following four claims. Let us start with the structural results.

Claim 69.1. *Let $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ and $i \in [5]$. We omit the C in the following notations and write for example A_i instead of $A_i(C)$.*

($S1$) *$E_G[A_i, B_{i+2} \cup B_{i+3}]$ is anticomplete.*

($S2$) *$E_G[A_i, A_{i+1} \cup A_{i+4} \cup B_{i+1} \cup B_{i+4} \cup Y_{i+1} \cup H_{i+1} \cup H_{i+3}]$ is complete.*

($S3$) *$E_G[A \cup B, N_G^2(C)]$ is anticomplete.*

($S4$) *If $B_i \neq \emptyset$, then $B_{i+1} = \emptyset$.*

($S5$) *If $B_i \neq \emptyset$, then $Y_{i+3} \cup Y_{i+4} = \emptyset$.*

($S6$) *If $B_i \neq \emptyset$, then $H_i \cup H_{i+4} = \emptyset$.*

($S7$) *$E_G[B_i, Y_i \cup Y_{i+2}]$ is anticomplete.*

($S8$) *$E_G[B_i, Y_{i+1}]$ is complete.*

($S9$) *$E_G[B_i, H_{i+1} \cup H_{i+3} \cup D]$ is anticomplete.*

($S10$) *Each $X \in \{Y_i \mid i \in [5]\} \cup \{H_i \mid i \in [5]\} \cup \{D\}$ is an independent set.*

($S11$) *$H \cup D$ is an independent set.*

($S12$) *$E_G[H_i, Y_{i+1} \cup Y_{i+2}]$ is complete and $E_G[H_i, Y_i \cup Y_{i+3} \cup Y_{i+4}]$ is anticomplete.*

($S13$) *$E_G[Y_i, Y_{i+1}]$ is complete.*

($S14$) *$E_G[D, Y]$ is anticomplete.*

($S15$) *There is no induced K_4 in $G[D \cup Y \cup H \cup N_G^2(C)]$ with $|V(K_4) \cap (D \cup Y \cup H)| \geq 2$.*

Proof. Proof of ($S1$): Suppose not then there is an $a \in A_i$, and a $b \in B_{i+2} \cup B_{i+3}$ with $ab \in E(G)$. If $b = b_{i+2} \in B_{i+2}$ then $[c_{i+1}, c_i, a, b, c_{i+3}]$ induces a P_5; a contradiction. If $b = b_{i+3} \in B_{i+3}$ then $[c_{i+1}, c_{i+2}, a, b, c_{i+4}]$ induces a P_5; a contradiction.

Proof of ($S2$): Suppose not then there is an $a \in A_i$, and a $b \in A_{i+1} \cup A_{i+4} \cup B_{i+1} \cup B_{i+4} \cup Y_{i+1} \cup H_{i+1} \cup H_{i+3}$ with $ab \notin E(G)$. If $b = a_{i+1} \in A_{i+1} \cup B_{i+1}$ then $[a, c_i, c_{i+4}, c_{i+3}, a_{i+1}]$

induces a P_5; a contradiction. If $b = a_{i+4} \in A_{i+4} \cup B_{i+4}$ then $[a_{i+4}, c_{i+4}, c_{i+3}, c_{i+2}, a]$ induces a P_5; a contradiction. If $b = y_{i+1} \in Y_{i+1} \cup H_{i+1}$ then $[a, c_i, c_{i+1}, y_{i+1}, c_{i+3}]$ induces a P_5; a contradiction. If $b = h_{i+3} \in H_{i+3}$ then $[a, c_{i+2}, c_{i+1}, h_{i+3}, c_{i+4}]$ induces a P_5; a contradiction.

Proof of $(S3)$: Suppose not then there is an $i \in [5]$, a $x \in A_i \cup B_i$, and an $n_2 \in N^2_G(C)$ with $xn_2 \in E(G)$. But now $[n_2, x, c_{i+2}, c_{i+3}, c_{i+4}]$ induces a P_5; a contradiction.

Proof of $(S4)$: Suppose not, so there is a $b_i \in B_i$ and a $b_{i+1} \in B_{i+1}$. If $b_i b_{i+1} \notin E(G)$, then $[b_{i+1}, c_{i+3}, c_{i+4}, c_i, b_i]$ induces a P_5; a contradiction. If $b_i b_{i+1} \in E(G)$, then $[c_i, b_i, c_{i+1}, c_{i+2}, b_{i+1}]$ induces a HVN; a contradiction.

Proof of $(S5)$: Suppose not, so there is a $b_i \in B_i$ and a $y \in Y_{i+3} \cup Y_{i+4}$. If $y = y_{i+3} \in Y_{i+3}$, then $[b_i, c_{i+1}, y_{i+3}, c_{i+3}, c_{i+4}]$ induces a P_5 if $b_i y_{i+3} \notin E(G)$, and $[c_{i+2}, c_{i+1}, b_i, c_i, y_{i+3}]$ induces a HVN if $b_i y_{i+3} \in E(G)$; a contradiction. Thus, $y = y_{i+4} \in Y_{i+4}$. But $[b_i, c_{i+1}, y_{i+4}, c_{i+4}, c_{i+3}]$ induces a P_5 if $b_i y_{i+4} \notin E(G)$, and $[c_i, c_{i+1}, b_i, c_{i+2}, y_{i+4}]$ induces a HVN if $b_i y_{i+4} \in E(G)$; a contradiction.

Proof of $(S6)$: Suppose not, so there is a $b_i \in B_i$ and a $h \in H_i \cup H_{i+4}$. If $h = h_i \in H_i$, then $[b_i, c_{i+1}, h_i, c_{i+3}, c_{i+4}]$ induces a P_5 if $b_i h_i \notin E(G)$, and $[c_{i+3}, c_{i+2}, h_i, c_{i+1}, b_i]$ induces a HVN if $b_i h_i \in E(G)$; a contradiction. If $h = h_{i+4} \in H_{i+4}$, then $[b_i, c_{i+1}, h_{i+4}, c_{i+4}, c_{i+3}]$ induces a P_5 if $b_i h_{i+4} \notin E(G)$, and $[c_{i+4}, c_i, h_{i+4}, c_{i+1}, b_i]$ induces a HVN if $b_i h_{i+4} \in E(G)$; a contradiction.

Proof of $(S7)$: Suppose not, so there is a $b_i \in B_i$ and a $y \in Y_i \cup Y_{i+2}$ with $b_i y \in E(G)$. If $y = y_i \in Y_i$, then $[c_{i+1}, b_i, y_i, c_{i+3}, c_{i+4}]$ induces a P_5; a contradiction. If $y = y_{i+2} \in Y_{i+2}$, then $[c_{i+1}, b_i, y_i, c_{i+4}, c_{i+3}]$ induces a P_5; a contradiction.

Proof of $(S8)$: Suppose not, then there is a $y_{i+1} \in Y_{i+1}$ and a $b_i \in B_i$ with $y_{i+1} b_i \notin E(G)$. But now $[c_i, b_i, c_{i+2}, c_{i+3}, y_{i+1}]$ induces a P_5; a contradiction.

Proof of $(S9)$: Suppose not, so there is a $b_i \in B_i$ and a $x \in H_{i+1} \cup H_{i+3} \cup D$ with $b_i x \in E(G)$. If $x = h_{i+1} \in H_{i+1}$, then $[c_i, c_{i+1}, b_i, c_{i+2}, h_{i+1}]$ induces a HVN; a contradiction. If $x \in H_{i+3} \cup D$, then $[c_{i+4}, c_i, x, c_{i+1}, b_i]$ induces a HVN; a contradiction.

Proof of $(S10)$: Suppose not, then there are $x, x' \in X$ with $xx' \in E(G)$. So there is an $i \in [5]$ with $x, x' \in Y_i$ or $x, x' \in H_i$ or $x, x' \in D$ and $[c_i, x, x', c_{i+2}, c_{i+3}]$ induces a HVN; a contradiction.

Proof of $(S11)$: For $i \in [5]$ H_i and D are independent sets, by (S10). Suppose $H \cup D$ is not an independent set, then there is an $i \in [5]$, a $j \in [5] \setminus \{i\}$ and a $f_i \in H_i \cup D$ and a $f_j \in H_j$ with $f_i f_j \in E(G)$. If $j = i + 1$, then $[c_i, f_i, c_{i+1}, c_{i+2}, f_j]$ induces a HVN; a contradiction. If $j = i + 2$, then $[c_i, f_i, f_j, c_{i+2}, c_{i+3}]$ induces a HVN; a contradiction. If $j = i + 3$, then $[c_{i+3}, f_i, f_j, c_i, c_{i+1}]$ induces a HVN; a contradiction. If $j = i + 4$, then $[c_{i+3}, f_i, c_{i+2}, c_{i+1}, f_j]$ induces a HVN; a contradiction.

Proof of $(S12)$: Suppose not, then there is a $h_i \in H_i$ and a $y \in Y_{i+1} \cup Y_{i+2}$ with $h_i y \notin E(G)$ or $y \in Y_i \cup Y_{i+3} \cup Y_{i+4}$ with $h_i y \in E(G)$. Let us look at the first case: If $y = y_{i+1} \in Y_{i+1}$, then $[y_{i+1}, c_{i+4}, c_i, h_i, c_{i+2}]$ induces a P_5; a contradiction. If $y = y_{i+2} \in Y_{i+2}$, then $[y_{i+2}, c_{i+4}, c_{i+3}, h_i, c_{i+1}]$ induces a P_5; a contradiction. Let us now look at the second case: If $y = y_i \in Y_i$, then $[c_i, y_i, h_i, c_{i+2}, c_{i+4}]$ induces a HVN; a contradiction. If $y = y_{i+3} \in Y_{i+3}$, then $[c_{i+3}, y_{i+3}, h_i, c_i, c_{i+1}]$ induces a HVN; a contradiction. If $y = y_{i+4} \in Y_{i+3}$, then $[c_i, h_i, c_{i+1}, c_{i+2}, y_{i+4}]$ induces a HVN; a contradiction.

Proof of $(S13)$: Suppose not, then there is a $y_i \in Y_i$ and a $y_{i+1} \in Y_{i+1}$ with $y_i y_{i+1} \notin E(G)$. But now $[y_i, c_{i+2}, c_{i+1}, y_{i+1}, c_{i+4}]$ induces a P_5; a contradiction.

Proof of $(S14)$: Suppose not, then there is an $i \in [5]$, a $y_i \in Y_i$, and a $d \in D$ with $dy_i \in E(G)$. But now $[c_i, y_i, d, c_{i+2}, c_{i+3}]$ induces a HVN; a contradiction.

Proof of $(S15)$: Suppose for the sake of contradiction there is such a K_4. If $|V(K_4) \cap (D \cup Y \cup H)| = 2$, there is a $j \in [5]$ such that $\{c_j\} \cup V(K_4)$ induces a HVN, by pigeonhole principle; a contradiction. So we may assume $|V(K_4) \cap (D \cup Y \cup H)| \geq 3$. Since $E_G[D, Y \cup H]$ is anticomplete and D is independent, by (S14) and (S11), $|V(K_4) \cap D| = 0$ Since H is independent, by (S11), $|V(K_4) \cap H| \leq 1$. Recall that, for $i \in [5]$, Y_i is independent, by (S10). Let us first look at the case $|V(K_4) \cap H| = 1$. Let $i \in [5]$ with $|V(K_4) \cap H_i| = 1$. By (S12) and (S10), $|V(K_4) \cap (Y \cup H)| \leq 3$. So $|V(K_4) \cap (Y \cup H)| = 3$, and there is a $y_{i+1} \in Y_{i+1} \cap V(K_4), y_{i+2} \in Y_{i+2} \cap V(K_4)$, and $V(K_4) \cup \{c_{i+1}\}$ induces a HVN; a contradiction. Let us lastly look at the case $|V(K_4) \cap H| = 0$. If $|V(K_4) \cap Y| = 4$, there is an $i \in [5]$ with $y_i \in Y_i, y_{i+1} \in Y_{i+1}, y_{i+2} \in Y_{i+2}, y_{i+3} \in Y_{i+3}$ and $V(K_4) = \{y_i, y_{i+2}, y_{i+3}, y_{i+4}\}$, since for each $j \in [5]$ Y_j is independent. But now $[c_{i+1}, y_{i+1}, y_{i+3}, y_{i+2}, y_i]$ induces a HVN; a contradiction. If $|V(K_4) \cap Y| = 3$, there is an $i \in [5]$ with $V(K_4) \cap Y = \{y_i, y_{i+1}, y_{i+2}\}$ or $V(K_4) \cap Y = \{y_i, y_{i+1}, y_{i+3}\}$. In the first case $V(K_4) \cup \{c_{i+2}\}$ and in the second case $V(K_4) \cup \{c_{i+1}\}$ induces a HVN, a contradiction. □

Claim 69.2. *Let $n_B^{\max} = 2$ in this case there is a $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ with $n_B(C) = 2$. We omit the C in the following notations and write for example A_i instead of $A_i(C)$. By (S4), there is an $i \in [5]$ with $B_i, B_{i+2} \neq \emptyset$.*

$(C1)$ *If $Y_{i+2} \neq \emptyset$, then $E_G[B_i, B_{i+2}]$ is anticomplete.*

$(C2)$ *If $H_{i+3} \cup D \neq \emptyset$, then $E_G[B_i, B_{i+2}]$ is complete.*

$(C3)$ *$N_G(C) = A \cup B \cup D$ or $N_G(C) = A \cup B \cup H_{i+3}$ or $N_G(C) = A \cup B \cup Y_{i+2}$.*

$(C4)$ *If $N_G(C) = A \cup B \cup D$ or $N_G(C) = A \cup B \cup H_{i+3}$, then $N_G^2(C) = \emptyset$.*

$(C5)$ *$A_{i+1}, A_{i+3}, A_{i+4}$ are independent sets.*

$(C6)$ *$E_G[A_{i+1}, A_{i+3}]$ is anticomplete.*

$(C7)$ *If* $N_G(C) = A \cup B \cup D$ *or* $N_G(C) = A \cup B \cup Y_{i+2}$*, then* G *is* $(\omega(G)+1)$*-colourable.*

$(C8)$ *If* $N_G(C) = A \cup B \cup H_{i+3}$*, then* G *is* $(\omega(G)+1)$*-colourable.*

$(C9)$ G *is* $(\omega(G)+1)$*-colourable.*

Proof. Proof of $(C1)$: Suppose not, then there is a $b_i \in B_i$, a $b_{i+2} \in B_{i+2}$, and a $y \in Y_{i+2}$ with $b_i b_{i+2} \in E(G)$. By (S7) $b_i y, b_{i+2} y \notin E(G)$ and $[c_{i+1}, b_i, b_{i+2}, c_{i+4}, y]$ induces a P_5; a contradiction.

Proof of $(C2)$: Suppose not, then there is a $b_i \in B_i$, a $b_{i+2} \in B_{i+2}$, and a $x \in H_{i+3} \cup D$ with $b_i b_{i+2} \notin E(G)$. By (S9) $b_i x, b_{i+2} x \notin E(G)$ and $[b_i, c_i, x, c_{i+3}, b_{i+2}]$ induces a P_5; a contradiction.

Proof of $(C3)$: By (S5) $Y_i \cup Y_{i+1} \cup Y_{i+3} \cup Y_{i+4} = \emptyset$. By (S6) $H_i \cup H_{i+1} \cup H_{i+2} \cup H_{i+4} = \emptyset$. So it remains to show that at most one of the three sets D, H_{i+3}, Y_{i+2} is non empty. If $H_{i+3} \cup D \neq \emptyset$, then $E_G[B_i, B_{i+2}]$ is complete by (C2). If $Y_{i+2} \neq \emptyset$, then $E_G[B_i, B_{i+2}]$ is anticomplete by (C1). Thus, $Y_{i+2} = \emptyset$ or $H_{i+3} \cup D = \emptyset$. In the latter case the claim is shown so we may assume the the first case. For the sake of contradiction we suppose $h_{i+3} \in H_{i+3}, d \in D$. By (S11) $h_{i+3} d \notin E(G)$. For $b_i \in B_i$ $[h_{i+3}, c_{i+4}, d, c_{i+2}, b_i]$ induces a P_5, by (S9); a contradiction.

Proof of $(C4)$: Suppose not, then there is an $n_2 \in N_G^2(C)$. Since G is connected, there is a $x \in D \cup H_{i+3}$ with $xn_2 \in E(G)$, by (S3). Now $[n_2, x, c_{i+3}, b_{i+2}, b_i]$ induces a P_5, by (C2) and (S9); a contradiction.

Proof of $(C5)$: Suppose not, then there is a $j \in \{i+1, i+3, i+4\}$ and $a, a' \in A_j$ with $aa' \in E(G)$. If $j = i+1$, $[c_i, b_i, c_{i+1}, a, a']$ induces a HVN, by (S2); a contradiction. If $j = i+3$, $[c_{i+2}, b_{i+2}, c_{i+3}, a, a']$ induces a HVN, by (S2); a contradiction. If $j = i+4$, $[c_{i+2}, b_i, c_{i+1}, a, a']$ induces a HVN, by (S2); a contradiction.

Proof of $(C6)$: Suppose not, then there is a $b_{i+2} \in B_{i+2}$, an $a_{i+1} \in A_{i+1}$, and an $a_{i+3} \in A_{i+3}$ with $a_{i+1} a_{i+3} \in E(G)$. We know that $a_{i+1} b_{i+2}, a_{i+3} b_{i+2} \in E(G)$, by (S2). Therefore, $[c_{i+2}, c_{i+3}, b_{i+2}, a_{i+3}, a_{i+1}]$ induces a HVN; a contradiction.

Proof of $(C7)$: We colour $N_G(c_{i+2})$ with the colours $1, \ldots, \omega(G)-1$, in such a way that $c(Y_{i+2}) \subseteq \{1\}$, which is possible by Corollary 68 and Lemma 69. By (C5) and (C6) we proper colour $G[A_{i+1} \cup A_{i+3} \cup A_{i+4} \cup \{c_i, c_{i+2}, c_{i+4}\}]$ with 2 colours as follows:

$$c(u) = \begin{cases} \omega(G), & \text{for } u \in A_{i+1} \cup A_{i+3} \cup \{c_{i+2}, c_{i+4}\}, \\ \omega(G)+1, & \text{for } u \in A_{i+4} \cup \{c_i\}. \end{cases}$$

So $N_G^2(C) \neq \emptyset$ and $N_G(C) = A \cup B \cup Y_{i+2}$ is the only remaining case, by (C4). Let $S_1, \ldots, S_k$ be the connected components of $G[V(G) \setminus N_G[C]]$. For each $j \in [k]$ there is a $y \in Y_{i+2}$ with $[y, S_j]$ is complete, since G is connected and P_5-free. So $\bigcup_{j=1}^{k} S_k$ is

$(\omega(G)-1)$-colourable, by Corollary 68. Using the colours $\{2,\dots,\omega\}$ on $\bigcup_{i=1}^{k} S_k$ admits an $(\omega(G)+1)$-colouring of G.

Proof of $(C8)$: We know by (C4) that $V(G) = N_G[C]$. We colour $G[A_{i+1} \cup A_{i+3} \cup A_{i+4} \cup \{c_i, c_{i+2}, c_{i+4}\}]$ with 2 colours as follows (identical as in (C7)):

$$c(u) = \begin{cases} \omega(G), & \text{for } u \in A_{i+1} \cup A_{i+3} \cup \{c_{i+2}, c_{i+4}\}, \\ \omega(G)+1, & \text{for } u \in A_{i+4} \cup \{c_i\}. \end{cases}$$

Thus, if we proper colour $N_G(c_{i+2}) \cup H_{i+3}$ with at most $\omega(G)-1$ colours, then the claim is proven. If $H_{i+3} = \emptyset$ we colour $N_G(c_{i+2})$ with at most $\omega(G)-1$ colours, which is doable by Corollary 68. So for the remainder of this claim let $h_{i+3} \in H_{i+3} \neq \emptyset$. We show next that for $j \in \{i, i+2\}$ $E_G[A_j, B_j]$ is complete and $E_G[A_j, B_{i+2-(j-i)}]$ is anticomplete: Suppose there is a $j \in \{i, i+2\}$ with $a_j b_j \notin E(G)$, then $[c_{i+4-2(j-i)}, h_{i+3}, a_j, c_{i+2}, b_j]$ induces a P_5, by (S2) and (S9); a contradiction. Suppose there is a $j \in \{i, i+2\}$ with $a_j b_{i+2-(j-i)} \in E(G)$, if $j = i$, this is a contradiction to (S1), if $j = i+2$ then $[c_{i+1}, c_{i+2}, b_i, b_{i+2}, a_{i+2}]$ induces a HVN, by (C2); a contradiction. We show next that A_i and A_{i+2} are independent sets. Suppose not then there is a $j \in \{i, i+2\}$ with $a, a' \in A_j$ with $aa' \in E(G)$. But now $[c_{i+1+(j-i)}, c_{i+2}, b_j, a, a']$ induces a HVN; a contradiction. Also B_i and B_{i+2} are independent sets. Suppose not then there is a $j \in \{i, i+2\}$ with $b, b' \in B_j$ with $bb' \in E(G)$. But now $[c_{i+2(j-i)}, b, b', c_{i+2}, b_{i+2-(j-i)}]$ induces a HVN, by (C2); a contradiction. Now $N_G(c_{i+2})$ is 2-colourable, as follows:

$$c(u) = \begin{cases} 1, & \text{for } u \in A_i \cup B_{i+2} \cup \{c_{i+1}\}, \\ 2, & \text{for } u \in A_{i+2} \cup B_i \cup \{c_{i+3}\}. \end{cases}$$

So colouring H_{i+3} in 3 admits a 3-colouring of $N_G(c_{i+2}) \cup H_{i+3}$. Since $3 \leq \omega(G) - 1$ the claim is proven.

Proof of $(C9)$: This follows directly from (C3), (C7) and (C8). □

Claim 69.3. *Let $n_B^{\max} \leq 1$ and $\chi(G[B(C)]) \leq 1$, for each $C \in \mathcal{C}_5(G)$. In this case we fix $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ with $|D(C)| = \min\{|D(C')| : C' \in \mathcal{C}_5(G)\}$. We omit the C in the following notations and write for example A_i instead of $A_i(C)$. In this setting we show the following claims:*

$(C1)$ *There is no $C' \in \mathcal{C}_5(G)$ with a vertex in $N_G^2(C)$, D and $V(C) \cup A \cup B$.*

$(C2)$ *For $i \in [5]$, $G[A_i]$ is K_3-free.*

$(C3)$ *$G[N_G[C]]$ is K_4-free.*

$(C4)$ *$N_G^3(C) = \emptyset$.*

$(C5)$ *$G[N_G^2(C)]$ is not K_3-free.*

By (C5) there is a component in $N_G^2(C)$ containing 3 pairwise adjacent vertices. We call the component K and the pairwise adjacent vertices $k_1, k_2, k_3 \in K$.

$(C6)$ *If $x \in H \cup Y$, then $E_G[\{x\}, K]$ is complete or anticomplete.*

$(C7)$ *$E_G[D, N_G^2(C)]$ is complete.*

$(C8)$ *There is a $C' \in \mathcal{C}_5(G)$ with $\chi(G[B(C')]) \geq 2$.*

Proof. Proof of $(C1)$: This is true if $D = \emptyset$. If $D \neq \emptyset$, for such a cycle C' $|D(C')| = 0$, by (S3), (S11), and (S14), which is a contradiction to the choice of C.

Proof of $(C2)$: Suppose not and $a, a', \tilde{a} \in A_i$ with $aa', a\tilde{a}, a'\tilde{a} \in E(G)$. But now $[a, c_{i+2}, c_{i+3}, c_{i+4}, c_i]$ induces a C_5, which we call C', with $a', \tilde{a} \in B(C')$; a contradiction to $\chi(G[B(C')]) \leq 1$.

Proof of $(C3)$: Suppose not, then there is a K_4, which we call K, with $n_C(K) \coloneqq |V(K) \cap V(C)|$. Clearly $n_C(K) \leq 2$. Suppose $n_C(K) = 2$ with $x, y \in V(K_4) \cap N_G(C)$, then there is an $i \in [5]$ with $c_i, c_{i+1} \in V(K_4)$. So $x, y \notin A$, and since $E_G[D, H \cup Y \cup B]$ is anticomplete and D is independent, by (S9),(S11),(S14), we know $x, y \notin D$. Suppose first $y \in Y_{i+3}$. Since Y_{i+3} is independent, $x \in H \cup B$. Since $xy \in E(G)$ $x \in H_{i+1} \cup H_{i+2} \cup B_{i+2}$, by (S12), (S7) and (S5), a contradiction to $xc_i, xc_{i+1} \in E(G)$. So $x, y \notin Y_{i+3}$ and $|\{x, y\} \cap B| = 1$, since H and B are independent sets. For the final contradiction in this case we suppose $x \in B_i \cup B_{i+4}$ and $y \in H$. If $x \in B_i$, then $y \in H_{i+2}$, by (S6) and (S9), a contradiction to $yc_{i+1} \in E(G)$. If $x \in B_{i+4}$, then $y \in H_{i+1}$, by (S6) and (S9); a contradiction to $yc_i \in E(G)$.

Suppose $n_C(K) = 1$, $V(K_4) \cap N_G(C) = \{x, y, z\}$. For $i \in [5]$ we define the integer n_i by $n_i \coloneqq |E_G[\{c_i\}, \{x, y, z\}]|$ and let $j \in [5]$ with $c_j \in V(K_4)$, so $n_j = 3$. We first argue that, for $i \in [5]$, if $n_i = 3$, then $n_{i+1} = n_{i-1} = 0$. We know that $n_{i+1}, n_{i-1} \leq 1$, since otherwise there is a K_4 K' with $n_C(K') \geq 2$; a contradiction to the previous case. Also $n_{i+1}, n_{i-1} \neq 1$, since otherwise $\{c_{i+1}, c_i, x, y, z, \}$ or $\{c_{i-1}, c_i, x, y, z\}$ induces a HVN; a contradiction. Which proves the just stated claim and we know $n_{j-1} = n_{j+1} = 0$. Also $n_{j+2} \neq 2$ and $n_{j-2} \neq 2$, since otherwise $V(K_4) \cup \{c_1\}$ or $V(K_4) \cup \{c_5\}$ induces a HVN. Since $\sum_{i=1}^{5} n_i \geq 6$, $n_{j-2} > 1$ or $n_{j+2} > 1$. Thus, by symmetry we may assume $n_{j+2} = 3$. But now $n_{j-2} = n_{j+2+1} = 0$, and $x, y, z \in A_3$; a contradiction to (C2).

Suppose last $n_C(K) = 0$. Since $|E_G[V(K_4), V(C)] \geq 8$, there is an $i \in [5]$ with $|E_G[\{c_i\}, V(K_4)]| \geq 2$, by the pigeonhole principle. Since the graph is HVN-free, this even implies $|E_G[\{c_i\}, V(K_4)]| > 2$. Thus, there is a K_4, called K', with $n_C(K') \geq 1$; a contradiction to the previous case.

Proof of $(C4)$: Suppose not, then there is an $n_3 \in N_G^3(C), n_2 \in N_G^2(C)$, and a $d \in D$ with $n_3n_2, n_2d \in E(G)$, since G is P_5-free. Since d is not a cutvertex, by Lemma 37, since G is critical, there is a $d' \in D \setminus \{d\}$. Since $[n_3, n_2, d, c_1, d']$ does not induces a P_5, $d'n_2 \in E(G)$. Since (d, d') and (d', d) are not comparable vertex pairs, by Lemma 34, since G is critical, there is a $p_d, p_{d'} \in V(G)$ with $p_d d, p_{d'} d' \in E(G)$ and $p_{d'} d, p_d d' \notin E(G)$. Clearly $p_d, p_{d'} \notin D$. If $p_{d'}, p_d \in N_G^2(C)$ then $[p_{d'}, d', c_1, d, p_d]$ induces a C_5, since G is P_5-free; a contradiction to (C1). Since $E_G[D, H \cup Y \cup B]$ is anticomplete, we may assume, by otherwise renaming, $p_d \in A$. If also $p_{d'} \in A$, $[p_{d'}, d', n_2, d, p_d]$ induces a C_5, since G is P_5-free; a contradiction to (C1). So $p_{d'} \in N_G^2(C)$ and there is an $i \in [5]$ with $p_d \in A_i$, and $[p_{d'}, d', c_{i+1}, d, p_d]$ induces a P_5; a contradiction.

Proof of $(C5)$: This follows from the fact that $\omega(G) \geq 4$, from (S15), and $G[N_G[C]]$ is K_4-free, by (C3).

Proof of $(C6)$: Suppose not, then there are $k, k' \in K$, an $i \in [5]$, and a $x \in H_i \cup Y_i$ with $xk \in E(G)$ and $xk' \notin E(G)$. By the connectivity of K we may assume $kk' \in E(G)$. But now $[k', k, h, c_i, c_{i-1}]$ induces a P_5; a contradiction.

Proof of $(C7)$: Suppose not, then there is a $d \in D$ and an $n_2 \in N_G^2(C)$ with $dn_2 \notin E(G)$. Since $n_2 \in N_G^2(C)$, there is a $x \in H \cup Y \cup D$ with $xn_2 \in E(G)$. If $x \in H \cup Y$, there is an $i \in [5]$ with $x \in H_i \cup Y_i$ and $[c_{i+4}, d, c_{i+2}, x, n_2]$ induces a P_5; a contradiction. So $x \in D$. Since (d, x) is not a comparable vertex pair, there is a $p_d \in V(G)$ with $p_d d \in E(G)$ and $p_d x \notin E(G)$. If $p_d \in N_G^2(C)$, then $[p_d, d, c_1, x, n_2]$ induces a C_5, since G is P_5-free; a contradiction to (C1). So $p_d \in A \cup B$ and there is an $i \in [5]$ with $p_d \in A_i \cup B_i$, and $[n_2, x, c_{i+1}, d, p_d]$ induces a P_5; the final contradiction.

Proof of $(C8)$: Since the graph is connected, there is a $x \in D \cup Y \cup H$ with $E_G[\{x\}, K]$ complete, by (C6) and (C7). Since x is not a cutvertex, by Lemma 37, there is a $y \in D \cup Y \cup H$ with $x \neq y$ such that $E_G[\{y\}, K]$ is complete. We see that $xy \notin E(G)$, since $xy \in E(G)$ is a contradiction to (S15). Since (x, y) and (y, x) is not a comparable vertex pair, there are $p_x, p_y \in V(G)$ with $p_x x, p_y y \in E(G)$ and $p_x y, p_y x \notin E(G)$. Clearly $p_x, p_y \notin K$. Also $E_G[\{p_x\} \cup \{p_y\}, K]$ is anticomplete, since otherwise $E_G[\{p_z\}, K]$ is complete, for a $z \in \{x, y\}$, and we end in a contradiction to (S15). Therefore, $p_x p_y \in E(G)$, since G is P_5-free. But now $C' : p_x x k_1 y p_y p_x \in \mathcal{C}_5(G)$ with $k_2, k_3 \in B(C')$. □

Claim 69.4. *Let $n_B^{\max} = 1$ and there be a $C \in \mathcal{C}_5(G)$ with $\chi(G[B(C)]) \geq 2$. We fix $C : c_1c_2c_3c_4c_5c_1 \in \mathcal{C}_5(G)$ with*

$$\chi(G[B(C)]) = \max\{\chi(G[B(C')]) : C' \in \mathcal{C}_5(G)\} \geq 2.$$

Let $i \in [5]$ with $B_i(C) \neq \emptyset$. We omit the C in the following notations and write for example A_i instead of $A_i(C)$. Since there is an edge $bb' \in E(G[B_i])$ quite some restrictions on $N_G(C)$ follow:

$(C1)$ $Y_{i+3}, Y_{i+4}, H_i, H_{i+4}, H_{i+1}, H_{i+3}, D = \emptyset$.

$(C2)$ $A_{i+1}, A_{i+2}, A_{i+4}$ are independent sets.

$(C3)$ $E_G[Y_{i+1}, A_{i+1} \cup A_{i+4}]$ *is anticomplete.*

$(C4)$ $A_{i+1} \cup A_{i+4} \cup Y_{i+1} \cup \{c_i, c_{i+2}\}$ *is an independent set.*

$(C5)$ G *is* $(\omega(G)+1)$*-colourable*

Proof. Recall that, by (S13), $E_G[Y_i, Y_{i+1}]$ is complete, by (S8), $E_G[B_i, Y_{i+1}]$ is complete, by (S7), $E_G[B_i, Y_i \cup Y_{i+2}]$ is anticomplete, by (S1), $E_G[B_i, A_{i+2} \cup A_{i+3}]$ is anticomplete, and, by (S2), $E_G[B_i, A_{i+1} \cup A_{i+4}]$ is complete.

Proof of $(C1)$: Recall that, by (S5), $Y_{i+3}, Y_{i+4} = \emptyset$ and, by (S6), $H_i, H_{i+4} = \emptyset$. By (S9), $E_G[B, H_{i+1} \cup H_{i+3} \cup D]$ is anticomplete. We first show that $D = \emptyset$. Suppose not, then there is a $d \in D$ with $db, db' \notin E(G)$, by (S9). But now $[d, c_i, c_{i+1}, b, b']$ induces a HVN; a contradiction. Suppose there is a $j \in \{i+1, i+3\}$ with $h \in H_j$, then, also by (S9), $[h, c_{i+3-(j-i)}, c_{i+1}, b, b']$ induces a HVN; a contradiction.

Proof of $(C2)$: Suppose not, then there is a $j \in \{i+1, i+2, i+4\}$ with $a, a' \in A_j$ and $aa' \in E(G)$. If $j = i+1$ or $j = i+4$, then $[c_i, b, c_{i+1}, a, a']$ induces a HVN, by (S2); a contradiction. If $j = i+2$, then $[a, c_{i+4}, c_i, c_{i+1}, c_{i+2}]$ induces a C_5 C' with $n_B(C') \geq 2$; a contradiction to $n_B^{\max} = 1$.

Proof of $(C3)$: Suppose not, then there is a $y_{i+1} \in Y_{i+1}$ and a $j \in \{i+1, i+4\}$ with $a \in A_j$ and $y_{i+1}a \in E(G)$. But now $[c_i, b, c_{i+1}, y_{i+1}, a]$ induces a HVN, by (S8); a contradiction.

Proof of $(C4)$: Suppose not, then there is a $x \in A_{i+1} \cup A_{i+4} \cup Y_{i+1} \cup \{c_i, c_{i+2}\}$ and a $y \in A_{i+1} \cup A_{i+4} \cup Y_{i+1} \cup \{c_i, c_{i+2}\}$ with $xy \in E(G)$. Note that $x, y \notin \{c_i, c_{i+2}\}$. Also x and y are not both in one of the 3 subsets, by (C2) and (S10). By (C3), $x, y \notin Y_{i+1}$. So $x \in A_{i+1}, y \in A_{i+4}$ but now $[c_i, b_i, c_{i+1}, x, y]$ induces a HVN; the final contradiction.

Proof of $(C5)$: Recall that by (S12) $E_G[H_{i+2}, Y_{i+2} \cup Y_i]$ is anticomplete. Thus, Y_i and $Y_{i+2} \cup H_{i+2}$ are two independent sets by (S10). We colour $N_G(c_i)$ with colours from $[\omega(G)-1]$ colours in such a way that $|c(H_{i+2} \cup Y_{i+2} \cup Y_i)| \leq 2$. Which is possible by Lemma 69, since $\omega(G) \geq 4$. The remainder of $G[N_G[C]]$ we colour as follows.

$$c(u) = \begin{cases} \omega(G), & \text{for } u \in A_{i+1} \cup A_{i+4} \cup Y_{i+1} \cup \{c_i, c_{i+2}\}, \\ \omega(G)+1, & \text{for } u \in A_{i+2} \cup \{c_{i+3}\}, \end{cases}$$

which is a proper colouring by (C4). So G is $(\omega(G)+1)$-colourable or $N_G^2(C) \neq \emptyset$. Clearly $N_G^3(C) = \emptyset$, since $D = \emptyset$. The trivial components in $N_G^2(C)$ we colour with colour $\omega(G)+1$. So let $S_1, \ldots, S_k$ be the non-trivial components of $G[N_G^2(C)]$. We choose $j \in [k]$ arbitrary. Observe first that if $x \in N_G(S_j)$, we know that $E_G[\{x\}, S_j]$ is complete, since G is P_5-free and $D = \emptyset$. Thus, there is a $y \in Y \cup H$ with $E_G[\{y\}, S_j]$ is complete, since G is connected. So $\chi(G[S_j]) \leq \omega(G)-1$, by Corollary 68. To prove our claim it suffices to show that if $\chi(G[S_j]) = \omega(G) - 1$, then $|c(N_G(S_j) \cap (Y \cup H))| \leq 2$. If $E_G[Y_{i+1}, S_j]$ is anticomplete the claim is proven, since $|c(Y_i \cup Y_{i+2} \cup H_{i+2})| \leq 2$. Thus, we may assume there is a $y_{i+1} \in Y_{i+1}$ with $E_G[\{y_{i+1}\}, S_j]$ is complete by the previous observation. But now $E_G[Y_i \cup Y_{i+2}, S_j]$ is anticomplete, by (S15), since $E_G[Y_j, Y_{j+1}]$ is complete for each $j \in [5]$, by (S13). So in this case $N_G(S_j) \subseteq Y_{i+1} \cup H_{i+2}$ and $E_G[H_{i+2}, Y_{i+1}]$ is anticomplete, by (S15). If $H_{i+2} \cap N_G(S_j) = \emptyset$ we are done so we suppose for the sake of contradiction, that there is a $h_{i+2} \in H_{i+2}$ with $E_G[\{h_{i+2}\}, S_j]$ is complete. Now $C' : n_2, h_{i+2}, c_{i+2}, c_{i+1}, y_{i+1}, n_2 \in \mathcal{C}_5(G)$ for every $n_2 \in S_j$. But also $n_B(C') \geq 2$, because $b \in B(C')$, since $by_{i+1}, bc_{i+1}, bc_{i+2} \in E(G)$, by (S8), and $n'_2 \in N_G(n_2) \cap N^2_G(C)$ is in $B(C')$ with $n'_2 c_{i+2} \notin E(G)$; the final contradiction. $\square$

10 Characterisation of graphs H with $f^\star_{\{P_5,H\}}(\omega) \leq \omega + c(H)$

Let us recall that $f^\star_{\{P_5,\text{HVN}\}}(\omega) \leq \omega + 2$ and $f^\star_{\{P_5,H\}}(\omega) = f^\star_{\{H\}}(\omega) \leq \omega = \omega + 0$, for every $\omega \in \mathbb{N}_{>0}$ and each $H \subseteq_{id} P_4$ (c.f. Theorem 7 and Observation 16). It is quite rare to find a graph H such that the family of (P_5, H)-free graphs has a binding function of that form.

In this section, we characterize all graphs H such that

$$f^\star_{\{P_5,H\}}(\omega) \leq \omega + c(H)$$

for some constant $c(H)$ – depending on H only – and each $\omega \in \mathbb{N}_{>0}$. To do that we define the following special graph. For $p \in \mathbb{N}_0$ the graph F_p is defined as $F_p := (K_1 \cup K_2) + K_p$. Note that $F_2 \cong HVN$, $F_1 \cong paw$, $F_0 \cong K_1 \cup K_2$, and F_p is the complementary graph of $pK_1 \cup P_3$ for each $p \in \mathbb{N}_0$. So this section is dedicated to the proof of Theorem 8 which states the following. For a graph H, there is a constant $c(H)$ such that $f^\star_{\{P_5,H\}}(\omega) \leq \omega + c(H)$, for $\omega \in \mathbb{N}_{>0}$, if and only if either $H \cong P_4$ or H is an induced subgraph of F_p for some $p \in \mathbb{N}_0$.

One direction we order in the following three lemmas.

Lemma 70. *Let $p \in \mathbb{N}_{>0}$ and G be a (P_5, K_p)-free graph. There exists a $c(K_p) = c(p) \in \mathbb{N}_0$ such that $\chi(G) \leq c(p)$.*

Proof. For $p \in \mathbb{N}_{>0}$ we define $c(p) := f^\star_{P_5}(p-1) \in \mathbb{N}_0$ and G be a (P_5, K_p)-free graph. Note that $f^\star_{P_5}$ is superadditive, by Lemma 43, and thus especially increasing. Since G is P_5-free and $\omega(G) \leq p-1$, we conclude $\chi(G) \leq f^\star_{P_5}(\omega(G)) \leq f^\star_{P_5}(p-1)$. □

We use the upcoming Section 10.1 to prove the following Lemma 71.

Lemma 71. *Let $p \in \mathbb{N}_0$ and G be a (P_5, F_p)-free graph. There exists a $c(F_p) = c(p) \in \mathbb{N}_0$ such that $\chi(G) \leq \omega(G) + c(p)$.*

Lemma 72. *Let $p \in \mathbb{N}_0$ and G be a $(P_5, 2K_1 + K_p)$-free graph. There exists a $c_1(2K_1 + K_p) = c_1(p) \in \mathbb{N}_0$ such that $\chi(G) \leq \omega(G) + c_1(p)$.*

Proof. Let $p \in \mathbb{N}_0$ be fixed. We define $c_1(p) := c(p) \in \mathbb{N}_0$, where c is the function from Lemma 71. Let G be an arbitrary $(P_5, 2K_1 + K_p)$-free graph. Since $2K_1 + K_p \subseteq_{\text{ind}} F_p$, we find G is (P_5, F_p)-free. Thus, we know that $\chi(G) \leq \omega(G) + c(p) = \omega(G) + c_1(p)$, by Lemma 71, which completes the proof. □

The following Lemma 73 states the reverse direction of Theorem 8. Note that in this lemma we use our Lemma 42 from Section 3.3.

Lemma 73. *Let H be a graph. If there exists a $c(H) \in \mathbb{N}_0$ such that $\chi(G) \leq \omega(G) + c(H)$ for all (P_5, H)-free graphs G, then $H \in \{F_p \mid p \in \mathbb{N}_0\}$ or $H \in \{2K_1 + K_p \mid p \in \mathbb{N}_0\}$ or $H \in \{K_p \mid p \in \mathbb{N}_{>0}\}$ or $H \cong P_4$.*

Proof. We prove this lemma by contraposition, thus, it suffices to show that

$$\lim_{\omega \to +\infty} (f^\star_{\{P_5,H\}}(\omega) - \omega) = +\infty$$

for each graph H which is neither isomorphic to P_4 nor an induced subgraph of F_p for some $p \in \mathbb{N}_0$. For all graphs H for which $\bar{H}$ is not a forest, we get that the class of (P_5, H)-free graphs does not even have a linear χ-binding function, by Lemma 42. Thus, it remains to assume that $\bar{H}$ is a forest. For each $t \geq 1$, let G_t be the graph which is the complementary graph of t pairwise vertex distinct cycles of length 5. Note that G_t has clique number $2t$, chromatic number $3t$, and G_t is P_5-free and $\bar{G}_t$ is $(P_5, K_{1,3})$-free. Consequently, if $\bar{H}$ contains an induced P_5 or $K_{1,3}$, then each graph $\bar{G}_t$ is $\bar{H}$-free, and so it follows that G_t is (P_5, H)-free and $\lim_{\omega \to +\infty}(f^\star_{\{P_5,H\}}\omega) - \omega) = +\infty$. In view of the desired result it remains to assume that $\bar{H}$ is $(P_5, K_{1,3})$-free. In other words, $\bar{H}$ is a linear forest each component of which is of order at most 4. Now, for each $t \geq 1$, let $G_t \cong C_5[K_t, K_t, K_t, K_t, K_t]$. It is easily seen that G_t is of clique number $2t$ but $\chi(G_t) \geq 5t/2$ as G_t is of independence number at most 2. Furthermore, G_t is P_5-free. As the complementary graph of G_t contains of 5 independent sets of size k and the complementary graph of C_5 is isomorphic to C_5, we find that $\bar{G}_t$ is $(K_1 \cup P_4, 2K_2)$-free. Consequently, if $\bar{H}$ contains an induced $K_1 \cup P_4$ or $2K_2$, then each graph $\bar{G}_t$ is $\bar{H}$-free, and so it follows that G_t is (P_5, H)-free and $\lim_{\omega \to +\infty}(f^\star_{\{P_5,H\}}(\omega) - \omega) = +\infty$. In view of the desired result it remains to assume that $\bar{H}$ is $(P_5, K_{1,3}, K_1 \cup P_4, 2K_2)$-free forest. In other words, $\bar{H}$ is isomorphic to P_4 or an induced subgraph of $(pK_1) \cup P_3$ for some $p \in \mathbb{N}_0$. Thus, H is either isomorphic to P_4 or an induced subgraph of F_p for some $p \in \mathbb{N}_0$, which completes this proof. □

Note that one direction of Theorem 8 follows from Lemma 70, Lemma 71, Lemma 72 and the fact that $f^\star_{\{P_5,P_4\}} \equiv \text{id}_{\mathbb{N}_0}$ by the Strong Perfect Graph Theorem. The reverse direction follows from Lemma 73. Therefore, it remains to show Lemma 71.

For (P_5, F_p)-free graphs we show quite a small χ-binding function. For that reason there is quite a bit of work to do.

10.1 Proof of Lemma 71

At the beginning of this section, let us introduce additional notation and terminology we specifically use in this section. A *hole* in a graph is an induced cycle of length at least four, and an *antihole* is an induced subgraph whose complementary graph is a hole in the complementary graph. Let for the following definitions G be connected graph that contains an induced odd antihole C. We let $A(C)$ be the set of vertices of $V(G) \setminus V(C)$ that have a neighbour and a non-neighbour in C, $B(C)$ be the vertices of $V(G) \setminus N_G[V(C)]$ that have a neighbour in $A(C)$, and $M(C)$ be the set of vertices which are adjacent to all vertices of C. Furthermore, let $X(C) := V(G) \setminus [A(C) \cup B(C) \cup M(C) \cup V(C)]$, and $Y(C)$ be the set of vertices of $X(C)$ such that for each $y \in Y(C)$ there exist two vertices $m_y \in M(C)$ and $x_y \in X(C)$ such that $m_y y \notin E(G)$ but $m_y x_y, x_y y \in E(G)$. In what follows, we may assume that C is in $\mathcal{C}_5(G)$. For the definition of the notation c^- and c^+ for a vertex c of C recheck Section 1.2. We further say that C *extends to a* $O[F]$ *in* G for some graph F if there is a vertex set $U \subseteq V(G)$ and a vertex $c \in V(C)$ such that $G[U]$ is isomorphic to F, $U \cap V(C) = \{c\}$, $E_G[U, \{c^-, c^+\}]$ is complete, and $E_G[U, \{c^{-2}, c^{+2}\}]$ is anticomplete. Moreover, U is the *extender* of C and $G[U \cup V(C)]$ is isomorphic to $O[F]$.

For each imperfect graph G, let

$$\varphi(G) := \min\{\chi(G[N_G[V(C)]]) : C \text{ is an odd antihole}\}$$

and, for $p \in \mathbb{N}_{\geq 2}$,

$$\vartheta(p) := \sup\{\varphi(G) : G \text{ is } (P_5, F_p, O[K_p])\text{-free and imperfect}\}.$$

Before we prove Lemma 71, we show some preliminary results. We note that F_p-free graphs have been studied in [14] as well, using these results we f.e. show in the upcoming Chapter 11 that $f^\star_{\{2K_2,F_p\}}$ is not non-decreasing, for some large $p \in \mathbb{N}_{>0}$. We show firstly that $f^\star_{\{P_5,F_p\}}$ is non-decreasing, for each $p \in \mathbb{N}_0$. Note that each F_0-free graph is perfect, and so each complete graph G is (P_5, F_0)-free and satisfies $\chi(G) = f^\star_{\{P_5,F_0\}}(\omega(G))$.

Lemma 74. *If* $p \geq 1$ *and* $r \geq 0$ *are integers, then*

$$x \leq f^\star_{\{P_5,F_p\}}(x) \leq f^\star_{\{P_5,F_p\}}(x+1) \quad \textit{and} \quad f^\star_{\{P_5,F_p\}}(x) + 2r \leq f^\star_{\{P_5,F_{p+r}\}}(x+r)$$

for each $x \geq 1$.

Proof. Since $f^\star_{P_5}$ exists, we find that $f^\star_{\{P_5,F_p\}}$ exists. We first show that $x \leq f^\star_{\{P_5,F_p\}}(x)$. Note that K_x is a (P_5, F_p)-free graph of clique number x and therefore $x \leq f^\star_{\{P_5,F_p\}}(x)$. This shows that for every $p \geq 1$ and every $x \geq 1$ there is always a graph $G' \in \text{For}(P_5, F_p)$ with $\omega(G') = x$ and $\chi(G') = f^\star_{\{P_5,F_p\}}(x)$.

The claim that $f^\star_{\{P_5,F_p\}}(x) \leq f^\star_{\{P_5,F_p\}}(x+1)$ follows directly from Lemma 45.

We prove the last inequality by induction on r. Trivially, we can assume $r \geq 1$. Let G_{r-1} be a $(P_5, F_{p+(r-1)})$-free graph of clique number $x + (r-1)$ such that $\chi(G_{r-1}) = f^\star_{\{P_5,F_{p+(r-1)}\}}(x + (r-1))$. Let $G_r := C_5[K_1, G_{r-1}, K_1, G_{r-1}, K_1]$. We see that G_r is (P_5, F_{p+r})-free and of clique number $x+r$. To figure out $\chi(G_r)$, we let C be a C_5 with vertex-weight function q fulfilling $\omega_q(C) = \chi(G_{r-1})+1$ and $q(C) = 2\cdot\chi(G_{r-1})+3$. Note that the chromatic number of a weighted C_5 only depends on the size of the largest clique and the sum of the weights, thus, by Corollary 46,

$$\chi(G_r) = \chi_q(C) = \max\left\{\omega_q(C), \left\lceil\frac{q(C)}{2}\right\rceil\right\} = \chi(G_{r-1}) + 2.$$

Thus, we obtain

$$f^\star_{\{P_5,F_p\}}(x)+2r \leq f^\star_{\{P_5,F_{p+(r-1)}\}}(x+(r-1))+2 = \chi(G_{r-1})+2 = \chi(G_r) \leq f^\star_{\{P_5,F_{p+r}\}}(x+r)$$

by induction hypothesis. □

In what follows is a series of lemmas culminating in the fact that $f^\star_{\{P_5,F_p\}}(x) \leq x + c(p)$ for some constant $c(p)$, and for each $p \geq 0$ and each $x \geq 1$.

Lemma 75. *Let $p \geq 1$ and G be a connected (P_5, F_p)-free graph.*

(i) *If C is an odd antihole in G, then $E_G[X(C), A(C) \cup B(C)]$ is anticomplete.*

(ii) *If C is an odd antihole in G, then $E_G[B, A(C) \cap N_G(B)]$ is complete for each set B of vertices that induces a component of $G[B(C)]$.*

(iii) *If C is an odd antihole in G with $Y(C) = \emptyset$, then $E_G[X, M(C) \cap N_G(X)]$ is complete for each set X of vertices that induces a component of $G[X(C)]$.*

(iv) *If C is an odd antihole in G, then $V(C) \setminus N_G(a)$ is an independent set for every vertex $a \in A(C)$ which has a neighbour in $B(C)$.*

(v) *If $S \subseteq V(G)$ is a clique of size at most p, then*

$$\chi(G[\bigcap_{s\in S} N_G(s)] \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) - 2(|S| - 1).$$

(vi) *If $S_1 \subseteq V(G)$ and $S_2 \subseteq \bigcap_{s\in S_1} N_G(s)$ are two sets of vertices, then $G[S_1]$ has clique number at most $p - 1$ or $G[S_2]$ is $(K_1 \cup K_2)$-free.*

Proof. We prove (i)-(iii) first. Let C be an odd antihole in G. By definition, no vertex of $A(C)$ has a neighbour in $X(C)$. We now suppose, for the sake of a contradiction, that a vertex $a \in A(C)$ has a neighbour in $b \in B(C)$ which is adjacent to a vertex x of $(B(C) \setminus N_G(a)) \cup X(C)$. As a has a neighbour and a non-neighbour on C and as C is connected, we find two adjacent vertices $c_1, c_2 \in V(C)$ such that $ac_1 \in E(G)$ but $ac_2 \notin E(G)$. It follows that $[x, b, a, c_1, c_2]$ induces a P_5. By this contradiction to our assumption on G, we find that our supposition is false. By the fact that $E_G[X(C), A(C)]$ is anticomplete and by the connectivity of $G[B]$, (i) and (ii) follow, respectively. It remains to assume that $Y(C) = \emptyset$ for (iii). We now find that by the connectivity of $G[X]$, each vertex of X is adjacent to each vertex of $M(C) \cap N_G(X)$ as otherwise $Y(C) \neq \emptyset$. Thus, (iii) follows.

We proceed with our proof for (iv). Let $a \in A(C)$ be a vertex with a neighbour $b \in B(C)$. As $\bar{C}$ is connected, we find two non-adjacent vertices $c_1, c_2 \in V(C)$ such that $ac_1 \in V(C)$ and $ac_2 \notin V(C)$. As $[b, a, c_1, c, c_2]$ does not induce a P_5, we find $ac \in E(G)$ for each $c \in V(C) \cap N_G(c_1) \cap N_G(c_2)$. Let $c_3 \in V(C) \setminus \{c_1, c_2\}$ such that $c_1c_3 \notin E(G)$ and $c_4 \in V(C) \setminus \{c_1, c_2, c_3\}$ be such that $c_3c_4 \notin E(G)$. We find that a is adjacent to c_4 as $c_4 \in V(C) \cap N_G(c_1) \cap N_G(c_2)$. As $[b, a, c_4, c_2, c_3]$ does not induce a P_5, it follows $ac_3 \in E(G)$. Thus, $V(C) \setminus N_G(a)$ consists of at most two vertices which, in particular, are c_2 and possibly a vertex that is distinct from c_1 but non-adjacent to c_2. Thus, (iv) follows.

We continue and prove (v). As $G[S \cup (\bigcap_{s \in S} N_G(s))]$ is an induced subgraph of G, we have that $G[S \cup (\bigcap_{s \in S} N_G(s))]$ is a (P_5, F_p)-free graph of clique number at most $\omega(G)$. Furthermore, $E_G[S, \bigcap_{s \in S} N_G(s)]$ is complete, and so

$$\omega(G[\bigcap_{s \in S} N_G(s)]) + |S| = \omega(G[\bigcap_{s \in S} N_G(s)]) + \omega(G[S]) \leq \omega(G).$$

Now, let us suppose for the sake of contradiction, that $G[\bigcap_{s \in S} N_G(s)]$ contains a vertex set U that induces a $F_{p-|S|}$. We find that $U \cup S$ induces a F_p in G as S is a clique S and $E_G[S, U]$ is complete. From this contradiction on our assumption on G, we find that $G[\bigcap_{s \in S} N_G(s)]$ is $F_{p-|S|}$-free, and so

$$\chi(G[\bigcap_{s \in S} N_G(s)]) \leq f^{\star}_{\{P_5, F_{p-|S|}\}}(\omega(G) - |S|).$$

By Lemma 74, we have

$$f^{\star}_{\{P_5, F_{p-|S|}\}}(\omega(G) - |S|) + 2(|S| - 1) \leq f^{\star}_{\{P_5, F_{p-1}\}}(\omega(G) - 1),$$

which completes our proof for (v).

Finally, we prove (vi). For the sake of a contradiction, let us suppose that $G[S]$ contains a clique W of size p and $\{u_1, u_2, u_3\} \subseteq S_2$ induces a $K_1 \cup K_2$. As $E_G[\{u_1, u_2, u_3\}, W]$ is complete, it follows that $\{u_1, u_2, u_3\} \cup W$ induces a F_p in G. By this contradiction to the fact that G is F_p-free, (vi) follows. □

Lemma 76. *Let $p \geq 2$ and G be a connected (P_5, F_p)-free graph. If C is an odd antihole in G, then*

(i) $\chi(G[X(C)]) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + f^\star_{P_5}(p-1)$ *or*

(ii) *there is a* $C' \in \mathcal{C}_5(G)$ *that extends to a* $O[K_p]$ *in* G *with* $|Y(C')| < |Y(C)|$.

Proof. By definition and by Lemma 75 (i), we find that no vertex of $A(C) \cup B(C) \cup V(C)$ is adjacent to a vertex of $X(C)$, that is, $M(C)$ is a cut-set which disconnects $X(C)$ from $V(G) \setminus (M(C) \cup X(C))$. Let $G^\star$ be a minimal induced subgraph of $G[X(C)]$ such that $\chi(G[X(C)]) = \chi(G^\star)$ and such that there is a vertex $m \in M(C)$ for which $G[V(G^\star) \cup \{m\}]$ is connected. Clearly, $G^\star$ is connected. We now partition $V(G^\star)$. Let S be the set of neighbours of m in $G^\star$, and let T_1 and T_2 be the sets of vertices of $G^\star - S$ which are in components of $G^\star - S$ with clique number at most $p-1$ and clique number at least p, respectively.

We first claim that any set, say, T of vertices that induces a component of $G^\star[T_1 \cup T_2]$ is a homogeneous set in $G^\star$ or consists of one vertex only. As $G^\star[T]$ is a connected graph, it suffices to prove that two arbitrarily chosen adjacent vertices of $G^\star[T]$ have the same neighbours in S. Let t_1 and t_2 be two such vertices and $c \in V(C)$. As neither $[t_1, t_2, s_2, m, c]$ nor $[t_2, t_1, s_1, m, c]$ induces a P_5 in $G^\star$ for each $s_1 \in S \cap N_{G^\star}(t_1)$ and each $s_2 \in S \cap N_{G^\star}(t_2)$, we find that t_1 and t_2 have the same neighbours in S, which shows our claim.

We next claim that (ii) follows or the neighbours in S of the vertices of any component of $G^\star[T_2]$ form a clique. Let G' be an arbitrary component of $G^\star[T_2]$ and $t \in V(G')$ be a vertex that is in a clique W of size p in G'. First of all, let us assume that t has two non-adjacent neighbours, say, s_1 and s_2 in S. Let $i \in [2]$. If $N_{G^\star}(s_i) \subseteq N_{G^\star}(s_{3-i})$, then $\chi(G^\star) = \chi(G^\star - s_i)$, by Lemma 34. Furthermore, as $N_{G^\star}(s_i) \subseteq N_{G^\star}(s_{3-i})$ and as $s_{3-i} \in N_G(m)$, we find that $G[V(G^\star - s_i) \cup \{m\}]$ is connected, which contradicts the minimality of $G^\star$. Thus, for each $i \in [2]$, we find that s_i has a neighbour, say, s'_i in $G^\star$ that is non-adjacent to s_{3-i}. Let us suppose, for the sake of a contradiction, that t is adjacent to some s'_i. As s'_i and s_{3-i} are non-adjacent and as t and its neighbours in $G^\star[T_2]$ have the same neighbours in S, we find $s'_i \in S$. By Lemma 75 (vi), we find that the component of $G^\star[T_1 \cup T_2]$ which contains t is of clique number at most $p-1$. From this contradiction to the fact $t \in T_2$, we find that t is adjacent to neither s'_1 nor s'_2 in $G^\star$, and thus in G. As $[s'_1, s_1, t, s_2, s'_2]$ does not induce a P_5, we find that the same vertex set induces a C_5 called C' in G. As the component of $G^\star[T_2]$ that contains t is a homogeneous set, it follows that $E_G[W, \{s_1, s_2\}]$ is complete and $E_G[W, \{s'_1, s'_2\}]$ is anticomplete, and so C' extends to a $O[K_p]$ in G. Thus, in order to prove (ii), it remains to show that $|Y(C')| < |Y(C)|$. We find $m \in A(C')$ as m is adjacent to s_1 and s_2 but non-adjacent to t. Furthermore, as neither $[t, s_1, m, c, a]$ for each $a \in A(C)$ and each $c \in N_G(a) \cap V(C)$, nor $[t, s_1, m, a, b]$ for each $b \in B(C)$ and each $a \in A(C) \cap N_G(b)$ induces a P_5, it follows that all vertices of $A(C)$ and $B(C)$ are adjacent to m, respectively. Thus, $A(C) \cup B(C) \cup \{m\} \subseteq A(C') \cup B(C')$. Furthermore, we find that any $c \in V(C)$ is a vertex of $A(C') \cup B(C')$ as c is adjacent to m, and so $V(C) \subseteq A(C') \cup B(C')$. By Lemma 75 (i), it follows that all vertices of $M(C)$ are not in $X(C')$ as each of them is adjacent to a vertex $c \in V(C) \subseteq A(C') \cup B(C')$. Moreover, we find $X(C') \subseteq X(C)$. We note that $t \in Y(C) \setminus Y(C')$. For the sake of a contradiction, let us suppose that there is a vertex $y' \in Y(C') \setminus Y(C)$. In particular, we find two vertices $m' \in M(C')$ and $x' \in X(C')$ such that $m'y' \notin E(G)$ but $m'x', x'y' \in E(G)$. Note that $x', y' \in X(C')$, and so $x', y' \in X(C)$. Furthermore, note that $m' \notin M(C)$ as otherwise $y' \in Y(C)$. As $m \in A(C')$, it follows by Lemma 75 (i) that x' and y' are non-neighbours of m, and so $[y', x', m', m, c]$ for some $c \in V(C)$ if $mm' \in E(G)$ and $[y', x', m', s_1, m]$ if $mm' \notin E(G)$ induces a P_5. By this contradiction to our assumption on G, we finally obtain that our supposition is false, and so $|Y(C')| < |Y(C)|$. We find that (ii) follows. In order to prove our claim, it remains to assume that $S \cap N_{G^\star}(t)$ is a clique. As the component G' of $G^\star[T_2]$ that contains t is a homogeneous set, it follows that $S \cap N_{G^\star}(V(G'))$ is a clique. By the arbitrariness of G', the neighbours in S of the vertices of any component of $G^\star[T_2]$ form a clique.

We proceed by assuming that (ii) does not hold and we colour the vertices of $G^\star$. As $G^\star[T_1]$ is of clique number at most $p-1$, we find $\chi(G^\star[T_1]) \leq f^\star_{P_5}(p-1)$ and, as $f^\star_{P_5}(p-1) \geq 1$, it suffices to show that there is a colouring $c\colon V(G^\star - T_1) \to [f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1]$ of $G^\star - T_1$ that uses colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1$ on vertices of T_2 only. Let $k = f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1$. As all vertices of S are adjacent to m, by Lemma 75 (v) there is a colouring $c\colon S \to [k-1]$ of $G^\star[S]$. Let G' be an arbitrary component of $G^\star[T_2]$ and $S' := N_{G^\star}(V(G')) \cap S$. Note that $|S'| \geq 1$ as $G[V(G^\star) \cup \{m\}]$ is connected. As S' is a clique and as $N_{G^\star}(t) \cap S = S'$ for each $t \in V(G')$, we find that there is a vertex $s' \in S$ that is adjacent to all vertices of $(S' \setminus \{s'\}) \cup V(G')$. Thus, by Lemma 75 (v) there is a colouring $c'\colon V((S' \setminus \{s'\}) \cup V(G')) \to [k] \setminus \{c(s')\}$ of $G^\star[(S' \setminus \{s'\}) \cup V(G')]$ such that $c(s) = c'(s)$ for each $s \in S' \setminus \{s'\}$. In particular, $c'(v) = k$ implies $v \in T_2$. The arbitrariness of G' completes our proof. □

We proceed by considering $\vartheta(p)$ for $p \geq 3$. As C_5 is $(P_5, F_p, O[K_p])$-free and imperfect, we find $\vartheta(p) \geq 3$ but possibly $\vartheta(p) = +\infty$. We next show that the latter fact cannot occur.

Lemma 77. *If $p \geq 3$, then $\vartheta(p) \leq \max\{10, 2p+3\} \cdot f^\star_{P_5}(p-1)$.*

Proof. Let G be an arbitrary imperfect $(P_5, F_p, O[K_p])$-free graph.

If G contains an induced C_5 C, then we partition $N_G[V(C)]$ into 10 sets of vertices, each of which induces a graph of clique number at most $p-1$. Let N be the vertices of $N_G(V(C))$ each of which has an independent non-neighbourhood in C. As there are at most 5 independent sets $I_1, I_2, \ldots, I_5$ of size 2 in C, we can partition the vertices of N into at most 5 sets $S_1, S_2, \ldots, S_5$ of vertices such that $E_G[S_i, V(C) \setminus I_i]$ is complete for each $i \in [5]$. As $V(C) \setminus I_i$ induces a $K_1 \cup K_2$, we find that $G[S_i]$ is K_p-free by Lemma 75 (vi). We now partition $N_G[V(C)] \setminus N$. By definition, each vertex of $N_G[V(C)] \setminus N$ has at most 3 neighbours on C. As G is P_5-free, we further find that each such vertex, say, u has at least 2 neighbours on C and there is a vertex $c \in V(C)$ such that u is adjacent to c and c^{+2}. As $u \notin N$, we find

$$\{c, c^{+2}\} \subseteq N_G(u) \subseteq \{c, c^+, c^{+2}\}.$$

We define for $c \in V(C)$ the set S'_c as all vertices $v \in N_G[V(C)] \setminus N$ with $E_G[v, \{c, c^{+2}\}]$ is complete. Thus, $\bigcup_{c \in V(C)} S'_c$ is a partition of $N_G[V(C)] \setminus N$. As G is $O[K_p]$-free, we find that $G[S'_c]$ is K_p-free for each $c \in V(C)$. Consequently, we partition $N_G[V(C)]$ into 10 sets of vertices, each of which induces a graph of clique number at most $p-1$, and so

$$\chi(G[N_G[V(C)]]) \leq 10 \cdot f^\star_{P_5}(p-1).$$

If G is C_5-free, then let C be an odd antihole of order at least 7 in G. As G is F_p-free, C contains at most $2p+3$ vertices. We first show that every vertex of $N_G[V(C)]$ is adjacent to two non-adjacent vertices of $V(C)$. For the sake of a contradiction, let us suppose that u is a counterexample to this claim. Clearly, $u \notin V(C)$. By the supposition on u and as C is of odd order, u has at most $(|V(C)|-1)/2$ neighbours on C. This fact particularly implies that u is non-adjacent to an independent set $\{c_1, c_2\}$ of C. As $u \in N_G(V(C))$, we may assume that a neighbour c_3 of u on C is non-adjacent to c_2. By definition, c_3 is adjacent to c_1. Let c_4 be the second non-neighbour of c_3. As u is adjacent to c_3, we find that u is non-adjacent to c_4 by our supposition on u, and so $[u, c_3, c_1, c_4, c_2]$ induces a P_5, a contradiction. Thus, we find that each vertex of $N_G[V(C)]$ is adjacent to two non-adjacent vertices of $V(C)$. Let u be such a vertex and c'_1, c'_2 be the two non-adjacent neighbours. We let c'_3 and c'_4 be the second non-neighbour of c'_1 and c'_2, respectively. As $[u, c'_1, c'_4, c'_3, c'_2]$ does not induce a C_5, we find that u is adjacent to three vertices of C which induce a $K_1 \cup K_2$. As C is an odd antihole on at most $2p+3$ vertices, there are at most $2p+3$ sets $I_1, I_2, \ldots, I_{2p+3}$ of vertices in C that induce copies of $K_1 \cup K_2$. We can partition $N_G[V(C)]$ into $2p+3$ sets $S_1, S_2, \ldots, S_{2p+3}$ such that $E_G[S_i, I_i]$ is complete for each $i \in [2p+3]$. As G is F_p-free, we find that $G[S_i]$ is K_p-free for each $i \in [2p+3]$ by Lemma 75 (vi). Consequently, we partition $N_G[V(C)]$ into $2p+3$ sets of vertices, each of which induces a graph of clique number at most $p-1$, and so

$$\chi(G[N_G[V(C)]]) \leq (2p+3) \cdot f^\star_{P_5}(p-1). \qquad \square$$

We are now in a position to prove our main preliminary result.

Lemma 78. *Let $p \geq 3$. If G is a connected (P_5, F_p)-free graph with $\omega(G) \geq p+2$, then*

$$\chi(G) \leq \max\{\vartheta(p), f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + f^\star_{P_5}(p-1)\} + f^\star_{P_5}(p-1).$$

Proof. Let $G^\star$ be the smallest induced connected subgraph of G such that $\chi(G^\star) = \chi(G)$ and $\omega(G^\star) \geq p+2$. We note that $G^\star$ is (P_5, F_p)-free, and the desired result follows if

$$\chi(G^\star) \leq \max\{\vartheta(p), f^\star_{\{P_5,F_{p-1}\}}(\omega(G^\star) - 1) + f^\star_{P_5}(p-1)\} + f^\star_{P_5}(p-1)$$

as $f^\star_{\{P_5,F_{p-1}\}}$ is non-decreasing by Lemma 74. We may assume, without loss of generality, that $G = G^\star$.

We begin by showing that $N_G(u) \not\subseteq N_G(v)$ and $N_G(v) \not\subseteq N_G(u)$ for each two non-adjacent vertices $u, v \in V(G)$. For the sake of a contradiction, let us suppose that u, v is a pair with $N_G(u) \subseteq N_G(v)$. We note that $\chi(G) = \chi(G-u)$ as we can safely assign the colour of v in a $\chi(G-u)$-colouring to u. As $G-u$ is connected and $\omega(G) = \omega(G-u)$, we find $G \neq G^\star$, a contradiction. Thus, $N_G(u) \not\subseteq N_G(v)$ and $N_G(v) \not\subseteq N_G(u)$ for each two non-adjacent vertices $u, v \in V(G)$.

If G is a perfect graph, then

$$\chi(G) = \omega(G) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) + f^\star_{P_5}(p-1)$$

as $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) \geq \omega(G)-1$ by Lemma 74 and $f^\star_{P_5}(p-1) \geq 1$ by definition. Thus, we assume that G is imperfect. By the Strong Perfect Graph Theorem, G contains an induced odd hole or induced odd antihole. As G is P_5-free, each odd hole is a C_5, and so an odd antihole as well. We continue with four cases arguably covering all possible situations.

Case 1: There is some odd antihole C in G such that $Y(C) \neq \emptyset$ but there is no C_5 C' in G that extends to a $O[K_p]$ in G with $|Y(C')| < |Y(C)|$.

We note that Lemma 76 immediately implies $\chi(G[X(C)]) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) + f^\star_{P_5}(p-1)$. Let $y \in Y(C)$ and $m \in M(C)$, $x \in X(C)$ be such that $my \notin E(G)$ but $mx, xy \in E(G)$. By Lemma 75 (i), $G - M(C)$ is disconnected and contains all components of $G[X(C)]$. As $[y, x, m, c, a]$ does not induce a P_5 for each $c \in V(C)$ and $a \in A(C)$, it follows that m is adjacent to all vertices of $A(C)$. Similarly, as $[y, x, m, a, b]$ does not induce a P_5 for each $a \in A(C)$ and $b \in B(C)$, it follows that m is adjacent to all vertices of $B(C)$. By Lemma 75 (v), we find that $\chi(G[A(C) \cup B(C) \cup V(C)]) \leq \chi(G[N_G(m)]) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$. As $M(C)$ is a cutset and, by Lemma 75 (vi), we have $\chi(M(C)) \leq f^\star_{P_5}(p-1)$, it follows

$$\begin{aligned}\chi(G) &\leq \chi(G[M(C)]) + \max\{\chi(G[A(C) \cup B(C) \cup V(C)]), \chi(G[X(C)])\} \\ &\leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) + 2f^\star_{P_5}(p-1),\end{aligned}$$

which completes our proof of Case 1. △

We may assume for the remaining cases that there is a $C' \in \mathcal{C}_5(G)$ that extends to a $O[K_p]$ in G with $|Y(C')| < |Y(C)|$ for each odd antihole C in G with $Y(C) \neq \emptyset$. In other words,

(a) there is a $C' \in \mathcal{C}_5(G)$ that extends to a $O[K_p]$ in G with $Y(C') = \emptyset$ or

(b) $Y(C) = \emptyset$ for each odd antihole C in G.

We consider situation (a) next. Let $\mathcal{C}(G)$ be the subset of $\mathcal{C}_5(G)$, such that each cycle, say C, of $\mathcal{C}(G)$ can be extended to a $O[K_p]$ in G and satisfies $Y(C) = \emptyset$. We now distinguish two cases – Case 2 and Case 3 of this proof.

Case 2: $\mathcal{C}(G) \neq \emptyset$ and, for some $C \in \mathcal{C}(G)$, there is a connected graph F with $\chi(F) \geq 2f^\star_{P_5}(p-1)$ such that C extends to a $O[F]$ in G called H whose extender is a homogeneous set in G.

Let us partition the vertices of $A(C) \cup M(C) \cup V(C)$. Let U be the extender of C to H. Recall that by definition, we have exactly two vertices, say, c_1 and c_2 in H with $E_G[\{c_1, c_2\}, U]$ is complete. Furthermore, we may assume $c_2 = c_1^{+2}$.

For each $i \in [2]$, we define two sets $A_{i,-}$ and $A_{i,+}$ such that

- $A_{i,-}$ contains all vertices of G that are adjacent to c_i and c_i^{-2} but non-adjacent to c_i^+ and c_i^{+2},
- $A_{i,+}$ contains all vertices of G that are adjacent to c_i and c_i^{+2} but non-adjacent to c_i^- and c_i^{-2}.

By definition, $U \subseteq A_{1,+}$ and $A_{1,+} = A_{2,-}$. We now let

$$o(C) := \max\{\omega(G[A_{1,-}]), \omega(G[A_{2,+}])\}.$$

We may assume, without loss of generality, that C maximizes $o(\cdot)$ among all cycles $C' \in \mathcal{C}(G)$ for which there is a connected graph F' with $\chi(F') \geq 2f^\star_{P_5}(p-1)$ such that C' extends to a $O[F']$ in G called H' whose extender is a homogeneous set in G.

We can now compare $\chi(G[A_{1,-}])$ and $\chi(G[A_{2,+}])$. Again without loss generality, let us assume $\chi(G[A_{1,-}]) \leq \chi(G[A_{2,+}])$. As $[a_{1,-}, c_1, c_1^+, c_2, a_{2,+}]$ does not induce a P_5 for each $a_{1,-} \in A_{1,-}$ and each $a_{2,+} \in A_{2,+}$, it follows that $E_G[A_{1,-} \cup \{c_2\}, A_{2,+}]$ is complete. By Lemma 75 (vi), $G[A_{1,-} \cup \{c_2\}]$ is $(K_1 \cup K_2)$-free or $\omega(G[A_{2,+}]) \leq p+1$. We conclude $\chi(G[A_{1,-}]) \leq f^\star_{P_5}(p-1)$ in both cases.

Let A_3 be the set of vertices which have a neighbour on C but which are non-adjacent to c_1 and c_2. As G is P_5-free, it follows that $E_G[A_3 \cup \{c_1, c_2\}, U]$ is complete. As $\chi(G[U]) > f^\star_{P_5}(p-1)$, there is a clique of size p in U. By Lemma 75 (vi), $G[A_3 \cup \{c_1, c_2\}]$ is $(K_1 \cup K_2)$-free. As $E_G[\{c_1, c_2\}, A_3]$ is anticomplete, we find that A_3, and so $A_3 \cup \{c_1, c_2\}$, is an independent set in G.

We next show that all vertices of $A(C) \cup M(C) \cup V(C)$ which are in none of the sets $A_{1,-}, A_{2,+}, A_3 \cup \{c_1, c_2\}$ are adjacent to c_1 and c_2. For the sake of a contradiction, let us assume that a with $a \in A(C) \cup M(C) \cup V(C)$ and $a \notin A_{1,-} \cup A_{2,+} \cup A_3 \cup \{c_1, c_2\}$

is a vertex which is not adjacent to c_i for some $i \in [2]$. We note that $a \notin U$ by definition, and $E_G[\{a\}, U]$ is complete or anticomplete as U is a homogeneous set. As $a \notin A_3 \cup \{c_1, c_2\}$ but $a \in A(C) \cup M(C) \cup V(C)$, we find that a and c_{3-i} are adjacent. As $\{a, c_1, c_2\} \cup W$ does not induce a F_p for some clique $W \subseteq U$ of size p, it follows that $E_G[\{a\}, U]$ is anticomplete. As $[a, c_2, c_1^+, c_1, c_1^-]$ if $i = 1$ or $[a, c_1, c_1^+, c_2, c_2^+]$ if $i = 2$ does not induce a P_5, it follows that a is adjacent to c_1^- or c_2^+, and so $a \in A_{2,+}$ or $a \in A_{1,-}$. From this contradiction to our assumption on a, we obtain the desired fact.

Let $A' := [A(C) \cup M(C) \cup \{c_1^+\}] \setminus [A_{1,-} \cup A_{2,+} \cup A_3]$. Note that

$$A' = [A(C) \cup M(C) \cup V(C)] \setminus [A_{1,-} \cup A_{2,+} \cup A_3 \cup \{c_1, c_2\}].$$

We partition A' into sets $A'_{1,+}$, A'_4, and A'_5. Let $a_1 \in A_{1,-}$ and $a_2 \in A_{2,+}$ be two vertices which are in maximum cliques of $G[A_{1,-}]$ and $G[A_{2,+}]$, respectively. Furthermore, let $A'_{1,+}$ be the set of those vertices of A' which are non-adjacent to a_1 and a_2, A'_4 be the set of those vertices of A' which are adjacent to a_2 but non-adjacent to a_1, and A'_5 be the set of those vertices of A' which are adjacent to a_1.

As a_1 and a_2 are adjacent, we find that $[c_1, c_1^+, c_2, a_2, a_1]$ induces a C_5, say, C'. Let $u \in N_G(V(C)) \setminus V(C')$ and $c \in V(C)$ be a neighbour of u. If $c \notin V(C')$, then $c \in A(C')$ and, as $V(C') \setminus N_G(c)$ is not independent, we have $u \notin B(C')$ by Lemma 75 (iv), and so $u \in N_G[V(C')]$ by definition. In other words, we conclude $N_G[V(C)] \subseteq N_G[V(C')]$ no matter whether or not $c \in V(C')$. Similarly, we find $N_G[V(C')] \subseteq N_G[V(C)]$, and so $N_G[V(C)] = N_G[V(C')]$ and $B(C) \cup X(C) = B(C') \cup X(C')$.

We now have to distinguish four subcases.

Case 2.1: $o(C) \geq p$

Let W_2 be a clique of size p in $G[A_{2,+}]$ that contains a_2. As $\omega(G[A_{2,+}]) \geq p$, as $E_G[\{c_2\}, A_{1,-}]$ is anticomplete, and as $E_G[A_{2,+}, A_{1,-} \cup \{c_2\}]$ is complete, we obtain that $A_{1,-}$ is independent by Lemma 75 (vi).

We first prove that $E_G[A_3, B(C) \cup X(C)]$ is anticomplete. Recall that $B(C) \cup X(C) = B(C') \cup X(C')$. For the sake of a contradiction, we suppose that $a_3 \in A_3$ is adjacent to a vertex $u \in B(C') \cup X(C')$. By Lemma 75 (i) and (iv), we find $u \in B(C')$ and $E_G[\{a_3\}, \{a_1, a_2\} \cup U]$ is complete, respectively. As $[w_2, a_2, a_3, c_1^+, c_1]$ does not induce a P_5 for each $w_2 \in W_2$, we find that $E_G[\{a_3\}, W_2]$ is complete. Thus, $\{a_1, a_3, c_2\} \cup W_2$ induces a F_p. From this contradiction to our assumption on G, we find that $E_G[A_3, B(C) \cup X(C)]$ is anticomplete.

We next prove that $\omega(G[A'_4]) \leq p - 2$. Suppose, for the sake of a contradiction, that $\omega(G[A'_4]) \geq p - 1$. Let W_4 be a clique of size $p - 1$ in $G[A'_4]$. As $\{a_1, c_2, w_4\} \cup W_2$ does not induce a F_p and as w_4 is adjacent to a_2, we find that $E_G[\{w_4\}, W_2]$ is mixed for each $w_4 \in W_4$. Recall that the extender U is a homogeneous set in G and $\chi(G[U]) \geq 2$.

Let $u_1, u_2 \in U$ be two adjacent vertices. We note that each $w_4 \in W_4$ is adjacent to u_1 and u_2 as otherwise $[u_i, c_1, w_4, a_2, w_2]$ induces a P_5 for some $w_2 \in W_2 \setminus N_G(w_4)$ and some $i \in [2]$. Thus, $\{c_2, a_2, u_1, u_2\} \cup W_4$ induces a F_p, a contradiction. We conclude that $\omega(G[A_4']) \le p-2$ and

$$\chi(G[A_4']) \le f^\star_{P_5}(p-2) = f^\star_{\{P_5,F_{p-2}\}}(p-2) \le f^\star_{\{P_5,F_{p-1}\}}(p-1) - 2 = f^\star_{P_5}(p-1) - 2$$

by Lemma 74.

We further have $\omega(G[A_5']) \le p-1$ by Lemma 75 (vi), and so $\chi(G[A_5']) \le f^\star_{P_5}(p-1)$. By Lemma 75 (v), there is a colouring of $G[N_G(c_2)]$ with at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$ colours. We recall that $A_3 \cup \{c_1, c_2\}$ and $A_{1,-}$ are independent sets. Thus, we use one additional colour for the vertices of $A_3 \cup \{c_1, c_2\}$, one additional colour for the vertices of $A_{1,-}$, and $f^\star_{P_5}(p-1) - 2$ additional colours for the vertices of A_4'. We use $f^\star_{P_5}(p-1)$ additional colours for the vertices of A_5'. We obtain a colouring of $G[N_G[V(C)]]$ with at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + 2f^\star_{P_5}(p-1)$ colours such that the vertices of $A_4' \cup A_5'$ are coloured by at most $2f^\star_{P_5}(p-1) - 1$ colours. By Lemma 75 (iv), all vertices of $A(C) \cup M(C)$ which have a neighbour in $B(C) \cup X(C)$ are indeed vertices of $A_3 \cup A_4' \cup A_5'$. However, we recall that $E_G[A_3, B(C) \cup X(C)]$ is anticomplete. Furthermore, by Lemma 75 (i), (ii) and (iii), each set X of vertices that induces a component of $G[B(C) \cup X(C)]$ has a vertex $a \in A(C) \cup M(C)$ with $E_G[\{a\}, X]$ is complete. Thus, we can reuse $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$ colours from $N_G[V(C)] \setminus (A_4' \cup A_5')$ to colour $G - N_G[V(C)]$, which completes the proof of this subcase.

Case 2.2: $o(C) \le p-1$ and $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) = 2f^\star_{P_5}(p-1)$

Note that $N_G[V(C)] = N_G(c_2) \cup A_{1,-} \cup (A_3 \cup \{c_1, c_2\})$. Recall that $\chi(G[A_{1,-}]) \le f^\star_{P_5}(p-1)$ and $A_3 \cup \{c_1, c_2\}$ is an independent set. By Lemma 75 (v), we also have $\chi(G[N_G(c_2)]) \le f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$. Thus,

$$\begin{aligned}\chi(G[N_G[V(C)]]) &\le \chi(G[N_G(c_2)]) + \chi(G[A_{1,-}]) + \chi(G[A_3 \cup \{c_1, c_2\}]) \\ &\le f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + f^\star_{P_5}(p-1) + 1.\end{aligned}$$

Consequently, we colour the vertices of $N_G(c_2)$ by $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$ colours, the vertices of $A_{1,-}$ by $f^\star_{P_5}(p-1)$ additional colours, and the vertices of $A_3 \cup \{c_1, c_2\}$ by again an additional colour. Let G' be an arbitrary component of $G[B(C) \cup X(C)]$. By Lemma 75 (iv), $V(G')$ has its neighbours in $A_3 \cup (N_G(c_2) \setminus U)$. Hence, we can reuse the colours of $A_{1,-}$ and $f^\star_{P_5}(p-1) - 1$ additional ones if $\chi(G') \le 2f^\star_{P_5}(p-1) - 1$ to colour the vertices of $V(G')$. Thus, we may assume $\chi(G') \ge 2f^\star_{P_5}(p-1)$. As $Y(C) = \emptyset$, we obtain from Lemma 75 (i), (ii), (iii), and (v) that $\chi(G') \le f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$. As $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) = 2f^\star_{P_5}(p-1)$, we have $\chi(G') = 2f^\star_{P_5}(p-1)$.

We now claim that $N_G(V(G'))$ is coloured by at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$ colours. For the sake of a contradiction, let us suppose that $V(G')$ has neighbours in all colours we assign to the vertices of $N_G(c_2)$ and $A_3 \cup \{c_1, c_2\}$. In particular, there is a vertex $a_3 \in A_3$ which has a neighbour in $V(G')$. By Lemma 75 (ii), $E_G[\{a_3\}, V(G')]$ is complete. Furthermore, there is a vertex $a \in N_G(c_2) \setminus U$ which has a neighbour in $V(G')$. Suppose for the sake of contradiction that a vertex $a' \in N_G(c_2) \setminus U$ exists with $E_G[\{a'\}, U]$ is complete, then

$$\begin{aligned} f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) = 2f^\star_{P_5}(p-1) \leq \chi(G[U]) \\ \leq \chi(G[N_G(a') \cap N_G(c_2)]) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) - 2 \end{aligned}$$

by Lemma 75 (v), a contradiction. Thus, as U is homogeneous, we find that $E_G[N_G(c_2) \setminus U, U]$ is anticomplete. In particular, $E_G[\{a\}, U]$ is anticomplete, and so $a \in A(C)$. Moreover, $E_G[\{a\}, V(G')]$ is complete by Lemma 75 (ii). If $aa_3 \in E(G)$, then $\{a_3\} \cup V(G') \subseteq N_G(a)$. It follows

$$\begin{aligned} f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) = 2f^\star_{P_5}(p-1) = \chi(G') \\ \leq \chi(G[N_G(a) \cap N_G(a_3)]) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) - 2 \end{aligned}$$

by Lemma 75 (v), a contradiction. We conclude $aa_3 \notin E(G)$. Furthermore, $U \subseteq N_G(a_3) \setminus N_G(a)$ and $c_2 \in N_G(a) \setminus N_G(a_3)$. Now $[c_1^+, a_3, v, a, c_2]$ for some $v \in V(G')$ induces a C_5 C'. Furthermore, $E_G[U, \{a_3, c_2\}]$ is complete and $E_G[U, \{a, v\}]$ is anticomplete. In other words, U is an extender of C' to a $O[F]$. Recall that $E_G[N_G(c_2) \setminus U, U]$ is anticomplete, and therefore $M(C') = \emptyset$. Thus, $Y(C') = \emptyset$ and so $C' \in \mathcal{C}(G)$. As $E_G[V(G'), \{a, a_3\}]$ is complete and $E_G[V(G'), \{c_1^+, c_2\}]$ anticomplete, and G' has a clique of size p, it follows $o(C') \geq p$. As $o(C) \leq p-1$, we have a contradiction to our choice of C. We conclude that our supposition is false and $N_G(V(G'))$ is coloured by at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$ colours. Thus, we find that we can reuse $f^\star_{P_5}(p-1)+1$ colours from $N_G[V(C)]$ and add new $f^\star_{P_5}(p-1)-1$ colours to colour the vertices of $B(C) \cup X(C)$. We conclude that G is coloured by at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + 2f^\star_{P_5}(p-1)$ colours, which complete the proof in this subcase.

Case 2.3: $o(C) \leq p-1$ and $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) \neq 2f^\star_{P_5}(p-1)$

Note that $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) \neq 2f^\star_{P_5}(p-1)$ implies $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) > 2f^\star_{P_5}(p-1)$ as

$$f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) \geq \chi(G[N_G(c_1)]) \geq \chi(G[U]) \geq 2f^\star_{P_5}(p-1).$$

As U is a homogeneous set and $E_G[c_1^+, A_{1,-} \cup A_{2,+}]$ is anticomplete, we conclude that $E_G[U, A_{1,-} \cup A_{2,+}]$ is anticomplete. In particular, we have $a_1, a_2 \notin N_G(U)$, and so $U \subseteq A'_{1,+}$. Now let $U' \subseteq A'_{1,+} \setminus U$ be the maximal set with $E_G[U', U]$ is complete and G_U be the component of $G[A'_{1,+}]$ that contains the vertices of U.

We now claim that $E_G[V(G_1) \setminus U', V(G_2)]$ is either complete or anticomplete for each two components G_1 of $G[A'_{1,+}]$ and G_2 of $G[A_{1,-}]$ or $G[A_{2,+}]$. We prove this claim in two steps. For the sake of a contradiction, let us suppose that a vertex $a'_{1,+} \in A'_{1,+} \setminus U'$ and a component G' of $G[A_{1,-}]$ or $G[A_{2,+}]$ exist with $E_G[\{a'_{1,+}\}, V(G')]$ is mixed. We may assume that $V(G') \subseteq A_{2,+}$. Recall that $E_G[U, A_{2,+}]$ is anticomplete. Thus, $a'_{1,+} \notin U$. As $a'_{1,+} \notin U'$, there is a vertex $u \in U$ that is non-adjacent to $a'_{1,+}$. As G' is connected, we find two vertices $a_{2,+}, a'_{2,+} \in A_{2,+}$ such that $a'_{1,+}a_{2,+}, a_{2,+}a'_{2,+} \in E(G)$ but $a'_{1,+}a'_{2,+} \notin E(G)$. Recall that $a_{2,+}u, a'_{2,+}u \notin E(G)$. Thus, $[u, c_1, a'_{1,+}, a_{2,+}, a'_{2,+}]$ induces a P_5, a contradiction. Consequently, for every vertex $a'_{1,+} \in A'_{1,+} \setminus U'$ and every component G' of $G[A_{1,-}]$ or $G[A_{2,+}]$ we have $E_G[\{a'_{1,+}\}, V(G')]$ is either complete or anticomplete. For the sake of a contradiction, let us suppose that a vertex $a_{2,+} \in A_{1,-} \cup A_{2,+}$ – by symmetry we may assume $a_{2,+} \in A_{2,+}$ – and a component G' of $G[A'_{1,+}]$ with $E_G[\{a_{2,+}\}, V(G') \setminus U']$ is mixed. As $E_G[\{a_2\}, A'_{1,+}]$ is anticomplete, we have $a_{2,+} \neq a_2$. Let $a'_{1,+} \in V(G') \setminus U'$ be a neighbour of $a_{2,+}$. As $E_G[\{a'_{1,+}\}, V(G'')]$ is complete, where G'' is the component of $G[A_{2,+}]$ that contains $a_{2,+}$, it follows that $a_2 \notin V(G'')$. In particular, we find $a_2a_{2,+} \notin E(G)$. As $E_G[\{a_{2,+}\}, V(G') \setminus U']$ is mixed but G' is connected, there are two vertices $a''_{1,+}, a'''_{1,+} \in V(G')$ such that $a''_{1,+}a'''_{1,+}, a''_{1,+}a_{2,+} \in E(G)$ and $a'''_{1,+}a_{2,+} \notin E(G)$. Recall that $a''_{1,+}$ and $a'''_{1,+}$ as vertices of $A'_{1,+}$ are non-adjacent to a_1 and a_2. But now $[a_2, a_1, a_{2,+}, a''_{1,+}, a'''_{1,+}]$ induces a P_5, a contradiction. Consequently, for every vertex $a_{2,+} \in A_{1,-} \cup A_{2,+}$ and every component G' of $G[A'_{1,+}]$ we have $E_G[\{a_{2,+}\}, V(G') \setminus U']$ is either complete or anticomplete. Moreover, we conclude that $E_G[V(G_1) \setminus U', V(G_2)]$ is either complete or anticomplete for each two components G_1 of $G[A'_{1,+}]$ and G_2 of $G[A_{1,-}]$ or $G[A_{2,+}]$.

We now colour the vertices of $A_{1,-}, A'_{1,+}$, and $A_{2,+}$. In particular, we define a vertex colouring of $G[A'_{1,+}]$ such that, for each component G' of $G[A'_{1,+}]$ with $\chi(G') < f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$ no vertex of G' receives colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$. To achieve that let $c_U\colon V(G_U) \to [\chi(G_U)]$ be a colouring of G_U such that c_U uses all colours of $[\chi(G[U])]$ on U. As $\chi(G[U]) \ge 2f^\star_{P_5}(p-1)$, we find that c_U uses all colours of $[2f^\star_{P_5}(p-1)]$ on U. Recall that $E_G[U, A_{1,-} \cup A_{2,+}]$ is anticomplete, and so we find that $E_G[V(G_U) \setminus U', A_{1,-} \cup A_{2,+}]$ is anticomplete. As $E_G[U', U]$ are complete, we find that all colours which are used by c_U on the vertices of U' are not in $[2f^\star_{P_5}(p-1)]$. As $o(C) \le p-1$, we further find $\chi(G[A_{1,-}]), \chi(G[A_{2,+}]) \le f^\star_{P_5}(p-1)$. Thus, we can extend the colouring c_U to the vertices of $A_{1,-} \cup A_{2,+}$ with colours from $[\chi(G_1)]$ for each component G_1 of $G[A_{1,-}]$ and from $[f^\star_{P_5}(p-1) + \chi(G_2)] \setminus [f^\star_{P_5}(p-1)]$ for each component G_2 of $G[A_{2,+}]$.

It remains to colour the components of $G[A'_{1,+}]$ that are distinct from G_U. Recall that $E_G[V(G'), V(G_2)]$ is either complete or anticomplete for such a component G' of $G[A'_{1,+}]$ and each component G_2 of $G[A_{1,-}]$ or $G[A_{2,+}]$. We now distinguish some simple cases depending on the edges between $V(G')$ and $A_{1,-} \cup A_{2,+}$. If $E_G[V(G'), A_{1,-} \cup A_{2,+}]$ is anticomplete, then we colour G' with colours from $[\chi(G')]$ which is a subset of

$[f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)]$ as

$$\chi(G') \leq \chi(G[N_G(c_1)]) \leq f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$$

by Lemma 75 (v). In what follows, we may assume that $E_G[V(G'), V(G_2)]$ is complete to component G_2 of $G[A_{1,-}]$ or $G[A_{2,+}]$. Let $a \in A_{1,-} \cup A_{2,+}$ be an arbitrary vertex with $E_G[\{a\}, V(G')]$ is complete. As $E_G[\{a, c_1, c_2\}, V(G')]$ is complete and $\{a, c_1, c_2\}$ induces a $K_1 \cup K_2$, Lemma 75 (vi) implies that G' has clique number at most $p-1$. If $E_G[V(G'), A_{1,-}]$ is anticomplete, we can colour the vertices of G' with colours from $[f^\star_{P_5}(p-1)]$. If $E_G[V(G'), A_{2,+}]$ is anticomplete, we can colour the vertices of G' with colours from $[2f^\star_{P_5}(p-1)] \setminus [f^\star_{P_5}(p-1)]$. Thus, it remains to assume that $E_G[V(G'), V(G_1) \cup V(G_2)]$ is complete, where G_1 and G_2 are a component of $G[A_{1,-}]$ and of $G[A_{2,+}]$, respectively. We may assume that G_1 and G_2 are chosen such that their chromatic number is maximum subject to the completeness to $V(G')$. Recall that $E_G[V(G_1), V(G_2)]$ is complete. If $\chi(G[V(G') \cup V(G_1)]) \leq f^\star_{P_5}(p-1)$, then we can use the colours of $[f^\star_{P_5}(p-1)] \setminus [\chi(G_1)]$ to colour the vertices of $V(G')$. If $\chi(G[V(G') \cup V(G_2)]) \leq f^\star_{P_5}(p-1)$, then we can use the colours of $[2f^\star_{P_5}(p-1)] \setminus [\chi(G_2) + f^\star_{P_5}(p-1)]$ to colour the vertices of $V(G')$. Thus, we may assume that $\chi(G[V(G') \cup V(G_i)]) > f^\star_{P_5}(p-1)$, and so $\omega(G[V(G') \cup V(G_i)]) \geq p$, for each $i \in [2]$. As $E_G[V(G') \cup V(G_i), \{c_i\} \cup V(G_{3-i})]$ is complete, Lemma 75 (vi) implies that $\{c_i\} \cup V(G_{3-i})$ does not induce a $K_1 \cup K_2$. As $E_G[\{c_i\}, V(G_{3-i})]$ is anticomplete, we have that $V(G_{3-i})$ is an independent set. Thus, $A_{1,-} \cap N_G(V(G'))$ and $A_{2,+} \cap N_G(V(G'))$ are independent sets. We can colour $V(G')$ by $f^\star_{P_5}(p-1)$ colours from $[f^\star_{P_5}(p-1)+2] \setminus \{1, f^\star_{P_5}(p-1)+1\}$, which is a proper subset of $[f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)]$ as $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) > 2f^\star_{P_5}(p-1) \geq f^\star_{P_5}(p-1)+2$, since $p \geq 3$.

We next colour the vertices of $A_3 \cup \{c_1, c_2\}$, A'_4, and A'_5. Firstly, let us colour the vertices of the independent set $A_3 \cup \{c_1, c_2\}$ by colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1$. By Lemma 75 (vi), we have $\chi(G[A'_4]), \chi(G[A'_5]) \leq f^\star_{P_5}(p-1)$. Let I be a (possibly empty) independent set of $G[A'_4]$ such that $\chi(G[A'_4] - I) < f^\star_{P_5}(p-1)$. Furthermore, as

$$\chi(G[(A'_4 \setminus I) \cup A'_5]) \leq \chi(G[A'_4] - I) + \chi(G[A'_5]) \leq 2f^\star_{P_5}(p-1) - 1,$$

we can use colours from $[f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + 2f^\star_{P_5}(p-1)] \setminus [f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1]$ to colour the vertices of $(A'_4 \setminus I) \cup A'_5$. At this point of our proof, let us note that all vertices of $G[N_G[V(C)]] \setminus I$ are coloured and there are no two adjacent ones which are coloured alike. Finally, let us colour the vertices of the independent set I by colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$. For the sake of a contradiction, we suppose that a vertex $i \in I$ is adjacent to a vertex, say, a' of $N_G[V(C)] \setminus I$ which is coloured by colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)$. As $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) > 2f^\star_{P_5}(p-1)$, we find that $a' \in A'_{1,+}$. Let G' be the component of $G[A'_{1,+}]$ that contains a'. As i has a neighbour in $V(G')$ and G' is connected, we find that either $E_G[\{i\}, V(G')]$ is complete or there are two adjacent vertices $a'_{1,+} \in V(G') \cap N_G(i)$ and $a''_{1,+} \in V(G') \setminus N_G(i)$. As $[a''_{1,+}, a'_{1,+}, i, a_2, a_1]$ does not induce a P_5, it follows indeed that $E_G[\{i\}, V(G')]$ is complete. Thus, $E_G[\{c_2, i\}, V(G')]$ is complete, and so

$$\chi(G') \le \chi(G[N_G(c_2) \cap N_G(i)]) \le f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) - 2$$

by Lemma 75 (v). By our colouring of $G[A'_{1,+}]$, no vertex of G' is coloured by colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$. From this contradiction to our supposition, we can safely assign colour $f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$ to all vertices of I and obtain a colouring $c\colon N_G[V(C)] \to [f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) + 2f^\star_{P_5}(p - 1)]$. Furthermore, we easily see that the vertices of $A_3 \cup A'_4 \cup A'_5$ are coloured by at most $2f^\star_{P_5}(p - 1) + 1$ colours.

We proceed and colour the vertices of $G[B(C) \cup X(C)]$. By Lemma 75 (i), (ii), and (iii), for each component G' of $G[B(C) \cup X(C)]$, there is a vertex $a \in A_3 \cup A'_4 \cup A'_5$ with $E_G[\{a\}, V(G')]$ is complete. Thus, $\chi(G') \le f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$ by Lemma 75 (v). For the sake of a contradiction, let us suppose that G' is a component of $G[B(C) \cup X(C)]$ with $\chi(G') = f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$ and the neighbours of $V(G')$ in $A_3 \cup A'_4 \cup A'_5$ are coloured by $2f^\star_{P_5}(p-1)+1$ colours. In other words, $V(G')$ has a neighbour $a_3 \in A_3$ and a neighbour $a'_4 \in A'_4$. Note that $\chi(G') = f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$ and $E_G[\{a_3, a'_4\}, V(G')]$ is complete. It follows that $a_3a'_4 \notin E(G)$ as otherwise

$$f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) = \chi(G') \le \chi(G[N_G(a_3) \cap N_G(a'_4)]) \le f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) - 2$$

by Lemma 75 (v). But now $[c_2, a'_4, v, a_3, a_1]$ for some $v \in V(G')$ induces a P_5, a contradiction. Thus, our supposition is false and each component G' of $G[B(C) \cup X(C)]$ satisfies $\chi(G') < f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1)$ or the neighbours of $V(G')$ in $A_3 \cup A'_4 \cup A'_5$ are coloured by at most $2f^\star_{P_5}(p - 1)$ colours. Thus, we can extend our colouring c to a colouring of G on $f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) + 2f^\star_{P_5}(p - 1)$ colours. This completes our proof in this subcase. △

Before we proceed with Case 3, we prove an auxiliary claim that we use in its proof as well as in the proof of Case 4.

Claim 78.1. *Let C be an odd antihole with $Y(C) = \emptyset$ and k, ℓ be integers with $k \ge f^\star_{\{P_5,F_{p-1}\}}(\omega(G) - 1) + 1$, $k > \ell$, and $\ell \ge f^\star_{P_5}(p - 1)$. If $c_N\colon N_G[V(C)] \to [k]$ is a vertex colouring such that all vertices in $N_G(V(G'))$ are coloured by at most $k - \ell$ colours for each component G' of $G - N_G[V(C)]$, then there is a vertex colouring $c\colon V(G) \to [k]$ or we find a $C' \in \mathcal{C}(G)$ that extends to a $O[F]$ in G called H for some connected graph F with $\chi(F) \ge \ell + 1$ and the extender is a homogeneous set in G.*

Proof. If, for each component G' of $G - N_G[V(C)]$, there is a colouring $c_{G'}\colon V(G') \to [k]$ such that $c_{G'}(u_1) \ne c_N(u_2)$ for each two adjacent vertices $u_1 \in V(G')$ and $u_2 \in N_G(C)$, then there is a vertex colouring $c\colon V(G) \to [k]$. In view of the desired result, let us assume that F is a component of $G - N_G[V(C)]$ that does not have such a colouring. Let S be the set $N_G(V(C)) \cap N_G(V(F))$. Trivially, $\chi(F) \geq \ell + 1$ as c_N colours the vertices of S by at most $k - \ell$ colours. In other words, F contains a clique of size p. Moreover, S is not a clique. We show this fact as follows: As $Y(C) = \emptyset$, there is a vertex $s \in S$ with $E_G[\{s\}, V(F) \cup (S \setminus \{s\})]$ is complete, and so $\chi(G[V(F) \cup S]) \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1) + 1 \leq k$ by Lemma 75 (v). If S is a clique, then there is a colouring $c_{F \cup S} \colon V(F) \cup S \to [k]$ such that $c_N(s') = c_{F\cup S}(s')$ for each $s' \in S$. We find that $c_{F \cup S}$ restricted to the vertices of F gives a colouring, say, c_F of F such that $c_F(u_1) \neq c_N(u_2)$ for each two adjacent vertices $u_1 \in V(F)$ and $u_2 \in N_G(C)$. Thus, S is not a clique, and so there are two non-adjacent vertices $s_1, s_2 \in S$. We now distinguish three cases.

Case a: There is a set $S' \subseteq A(C) \cup M(C)$ such that $\chi(G[S']) \leq 2f^\star_{P_5}(p-1)$ and $N_G(V(H)) \subseteq S'$ for some component H of $G - S'$ with $V(H) \subseteq B(C) \cup X(C)$, and there is a vertex $s \in S'$ with $E_G[\{s\}, V(H)]$ is mixed.

By the connectivity of H, there are two adjacent vertices $u_1, u_2 \in V(H)$ such that $u_1 s' \in E(G)$ but $u_2 s' \notin E(G)$. Let G' be an arbitrary component of $G - S'$.

If G' is a component whose all vertices are in $B(C) \cup X(C)$, then there is a vertex in $s'' \in S'$ with $E_G[\{s''\}, V(G')]$ is complete as $Y(C) = \emptyset$. Hence, $\chi(G') \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1)$ by Lemma 75 (v). If G' is a component that has a vertex which is not in $B(C) \cup X(C)$, then the vertices of C are also vertices of G'. Furthermore, G' has a vertex which is adjacent to s'. Now all vertices of G' are adjacent to s' as $[u_2, u_1, s', v_1, v_2]$ does not induce a P_5 for each two vertices $v_1 \in N_G(s') \cap V(G')$ and $v_2 \in N_G(v_1) \cap V(G')$. Consequently, $\chi(G') \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1)$ by Lemma 75 (v).

We find $\chi(G') \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1)$ for each component G' of $G - S'$, and $\chi(G[S']) \leq 2f^\star_{P_5}(p-1)$. Therefore, $\chi(G) \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1) + 2f^\star_{P_5}(p-1)$.

Case b: $V(F)$ is a homogeneous set.

Let $u \in V(F)$. As we assume $N_G(s_1) \not\subseteq N_G(s_2)$ and $N_G(s_2) \not\subseteq N_G(s_1)$, there are two vertices $s'_1 \in N_G(s_1) \setminus N_G(s_2)$ and $s'_2 \in N_G(s_2) \setminus N_G(s_1)$. As $V(F)$ is a homogeneous set in G, as F has a clique of size p, and as $\{s_1, s_2, s'_i\}$ induces a $K_1 \cup K_2$ for each $i \in [2]$, we find that neither s'_1 nor s'_2 has a neighbour in $V(F)$ by Lemma 75 (vi). As G is P_5-free, $[s'_1, s_1, u, s_2, s'_2]$ induces a C_5, say, C' that extends to a $O[F]$ in G and the extender $V(F)$ is a homogeneous set in G. It remains to show $Y(C') = \emptyset$.

For the sake of a contradiction, let us suppose $y' \in Y(C')$. Thus, there are vertices $m' \in M(C')$ and $x' \in X(C')$ such that $m'y' \notin E(G)$ but $m'x', x'y' \in E(G)$. As $E_G[M(C'), V(C')]$ is complete, we find that $M(C') \subseteq S \setminus V(C')$. Thus, $M(C') \subseteq A(C) \cup M(C)$. Let X' be the set of vertices that induces the component of $G - M(C')$ which contains x and y. Note that $M(C')$ separates $V(C')$ and X' but does not separate $V(C)$ and $V(C')$. Thus, every x-u-path contains a vertex of $M(C')$ for every $u \in V(C)$. As an immediate consequence, we find $X' \subseteq B(C) \cup X(C)$. Furthermore, $\chi(G[M(C')]) \leq f^\star_{P_5}(p-1)$ by Lemma 75 (vi). As we are not in Case a, we find that $E_G[\{m'\}, X']$ is complete. By this contradiction to the fact that $m' \in M(C')$ is non-adjacent to $y' \in X'$, we obtain $Y(C') = \emptyset$.

Case c: $V(F)$ is not a homogeneous set.

As $V(F)$ is not a homogeneous set, there is a vertex in $s \in S$ with $E_G[\{s\}, V(F)]$ is mixed. Thus, as we are not in Case a, we find that $\chi(G[S]) > 2f^\star_{P_5}(p-1)$. In particular, $\chi(G[A(C) \cap S]) > f^\star_{P_5}(p-1)$ as $S \subseteq A(C) \cup M(C)$ and $\chi(G[M(C) \cap S]) \leq f^\star_{P_5}(p-1)$ by Lemma 75 (vi). Moreover, the fact $\chi(G[A(C) \cap S]) > f^\star_{P_5}(p-1)$ implies $\omega(G[A(C) \cap S]) \geq p$.

Let $M_0, M_1, M_2 \subseteq M(C)$ with $M_0 \cup M_1 \cup M_2 = M(C)$ such that for all $m_0 \in M_0$ we have $E_G[\{m_0\}, V(F)]$ is anticomplete, for all $m_1 \in M_1$ we have $E_G[\{m_1\}, V(F)]$ is complete, and for all $m_2 \in M_2$ we have $E_G[\{m_2\}, V(F)]$ is mixed. We next show that $E_G[M_2, A(C) \cap S]$ is complete. For the sake of a contradiction, let us suppose that $m_2 \in M_2$ is non-adjacent to $a \in A(C) \cap S$. As $E_G[\{a\}, V(C)]$ is mixed and as $E_G[\{m_2\}, V(F)]$ is mixed, there are three pairwise non-adjacent vertices $u_1, u_2 \in V(C)$ and $v \in V(F)$ such that $au_2 \in E(G)$ and $au_1, m_2v \notin E(G)$. Thus, $[v, a, u_2, m_2, u_1]$ induces a P_5, a contradiction. Consequently, $E_G[M_2, A(C) \cap S]$ is complete. As there is a clique of size p in $G[A(C) \cap S]$, $G[M_2 \cup V(F)]$ is $(K_1 \cup K_2)$-free by Lemma 75 (vi), and so complete multipartite. Let I be an independent set in F. We note that $E_G[(A(C) \cap S) \cup M_1, V(F)]$ is complete. So, $N_G(v_1) = N_G(v_2)$ for each two vertices in I. As we assume $N_G(v_1) \not\subseteq N_G(v_2)$, F is a complete graph of order $\chi(F)$. In particular, $|V(F)| \geq f^\star_{P_5}(p-1) + 1$. As $p \geq 3$, it follows $|V(F)| \geq p + 1$.

Let $m_2 \in M_2$ be arbitrary. As $E_G[\{m_2\}, V(F)]$ is mixed, there is a vertex $v \in V(F)$ that is non-adjacent to m_2. For the sake of a contradiction, let us suppose that $m_1 \in M_1$ is non-adjacent to m_2. As $G[V(F) \cup \{m_2\}]$ is a complete multipartite graph, we find that $E_G[\{m_2, v\}, W]$ is complete for a clique $W \subseteq V(F)$ of size p. Hence, $\{m_1, m_2, v\} \cup W$ induces a F_p. From this contradiction, we find that $E_G[\{m_2\}, M_1]$ is complete. Thus, $N_G(v) \subseteq N_G(m_2)$, which is a contradiction to our assumption on non-adjacent vertices. Thus, the claim is proven. (□)

Next let us focus on what is left in situation (a). In particular, let us assume we are not in Case 2. However, as we are still in situation (a), we find $\mathcal{C}(G) \neq \emptyset$ and $Y(C) = \emptyset$ for each $C \in \mathcal{C}(G)$.

Case 3: $\mathcal{C}(G) \neq \emptyset$ and, for each $C \in \mathcal{C}(G)$, there is no connected graph F with $\chi(F) \geq 2f^\star_{P_5}(p-1)$ such that C extends to a $O[F]$ in G called H whose extender is a homogeneous set in G.

Let $C \in \mathcal{C}(G)$. Similarly as in Case 2, we partition $A(C) \cup M(C) \cup V(C)$. Let U be the extender that extends C to a $O[K_p]$ in G. We define the two vertices c_1 and c_2 as well as the sets $A_{1,-}, A_{1,+}, A_{2,-}$, and $A_{2,+}$ as in Case 2. In particular, we have $U \subseteq A_{1,+} = A_{2,-}$, and that c_1 and c_2 are the neighbours of $A_{1,+}$ on C. We also assume $\chi(G[A_{1,-}]) \leq \chi(G[A_{2,+}])$, which implies as in Case 2 that $\chi(G[A_{1,-}]) \leq f^\star_{P_5}(p-1)$. Let us define A_3 as the set of vertices which have a neighbour on C but which are non-adjacent to c_1 and c_2. As in Case 2, we find that $E_G[A_3 \cup \{c_1, c_2\}, U]$ is complete, and so this set is independent.

We let A_4 be the set of vertices of $N_G(V(C))$ which are non-adjacent to c_2 and which do not belong to $A_{1,-} \cup A_3 \cup V(C)$. We show that $E_G[A_4, V(C) \setminus \{c_2\}]$ is complete. Let $a \in A_4$ be arbitrary. As $a \notin A_3$, we find that a is adjacent to c_1. As $\{a, c_1, c_2\} \cup U$ does not induce a F_p, we find that there is a vertex $u \in U$ that is non-adjacent to a. As $[a, c_1, u, c_2, c_2^+]$ does not induce a P_5, we find that a is adjacent to c_2^+. As $a \notin A_{1,-}$, it follows that a is a neighbour of c_1^+. Observe that $c_1^+ \in U$. As $[u, c_1^+, a, c_2^+, c_2^{+2}]$ does not induce a P_5, it follows that a is adjacent to c_2^{+2}. Consequently, $E_G[\{a\}, V(C) \setminus \{c_2\}]$ is complete, which proves that $E_G[A_4, V(C) \setminus \{c_2\}]$ is complete by the arbitrariness of a.

We next show that $\chi(G[A_{1,-} \cup A_4]) \leq 2f^\star_{P_5}(p-1) - 1$, which implies

$$\begin{aligned}\chi(G[N_G[V(C)]]) &\leq \chi(G[A_3 \cup \{c_1, c_2\}]) + \chi(G[N_G(c_2)]) + \chi(G[A_{1,-} \cup A_4]) \\ &\leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1) + 2f^\star_{P_5}(p-1)\end{aligned}$$

as $A_3 \cup \{c_1, c_2\}$ is an independent set and as $\chi(G[N_G(c_2)]) \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G) - 1)$ by Lemma 75 (v). Let $W \subseteq A_4$ be a clique of size $\omega(G[A_4])$. As $\{c_1, c_2, a_4\} \cup U$ does not induce a F_p for some $a_4 \in A_4$ and as $a_4 c_1^+ \in E(G)$, we get for each $a_4 \in A_4$ that $E_G[\{a_4\}, U]$ is mixed. Let $w \in W$ be arbitrary and $u \in U$ be a non-neighbour of w. We show next that $E_G[\{w\}, A_{1,-} \setminus \{c_1^-\}]$ is complete. We suppose for the sake of contradiction, that there is an $a \in A_{1,-} \setminus \{c_1^-\}$ with $wa \notin E(G)$. Firstly in this case $au \in E(G)$, since $[u, c_1^+, w, c_2^+, a]$ does not induces a P_5. But now $[c_1^+, u, a, c_2^+, c_1^-]$ if $ac_1^- \notin E(G)$, and $[a, c_1^-, w, c_1^+, c_2]$ if $ac_1^- \in E(G)$ induces a P_5, a contradiction. Therefore, $E_G[\{w\}, A_{1,-} \setminus \{c_1^-\}]$ and, thus, $E_G[\{w\}, A_{1,-}]$ is complete. Now $|W| \leq p-2$ or $\omega(G[A_{1,-}]) \leq 1$ as otherwise $\{a_1, a_2, c_1, c_1^+\} \cup W$ induces a F_p for two adjacent $a_1, a_2 \in A_{1,-}$. If $|W| \leq p-2$, then

$$\begin{aligned}\chi(G[A_{1,-} \cup A_4]) &\leq f^\star_{\{P_5, F_p\}}(p-1) + f^\star_{\{P_5, F_p\}}(p-2) \\ &\leq f^\star_{P_5}(p-1) + (f^\star_{\{P_5, F_{p+1}\}}(p-1) - 2) = 2f^\star_{P_5}(p-1) - 2\end{aligned}$$

by Lemma 74. If $\omega(G[A_{1,-}]) \leq 1$, then

$$\chi(G[A_{1,-} \cup A_4]) \leq 1 + f^\star_{P_5}(p-1) \leq 2f^\star_{P_5}(p-1) - 1$$

as $p \geq 3$ and so $f^\star_{P_5}(p-1) \geq 3$.

We now show that the neighbours of a component G' of $G - N_G[V(C)]$ are coloured by at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1$ colours. Let $b \in B(C) \cup X(C)$ and $a \in A(C) \cup M(C)$ be adjacent. By Lemma 75 (iv), we find $a \notin A_{1,-} \cup U$. Recall that for each vertex of $a_4 \in A_4$ we know that $E_G[\{a_4\}, U]$ is mixed. Thus, it follows $a \notin A_4$ as otherwise $[b, a, c_2^+, c_2, u]$ induces a P_5 for some $u \in U$ that is non-adjacent to a. We conclude $N_G(V(G')) \subseteq A_3 \cup N_G(c_2)$, which gives the desired result as the vertices of $A_3 \cup N_G(c_2)$ are coloured by at most $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+1$ colours.

Finally, we apply Claim 78.1 with

$$k := f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + 2f^\star_{P_5}(p-1) \quad \text{and} \quad \ell := 2f^\star_{P_5}(p-1)-1.$$

As we are not in Case 2, we obtain a k-colouring of G and, thus, the proof of Case 3 is complete. △

It remains to assume that we are not in situation (a) but in situation (b). Recall that the latter means $Y(C) = \emptyset$ for each odd antihole C in G. We immediately find that G is $O[K_p]$-free as otherwise we would be in situation (a) as we could find a $C \in \mathcal{C}_5(G)$ that extends to a $O[K_p]$ in G with $Y(C) = \emptyset$.

Case 4: G is $O[K_p]$-free and $Y(C) = \emptyset$ for each odd antihole C in G.

Let C be an odd antihole that satisfies $\chi(C) \leq \vartheta(p)$, which exists by the definition of ϑ. We colour $N_G[V(C)]$ by at most $\vartheta(p)$ colours, and apply Claim 78.1 with

$$k := \max\{\vartheta(p), f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1) + f^\star_{P_5}(p-1)\} + f^\star_{P_5}(p-1) \quad \text{and} \quad \ell := f^\star_{P_5}(p-1).$$

As every $C' \in \mathcal{C}(G)$ does not extend to a $O[K_p]$ called H, we obtain the desired result. Thus, the proof of Case 4 and the proof of this Lemma are complete. □

We are now in a position to prove our main result, namely Lemma 71. Let $\vartheta' \colon \mathbb{N}_0 \to \mathbb{N}_0$ be a function with

$$p \mapsto \begin{cases} 0 & \text{if } p \leq 1, \\ 4 & \text{if } p = 2, \\ \max\{10, 2p+3\} \cdot f^\star_{P_5}(p) & \text{if } p \geq 3. \end{cases}$$

Note that ϑ' can be thought of as an upper bound to ϑ, for $p \in \mathbb{N}_{\geq 3}$.

In view of simplicity let $f^\star_{P_5}(0) = f^\star_{P_5}(-1) = 0$. We first claim that each (P_5, F_p)-free graph G with $\chi(G) > \max\{f^\star_{P_5}(p+1), \vartheta'(p) + f^\star_{P_5}(p-1)\}$ satisfies

$$\chi(G) \leq \omega(G) + \sum_{i=1}^{p-1} (2f^\star_{P_5}(i) - 1)$$

for each $p \in \mathbb{N}_0$. We prove this claim by induction hypothesis on p. Let G_c be a component of G with $\chi(G_c) = \chi(G)$. Note that G_c is (P_5, F_p)-free with $\chi(G_c) > \max\{f^\star_{P_5}(p+1), \vartheta'(p) + f^\star_{P_5}(p-1)\}$, and so $\omega(G_c) \geq p+2$. In view of the desired result it suffices to show that

$$\chi(G_c) \leq \omega(G_c) + \sum_{i=1}^{p-1}(2f^\star_{P_5}(i) - 1)$$

as $\chi(G_c) = \chi(G)$ and $\omega(G_c) \leq \omega(G)$. For $p = 0$ this follows from Observation 16. For $p = 1$ and $p = 2$ this follows from Corollary 68 of Chapter 9 and Theorem 7, respectively.

So we may assume $p \geq 3$. Thus, $\vartheta(p) \leq \vartheta'(p)$, by Lemma 77, and Lemma 78 implies

$$\chi(G_c) \leq f^\star_{\{P_5, F_{p-1}\}}(\omega(G_c) - 1) + 2f^\star_{P_5}(p-1).$$

Now, let G' be a (P_5, F_{p-1})-free graph with $\chi(G') = f^\star_{\{P_5, F_{p-1}\}}(\omega(G_c) - 1)$ and $\omega(G') = \omega(G_c) - 1$. The existence of G' follows from Lemma 74. If

$$\chi(G') \leq \omega(G') + \sum_{i=1}^{p-2}(2f^\star_{P_5}(i) - 1),$$

then

$$\begin{aligned}\chi(G_c) &\leq \chi(G') + 2f^\star_{P_5}(p-1)\\ &\leq \left(\omega(G_c) - 1 + \sum_{i=1}^{p-2}(2f^\star_{P_5}(i) - 1)\right) + 2f^\star_{P_5}(p-1) \leq \omega(G) + \sum_{i=1}^{p-1}(2f^\star_{P_5}(i) - 1).\end{aligned}$$

Thus, it remains to suppose, for the sake of a contradiction, that

$$\chi(G') > \omega(G') + \sum_{i=1}^{p-2}(2f^\star_{P_5}(i) - 1).$$

As $\omega(G_c) \geq p+2$, it follows $\omega(G') \geq p+1$. By induction we find

$$\chi(G') \leq \max\{f^\star_{P_5}(p), \vartheta'(p-1) + f^\star_{P_5}(p-2)\}.$$

We consider first the case where $\chi(G') \leq \vartheta'(p-1) + f^\star_{P_5}(p-2)$. Thus,

$$\vartheta'(p) + f^\star_{P_5}(p-1) < \chi(G_c) \leq \chi(G') + 2f^\star_{P_5}(p-1) \leq \vartheta'(p-1) + f^\star_{P_5}(p-2) + 2f^\star_{P_5}(p-1).$$

In other words,

$$\vartheta'(p) < \vartheta'(p-1) + f^\star_{P_5}(p-2) + f^\star_{P_5}(p-1).$$

As $f^\star_{P_5}(1) = 1$ by definition, $f^\star_{P_5}(2) = 3$ by [66], $f^\star_{P_5}(3) = 5$ by Theorem 15, and as $f^\star_{P_5}(4) \geq 7$ by the facts $f^\star_{P_5}(3) = f^\star_{\{P_5,F_2\}}(3)$, $f^\star_{P_5}(4) = f^\star_{\{P_5,F_3\}}(4)$ and by Lemma 74, it follows by putting in the numbers that $p \geq 5$. Thus, by the definition of $\vartheta'(p)$, it follows

$$\max\{9, 2p+2\} \cdot f^\star_{P_5}(p) \leq \max\{10, 2p+3\} \cdot f^\star_{P_5}(p) - f^\star_{P_5}(p-2) < \max\{11, 2p+2\} \cdot f^\star_{P_5}(p-1).$$

As $f^\star_{P_5}(p) \geq f^\star_{P_5}(p-1)$, we find $2p+2 < 11$, which is a contradiction to the fact $p \geq 5$. Thus, it remains to consider the case where $\chi(G') > \vartheta'(p-1) + f^\star_{P_5}(p-2)$, and so $\chi(G') \leq f^\star_{P_5}(p)$. Now, the fact that $\chi(G_c) \leq \chi(G') + 2f^\star_{P_5}(p-1)$ implies

$$\begin{aligned} 10 \cdot f^\star_{P_5}(p) &\leq \vartheta'(p) + f^\star_{P_5}(p-1) < \chi(G_c) \\ &\leq \chi(G') + 2f^\star_{P_5}(p-1) \leq f^\star_{P_5}(p) + 2f^\star_{P_5}(p-1) \leq 3 \cdot f^\star_{P_5}(p), \end{aligned}$$

a contradiction. Therefore, our supposition is false, and our claim follows. In particular, we have

$$\chi(G) \leq \max\left\{\omega(G) + \sum_{i=1}^{p-1}(2f^\star_{P_5}(i) - 1), \vartheta'(p) + f^\star_{P_5}(p-1), f^\star_{P_5}(p+1)\right\} \tag{1}$$

for each (P_5, F_p)-free graph G and each $p \geq 0$.

11 Open questions and outlook

In this concluding chapter we talk about open questions related to our research field and give an outlook for future research. This chapter is subdivided into two sections. In Section 11.1 we talk in some detail about a question related to the not non-decreasing χ-binding functions. In Section 11.2 we talk about some of our χ-binding functions and closely related open questions.

11.1 Non-decreasing χ-binding function

Let $\mathcal{G}$ be a family of graphs and we are interested in the optimal χ-binding function of $\mathcal{G}$. If this χ-binding function is known to be non-decreasing, it is sufficient to just research the critical graphs of $\mathcal{G}$, by Lemma 1. Since we are interested in P_5-free graphs, this raises the question, for which subfamilies of $\mathrm{For}(P_5)$ we know that their optimal χ-binding function is non-decreasing. Or reversely stated, we are interested in a complete characterisation of subfamilies of $\mathrm{For}(P_5)$ with not non-decreasing optimal χ-binding functions. To partially answer this question, let $I \subseteq \mathbb{N}_{>0}$ with $1 \in I$ and $\mathcal{H} = \bigcup_{i \in I} \{H_i\}$ be a family of forbidden graphs where $H_1 \subseteq_{\text{ind}} P_5$. Let us also assume that the graph H_i is not an induced subgraph of H_j for $i, j \in I$ with $i \neq j$. Since otherwise $f^\star_{\mathcal{H}} \equiv f^\star_{\mathcal{H} \setminus \{H_j\}}$ and the graph H_j has no influence on the optimal χ-binding function. In this setting we know that $f^\star_{\mathcal{H}}$ exists, since $f^\star_{P_5}$ exists (cf. Theorem 12, [31]). Also note that if $H_1 \subseteq_{\text{ind}} P_4$ this optimal χ-binding function is easy to determine, since $f^\star_{\{P_4\}}(\omega) = \omega$ for $\omega \in \mathbb{N}_{>0}$ [66]. So from now on we may assume $H_1 \not\subseteq_{\text{ind}} P_4$. We now collect sufficient conditions on $\mathcal{H}$ such that $f^\star_{\mathcal{H}}$ is non-decreasing and state some examples of $\mathcal{H}$ such that $f^\star_{\mathcal{H}}$ is not non-decreasing.

We prove a positive result for our current aim in Lemma 44 of Section 3.3. It states that as long as each forbidden subgraph $H \in \mathcal{H}$ does not contain a universal vertex, the function $f^\star_{\mathcal{H}}$ is strictly increasing. In the same section we also show in Lemma 45 the following positive result. As long as for all $H \in \mathcal{H}$ each connected component of H is non-isomorphic to a complete graph, we prove that the χ-binding function is non-decreasing.

So to find a family $\mathcal{H}$ such that $f^\star_{\mathcal{H}}$ is not non-decreasing, there are minimal $i_u, i_c \in I$

such that $H_{i_u} \in \mathcal{H}$ contains a universal vertex, and one component of $H_{i_c} \in \mathcal{H}$ is isomorphic to a complete graph. Note that the complete graph is the only graph with a universal vertex and a component which is a complete graph. Thus, if $i_u = i_c$, there is an $n \in \mathbb{N}_{>0}$ with $K_n \in \mathcal{H}$. For $n = 1$ the family of K_n-free graphs is empty and for $n > 1$ we see that $f^{\star}_{\mathcal{H}}(1) = 1$, $f^{\star}_{\mathcal{H}}(n) = 0$, since there is no graph of clique size n in this family, which implies that the function $f^{\star}_{\mathcal{H}}$ is not non-decreasing. So in this case $f^{\star}_{\mathcal{H}} \equiv 0$ or $f^{\star}_{\mathcal{H}}$ is not non-decreasing.

So from now on we may assume that $i_u \neq i_c$. The case $i_u = 1$ leads to a contradiction to our assumption that $H_1 \not\subseteq_{\text{ind}} P_4$, since $H_1 \subseteq_{\text{ind}} P_5$. So in all interesting cases we have either $i_c = 1 < i_u$ or without loss of generality $1 < i_u < i_c$.

Let us firstly note that in the latter case $H_1 \cong P_5$, since $1 < i_c$. So the first open question is for which graphs H_{i_c}, H_{i_u} the function $f^{\star}_{\{P_5,H_{i_u},H_{i_c}\}}$ is not non-decreasing. In this situation there is no easy way to show that this function is non-decreasing, but it still could be as the following example shows. Let us choose $H_{i_u} \cong \mathit{dart}$ and $H_{i_c} \cong 4K_1$. We know $f^{\star}_{\{P_5,\mathit{dart}\}} \equiv f^{\star}_{\{3K_1\}}$, by Theorem 4. It follows $f^{\star}_{\{P_5,\mathit{dart},4K_1\}} \leq f^{\star}_{\{P_5,\mathit{dart}\}} \equiv f^{\star}_{\{3K_1\}}$ and $f^{\star}_{\{P_5,\mathit{dart},4K_1\}} \geq f^{\star}_{\{3K_1\}}$, since $3K_1 \subseteq_{\text{ind}} P_5, \mathit{dart}, 4K_1$. It becomes clear that the function $f^{\star}_{\{P_5,\mathit{dart},4K_1\}} (\equiv f^{\star}_{\{3K_1\}})$ is non-decreasing even though both necessary conditions are fulfilled. Thus, the stated necessary conditions are not sufficient to grant a not non-decreasing χ-binding function.

Let us now look at the first case: In this case $H_1 \cong P_3 \cup K_1$, $H_1 \cong 3K_1$, or $H_1 \cong 2K_2$. In the next paragraph we argue that $f^{\star}_{\{3K_1\}\cup\mathcal{H}} = f^{\star}_{\{P_3\cup K_1\}\cup\mathcal{H}}$ for each graph family $\mathcal{H}$. Proving this claim shows that it suffices to consider $H_1 \in \{3K_1, 2K_2\}$ in this case.

Since $3K_1 \subseteq_{\text{ind}} P_3 \cup K_1$, we find $f^{\star}_{\{3K_1\}\cup\mathcal{H}} \leq f^{\star}_{\{P_3\cup K_1\}\cup\mathcal{H}}$. To prove the other direction let G be an arbitrary $(P_3 \cup K_1, \mathcal{H})$-free graph. Thus, the complementary graph $\bar{G}$ is *paw*-free. Let I be a finite set and $\emptyset \neq V_i \subseteq V(G)$ for each $i \in I$ such that $V(G) = \bigcup_{i\in I} V_i$, V_j induces a connected component in $\bar{G}$, V_j and V_k are pairwise disjoint, and $E_{\bar{G}}[V_j, V_k]$ is anticomplete, for $j \neq k$. Thus, for each $i \in I$ the graph $\bar{G}[V_i]$ is complete multipartite or K_3-free, by Olariu (cf. Theorem 20, [48]). Let I_1 be the maximum subset of I with $\bar{G}[V_i]$ is K_3-free, for each $i \in I_1$ and $I_2 = I \setminus I_1$. Note that $\bar{G}[\bigcup_{i\in I_1} V_i]$ is K_3-free if $I_1 \neq \emptyset$. Thus, $G[\bigcup_{i\in I_1} V_i]$ is $3K_1$-free if $I_1 \neq \emptyset$. Since $\bar{G}[V_j]$ is completely multipartite for each $j \in I_2$, we obtain $G[V_j]$ is a disjoint union of complete graphs. For $j \in I_2$, let V_j^m be a subset of V_j such that $G[V_j^m]$ is a complete graph and $\omega(G[V_j^m]) = \omega(G[V_j])$. Let us define the set V' and the graph G' by

$$V' := \bigcup_{i\in I_1} V_i \cup \bigcup_{j\in I_2} V_j^m \quad \text{and} \quad G' = G[V'].$$

Then, we find $\omega(G') = \omega(G)$ and $\chi(G') = \chi(G)$. If $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$, we recall that $G[\bigcup_{i\in I_1} V_i]$ is $3K_1$-free, $G[\bigcup_{j\in I_2} V_j^m]$ is a complete graph, and $E_G[\bigcup_{i\in I_1} V_i, \bigcup_{j\in I_2} V_j^m]$ is complete. Thus, the graph G' is $3K_1$-free. Otherwise, the graph G' is also $3K_1$-free.

Therefore, G' is $3K_1$-free in both cases and as $G' \subseteq_{\text{ind}} G$ especially $\mathcal{H}$-free. Thus,

$$\chi(G) = \chi(G') \leq f^{\star}_{\{3K_1\}\cup\mathcal{H}}(\omega(G')) = f^{\star}_{\{3K_1\}\cup\mathcal{H}}(\omega(G)),$$

which completes the proof.

We lastly introduce two non-trivial examples of a set $\mathcal{H}$ with $|\mathcal{H}| = 2$ such that $f^{\star}_{\mathcal{H}}$ is not non-decreasing. Firstly let us look at the set of $(2K_2, (K_1 \cup K_2) + K_p)$-free graphs for some large $p \in \mathbb{N}_{>0}$. To use the following Theorem 79 by Brause et al. [14] let us introduce the following definition. A graph G is a *multisplit graph* if its vertex set $V(G)$ can be divided into two vertex disjoint sets S_1 and S_2 such that S_1 induces a complete multipartite graph and S_2 is an independent set in G.

Theorem 79 (Brause et al. [14])**.** *If G is a connected $(2K_2, (K_1 \cup K_2) + K_p)$-free graph with $\omega(G) \geq 2p$ for some integer $p \geq 2$, then G is a multisplit graph.*

Let us shortly argue that this statement is also true for the disconnected graphs G. In a $2K_2$-free graph there is at most one connected component consisting of at least two vertices. Additionally the disjoint union of a multisplit graph and a K_1 is still a multisplit graph, by the definition of a multisplit graph. Thus, each graph $G \in \text{For}(2K_2, (K_1 \cup K_2) + K_p)$ with $\omega(G) \geq 2p$ is a multisplit graph, by Theorem 79. In the same paper they also prove that multisplit graphs are perfect. On the other hand, let us recall Theorem 29 by Gyárfás [31], which states that there exists an $\epsilon > 0$ such that $\frac{\omega^{1+\epsilon}}{3} \leq f^{\star}_{\{2K_2\}}(\omega)$, for each $\omega \in \mathbb{N}_{>0}$. Let $p = \lceil 6^{1/\epsilon} + 2 \rceil$, then $p \geq 2$ and $\frac{p^{1+\epsilon}}{3} > 2p$. So using these two results and the definition of p we find

$$f^{\star}_{\{2K_2,(K_1\cup K_2)+K_p\}}(p) = f^{\star}_{\{2K_2\}}(p) \geq \frac{p^{1+\epsilon}}{3} > 2p = f^{\star}_{\{2K_2,(K_1\cup K_2)+K_p\}}(2p).$$

Thus, this optimal χ-binding function is not non-decreasing. Brause et al. [14] also research the family of $(2K_2, 2K_1 + K_p)$-free graphs and prove a similar result as Theorem 79 for this family. One can argue analogously to our previous argumentation that for large $p \in \mathbb{N}_{>0}$ the function $f^{\star}_{\{2K_2,2K_1+K_p\}}$ is not non-decreasing.

Let us shortly summarize some results of this section. We prove two necessary conditions on a graph family $\mathcal{H}$ such that $f^{\star}_{\mathcal{H}}$ is not non-decreasing. Additionally, we introduce the graph family $\text{For}(P_5, dart, 4K_1)$ fulfilling both conditions whose optimal χ-binding function is still non-decreasing. In the final part we state two families with not non-decreasing χ-binding function.

11.2 Improvable χ-binding functions

In this thesis we have shown a χ-binding function for the graph family of (P_5, H)-free graphs (cf. Theorem 2-8), for several graphs H. Moreover, several of these binding functions are exact or achieve the right order of magnitude. For example from Theorem 4 and Corollary 27, which uses the landslide result by Kim (cf. Theorem 26, [42]), we obtain

$$f^{\star}_{\{P_5,dart\}}(\omega) = f^{\star}_{\{3K_1\}}(\omega) \in \Theta\left(\frac{\omega^2}{\log(\omega)}\right).$$

Achieving the right order of magnitude is generally a great achievement and there is further research by Pontiveros et al. [51] improving the constants of Kim. Note that asymptotically the functions $f_R : \mathbb{N}_0 \to \mathbb{N}_0$ defined by $k \to R(3, k+1)$ and $f^{\star}_{\{3K_1\}}$ behave the same and for $k \leq 9$ the Ramsey number $R(3, k)$ is known but to the best of our knowledge $f^{\star}_{\{3K_1\}}(k)$ is unknown for $k \geq 6$. Pedersen just mentions the following concrete upper bounds on $f^{\star}_{\{3K_1\}}$ in the concluding remarks of [49]. They claim that using data from the Ramsey numbers, they calculate that $f^{\star}_{\{3K_1\}}(4) \leq 7$ and $f^{\star}_{\{3K_1\}}(5) \leq 9$. Note that by Corollary 27 this implies $f^{\star}_{\{3K_1\}}(4) = 7$ and $f^{\star}_{\{3K_1\}}(5) = 9$, since $R(3, 5) = 14$ and $R(3, 6) = 18$. Which raises the question, whether or not the lower bound in Corollary 27 is always achieved with equality.

Question 1. *For $\omega \in \mathbb{N}_{>0}$,*

$$\left\lceil \frac{R(3, \omega + 1) - 1}{2} \right\rceil = f^{\star}_{\{3K_1\}}(\omega).$$

Assuming this question to be answered positively, calculating for $\omega \in \mathbb{N}_{>0}$ values of $f^{\star}_{\{3K_1\}}(\omega)$ reduces to the problem of calculating $R(3, \omega+1)$. Proving Ramsey numbers is a widely considered computational problem which sharply increases in difficulty when increasing the input. Recall that the Ramsey Number $R(3, k)$ is known for $k \in [9]$. Thus, proving the question grants multiple new values of the function $f^{\star}_{\{3K_1\}}$.

We also prove $f^{\star}_{\{P_5,banner\}} \equiv f^{\star}_{\{2K_2\}}$ in Theorem 4. As opposed to $f^{\star}_{\{3K_1\}}$ the asymptotically behaviour of $f^{\star}_{\{2K_2\}}$ is unknown and the best known general bound is still by Wagon (cf. Theorem 30, [67]). By using the result of Gasper and Huang (cf. Theorem 31, [29]) we improve the bound by Wagon by a linear factor in Corollary 32. Note that the asymptotic behaviour is not improved by this proof. This raises the open question for the asymptotic behaviour of $f^{\star}_{\{2K_2\}}$. This seems to be a difficult problem, since there has been no significant improvement on the bound by Wagon from 1980.

In multiple cases we reduce the problem of finding a χ-binding function for the graph family $\mathcal{G}$ to the problem of finding a χ-binding function for a real subfamily $\mathcal{G}'$. Like we argue previously to calculate $f^{\star}_{\{3K_1\}}$ or $f^{\star}_{\{2K_2\}}$ is a challenging problem. Naturally there are still open cases to solve and one question which arises from this thesis regards the function $f^{\star}_{\{P_5,kite\}}$. In Chapter 8 we show

$$\left\lfloor \frac{3\omega}{2} \right\rfloor \leq f^{\star}_{\{P_5,kite\}}(\omega) = f^{\star}_{\{2K_2,K_1\cup K_3,K_1\cup C_5\}}(\omega) \leq \begin{cases} \lfloor \frac{3\omega}{2} \rfloor & \text{if } \omega \leq 4, \\ 2\omega - 2 & \text{if } \omega \geq 5, \end{cases}$$

for $\omega \in \mathbb{N}_{>0}$. Our guess is also that the lower bound is sharp, but it seems to be a challenging problem to even show $f^\star_{P_5,kite}(5) = \lfloor 3 \cdot 5/2 \rfloor = 7$. Therefore, we formulate it as an open question.

Question 2. *For $\omega \in \mathbb{N}_{>0}$,*

$$f^\star_{P_5,kite}(\omega) = \left\lfloor \frac{3\omega}{2} \right\rfloor .$$

We prove the optimal χ-binding function for (P_5, F_2)-free graphs (cf. Chapter 9). We also put quite some effort in proving a small χ-binding function for (P_5, F_p)-free graphs and $p \in \mathbb{N}_{>2}$ (cf. Chapter 10). In the proof of Lemma 78 there are multiple cases in which $f^\star_{\{P_5,F_{p-1}\}}(\omega(G)-1)+2f^\star_{P_5}(p-1)$ colours are needed. Thus, to improve the bound multiple new colourings are needed. Still, it is a interesting question to ask whether or not these χ-binding functions can be improved.

These are just a few questions which arise while working in this mathematical field. Clearly just looking at the results of this research field collected in Chapter 2 creates a lot of interesting open questions especially in regards to optimal χ-binding functions. However, the stated questions are the ones which are closest related to this thesis.

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