INVESTIGATION OF NONLINEAR CONTROL SYSTEM STABILITY BY PHASE SPACE PARTITION

A Thesis

Presented to

The Faculty of the Department of Electrical Engineering

The University of Houston

In Partial Fulfillment

of the Reguirements for the Degree

Master of Science in Electrical Engineering

by

Arun Kumar Bidani

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ABSTRACT

One of the most important areas of nonlinear control system study is system stability. Unlike linear systems, it is quite complex and it has been defined in various ways in literature. The only powerful general tool available for determining regions of asymptotic stability is Liapunov's direct method. But the finding of the Liapunov function is quite difficult in most cases and there are few general rules available though continuing research has broadened the class of functions for which it can be used.

This research concentrates on an effort to establish whether the suggested criterions for phase space partition, discriminant, generalised Hurwitz and Ku Shen, can be made to yield areas of asymptotic stability for a class of functions which could be mathematically defined.

It finds that though in general, they are quite useful in building Multilinear Models for non-linear systems, they are of little use in defining stability. A new separate use criterion is suggested, which does furnish the regions of stability but for a very restricted class of simple control systems.

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CHAPTER 1

INTRODUCTION

Whenever confronting any control system, the most important question is that of its stable equilibrium states. This may consist of a position of asymptotic stability in the Liapunov sense or of a stable oscillatory motion of finite amplitude. For a linear system, if stability exists at all, it is global monotonic stability.[1] This is defined as the stability condition for which a state point, anywhere in the entire finite region of phase plane, tends to the singularity and approaches arbitrarily close to it as time approaches infinity. This is not true of nonlinear control systems at all, and it is of great importance to prove stability in a defined finite region of state space , when global stability does not exist or cannot be proved. This defined area furnishes us the constraints for the system parameters which can only be exceeded at the cost of instability.[1]

The nature of the phase space trajectories were studied in detail by Poincaré in his investigation of nonlinear mechanics and he developed the classification for different kinds of simple singularities that can exist. If first degree or linear terms are present in the state space equations, then the singularity is defined as simple. For a small region around these points in the phase plane, for a structurally stable system (any physical system almost automatically meets this requirement)[2], Liapunov showed that the linear approximation will determine the stability. However, frequently, practical interest dictates the exact delineation of as large a region of asymptotic stability around the singularity as possible.

Liapunov's second or direct method is a powerful and general approach to the stability of control systems and defining this area. The major problem in applying it is of construction of the Liapunov function. This is further complicated by the fact that the Liapunov function is not unique. Thus:

(1) Should a particular function fail to show that a specific system is stable or unstable, there is no assurance that another function cannot be defined that does demonstrate stability or instability.

(2) Should a particular function fail to show that a particular system is stable or unstable, there is no assurance that exceeding these limits will actually cause the system to be unstable. In other words, stability requirements are almost without exception, overly

rigorous.

There are few general rules for choosing the Liapunov function though more and more attention has been paid in recent years, to developing and broadening the classes for which it is available. Chetaev,[3] Lüre,[4] Letov[5] and Zubov[6] have done important pioneering work in the area, with regional studies around singularities by Zubov[7] and Vogt,[8] among others. But it is difficult even now to find a Liapunov function, quickly and conveniently for a specific control system. Some of the simplest ones that can be found give a quite small defined area out of a much larger possible one.

The great usefulness of the Liapunov approach, apart from its generality, is that the system equations do not have to be solved. Therefore if some other criterion were developed, which was much less general, but was simpler and systematic, these criterion would still have some important practical utility. It is towards investigating this possibility that this research addresses itself.

Since each singularity in a state plane dominates the behavior of the neighborhood trajectories, it seems feasible, at least intuitively, to partition state space into single singularity dominated regions. Then if the singularity was stable by nature, the defined area would be one of asymptotic stability.

One such approach was investigated by Chung.[9] The scope of the discriminant criterion and generalized Hurwitz criterion suggested by Chung are examined for this purpose, as is the work of Ku and Shen.[10] The latter work is of some importance as it leads to a possible unified treatment with singularities of the second kind. It will be shown that the first two criterion (Chung) are quite weak as generally the defined area of stability always contains an undefined area of instability in it which almost completely invalidates its usefulness. A new criterion is also suggested for a very restricted class of functions using the property of the separatrix.

CHAPTER II

SINGULARITIES

(1) Definitions of terms

Just as poles and zeros exist in the complex plane, similarly there exist points on the phase planes called singularities which characterize the system response.[10]

Limiting our attention to the second order system, the state space equations can be represented in the general form by:

$$\dot{x}_1 = P(x_1, x_2) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\dot{x}_2 = Q(x_1, x_2) \quad \ldots\ldots\ldots\ldots\ldots (2)$$

where P and Q are not restricted to linear terms but are ascending polynomials in in the general case.

Then, dividing (2) by (1):

$$\frac{\dot{x}_1}{\dot{x}_2} = \frac{Q(x_1, x_2)}{P(x_1, x_2)} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

The integral curves of equation (3) on the plane, called the state plane, determine the phase portrait of the system[11] which represents all possible histories[12] of it. If $\dot{x}_1 = x_2$, then the state plane is referred to as the phase plane.[11,13]

Singularities of this function are said to occur where

$$P(x_{10}, x_{20}) = Q(x_{10}, x_{20}) = 0 \ldots\ldots\ldots\ldots\ldots (4)$$

that is where the slope is indeterminate.

Graphically speaking, there is a unique trajectory which passes through each point with the exception of the points of singularities through which either an infinite number, or none, pass.[11] (Poincaré) The singularity is classified as simple, which are the ones dealt with here, if the lowest degree terms present in (3) in the numerator and denominator are of first degree.

(2) Location of Singularities

The singularities can thus be obtained by forcing the rate variables in the state equations to zero (equations 2 and 4).

Illustrative Example

1. Consider the following nonlinear differential equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^2 - x_2$$

By (4) for singularities

$$0 = x_2 \quad \text{...............(5)}$$
$$0 = -x_1 + x_1^2 - x_2 \quad \text{...............(6)}$$

From (6) $x_1^2 - x_1 = 0$ or $x_1 = 0, 1$

Therefore singular points are (0,0), (1,0)

2. For the linear case:

$$xi = ax_1 + bx_2$$
$$xi = cx_1 + dx_2$$

The singularities are determined by:

$$0 = a x_1 + b x_2$$
$$0 = c x_1 + d x_2$$

which is possible only if $x_1 = 0, x_2 = 0$

Therefore linear systems have only one singularity which is located at the origin.

(3) <u>Classification of Singularities of Second Order System</u> (Linear)

Starting with a linear second order system we have the space equations:

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2$$
$$\dot{x}_2 = a_{21} x_1 + a_{22} + x_2 \quad \ldots\ldots\ldots\ldots\ldots (7)$$

a_{11}, a_{12}, a_{21} and a_{22} being constants, the particular form being chosen because of the facility with which it can be generalized to higher order systems, if necessary. The matrix form for (7) being:

$$[\dot{x}] = [A][x] \quad \ldots\ldots\ldots\ldots\ldots (8)$$

Then the singularity of this system exists at (0,0) as mentioned earlier.

The solution of equations of the form (7) is known to involve exponential functions and can be represented by $[C x]$ exponential(λt) Non-trivial solutions exist only if λ's are the roots of the characteristic equations.[1,13,14]

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \quad \text{OR} \quad [A] - \lambda [I] = 0$$

This roots or eigenvalues determine completely the nature of the singularity at the origin and the solution will be of the form:[1,13]

$$x_1 = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$
$$x_2 = C_3 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

The following is the classification developed by Poincaré:[1,11]

(1) If the roots are real and of the same sign, the singularity is called a node.

(2) If the roots are real and of opposite sign, the singularity is called a saddle.

(3) If the roots are purely imaginary and conjugate, the singularity is called a vortex or a center.

(4) If the roots are complex conjugate, the singularity is called a focus.

(1) and (4) have two sub-classes:

1(a) If the roots are real, of the same sign and positive, the singularity is classified as an unstable node, as with $t \to \infty$, $x_1 \to \infty$, $x_2 \to \infty$

(b) If the roots are real, of the same sign and negative, the singularity is classified as a stable node (as with $t \to \infty$, $x_1 \to \infty$, $x_2 \to \infty$ or the trajectory approaches [0,0] the singularity).

Similarly for (4),

4(a) If the real part of the complex root is positive, the singularity is classified as an unstable focus.

(b) If the real part of the complex root is negative, the singularity is classified as a stable focus.

Some Illustrative Examples:

Example 1:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 + 5x_2$$

Therefore characteristic equation is given by:

$$\begin{vmatrix} 0-\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = 0$$
$$\text{OR } \lambda_1 = 4 \quad \lambda_2 = 1$$

The roots have the same sign, are real and positive.

The singularity is unstable node. The solution is of the form:

$$x_1 = Ae^{4t} + Be^{t}$$
$$x_2 = 4Ae^{4t} + Be^{t}$$

A and B being constants which are determined by the initial conditions. A plot of the trajectories with different initial conditions are shown in Fig. 1. The lines with slope λ_1, and λ_2 are called eigenvectors and are special solution curves. The eigenvector nearer the $\dot{x}_1$ axis (or x_2 axis) is called the fast eigenvector and the other the slow eigenvector because of the relative phase velocities of the state point when moving along them.[6]

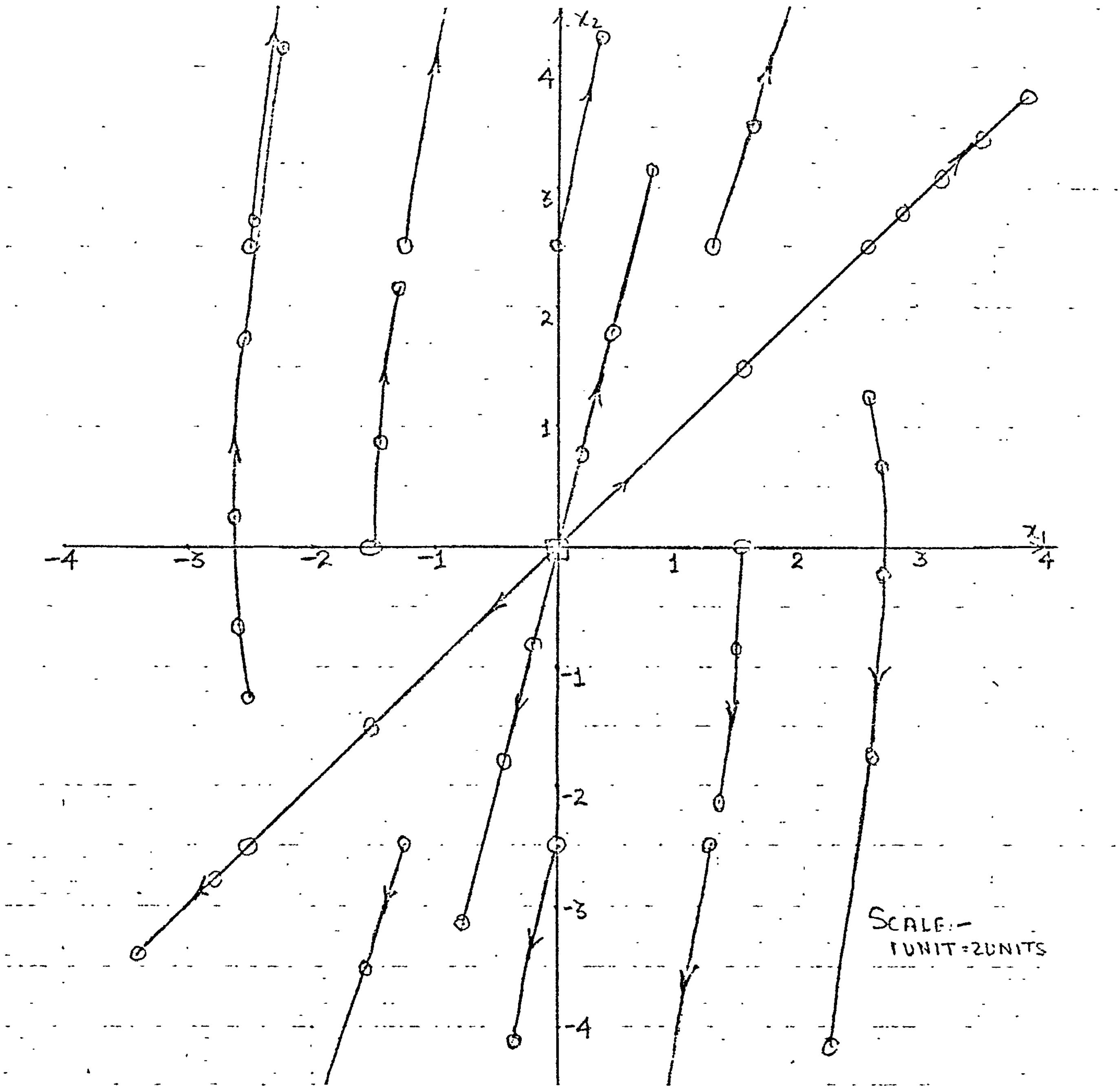

Fig. 1 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4x_1 - 5x_2 \end{cases}$$

The singularity is a unstable node at the origin.

Example 2:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 4x_2$$

The characteristic equation is given by:

$$\begin{vmatrix} 0-\lambda & +1 \\ -4 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = 0$$
$$\therefore \lambda_1 = -2 \qquad \lambda_2 = -2$$

The roots have the same sign, are real and negative.

The singularity is a stable node. The solution is of the form:

$$x_1 = Ae^{-2t} + Bte^{-2t}$$
$$x_2 = -2Ae^{-2t} - 2Bte^{-2t}$$

The different trajectories, for different initial conditions, and the eigenvectors are shown in Figure 2.

Example 3:

$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = x_2$$

The characteristic equation is given by:

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$
$$\therefore \lambda_{1,2} = \pm 1$$

The roots are real and of opposite sign, therefore the singularity is a saddle. The phase portrait of the system is shown in Figure 3.

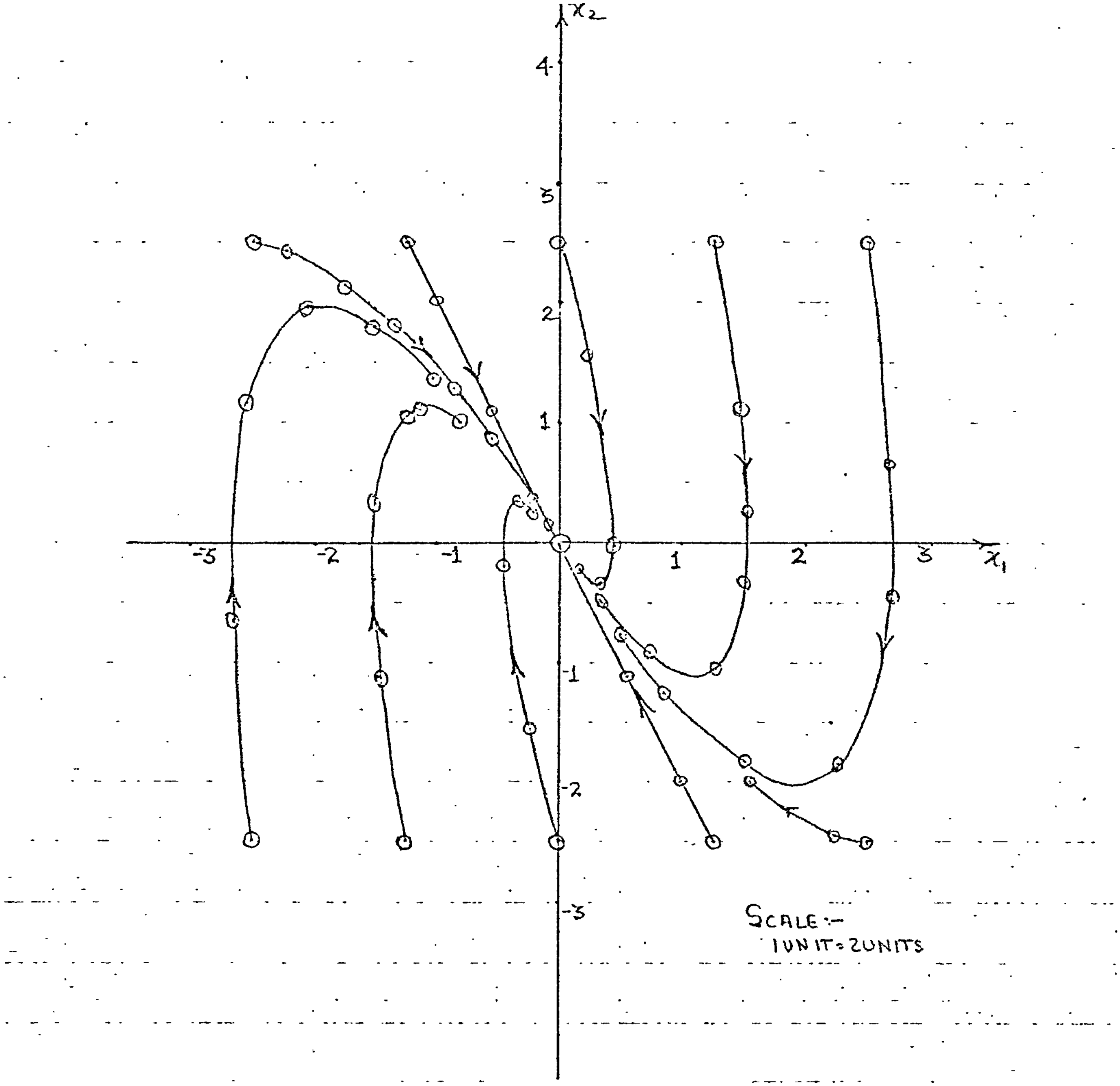

Fig. 2 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4x_1 - 4x_2 \end{cases}$$

The singularity is a stable node at the origin.

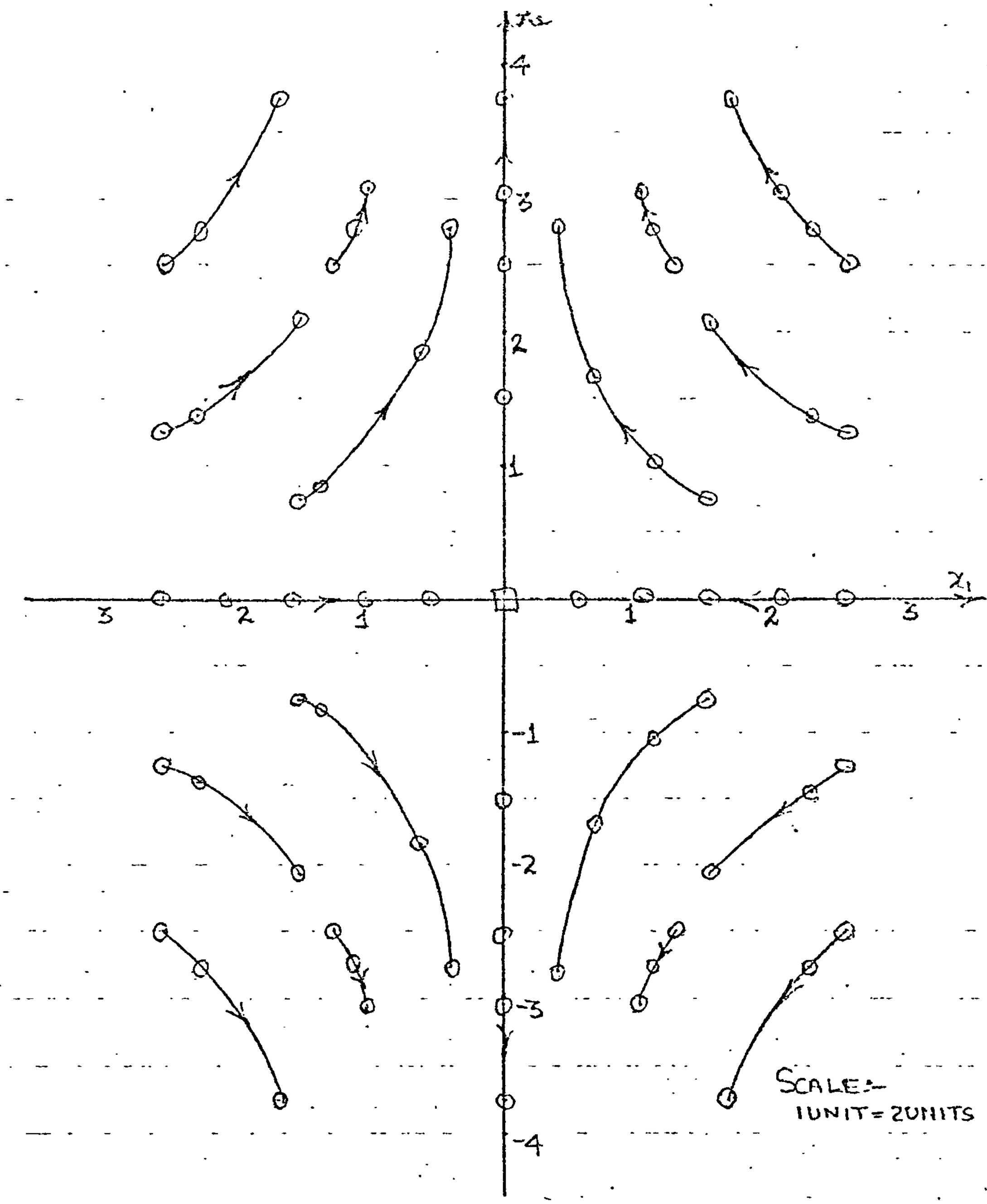

Fig. 3 Trajectories of

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 \end{cases}$$

The singularity is a saddle at the origin.

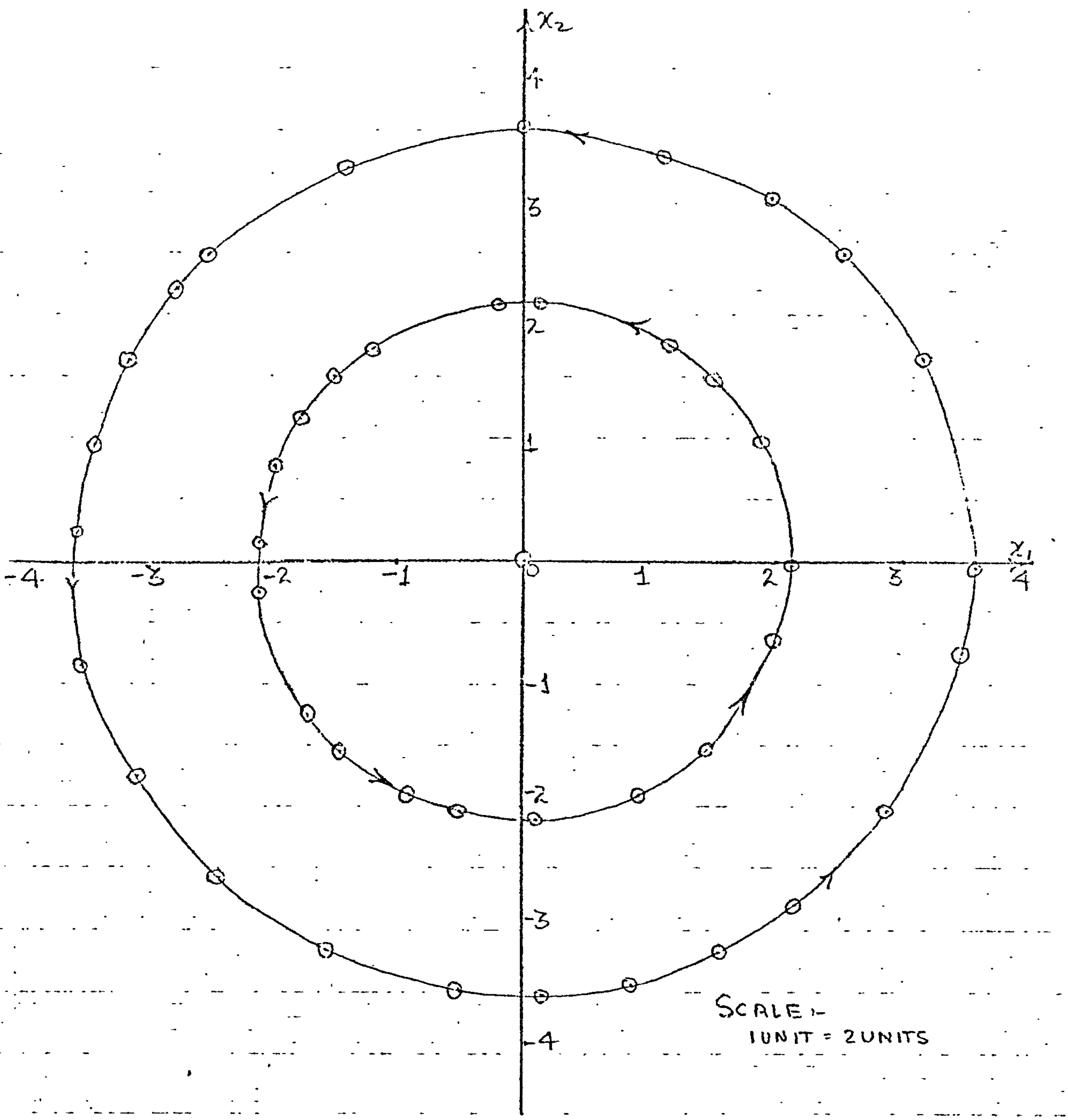

Fig. 4 Trajectories of

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases}$$

The singularity is a center at the origin.

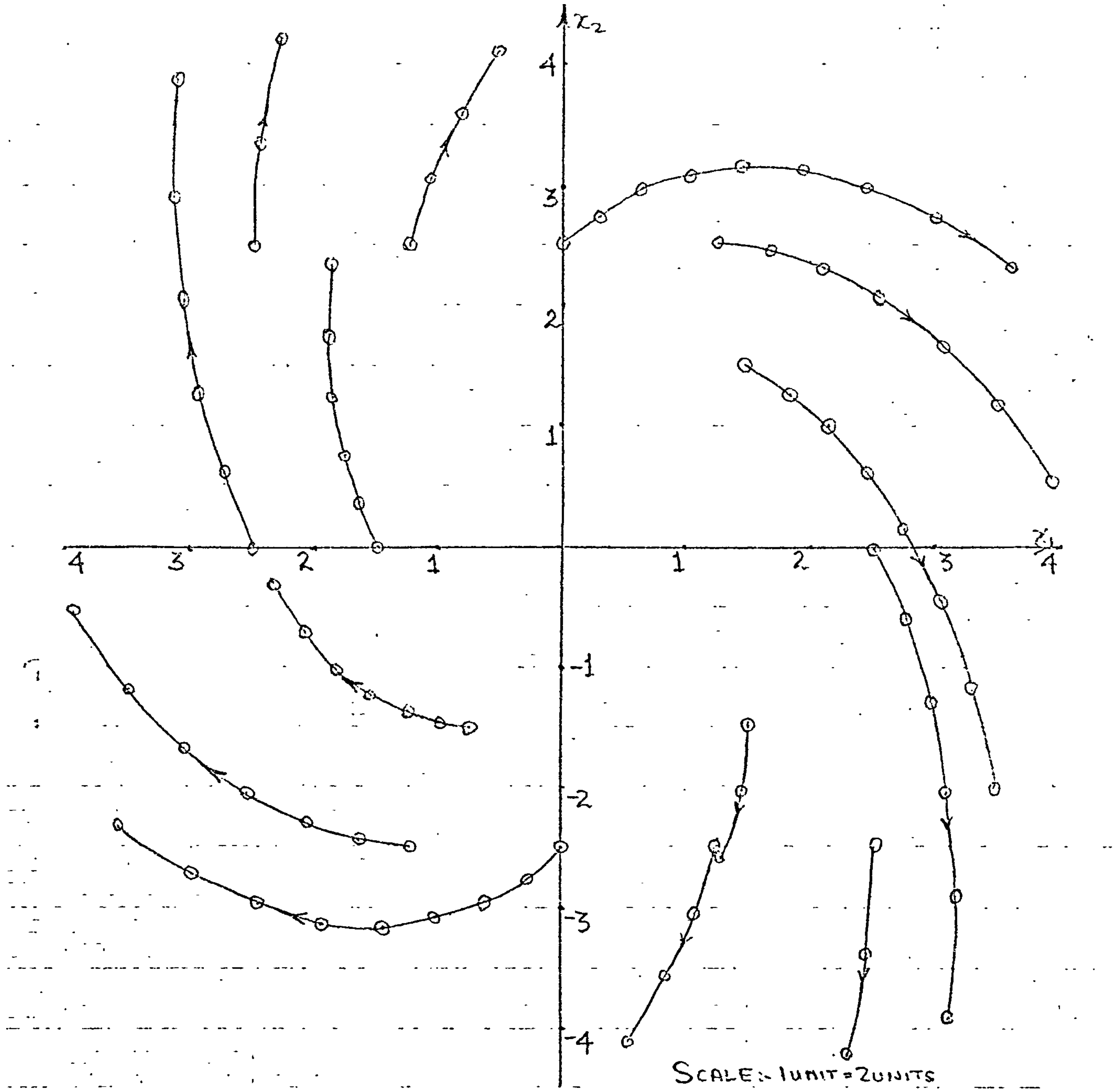

Fig. 5 Trajectories of

$$\begin{cases} \dot{x}_1 = x_1 + x_2 \\ \dot{x}_2 = -2x_1 + x_2 \end{cases}$$

The singularity is an unstable focus at the origin.

Example 4:

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1$$

The characteristic equation is given by:

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_1 = j1$$

$$\lambda_2 = -j1$$

The roots are purely imaginary and conjugate, therefore the singularity is defined as a center or vortex.

Example 5:

The characteristic equation is given by:

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

or $\lambda 2 - 2\lambda + 2 = 0$

$\therefore \lambda_1 = 1 + j1$

and $\lambda_2 = 1 - j1$

The roots are complex conjugates and the real part is positive, therefore the singularity according to Poincare's classification is an unstable focus.

The various trajectories for different initial conditions are plotted in Fig. 5

Example 6:

Consider the system represented by:

$$\dot{x}_1 = -3x_1 - 5x_2$$
$$\dot{x}_2 = x_1 + x_2$$

The characteristic equation is given by:

$$\begin{vmatrix} -3-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$
$$\therefore \lambda_1 = (-1 + j1),\ \lambda_2 = (-1 - j1)$$

The roots are again complex conjugate, only the real part is negative, therefore the singularity is a stable focus. The phase plot is shown in Fig. 6.

(4) Singularities of Nonlinear Systems

Consider the following equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^2 - x_2 \qquad \text{...............(8)}$$

The singularities are obtained by the usual method of forcing the rate variables to zero. Thus:

$$0 = x_2$$
$$0 = x_1 - x_1^2 - x_2$$

Therefore singularities are (0,0) and (1,0). Since we cannot find the eigenvalues in the conventional sense, the problem arises as to the classification of the singularity.

However when the phase plot is obtained by a numerical solution of the above equations by the well known

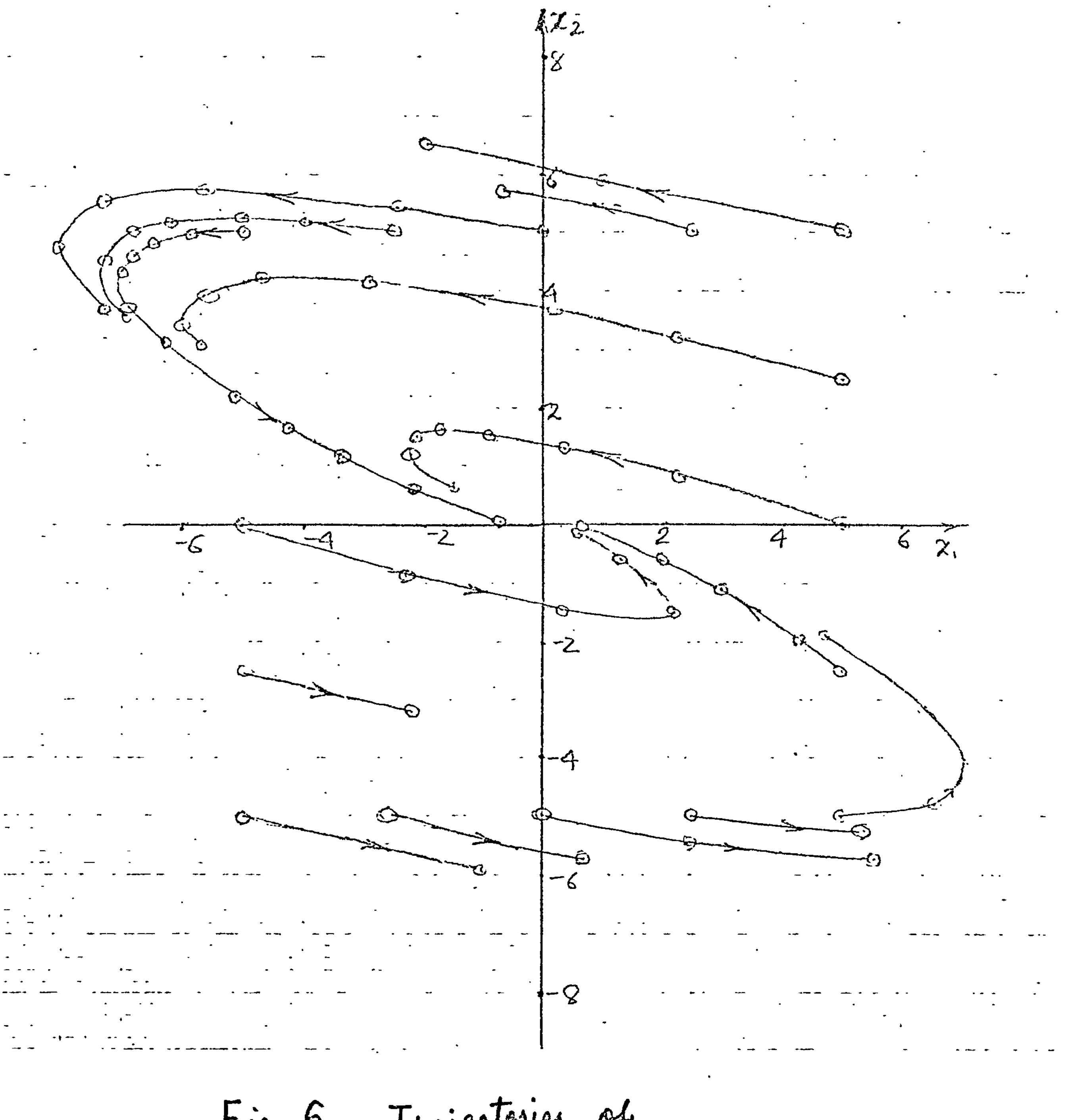

Fig. 6 Trajectories of

$$\begin{cases} \dot{x}_1 = -3x_1 - 5x_2 \\ \dot{x}_2 = x_1 + x_2 \end{cases}$$

The singularity is a stable focus at the origin.

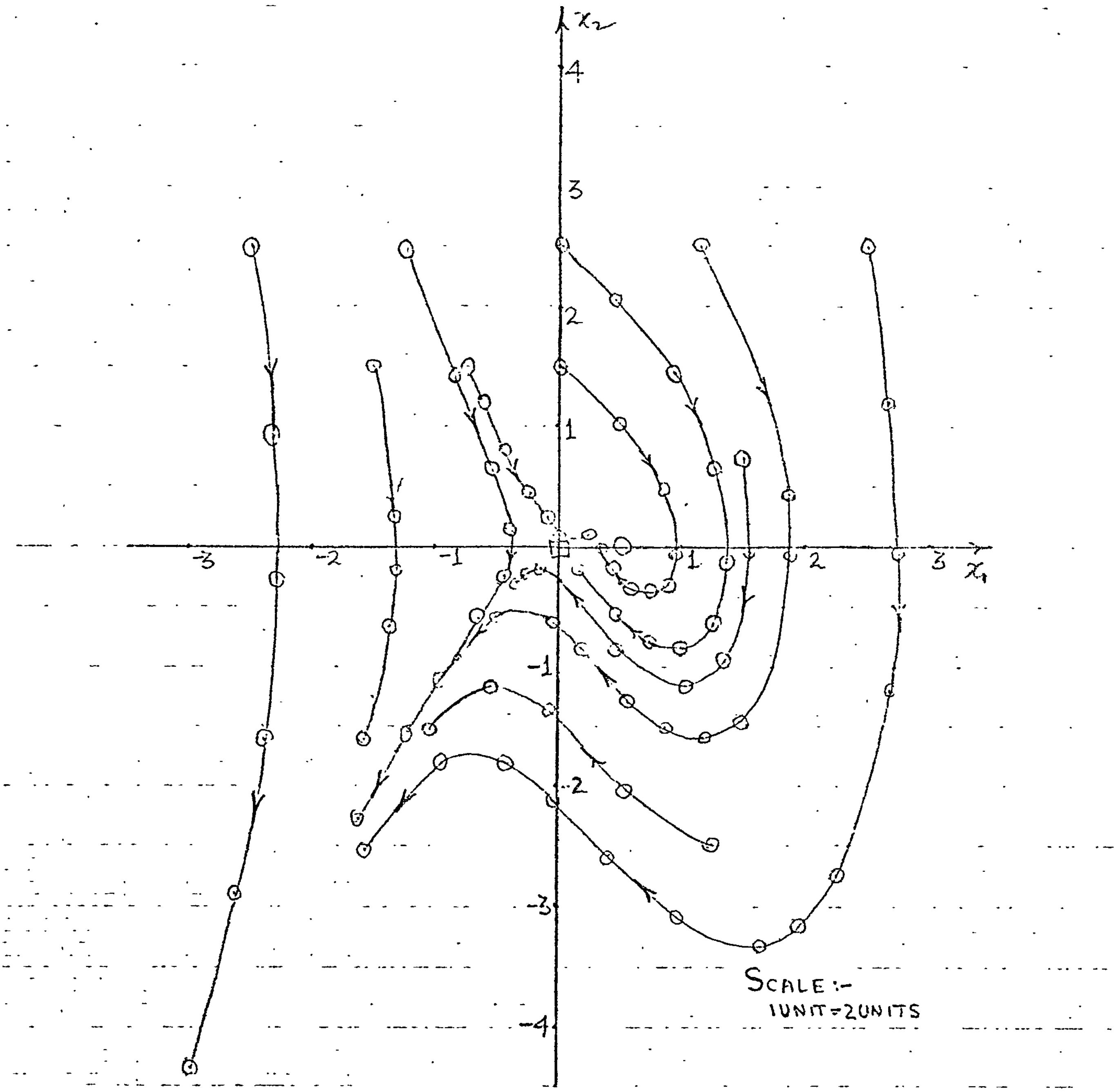

Fig. 7 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_1^2 - x_2 \end{cases}$$

The singularities are at (1) saddle point at (0,0)
(2) stable focus at (1,0)

Runge Kutta method, the trajectories close to (0,0) very closely resemble a saddle. (Fig. 3) and the trajectories close to (1,0) are very much like that of a stable focus, (Fig. 6) The phase plot is in Fig. 7.

This is not entirely unexpected as Liapunov pointed out that in the small, the linear terms approximate this behavior as of the system. Therefore we will henceforth use the linear approximation to define nonlinear singularities.

CHAPTER III

LINEARIZATION

It has been mentioned in Chapter II that the system behaviors for nonlinear system is approximated by the linear terms near the singularity. However, before proceeding to a study of stability of nonlinear control systems, the general approach to linearization of a control system must be considered. This is especially important as the ideas suggested for phase partition stem from this general concept.

Let the nonlinear system be represented by the space equations:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2 \ldots\ldots x_n)\\ \dot{x}_2 &= f_2(x_1, x_2 \ldots\ldots x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, x_2 \ldots\ldots x_n)\end{aligned}$$

Then the linear model can be obtained by using the Jacobian matrix:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & & \vdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{bmatrix}\Bigg|_{(x_1, x_2 \ldots x_n) \to s} \begin{bmatrix}x_1\\ \vdots\\ x_2\\ \vdots\\ x_n\end{bmatrix}$$

where "s" denotes the singularity point.

Restricting our attention to the second order case:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}\Bigg|_{x_1, x_2 \to s} \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Illustrative Example:

Take the nonlinear case in Chapter II (Equation 7) mentioned in connection with classification of singularites:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^2 - x_2$$

The linear models can be obtained by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-2x_1 & -1 \end{bmatrix}_{x_1, x_2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The singularities of the system were (0,0) and (1,0).

Therefore the two linear models governing the system behaviors are:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{near the singularity (0,0)...(8)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{near the singularity (1,0)...(9)}$$

The characteristic equation of the system is given by:

$$\begin{bmatrix} -\lambda & -1 \\ 1-2x_1 & -1-\lambda \end{bmatrix} = \lambda^2 + \lambda - (1-2x_1) = 0 \quad \text{...............(10)}$$

It must be pointed out that, strictly speaking, this is valid near the singular point only.

The characteristic equation of (8) and (9) can be obtained by substituting the singularities in (10), which gives:

$$\lambda^2 + \lambda - 1 = 0 \quad \text{...............(11)}$$

which indicates a saddle point and:

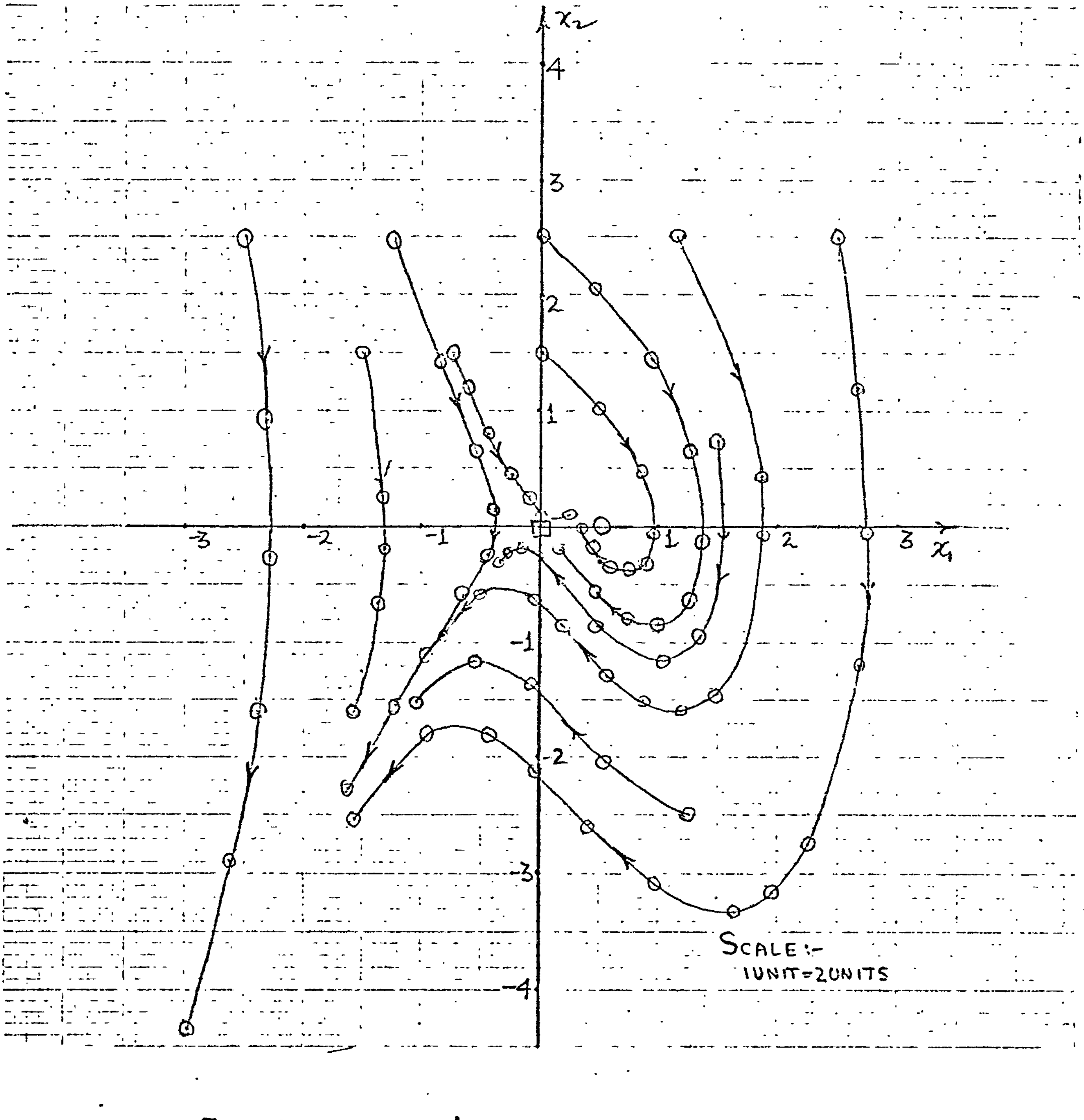

Fig. 8 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - x_1^2 - x_2 \end{cases}$$

The singularities are at (1) saddle point at (0,0)
(2) stable focus at (1,0)

$$\lambda^2 + \lambda + 1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots (12)$$

which indicates a focal point.

Thus the nonlinear system has two singularities, a saddle point at (0,0) and a stable focal point at (1,0). The behavior near these points are approximated by the linear models (8) and (9).

This brings us to the interesting question as to whether there is a way of easily determining an area of asymptotic stability around the stable focal point. It is on this problem that this research concentrates.

CHAPTER IV

AN INVESTIGATION INTO STABILITY OF NONLINEAR CONTROL SYSTEMS

In Chung's thesis,[9] it was pointed out that, on occasion, the area dominated by a stable singularity determined by the discriminant criterion or his generalized Hurwitz criteria was one of asymptotic stability. It will now be investigated whether this is true for any significant number of control systems and if so, whether there is any scope for generalization to a class of functions.

Attention here will be concentrated on the simple control systems, ie., those of the second order containing one or two singularities.

1. Stability: The stability of a nonlinear control system is a more complex matter than linear systems and a number of definitions (more than 28)[1] have been used in the literature. However, the definitions we will be concerned with will be the following:[1]

> <u>Asymptotic Stability</u>: If for any initial conditions within the region under consideration the state point approaches arbitrarily close to the singularity as time approaches infinity, the system is said to exhibit asymptotic stability for that region.

Local Stability: or stability in the small strictly applies only in the infinite small region about a singular point.

The first degree approximation yields information for this only, rigorously speaking.

Global Stability: This refers to the entire finite region of the state space.

Finite Stability: It lies in between global and local stability and applies to a finite region of the state space.

Stability, local, global, or finite does not rule out limit cycles but only asymptotic local, global and finite stabilities will be dealt with here.

2. Generalized Routh Hurwitz and Discriminant Criterion

Both the above criterion try to partition phase space into single singularity dominated regions, ie., regions in which all the trajectories are influenced by one singularity alone.

Any second order control system can be represented by the equations:

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

Then the Jacobian is

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial f_2} \end{vmatrix}$$

Therefore the characteristic equation is:

$$\left|\begin{matrix} \frac{\partial f_1}{\partial x_1}-\lambda & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}-\lambda \end{matrix}\right|_{(x_1,x_2)} = 0$$

or

$$\lambda^2 - \lambda\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) - \frac{\partial f_1}{\partial x_2}\cdot\frac{\partial f_2}{\partial x_1} = 0 \quad \ldots\ldots(1)$$

(a) According to the discriminant criterion the boundary is determined by the equation:

$$\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)^2 + 4\,\frac{\partial f_1}{\partial x_2}\cdot\frac{\partial f_2}{\partial x_1} = 0$$

This gets more and more difficult to use as the system gets complex. For more than two singularities or for a higher order system it becomes unworkable. Even in the two singularity case it gives the boundary between a center dominated region, or between a focus dominated region and a saddle dominated region, but not between stable and unstable focus dominated regions. The limit cycle, if any, is also not confirmed or ruled out.[1]

(b) The Generalized Routh Hurwitz criterion suggested by Chung[9] is that the following equations determine the partition:

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial f_2} = 0 \quad \ldots\ldots\ldots(2)$$

$$\frac{\partial f_1}{\partial f_2}\cdot\frac{\partial f_2}{\partial x_1} = 0 \quad \ldots\ldots\ldots(3)$$

The boundary partition is actually given by equation (3) and for a certain broad class of functions as demonstrated by Ku and Shen's[10] more rigorous mathematical treatment which will be taken up in the next section.

3. Ku and Shen's Partition of Phase Space:[10]

They established that just as there is separation property for poles and zeros in the complex plane, there is a separation property of singularities in the phase plane in that the focal points, centers, or nodal points alternate with saddle points. Also for systems represented by the following equations is defined in the following way:

If $\dot{x}_1 = x_2$

and $\dot{x}_2 = -A(x_2)_n - B(x_1)$

where $A(x_2) = c_1 x_2 \left(\frac{1 + x_2^2}{a_i^2}\right)$

a real number and $\longrightarrow a_i > 0 \ (i = 1, 2 \ldots n)$

and $B(x_1)$ assumes one of the forms:

$$B(x_1) = C_2 x_1 \prod_{i=1}^{n} \left(1 - \frac{x_1}{b_i}\right), \quad b_i \neq b_j \neq 0$$

$$B(x_1) = C_2 \sin X_1$$

$$B(x_1) = C_2 \sin X_1, \prod_{i=1}^{n} \left(1 - \frac{\sin X_1}{\sin Y_1}\right)$$

when C_2 is a real number $\neq 0$, Sin = Sin$y \neq 0$.

Then the boundary is given by:

$$\frac{d}{dx_1} B(x_1) = 0$$

Separation of singularity regions leads to a unified treatment of the singularites of the second kind, a fact that will be of no concern here, as only simple singularity situations are dealt with.

So Chung's thesis[9] is only a less general variation of Ku and Shen's work.[10]

Now the structure of the entire phase plane is determined by putting the regions together. The transition of a phase trajectory from the region of one singularity to the other may be interpreted as a jumping phenomena in a fictitious singular point associated with the trajectory at the boundary point of the two regions.

Chung showed that this has scope for higher order systems also. This or the more restricted discriminant criterion can lead to multilinear models for nonlinear systems, but , however, the point of interest here is the region of asymptotic stability. Obviously whenever a trajectory from a stable singularity region hits the boundary of the unstable region (separation property), then whether it will return to the first singularity region (double singularity case) depends on the orientation of the second singularity. So a number of cases will be studied to see whether any criterion does give finite asymptotic stability for a defined class of functions.

It must be pointed out that neither method rules out limit cycles and Bendixson's criterion is a sufficient but not necessary condition for their existence.

4. Single Singularity Systems:

The determination of stability of single singularity control systems turns out to be quite simple. The nature of the singularity controls the behaviors of the system and since it is the only singularity, local stability turns out to be global stability. Thus stable node, stable focus systems are globally stable while unstable node, unstable focus and saddle systems are globally unstable.

Example 1

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -5x_2 - \tfrac{5}{3}x_2^3 - x_1$$

The singularity is at (0,0)

The Jacobian is given by:

$$\begin{bmatrix} 0 & 1 \\ 1 & -5-5x_2^2 \end{bmatrix}$$

Therefore the characteristic equation is:

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & -5-5x_2^2-\lambda \end{bmatrix} = 0 \qquad x_1, x_2 \rightarrow s$$

or $\lambda^2 + (5x_2^2 + 5)\lambda + 1 = 0$

At (0,0)

$$\lambda^2 + 5\lambda + 1 = 0$$

or $\lambda = \dfrac{-5 \pm \sqrt{21}}{2}$

The singularity is a stable node.

Therefore the system is globally stable.

This could be verified by the phase plot obtained by numerical Runge Kutta method or Liapunov's direct method, the first one being preferred because it will be used in all the other cases. The plot is shown in Fig. 9.

The discriminant criterion suggests:

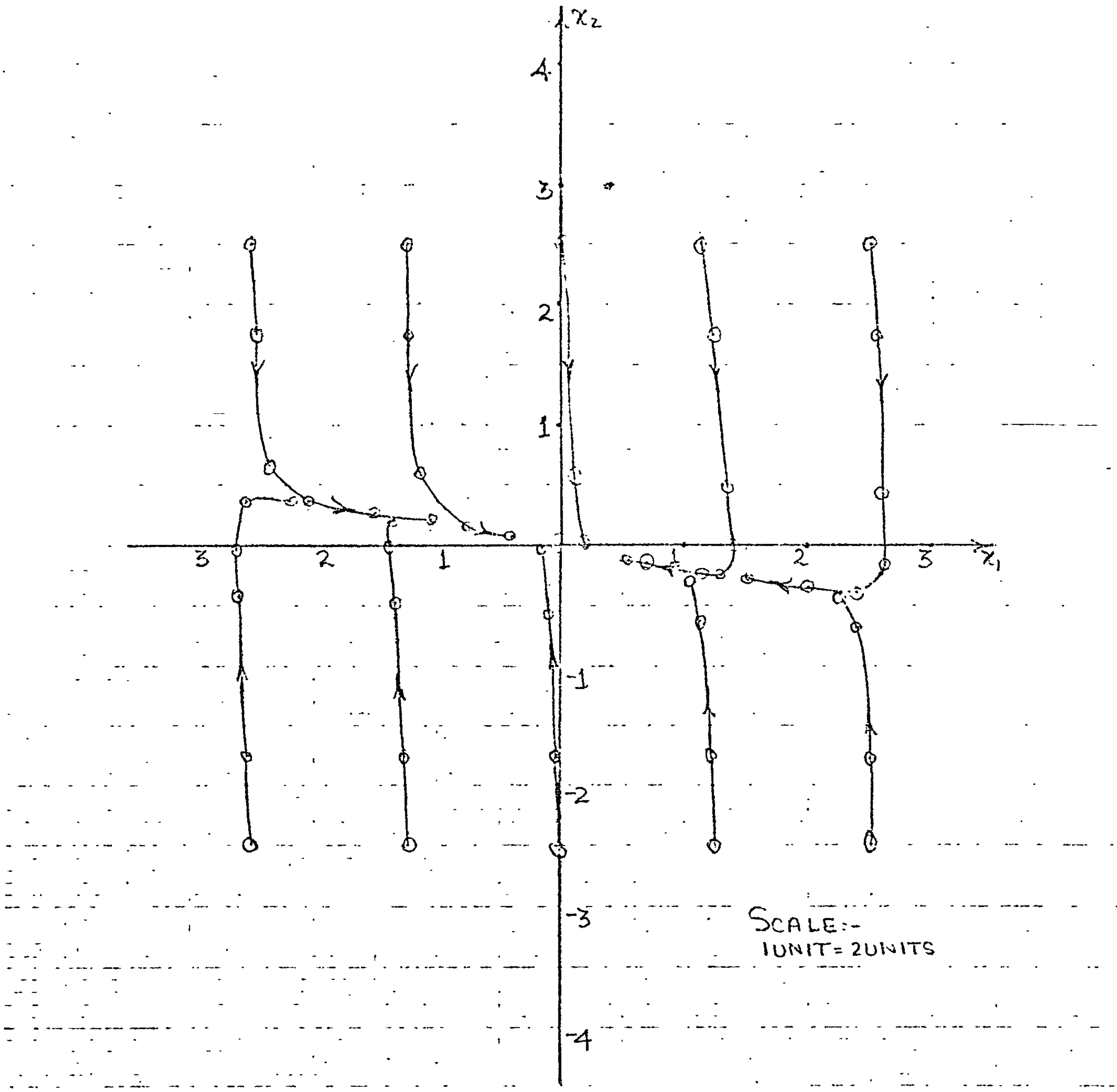

Fig. 9 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -5x_2 - \frac{5}{3}x_2^3 - x_1 \end{cases}$$

The single singularity is a stable node at the origin.

$$(5x_2^2+5)-4=0$$
$$25x_2^4+25+50x_2^2-21=0$$

OR $25x_2^4+50x_2^2+21 > 0$ as the boundary line but $25x_2^4+50x_2^2+21 > 0$ always.

Therefore no boundary exists.

Therefore local stability implies global stability.

The Generalized Hurwitz Criterion suggests:

$$5x_2^2+5 > 0$$
$$1 > 0$$

which are both true irrespective of x_1 and x_2

Therefore system is globally stable.

Example 2

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -0.5x_2 - 0.166x_2^3 - x_1$$

The singularity is at (0,0)

The Jacobian is given by:

$$\begin{vmatrix} 0 & 1 \\ -1 & -0.5-.498x_2^2 \end{vmatrix}$$

The characteristic equation is:

$$\lambda^2+\lambda(0.5+.498x_2^2)+1 \Big|_{0,0} = 0$$

Therefore at singularity:

$$\lambda^2+\lambda(0.5)+1=0$$

or $\lambda = \dfrac{-0.5 \pm \sqrt{.25-4}}{2}$

Therefore singularity is a stable focus.

System is globally asymptotically stable. Fig. 10

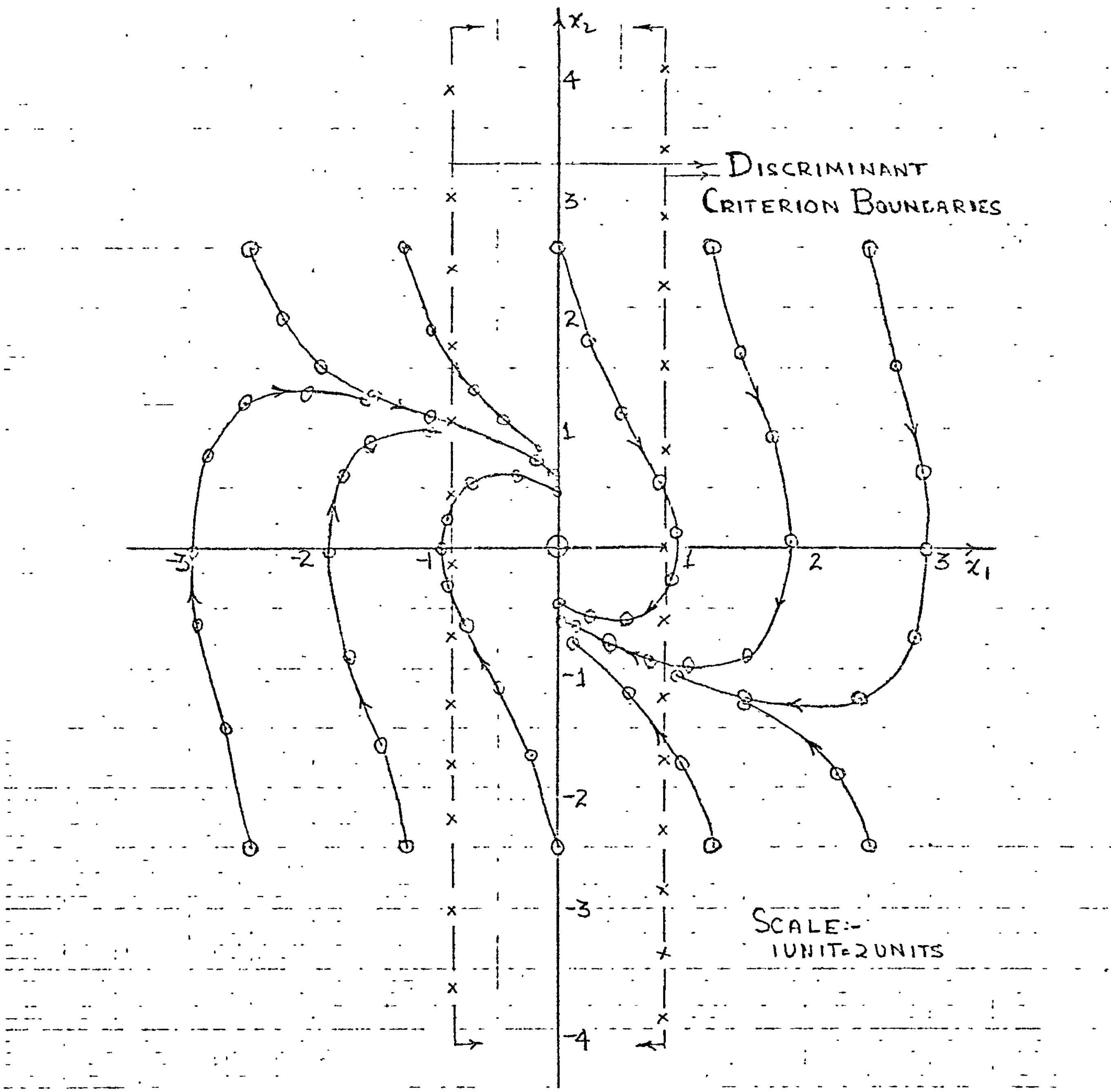

Fig 10 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -0.5\,x_2 - 0.166\,x_2^3 - x_1 \end{cases}$$

The single singularity is a stable focus at the origin.

shows the phase plot.

The discriminant criterion suggest the boundary:

$$(0.5 + .498x_2^2)^2 - 4 = 0$$

or

$$.25 + (.498)^2 x_2^4 + .498 x_2^2 - 4 = 0$$ which exists.

Therefore Discriminant gives a portion of the region that is asymptotically stable.

The Routh Hurwitz suggests the criterion of:

$$.05 + .498 x_2^2 > 0$$
$$1 > 0$$

both of which are automatically fulfilled.

Therefore it gives global asymptotic stability.

It might be pointed out that both of the above examples originate from Rayleigh's well known equation:[14]

$$\ddot{x}_1 + K\dot{x}_1 + m\dot{x}_1^3 + n^2 x_1 = 0$$

Example 3

$$\dot{x}_1 = -x_2 - x_1^3$$
$$\dot{x}_2 = +x_1 - x_2$$

The singularity is at the origin.

The characteristic equation is:

$$\left.\begin{bmatrix} -\lambda - 3x_1^2 & -1 \\ 1 & -\lambda - 1 \end{bmatrix}\right|_{0,0} = 0$$

or $\lambda^2 + (1 + 3x_1^2)\lambda + 3x_1^2 + 1 = 0$

At the singularity:

$$\lambda^2 + \lambda + 1 = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

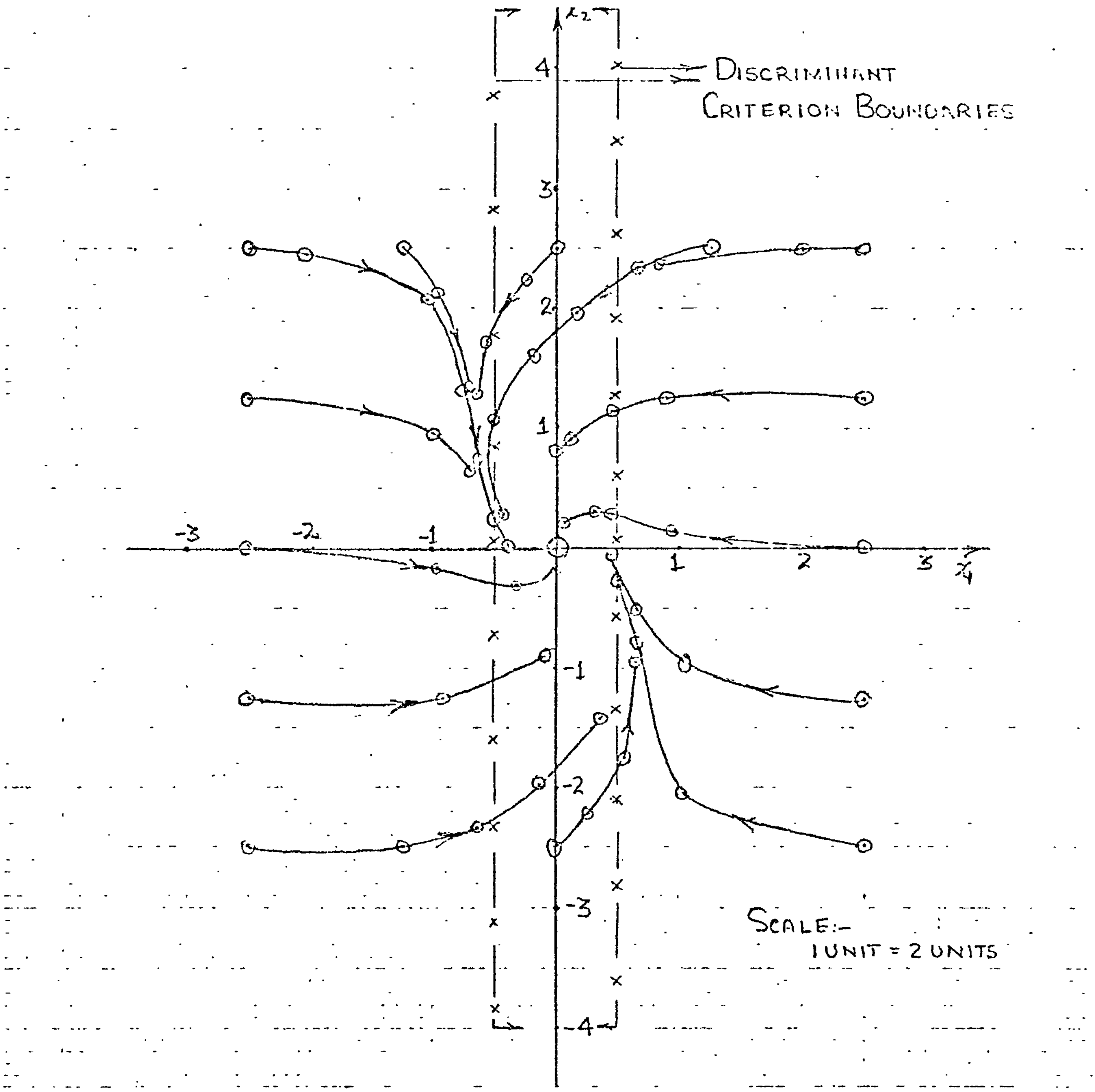

Fig. 11 Trajectories of

$$\begin{cases} \dot{x}_1 = -x_2 - x_1^3 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$$

The single singularity is a stable focus at the origin.

Therefore the singularity is a stable focus.

The system is globally stable.

Phase plot is shown in diagram 11.

Discriminant criterion suggests the boundary:

$$(1+3x_1^2)^2 - 4(3x_1^2+1) = 0$$

$$(1+3x_1^2)(1+3x_1^2-4) = 0$$

or

$$x_1^2 = 1$$ which exists.

Therefore it gives only a portion of the area of asymptotic stability.

Generalized Hurwitz criterion evidently also gives global asymptotoic stability. It must be added that the singularity at the origin is not a special case as translation of axis leaves the system stability obviously undisturbed.

5. Double Singularity Control Systems

Since the interest lies in asymptotic stability, the center or vortex will not be considered. As it is it never occurs in physical systems except as a limiting case. Restricting the investigation to saddles, nodes, and foci, it is also evident that two stable singularities or two unstable singularities cannot occur because of the separation property.

(c) Saddle-Node Systems

Example 4

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - x_2^2 - 2x_2$

The singularities are (0,0) and (-1,0).

The characteristic equation is:

$$\begin{vmatrix} 0-\lambda & 1 \\ -1-2x_1 & -2-\lambda \end{vmatrix} = 0$$

or $\lambda^2 + 2\lambda + (1 + 2x_1) = 0$

For (0,0) it becomes:

$\lambda_1^2 + 2\lambda + 1 = 0$

or $\lambda_1 = -1, -1$

(0,0) is a stable node.

For (-1,0) it becomes:

$\lambda^2 + 2\lambda - 1 = 0$

or

$$\lambda = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2} \doteq .414, -2.414$$

The singularity is a saddle.

Discriminant criterion:

$4 - 4(1 + 2x_1) = 0$ OR $x_1 = 0$

Generalized Routh ~~Hurwitz~~ Criterion:

$1 + 2x_1 > 0$ OR $x_1 > -1/2$

Both criterion delimited stable singularity regions have positions of instability. (Fig. 12)

Example 5

$\dot{x}_1 = -5x_1 - x_2$

$\dot{x}_2 = x_2 + x_1x_2$

Singularities are (0,0) and (-1,5).

Characteristic equation is given by:

$$\begin{vmatrix} -5-\lambda & -1 \\ x_2 & -\lambda+1+x_1 \end{vmatrix} = 0$$

OR $\lambda^2 + \lambda(5 - 1 - x_1) - 5 - 5x_1 + x_2 = 0$

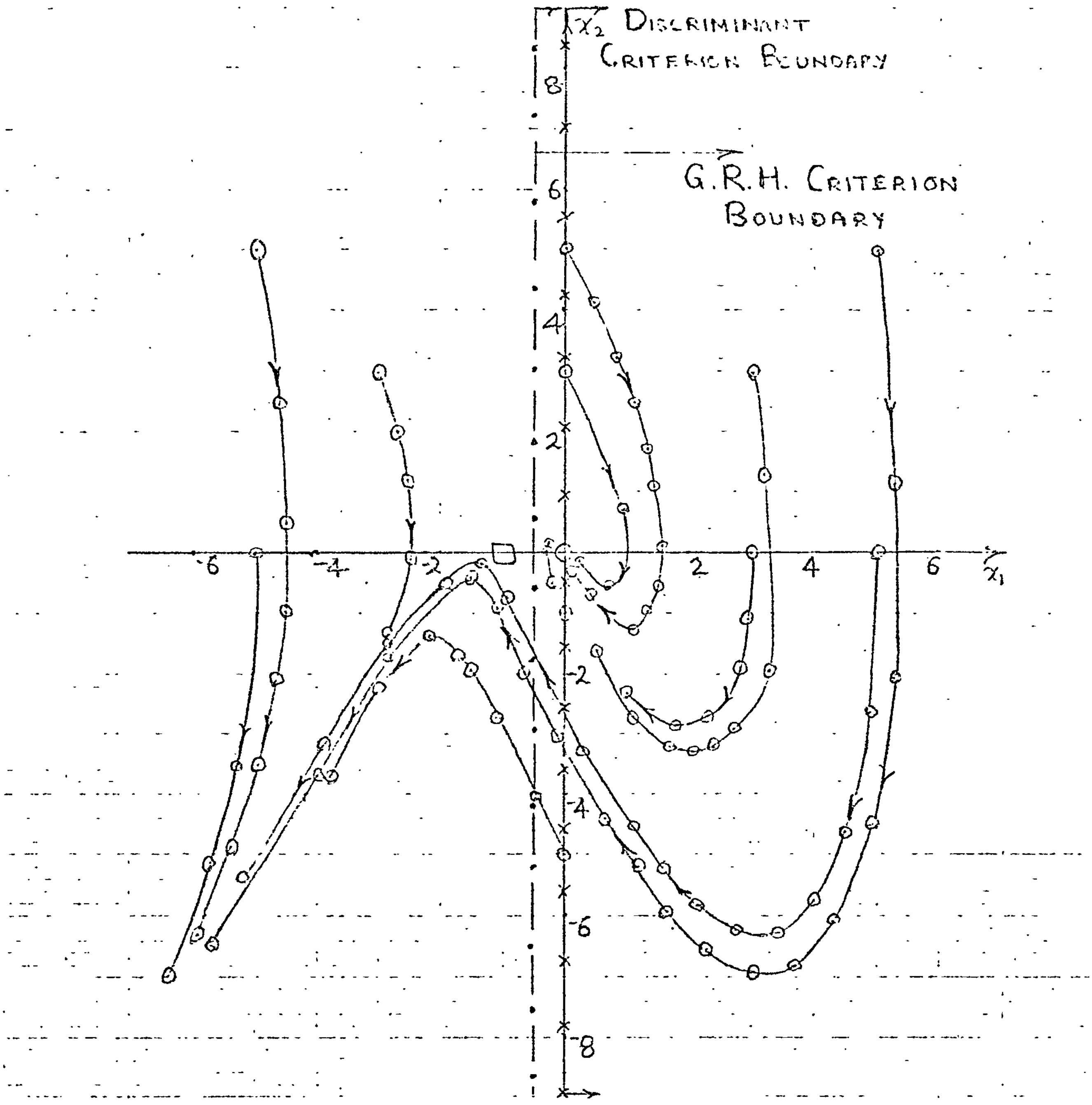

Fig. 12 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1^2 - 2x_2 \end{cases}$$

The singularities are at (1) saddle point at (-1,0)
(2) stable node at (0,0)

For (0,0)

$$\lambda^2 + \lambda(4) - 5 = 0$$

or

$$\lambda = -1, -5 \longrightarrow \text{saddle}$$

For (-1,5)

$$\lambda^2 + 5\lambda + 5 = 0$$

or

$$\lambda = \frac{-5 \pm \sqrt{25-20}}{2} \longrightarrow \text{stable node}$$

Discriminant Criterion

$$(4 - x_1)^2 - 4(x_2 - 5x_1 - 5) = 0$$
$$x_1^2 + 16 - 8x_1 - 4x_2 + 20x_2 + 20 = 0$$
$$x^2 + 12x_1 + 36 - 4x_2 = 0$$
$$\text{or} \quad (x_1 + 6)^2 - 4x_2 = 0$$

which is a parabola and does give the area of stability.

Generalized Routh Hurwitz Criterion

$$4 - x_1 > 0 \quad \text{or} \quad x_1 < 4$$
$$\text{and} \quad x_2 - 5x_1 - 5 > 0$$

which contains portions which are unstable.

The trajectories for the state space are plotted in Fig. 13 along with the boundary predicted by the above two methods.

Example 6

$$\dot{x}_1 = -3x_1 - x_2$$
$$\dot{x}_2 = x_1 + x_1 x_2$$

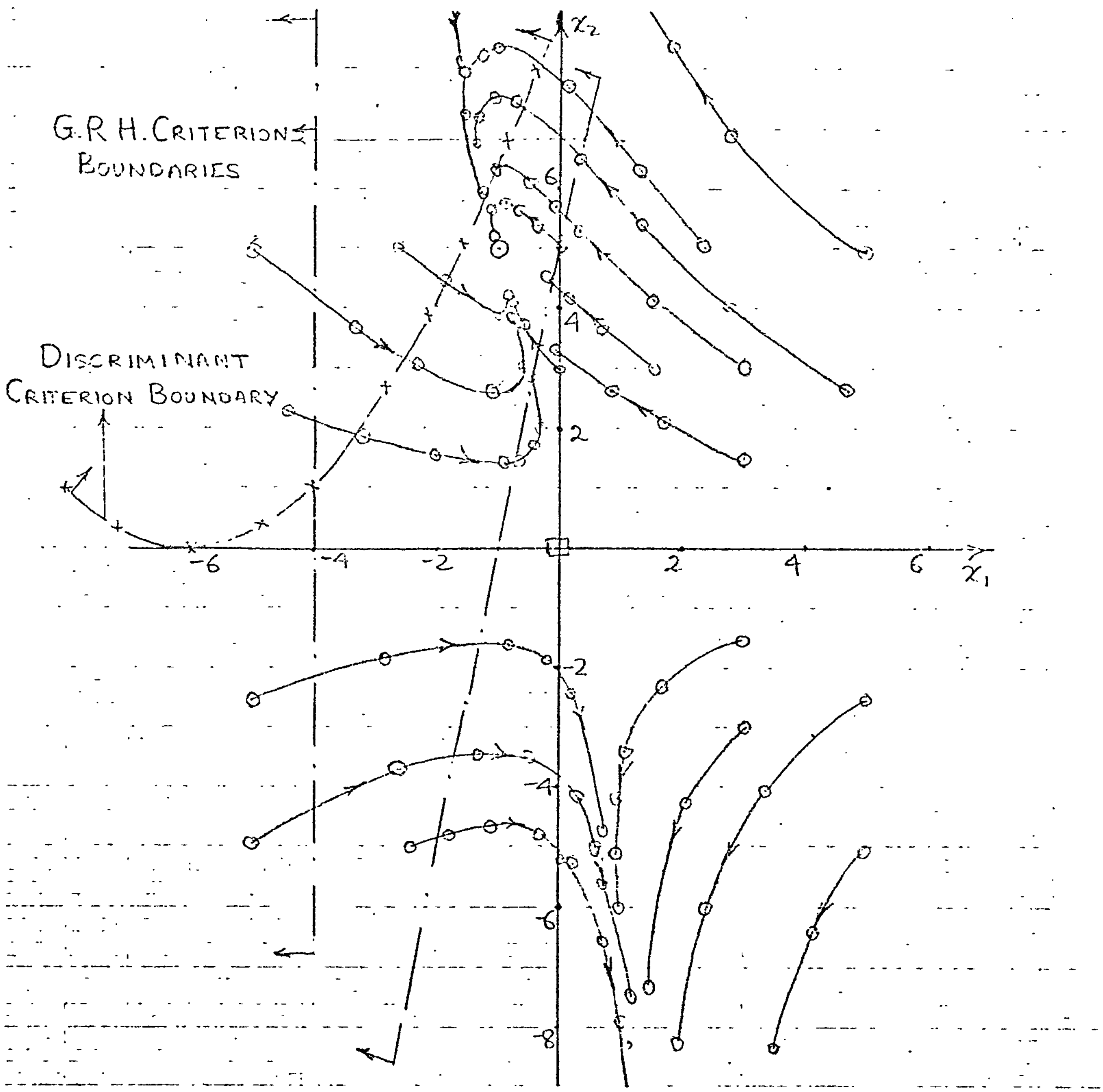

Fig. 13 Trajectories of

$$\begin{cases} \dot{x}_1 = -5x_1 - x_2 \\ \dot{x}_2 = x_2 + x_1 x_2 \end{cases}$$

The singularities are at (1) saddle point at (0,0)
(2) stable node at (-1,5)

Singularities are (0,0) and ($\frac{1}{3}$,-1).

Characteristic equation is given by:

$$\begin{vmatrix} -3-\lambda & 1 \\ 1+x_2 & x_1-\lambda \end{vmatrix} = 0$$

$$\text{OR } \lambda^2+\lambda(3-x_1)+(1+x_2-3x_1)=0$$

For (0,0):

$$\lambda^2+3\lambda+1=0$$

$$\lambda = \frac{-3 \pm \sqrt{9-4}}{2} \longrightarrow \text{Stable node}$$

For ($\frac{1}{3}$, -1):

$$\lambda^2+\lambda(3-1/3)+1-1-1=0$$

$$\text{OR } \lambda^2+8/3\,\lambda-1=0 \quad \text{OR } \lambda = \frac{-8/3 \pm \sqrt{64/9+9}}{2} \longrightarrow \text{saddle}$$

Discriminant Criterion

$$(3-x_1)^2-4(1+x_2-3x_1)=0$$

$$(x_1+3)^2-4(1+x_2)=0$$

which is a parabola and the area is asymptotically stable.

Generalized Routh Hurwitz Criterion

$$3-x_1>0 \quad \text{OR } x_1<3$$

$$\text{OR } 1+x_2-3x_1>0$$

All the area demarcated under the influence of stable node is not asymptotically stable. (Fig. 14)

Example 7

$$\dot{x}_1=x_2$$

$$\dot{x}_2=-x_1+x_1^2+x_2^2-2x_2$$

Singularities are (0,0) and (1,0)

Characteristic equation is given by:

$$\begin{vmatrix} 0-\lambda & 1 \\ -1+2x_1 & 2x_2-2-\lambda \end{vmatrix} = 0$$

$$\text{OR } \lambda^2-\lambda(2x_2-2)-(2x_1-1)=0$$

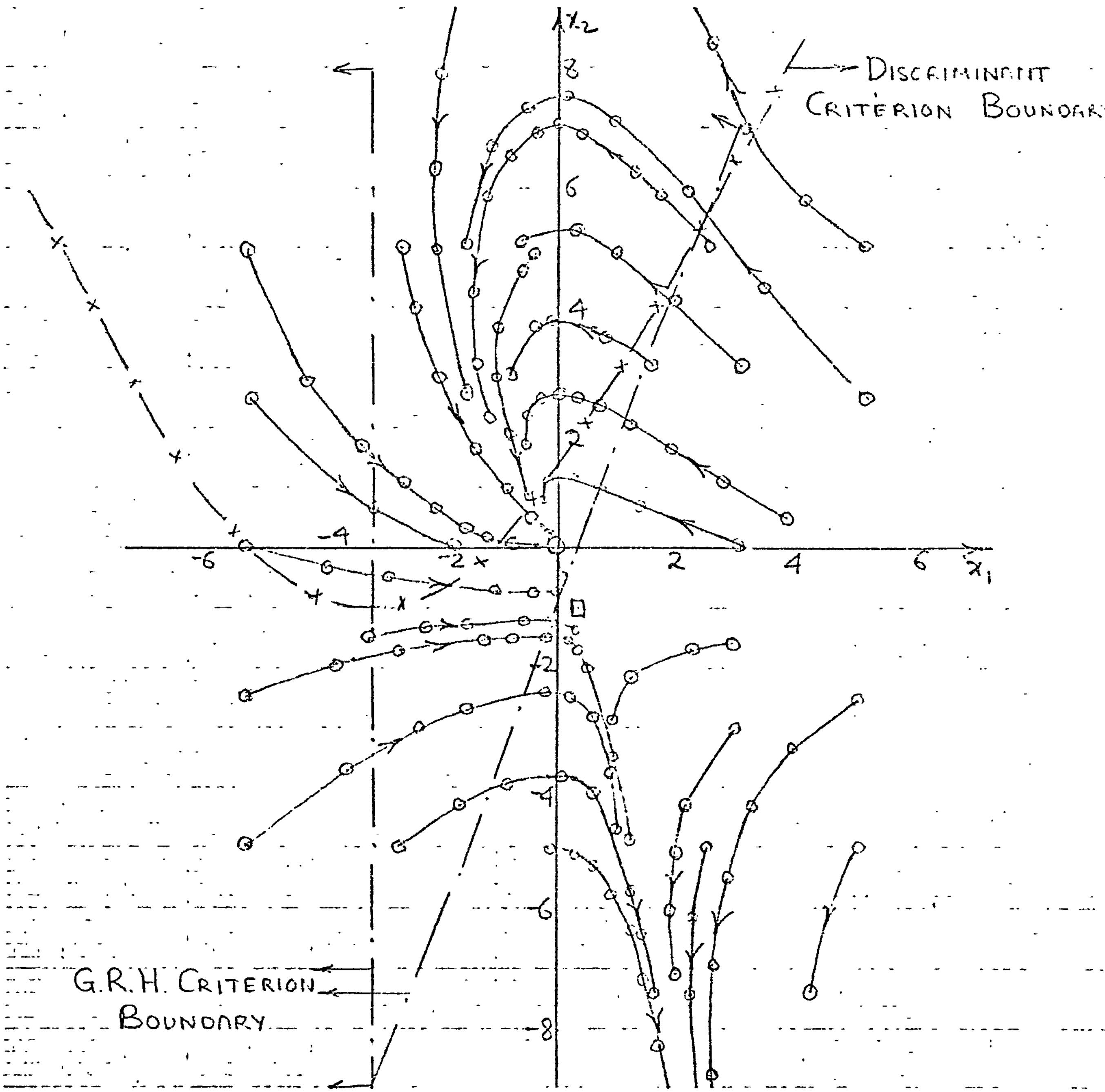

Fig. 14 Trajectories of

$$\begin{cases} \dot{x}_1 = -3x_1 - x_2 \\ \dot{x}_2 = x_1 + x_1 x_2 \end{cases}$$

The singularities are at (1) saddle point at $(\frac{1}{3}, -1)$
(2) stable node at $(0,0)$

At (0,0)

$\lambda^2+2\lambda+1=0$

$\lambda=-1,-1$ stable node

At (1,0)

$\lambda^2+2\lambda-1=0$

OR $\lambda=\frac{-2\pm\sqrt{4+4}}{2}=-1\pm\sqrt{2}\longrightarrow$ saddle

Discriminant Criterion

$$(2x_2-2)^2+4(2x_1-1)=0$$

which is a parabola and area bounded by it contains area of instability. (Fig. 15)

Generalized Routh Hurwitz Criterion

$-(2x_2-2)>0$ OR $x_2<1$

$2x_1-1<0$ OR $x_1<1/2$

Area contains portions of unstable region as shown in the phase plot. (Fig. 15)

Example 8

$\dot{x}_1=x_2$

$\dot{x}_2=x_1+x_1^2+4x_1x_2+2x_2+x_2^2$

Singularities are (0,0) and (-1,0)

Characteristic equation is given by:

$$\begin{vmatrix} -\lambda & 1 \\ 1+2x_1+4x_2 & 2+2x_2+4x_1-\lambda \end{vmatrix}=0$$

or $\lambda^2-\lambda(2+2x_2+4x)-(1-2x_1+4x_2)=0$

At (0,0)

$\lambda^2-2\lambda-1=0$

OR $\lambda=\frac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}\longrightarrow$ saddle

At (-1,0)

$\lambda^2-\lambda(2-4)-(1-2)=0$

OR $\lambda^2+2\lambda+1=0$ OR $\lambda=-1,-1\longrightarrow$ stable node

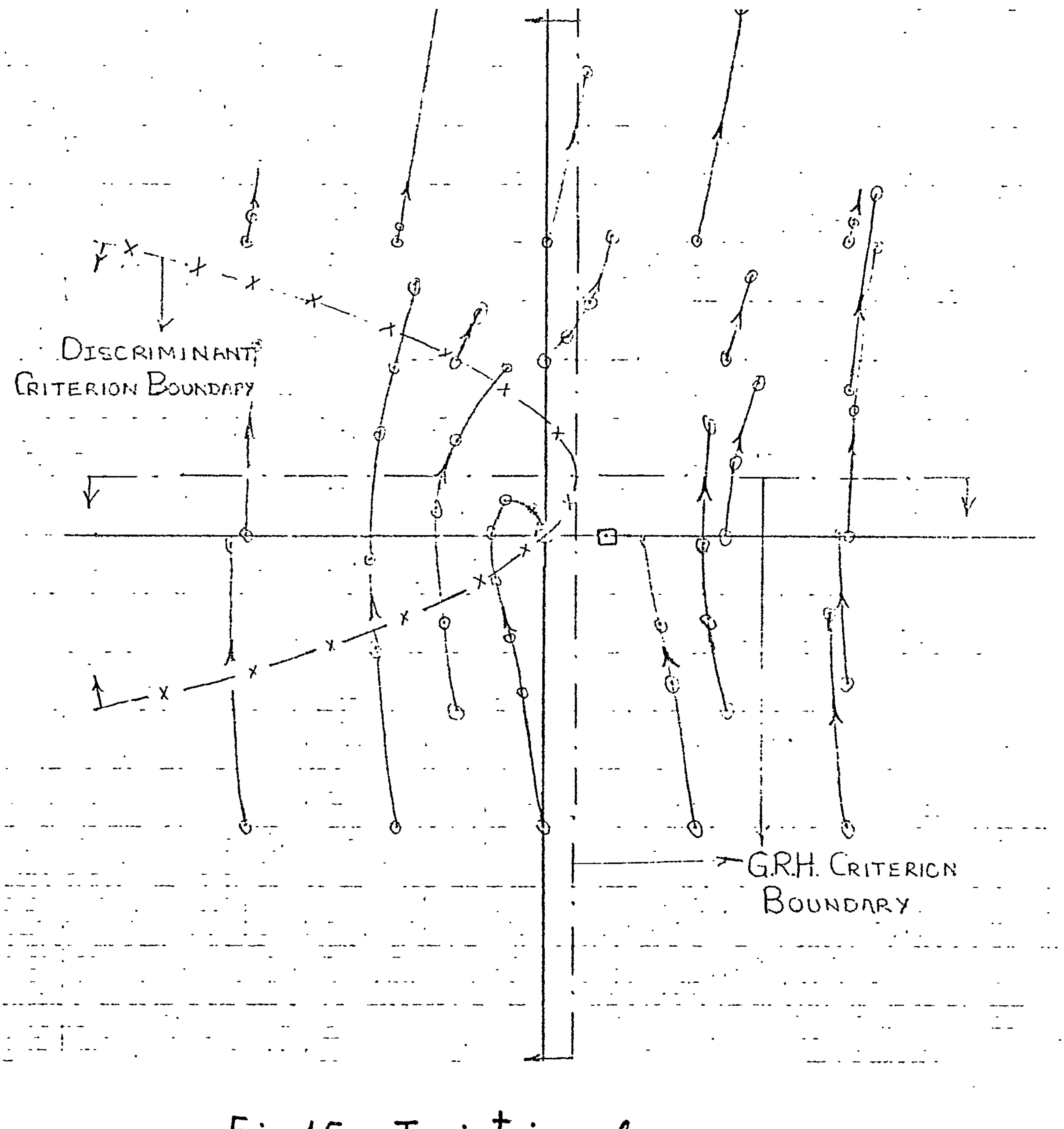

Fig. 15 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + x_1^2 + x_2^2 - 2x_2 \end{cases}$$

The singularities are at (1) saddle point at (1,0)
(2) stable node at (0,0)

Discriminant Criterion

$$(2+2x_2+4x_1)^2 - 4(1+2x_1+4x_2)=0$$
$$4x_2+16x_1^2+8x_2+16x_1+16x_1x_2-8x_1-16x_2=0$$
$$4x_2^2-8x_2+16x_1^2+8x_1+16x_1x_2=0$$

which is plotted on the phase plot and not all this area demarcated is asymptotically stable.

Generalized Routh Hurwitz Criterion

$$4x_1+2x_2+2 < 0$$
and $$1+2x_1+4x_2 < 0$$

The area defined has portions of instability in it. (Fig. 16)

Therefore both methods do not yield the area of asymptotic stability.

(b) Saddle Focus Systems:

These are the ones most often found among physical systems.

Example 9

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -3x_1 - x_1^2 - 2x_2$$

The singularities are (0,0) and (-3,0)

The characteristic equation is given by:

$$\begin{vmatrix} -\lambda & 1 \\ -3-2x_1 & -2-\lambda \end{vmatrix} = 0$$

or $\lambda^2+2\lambda+(3+2x_1)=0$

At (0,0)

$$\lambda^2+2\lambda+3=0 \quad \text{OR } \lambda = \frac{-2\pm\sqrt{4-12}}{2} \longrightarrow \text{stable node}$$

At (-3,0)

$$\lambda^2+2\lambda-3=0 \quad \text{OR } \lambda = \frac{-2\pm\sqrt{4+12}}{2} = -1, 3 \longrightarrow \text{saddle}$$

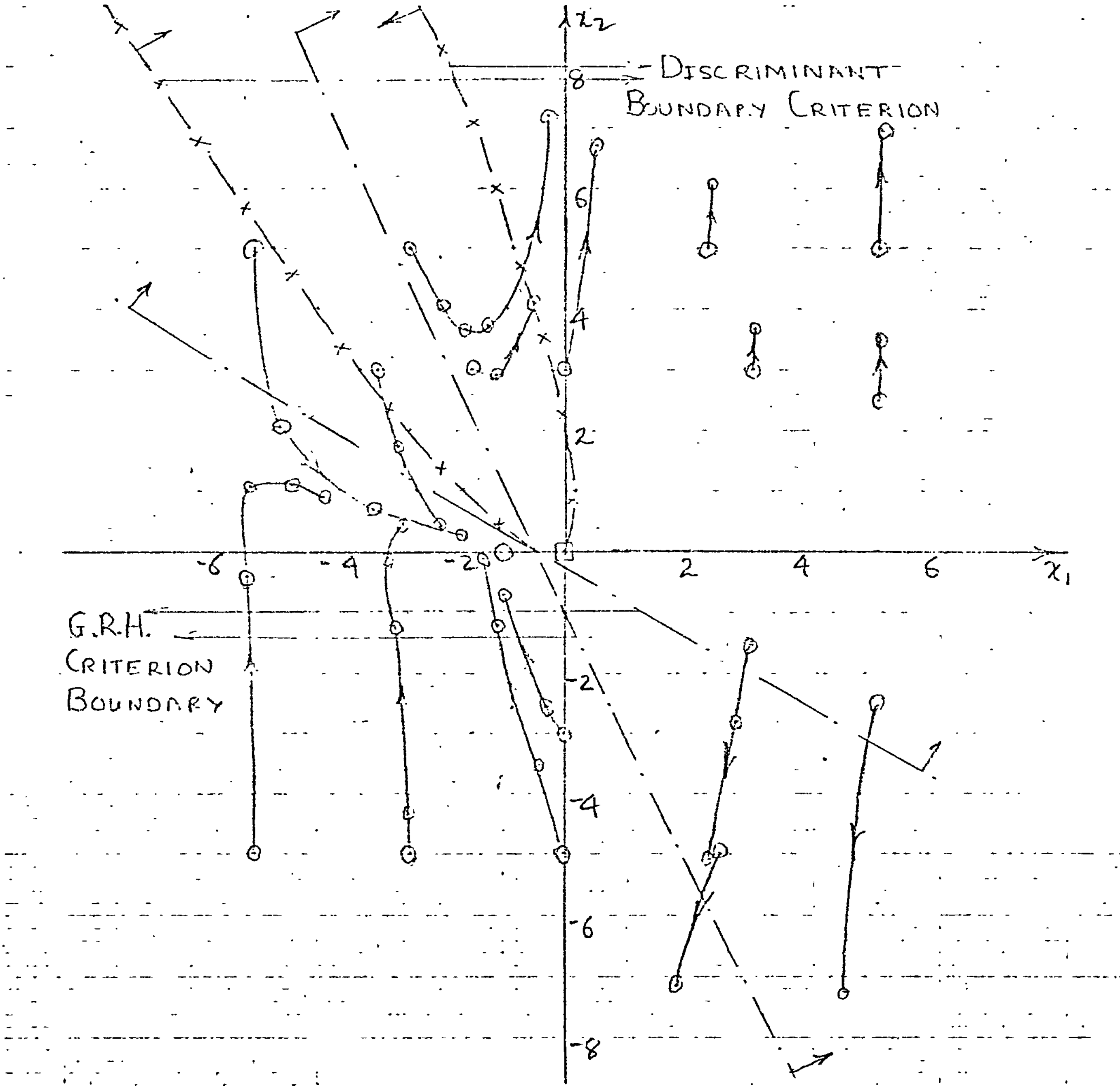

Fig. 16 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + x_1^2 + 4x_1x_2 + 2x_2 + x_2^2 \end{cases}$$

The singularities are at (1) saddle point at (0,0)

(2) stable node at (-1,0)

Discriminant Criterion

$$4-12-8x_1=0$$

or $x_1=-1$

which fails to yield the region of asymptotic stability.

Generalized Routh Hurwitz Criterion

$$3+2x_1>0 \quad \text{OR} \quad x_1>-1.5$$

which also fails as far as defining the area of asymptotic stability is concerned. (Fig. 17)

Example 10

$$\dot{x}_1=-x_1-x_2$$
$$\dot{x}_2=x_2+x_1x_2$$

The singularities are (0,0) and (-1,1)

Therefore the characteristic equation is:

$$\begin{vmatrix} -1-\lambda & -1 \\ x_2 & 1+x_1-\lambda \end{vmatrix}=0$$

or

$$\lambda^2-\lambda(1+x_1-1)-1-x_1+x_2=0$$

or

$$\lambda^2-\lambda(x_1)+x_2-x_1-1)=0$$

At (0,0)

$$\lambda^2-1=0 \quad \text{OR} \quad \lambda=\pm 1 \longrightarrow \text{saddle}$$

At (-1,1)

$$\lambda^2+\lambda+1=0 \quad \text{OR} \quad \lambda=\frac{-1\pm\sqrt{1-4}}{2} \longrightarrow \text{stable focus}$$

Discriminant Criterion

$$(x_1)^2-4x_2+4x_1+4=0$$
$$(x_1+2)^2-4x_2=0$$

which is a parabola that does furnish the area of asymptotic stability. (Fig. 18)

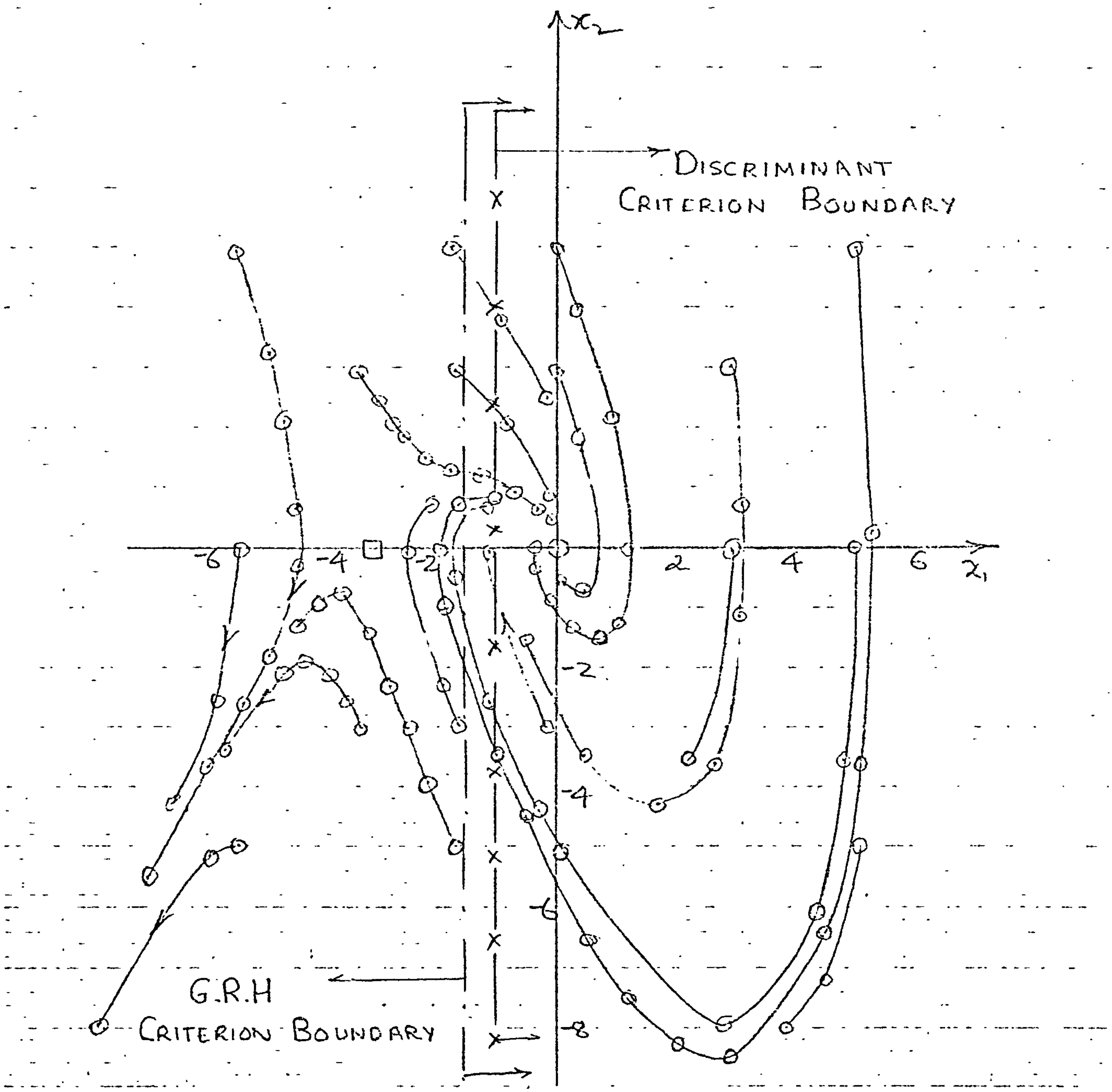

Fig. 17 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -3x_1 - x_1^2 - 2x_2 \end{cases}$$

The singularities are (1) saddle point at (-3,0)

(2) stable focus at (0,0)

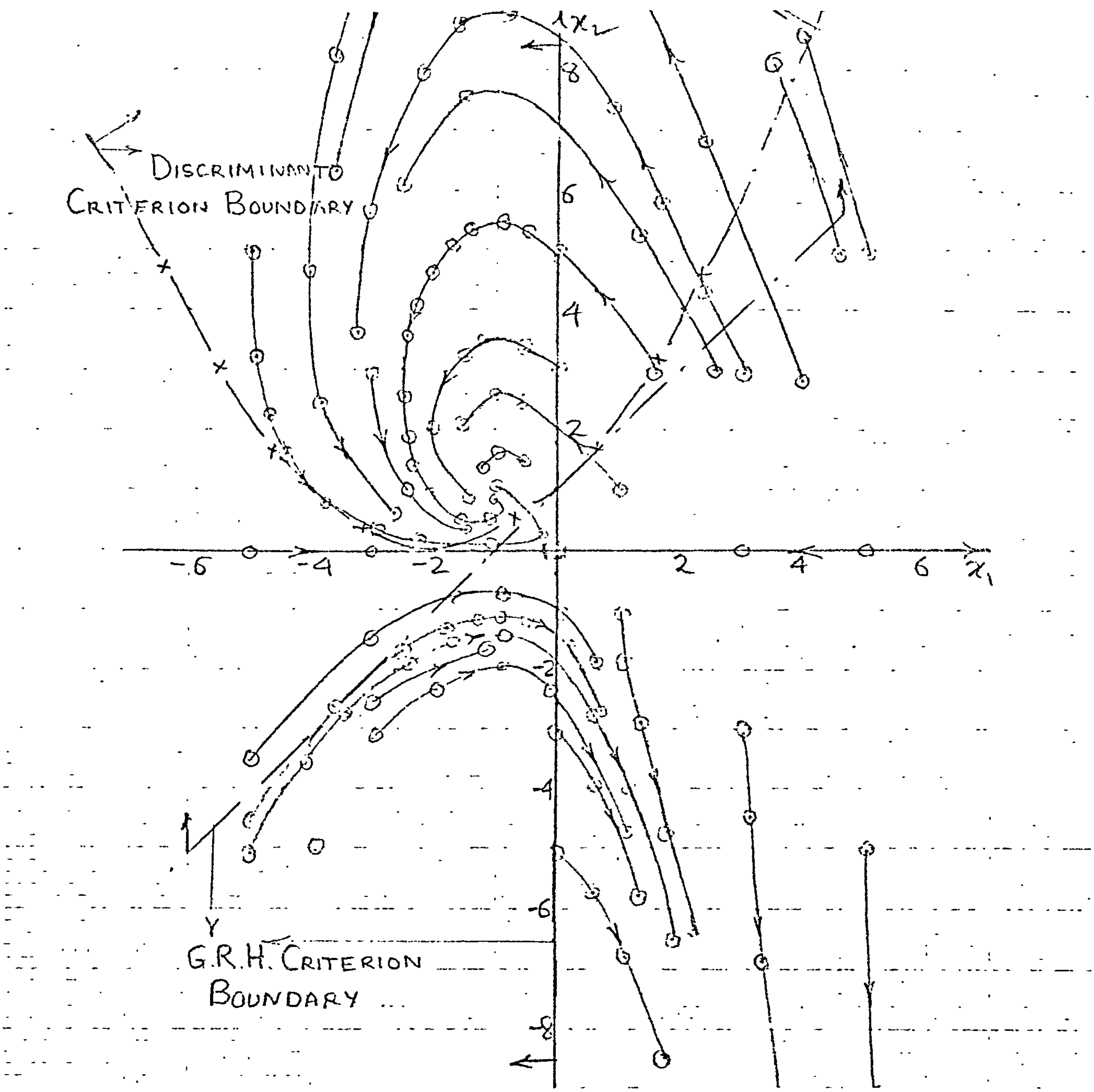

Fig. 18 Trajectories of

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_2 + x_1 x_2 \end{cases}$$

The singularities are at (1) saddle point at (0,0)
(2) stable focus at (-1,1)

Generalized Routh Hurwitz Criterion

$x_1 < 0$

and $x_2 - x_1 - 1 > 0$

The test fails (areas of instability in it)

The state space plot is shown in Fig. 18 with both criterion boundaries shown.

Example 11

$\dot{x}_1 = x_2$

$\dot{x}_2 = -2x_1 - x_1^2 - x_2$

The singularities are (-2,0) and (0,0).

The characteristic equation is:

$$\begin{bmatrix} 0-\lambda & 1 \\ -2-2x_1 & -1-\lambda \end{bmatrix}$$

or $\lambda^2 + \lambda + 2 + 2x_1 = 0$

At (0,0)

$\lambda^2 + \lambda + 2 = 0 \quad \text{OR} \quad \lambda = \dfrac{-1 \pm \sqrt{1-8}}{2} \longrightarrow$ stable focus

At (-2,0)

$\lambda^2 + \lambda - 2 = 0 \quad \text{OR} \quad \lambda = \dfrac{-1 \pm \sqrt{1+8}}{2} = 1, -2 \longrightarrow$ saddle

Discriminant Criterion

$(1)^2 - 4(1 - 2x_1) = 0 \quad \text{OR} \quad x_1 = 3/8$

Generalized Routh Hurwitz Criterion

$2 + 2x_1 > 0 \quad \text{OR} \quad x_1 > -1$

Both of these criteria do not define an area of asymptotic stability. (Fig. 19)

Example 12

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_1 + x_1^2 + x_2^2 - x_2$

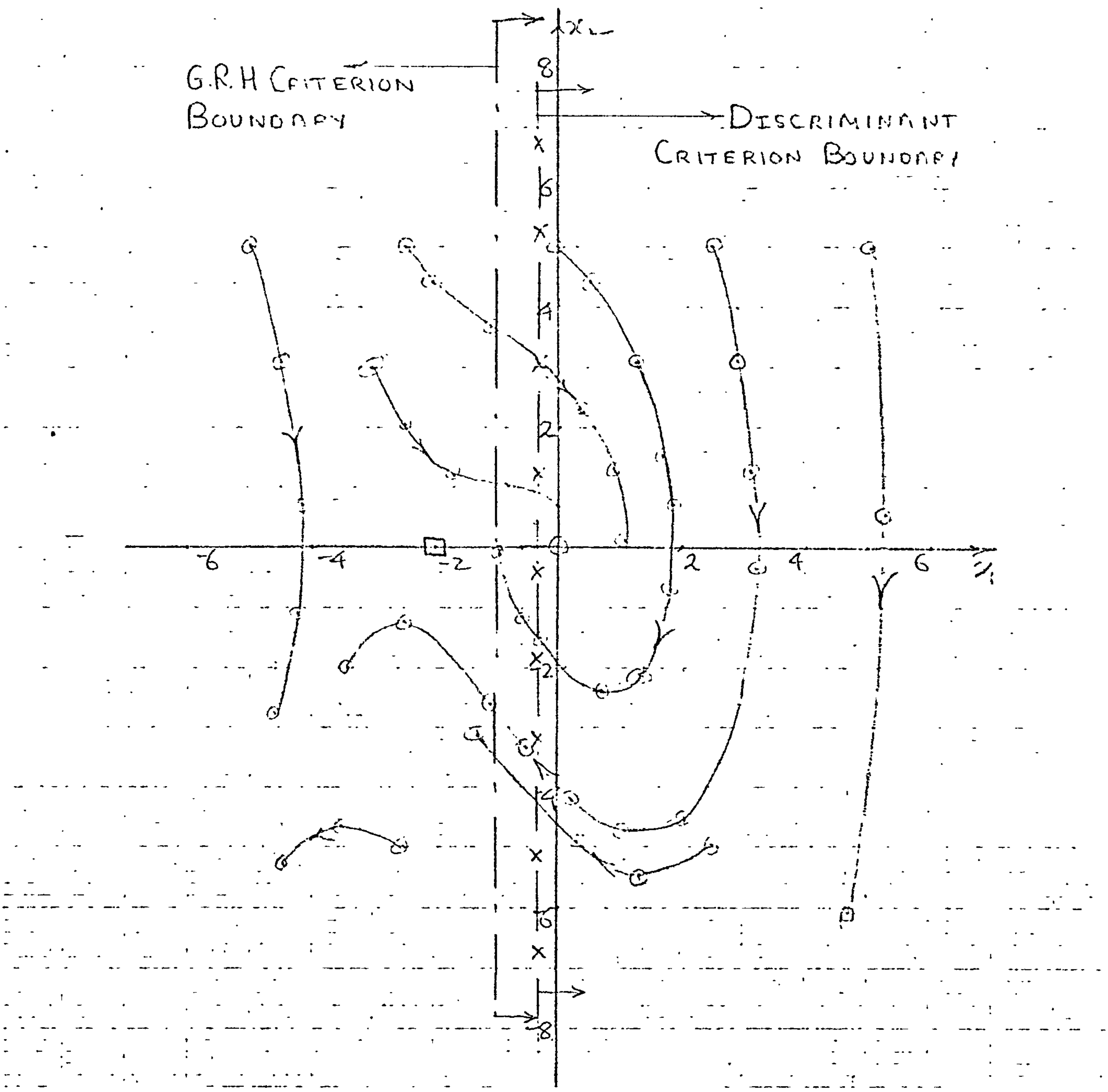

Fig. 19 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - x_1^2 - x_2 \end{cases}$$

The singularities are (1) saddle point at (-2,0)
(2) stable focus at (0,0)

Singularities are (0,0) and (-1,0)

The characteristic equation is:

$$\begin{vmatrix} -\lambda & 1 \\ 1+2x_1 & 2x_2-1-\lambda \end{vmatrix} = 0$$

or $\lambda^2 - \lambda(2x_2-1) - (1+2x_1) = 0$

For (0,0)

$\lambda^2 + \lambda - 1 = 0$

or

$$\lambda = \frac{-1 \pm \sqrt{1+4}}{2} \longrightarrow \text{saddle}$$

For (-1,0)

$\lambda^2 + \lambda + 1 = 0$

or

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} \longrightarrow \text{stable focus}$$

Discriminant Criterion

$$(2x_2-1)^2 + 4(1+2x_1) = 0$$

Generalized Routh Hurwitz Criterion

$2x_2 - 1 < 0$ OR $x_2 < 1/2$

$1 + 2x_1 < 0$ OR $x_1 < 1/2$

Both the boundaries are plotted in the phase plot but fail to give the desired area. (Fig. 20)

Thus it is seen that though both criterion are valuable in multilinear model simulation, especially the second, neither one defines consistently, the area of asymptotic stability.

Only in Examples 5, 6, and 10 the area furnished by the discriminant criterion is such an area, and at first it seems hard to find the common element in them. In the next chapter, a criterion with very restricted application will be developed, which explains the above as well

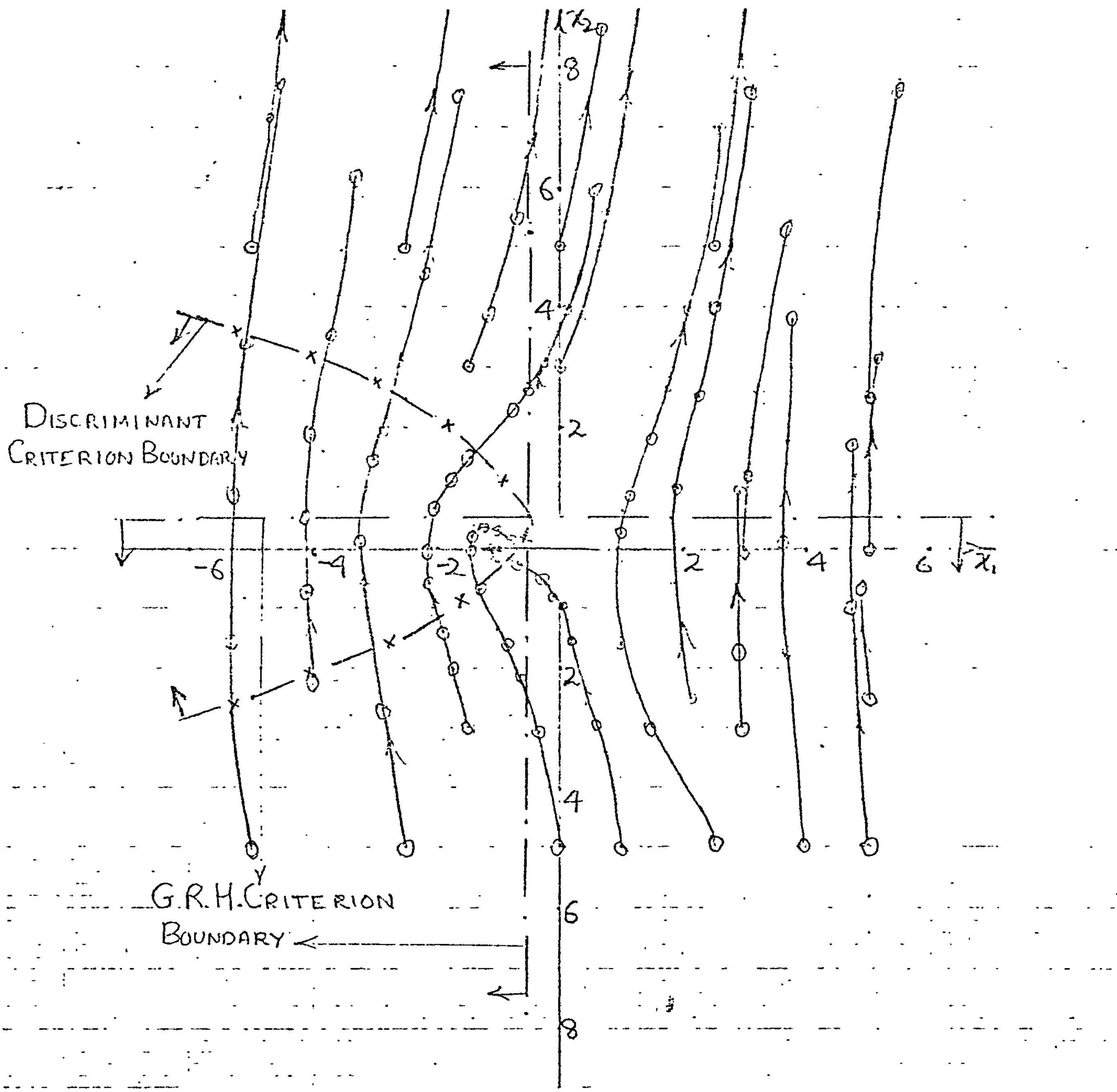

Fig. 20 Trajectories of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + x_1^2 + x_2^2 - x_2^2 \end{cases}$$

The singularities are (1) saddle point at (0,0)
(2) stable focus at (-1,0)

as defines the narrow class of functions for which it will give the desired area of asymptotic stability.

CHAPTER V

SEPERATRIX CRITERION FOR THE PHASE-SPACE PARTITION

Before dealing with the criterion, a seperatrix will be defined, so as to make the following discussion more clear.

Seperatrix: A seperatrix is a path tending to a singular point (generally a saddle point) as $t \to +\infty$ (or $-\infty$), such that the neighboring paths do not tend to that point under the same conditions and so part from it as $t \to \infty$ (or $-\infty$). Or more simply, seperatrices can be defined as curves passing through singular points.[1,12]

The role of seperatrices as dividing curves which separate regions with paths of different types is well known.[12,13,14] Andronov in his classical work on the theory of oscillations gives a detailed treatment on their importance and existence, and deals with a number of idealized mathematical cases. Gibson points it out in connection with a damped single pendulum under constant driving force or torque and an illustration of Bendixson's Theorem.[1]

It is known that they correspond to the path when a line is tangent to the potential energy curve[12] but though they can be established near the saddle point, an equation for them can be generally found only if the differential equation representing the system can be solved.

However, in the special case, when the seperatrix is a straight line, one would expect to obtain a division into regions of opposing tendencies. For a saddle, the eigenvectors correspond to special solutions which pass through the singularity and hence would be the seperatrices. Of these two seperatrices, one moves away from the singularity and goes to the singularity dominated regions while the other tends to divide the phase space. This will not only furnish us a portion of the region of asymptotic stability where the above conditions hold, but define the whole of it. Some examples will now be taken to demonstrate the working of this criterion, and in particular Examples 5, 6, and 10 of Chapter IV will be dealt with.

Example 1: Considering Ex. 5 of Chapter IV again:

$$\begin{aligned} \dot{x}_1 &= -5x_1 - x_2 \\ \dot{x}_2 &= x_2 + x_1 . x_2 \end{aligned} \qquad ....(1)$$

Singularities are (0,0), saddle and (-1,5), node.

Then the eigenvectors for the saddle are given by the equations:

$$x_1 = m_1 x_1 \quad \text{and} \quad x_2 = m_2 x_2 \qquad ....(2)$$

where $m_1 = \dfrac{\lambda_1 - a}{b} = \dfrac{c}{\lambda_2 - d}$

$m_2 = \dfrac{\lambda_2 - a}{b} = \dfrac{c}{\lambda_2 - d}$

λ_1, λ_2 are roots of characteristic equations for saddle equal to 1,-5. and a, b, c, and d are the coefficients of the equations that represent the linear approximation to the control system, ie.:

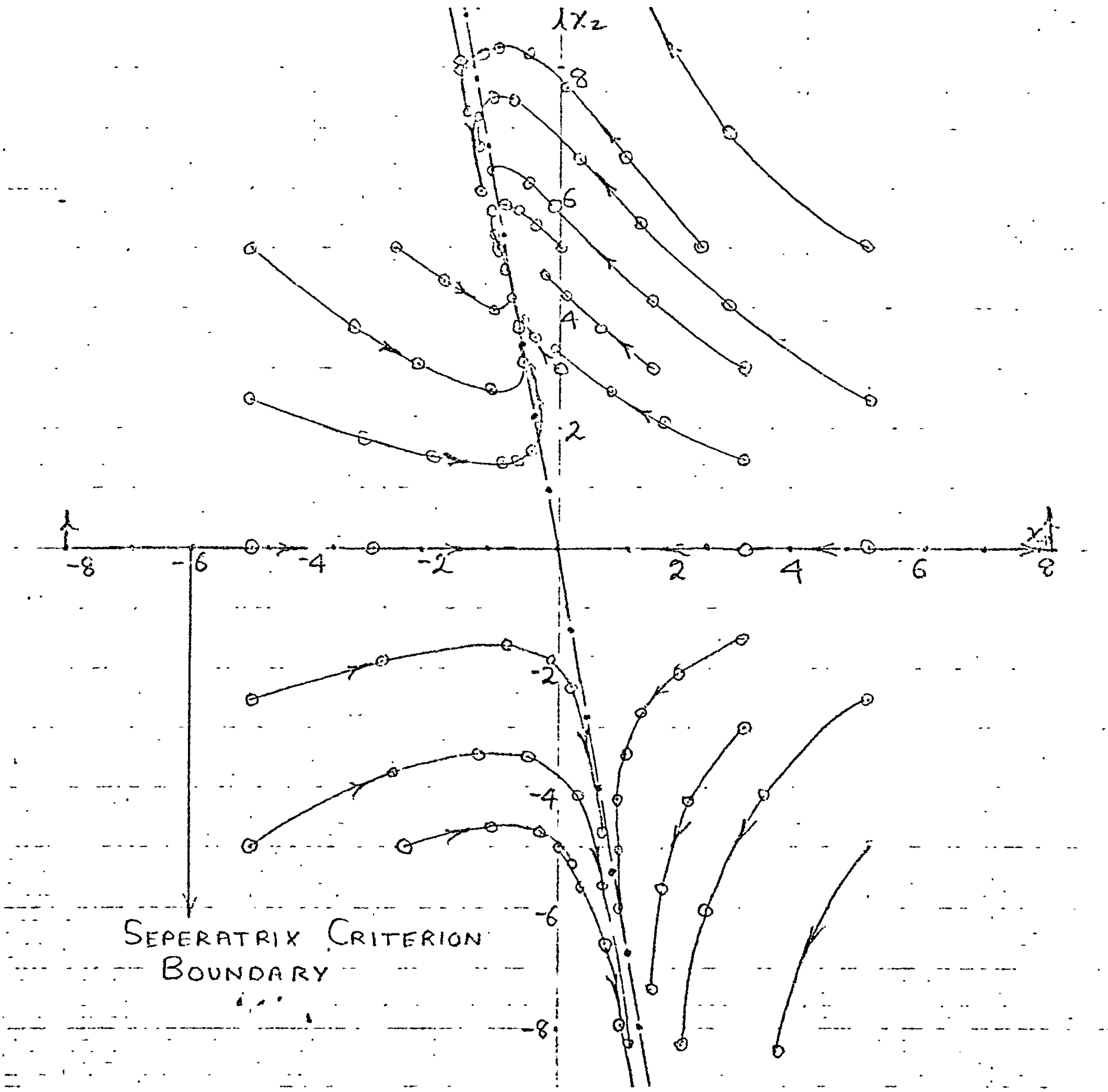

Fig. 21 Trajectories of

$$\begin{cases} \dot{x}_1 = -5x_1 - x_2 \\ \dot{x}_2 = x_2 + x_1 x_2 \end{cases}$$

The singularities are (1) saddle point at (0,0)
(2) stable node at (-1,5)

$$\dot{x}_1 = ax_1 + bx_1$$
$$\dot{x}_2 = cx_1 + dx_2$$

Here a=-5, b=-1, c=0, d=1

Therefore $m_1 = \frac{1+5}{-1} = -6$ $\quad m_2 = \frac{-5+5}{-1} = 0$

λ_2= negative, therefore m_2 corresponds to the path that tends <u>to</u> the singularity.

Therefore, $x_2 = 0$ (from 2) is the equation of the seperatrix near the saddle. But the equation for $\dot{x}_2$ indicates it will remain zero if $x_2 = 0$, no matter what value x_1 assumes.

Therefore, the seperatrix is a straight line.

Therefore, the region dominated by the stable singularity (node) bounded by this seperatrix (upper half of the phase plane) will be asymptotically stable.

A look at the phase plot confirms this. (Fig. 21)

<u>Example 2</u>: Considering Ex.6 of Chapter IV:

$$\dot{x}_1 = -3x_1 - x_2$$
$$\dot{x}_2 = x_1 + x_1 x_2 \qquad \text{.......(3)}$$

Singularities are (0,0), node, and $(+\frac{1}{3},-1)$, saddle. Shifting the origin to $(\frac{1}{3},-1)$, because the seperatrix passes through the saddle point and equations for the eigenvectors (which turn out to be the seperatrix) are given by the coefficients

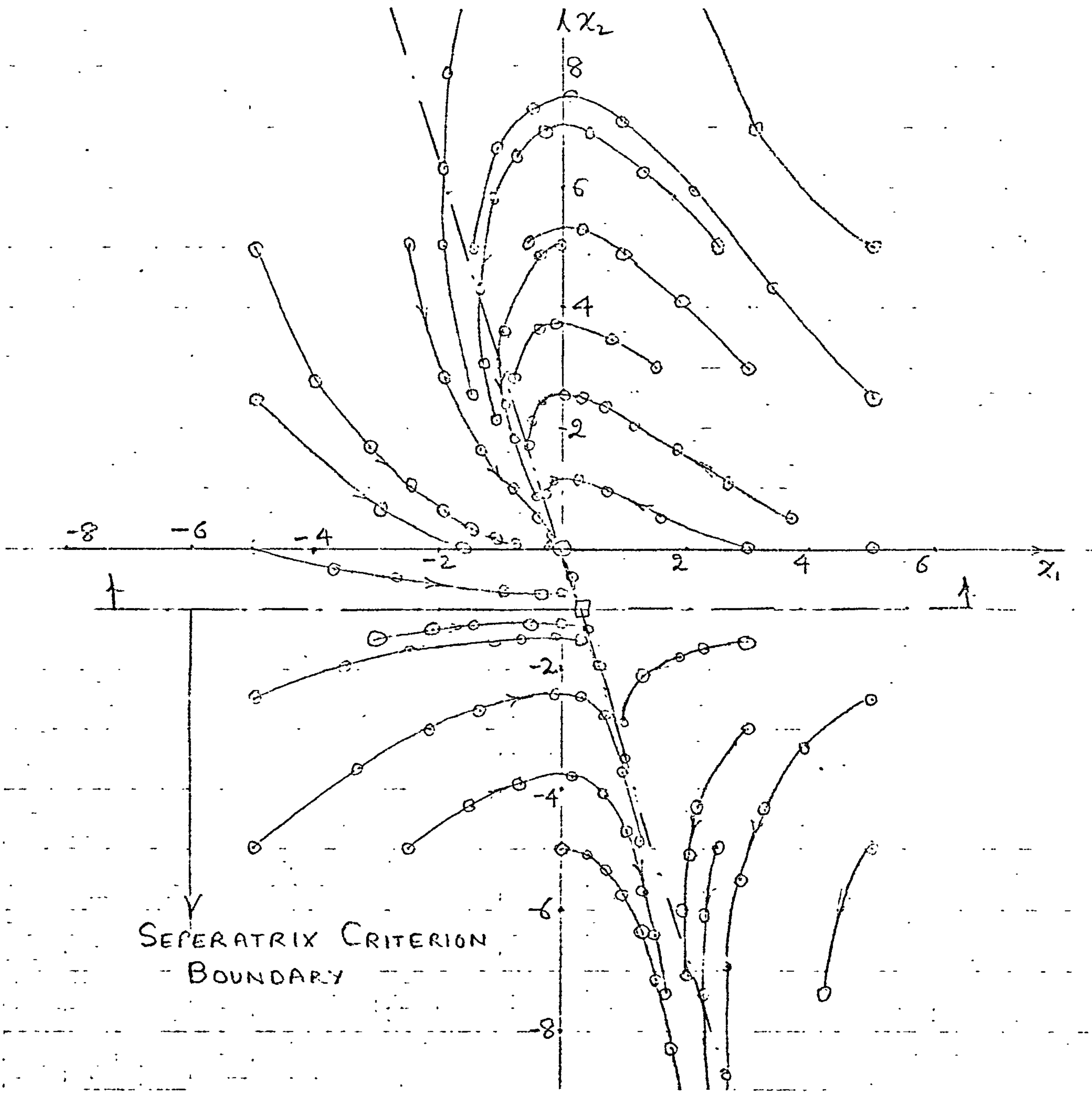

Fig. 22 Trajectories of

$$\begin{cases} \dot{x}_1 = -3x_1 - x_2 \\ \dot{x}_2 = x_1 + x_1 x_2 \end{cases}$$

The singularities are (1) saddle point at $(\frac{1}{2}, -1)$
(2) stable node at $(0,0)$

representing the system near the singularity in question.

$$\therefore \dot{x}_1' = -3x_1' - x_2'$$
$$\dot{x}_2' = x_1' x_2' + \tfrac{1}{3} x_2'$$

where

$$x_1 = x_1' + \tfrac{1}{3} \qquad x_2 = x_2' - 1$$
$$\therefore \; a = -3 \qquad b = -1$$
$$c = 0 \qquad d = \tfrac{1}{3}$$

(Roots of the characteristic equation and nature of trajectories remain unchanged under the operation of translation of axes.) The seperatrix corresponding to the path tending to the singularity is given by the equation $\dot{x}_2' = 0$ OR $x_2 = -1$

But again from equation 3:

$$\dot{x}_2 = x_1 (1 + x_2)$$

And if $x_2 = -1$, no matter what x_1 is, $\dot{x}_2 = 0$.

Therefore the seperatrix is a straight line which should then define the area of asymptotic stability, which it does. (Fig. 22)

<u>Example 3</u>: Considering Ex. 10 of Chapter IV:

$$\dot{x}_1 = -x_1 - x_2$$
$$\dot{x}_2 = x_2 + x_1 x_2 \qquad \text{.......(4)}$$

The singularities are (0,0), saddle, and (-1,1), focus.

The roots of the characteristic equation for the saddle are ± 1 .

a=-1 b=-1

c=0 d=1

$\therefore\ m_1 = \frac{\lambda_1 - a}{b} = -2 \qquad m_2 = -\frac{1+1}{1} = 0$

Therefore the equation for eigenvectors:

$x_2 = -2x_1$ and $x_2 = 0$

Therefore the seperatrix tending to (0,0) the saddle point is given by:

$x_2 = 0$

But $\dot{x}_2 = 0$ if $x_2 = 0$ from (4) and remains so for any x_1.

Therefore seperatrix is a straight line.

The phase plot in Fig. 23 confirms that the seperatrix criterion gives the area of asymptotic stability.

Example 4:

Let $\dot{x}_1 = 2x_1 - 4x_2$

$\dot{x}_2 = -3x_2 - x_1 x_2 + x_2^2$

Singularities are (0,0) and (-6,-3)

Characteristic equation is:

$$\begin{vmatrix} 2-\lambda & -4 \\ -x_2 & -3-x_1+2x_2-\lambda \end{vmatrix} = 0$$

or $\lambda^2 + \lambda(1 + x_1 - 2x_2) - 6 - 2x_1 + 4x_2 - 4x_2 = 0$

or $\lambda^2 + \lambda(1 + x_1 - 2x_2) - 6 - 2x_1 = 0$

At (0,0)

$\lambda^2 + \lambda - 6 = 0$ OR $\lambda = \frac{-1 \pm \sqrt{1+24}}{2} = 2, -3 \rightarrow$ saddle.

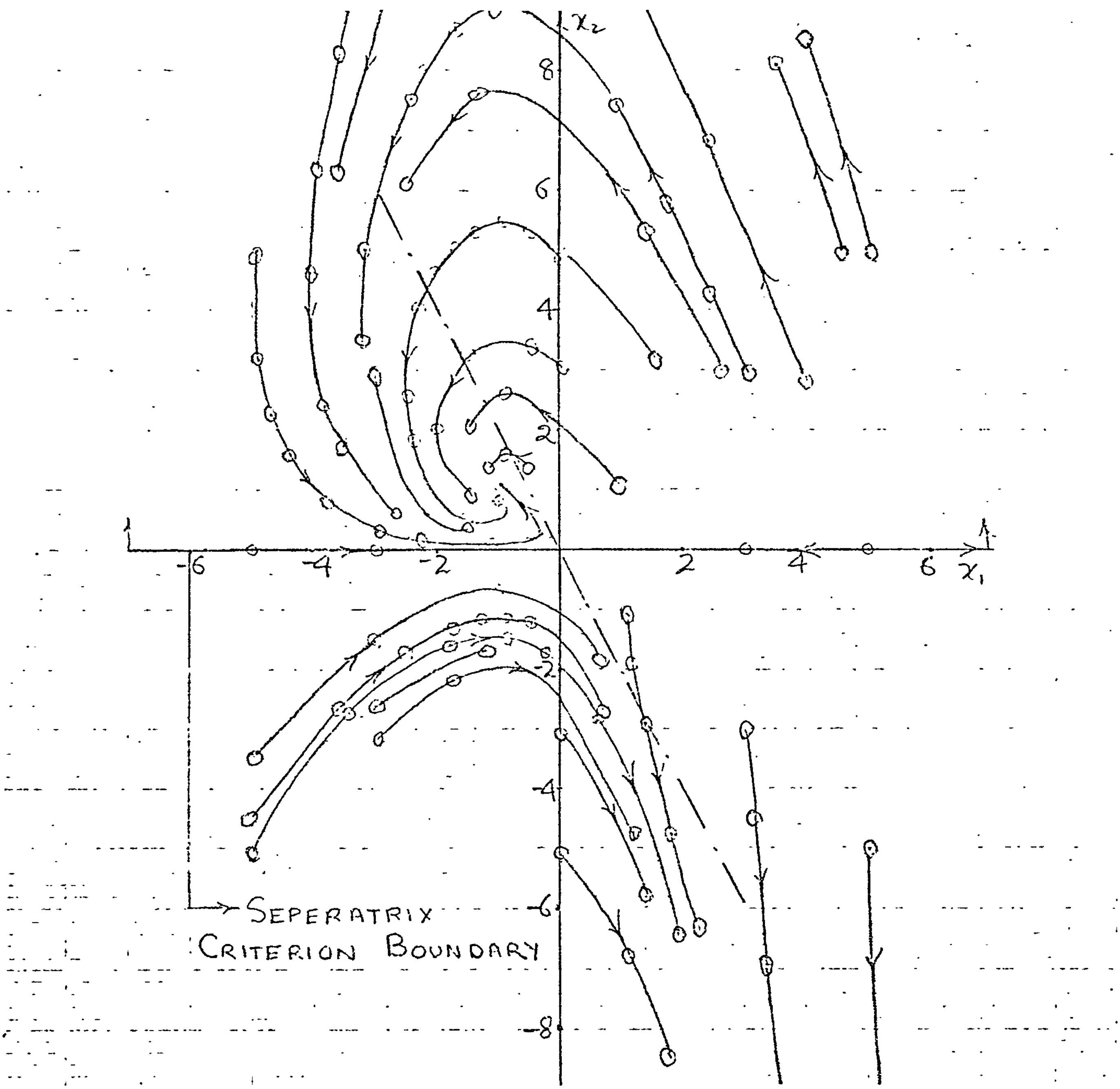

Fig. 23 Trajectories of

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_2 + x_1 x_2 \end{cases}$$

The singularities are (1) saddle point at (0,0)
(2) stable focus at (-1,1)

For (-6,-3)

$$\lambda^2 + \lambda(1-6+6) - 6 + 12 = 0$$

$$\lambda^2 + \lambda + 6 = 0$$

or

$$\lambda = \frac{-1 \pm \sqrt{1-24}}{2} = \frac{-1 \pm j\sqrt{23}}{2} \longrightarrow \text{saddle}$$

a=2 b=-4

c=0 d=-3

$$m_1 = \frac{2-2}{-4} = 0 \qquad m_2 = \frac{-3-2}{-4} = \frac{5}{4}$$

The seperatrix which is a straight line is $x_2 = 0$ but this corresponds to the path going away from the singularity. The eigenvector corresponding to the stable path is not a straight line and thus it could be expected that area of asymptotic stability will not be yielded. The Figure 24 confirms this contention.

Thus we see that the seperatrix which is linear does demarcate the state space into regions of opposing tendencies furnishing us with the area of asymptotic stability. However the limitation of this criterion is that since it gives the exact demarcation where the region nature changes from stability to instability, the linear approximation to the seperatrix will not in general give the desired area. But if the boundary is uniformly convex to the stable single singularity region then the criterion would still hold. As in general the nature of the equations of the seperatrices are not known as they imply a knowledge of the solution of the system differential equations, the power of the criterion

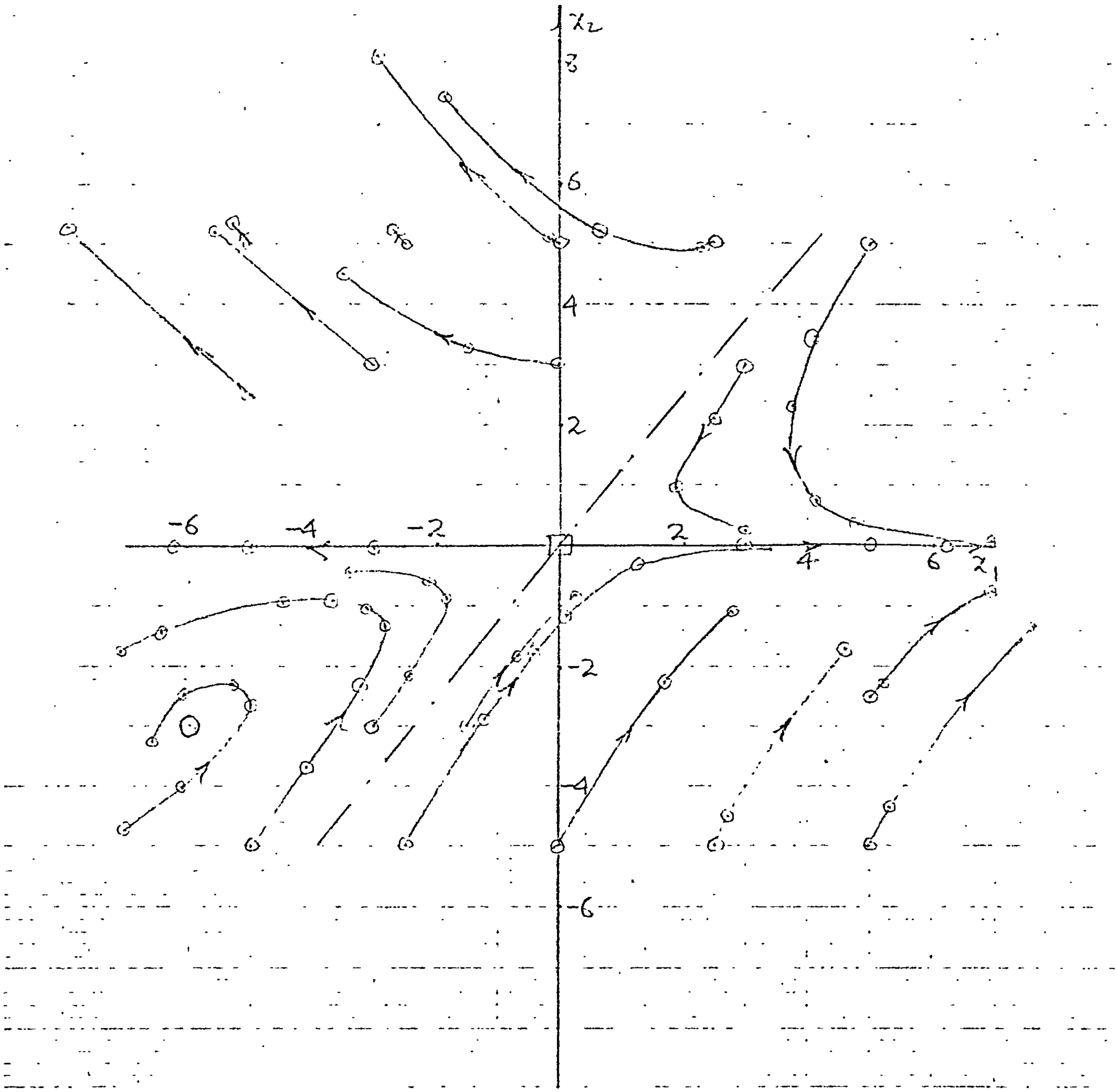

Fig. 24 Trajectories of

$$\begin{cases} \dot{x}_1 = 2x_1 - 4x_2 \\ \dot{x}_2 = -3x_2 - x_1 x_2 + x_2^2 \end{cases}$$

The singularities are (1) saddle point at (0,0)

(2) stable focus at (-6,-3)

is strictly limited.

CHAPTER VI

CONCLUSION

Summing up, it becomes clear that though the discriminant criterion and Ku and Shen[10] criterion for phase partition are useful for simulating non-linear control systems by multilinear models, they do not yield information leading to the definition of the area of asymptotic stability. The Generalized Hurwitz Criterion of Chung[9], a weak and slightly erroneous restatement of the more powerful Ku and Shen criterion, also gives no valuable information in this regard.

The discriminant criterion does furnish such an area for an undefined class of functions but the better defined seperatrix criterion gives the whole area in all such cases. This new criterion, though useful, has, however, a strictly limited range of applicability.

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