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Abstract

The central issue with which this paper deals is the effectiveness of alternative air pollution control standards presently in use in the United States. More specifically, an analysis and comparison of effluent air standards versus ambient air standards will be performed. The question of effectiveness will be in the context of how well the alternative pollution control measures achieve society's expectations as goals when the standards are imposed. Society's views are assumed to be reflected through a regional (state) planner. The analysis is performed through the use of optimal control techniques. Initially the effluent air standards model will be examined. Next, the ambient air standard model will be analyzed. Finally, a summary and conclusion section will be presented.

In general the results suggest that there is a possibility that the optimal path may explode or fall toward zero. In some cases, finite amounts of pollution may occur as equilibria. In any case, the imposition of either type of pollution constraint will effectively reduce the level of social pollution and in some cases the pollution will naturally fall to zero under the constraint. It is also suggested that selective antipollution laws will not, in general, aid in attempts to clean the air.

1. INTRODUCTION

The central issue with which this paper deals is the effectiveness of alternative air pollution control standards presently in use in the United States. More specifically a comparison of effluent standards vs. ambient air standards will be performed. The question of effectiveness will be in the context of how well the alternative pollution control measures achieve society's expectations or goals when it imposes the standards. The analysis is performed through the use of optimal control theory. The next section of the paper consists of the analysis of the effluent air standard model by optimal control techniques, the analysis of the ambient air standards model by optimal control techniques, a comparison of the results yielded in each case and finally a summary of the analysis and results.

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2.1 INTRODUCTION

In this section a model of the economic behavior of firms operating under effluent air standards will be posited, and the economic implications of the model will be examined. Initially the model will be stated verbally, after which a mathematical formulation of the model will be presented. The optimal path will then be characterized and phase diagrams will be employed to investigate the nature of the optimal path in relation to the steady state.

2.2.1 Effluent Model

The model takes the point of view of a regional or state planner attempting to see how the firms in his area should optimally respond to air pollution standards. The model employed is a partial equilibrium model involving two firms. Each firm is assumed to produce one type of output and pollution. Competition is assumed for both firms in the sense that the prices are taken as parameters. The pollution that each firm generates in production combines to give a net addition to the existing stock of social pollution. The stock of social pollution, Q, over time changes due to the net change in pollution from current product less the reduction in pollution resulting from the air's ability to clean itself. The law is assumed to limit the amount of pollution each firm is allowed to produce per time period. Each firm recognizes that pollution affects the cost of production and, as a result, may purchase equipment to abate the level of their pollution. This means that a firm's cost of production is both the cost of producting output and the cost of any abatement equipment used. This assumption implies that the addition of abatement equipment is simply an add-on process and does not affect the present technology being utilized in production. Finally it is assumed that the individual firms act to keep their actual level of pollution below or equal to the legal limit. The planner only wishes to consider those cases where the pollution laws are obeyed.

The problem is for the planner to choose (for each firm) output, abatement equipment and flows of pollution over time so that the profit in his area is as large as possible over some time interval subject to production constraints and the air quality standards.

The notation is now given. The subscripts 1 and 2 will denote the two firms, output is denoted by x. Let P_1 and P_2 be the price of the outputs. The pollution generated by the individual firms is given by Q_1 and Q_2. Q is the level of social pollution. The price for abatement equipment is r/unit and the quantity purchased will be K_1 and K_2 respectively. The cost of production for each firm depends on both the level of output and the level of social pollution, hence $c_1(x_1, Q)$ and $c_2(x_2,Q)$. The amount of pollution generated by the firms is a function of both the level of output and the abatement equipment utilized. Thus $Q_1 = h_1(x_1,K_1)$ and $Q_2 = h_2(x_2,K_2)$. It follows that the total cost of production for the firms will be $c_1(x_1,Q) + rK_1 + c_2(x_2,Q) + rK_2$. The net addition to social pollution over time will be the additions to pollution as a consequence of output production $g(Q_1, Q_2)$, less the ability of the air to clean itself, $D(Q)$. Hence $\dot{Q} = g(Q_1, Q_2) - D(Q)$ denotes the change in social pollution with respect to time.

There is some question as to how Q will be measured. The measurement problem is particularly thorny when more than one pollutant is involved and interaction of the pollutants may occur. Clearly with more than one pollutant some index for Q must be specified which requires that values must be assumed for the relative importance of each kind of pollutant (and any interaction that is forthcoming) to the level of social pollution. The point is that, in terms of the model at hand, the index used to measure social pollution will imply signs for the derivations of the $g(Q_1,Q_2)$ function. In particular no obvious signs are available for the second partials of $g(Q_1, Q_2)$; the signs of $\frac{\partial g}{\partial Q_1}$ and $\frac{\partial g}{\partial Q_2}$ will of course be positive. The upper limit on the firms pollution levels is denoted by $\bar{Q}_1$ and $\bar{Q}_2$ so that the legal requirement is that $\bar{Q}_1 - Q_1 \geq 0$ and $\bar{Q}_2 - Q_2 \geq 0$ for all time.

The mathematical formulation of the model is now given. The problem is for the firms total profits to be maximized over the horizon.

$$\int_0^T [P_1(t)x_1(t) - c_1(x_1,Q) - r(t)K_1(t) + P_2(t)x_2(t) - c_2(x_2, Q) - r(t)K_2(t)]\, dt$$

subject to: $\dot{Q}_t = g(Q_1,Q_2) - D(Q)$

$$\bar{Q}_1 - Q_1 \geqq 0$$
$$\bar{Q}_2 - Q_2 \geqq 0$$
$$Q_1 = h_1(x_1,K_1)$$
$$Q_2 = h_2(x_2,K_2)$$

In order to reduce the number of variables in the problem Q_1 and Q_2 are eliminated by substituting into the differential equation the values of Q_1 and Q_2 respectively. The resulting constraint set becomes

$$\dot{Q} = g(h_1(x_1,K_1),\ h_2(x_2,\ K_2)) - D(Q)$$
$$\bar{Q}_1 - h_1(x_1,K_1) \geqq 0$$
$$\bar{Q}_2 - h_2(x_2,K_2) \geqq 0$$

The Hamiltonian and first order conditions are given next

$$H = P_1x_1 - c_1(x_1,Q) - rK_1 + P_2x_2 - c_2(x_2,Q) - rK_2 + \lambda\,[g(h_1(x_1,K_1),\ h_2(x_2,K_2)) - D(Q)] + \mu_1[Q_1 - h_1(x_1,\ K_1)] + \mu_2[Q_2 - h_2(x_2,\ K_2)].$$

$$(1)\quad \frac{\partial H}{\partial x_1} = P_1 - \frac{\partial c_1}{\partial x_1} + \lambda\frac{\partial g}{\partial Q}\frac{\partial h_1}{\partial x_1} - \mu_1\frac{\partial h_1}{\partial x_1} = 0$$

$$(2)\quad \frac{\partial H}{\partial K_1} = -r + \lambda\frac{\partial g}{\partial Q_1}\frac{\partial h_1}{\partial K_1} - \mu_1\frac{\partial h_1}{\partial K_1} = 0$$

$$(3)\quad \frac{\partial H}{\partial x_2} = P_2 - \frac{\partial c_2}{\partial x_2} + \lambda\frac{\partial g}{\partial Q_2}\frac{\partial h_2}{\partial x_2} - \mu_2\frac{\partial h_2}{\partial x_2} = 0$$

$$(4)\quad \frac{\partial H}{\partial K_2} = -r + \lambda\frac{\partial g}{\partial Q_2}\frac{\partial h_2}{\partial K_2} - \mu_2\frac{\partial h_2}{\partial K_2} = 0$$

$$(5)\quad \dot{\lambda} = -\frac{\partial H}{\partial Q} = -\left[-\frac{\partial c_1}{\partial Q} - \frac{\partial c_2}{\partial Q} - \lambda D'(Q)\right]$$

$$(6)\quad \dot{Q} = g[h_1(x_1,\ K_1),\ h_2(x_2,\ K_2)] - D(Q)$$

$$(7)\quad \mu_1[\bar{Q}_1 - h_1(x_1,K_1)] = 0 \quad \mu_1 \geqq 0$$

$$(8)\quad \mu_2[\bar{Q}_2 - h_2(x_2,K_2)] = 0 \quad \mu_2 \geqq 0$$

(9) and transversal

The initial two first order conditions (eqs. 2 and 3) yield the following equations: $p = \frac{\partial c_1}{\partial x_1} + \frac{dK_1}{dx_1}$. This equation says that firm 1 produces up to the point where the marginal revenue of the sale of the last unit (P) is equal to the marginal cost of producing the last unit $(\frac{\partial c_1}{\partial x_1} + \frac{dK_1}{dx_1})$. The second part of the marginal cost term $(r\frac{dK_1}{dx_1})$ is the cost to the firm of the addition pollution generated by producing x. Equations 3 and 4 yield similar results and interpretation for firm 2.

The phase representation of the problem will now be given. Equations 1-4 together with the constraints determine the variables x_1, x_2, K_1, K_2, μ_1 and μ_2 in terms of P_1, P_2, r, Q and λ. The motion of the phase space $(\lambda,\ Q)$ is governed by the equations

$$(10)\quad \dot{\lambda} = \frac{\partial c_1}{\partial Q} + \frac{\partial c_2}{\partial Q} + \lambda D'(Q)$$

$$(11)\quad \dot{Q} = g[h_1(x_1,K_1),\ h_2(x_2,K_2)] - D(Q)$$

for given P_1, P_2 and r.

The presentation of the phase space in the effluent air standards model will be given in three seperate cases. The cases are where A) neither constraint is effective, B) one constraint is effective, and C) both constraints are effective. The following signs for the derivatives of the functions of g, h and c assumed and will hold for all of the above cases:

$$\frac{\partial^2 c_i}{\partial x_i^2} > 0,\quad \frac{\partial^2 c_i}{\partial x\,\partial Q} > 0,\quad \frac{\partial c_i}{\partial x_i} > 0,\quad \frac{\partial c_i}{\partial Q} > 0,$$

$$\frac{\partial g}{\partial Q_i} > 0,\quad \frac{\partial^2 g}{\partial Q_i^2} > 0,\quad \frac{\partial^2 g}{\partial Q_1 \partial Q_2} > 0,\quad \frac{\partial h_i}{\partial x_i} > 0,\quad \frac{\partial h_i}{\partial K_i} < 0,$$

$\frac{\partial^2 h_i}{\partial x_i^2} > 0$, $\frac{\partial^2 h_i}{\partial K_i^2} > 0$ and $\frac{\partial^2 h_i}{\partial x_i \partial K_i} > 0$. Further,

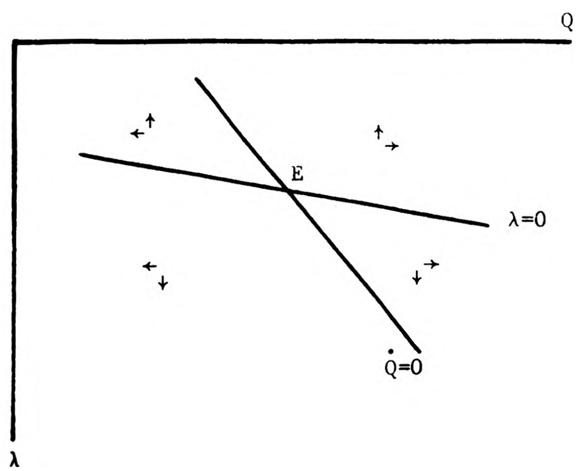
$$\left|\left(\lambda\frac{\partial g}{\partial Q_i} - \mu_i\right)\frac{\partial^2 h_i}{\partial x_i \partial K_i}\right| > \left|\lambda\frac{\partial h_i}{\partial x_i}\frac{\partial^2 g}{\partial Q_i^2}\frac{\partial h_i}{\partial K_i}\right| \text{ holds.}$$

Case A: Neither Constraint Effective

In addition to the above signs, the matrix operations given in the appendix yielded the following signs for Case A:

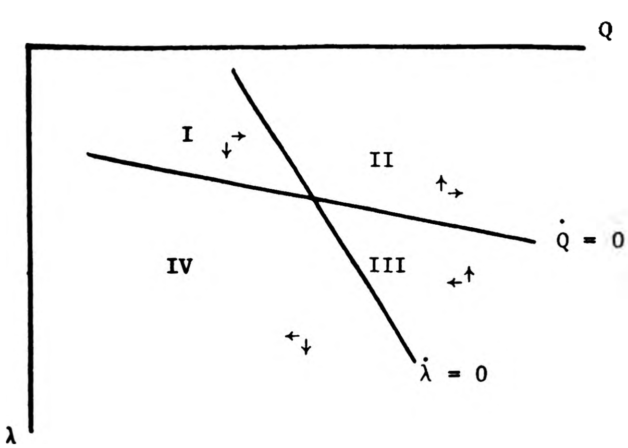
$\frac{dx_1}{dQ}$, $\frac{dx_2}{dQ} < 0$; $\frac{dK_1}{dQ}$, $\frac{dK_2}{dQ} > 0$; $\frac{dx_1}{d\lambda}$, $\frac{dx_2}{d\lambda} > 0$ and $\frac{dK_1}{dQ}$ $\frac{dK_2}{dQ} < 0$. The combined set of signs allows the determation of the sign of $\frac{d\lambda}{dQ}$ along $\dot{Q} = 0$. The derivative $\frac{d\lambda}{dQ}$ is obtained by finding the total derivative of $\dot{Q}$ where $Q^{\bullet} = 0$. This procedure suggests that $\frac{d\lambda}{dQ} < 0$ if $D'(Q)$ is small. The given signs also determine the sign slope of $\frac{d\lambda}{dQ}$ along $\dot{\lambda} = 0$. The analysis gives $\frac{d\lambda}{dQ} < 0$ if $\frac{\partial^2 c_2}{\partial x_1 \partial Q}$ dominates. It is now necessary to ascertain the motion in the phase space. Since $\dot{Q} = g[h_1(x_1, K_1), h_2(x_2,K_2)] - D'(Q)$, then $\frac{d\dot{Q}}{dQ} = - D'(Q) > 0$. What this suggests is that as Q increases, given no change in g, D'(Q) will decrease. As the ability of the air to clean itself falls, $\dot{Q}$ must rise. For $\dot{\lambda} = \frac{\partial c_1}{\partial Q} + \frac{\partial c_2}{\partial Q} + D'(Q)$, $\frac{d\dot{\lambda}}{d\lambda} = -D'(Q) > 0$. While the signs of the slopes of $\dot{Q} = 0$ and $\dot{\lambda} = 0$ are known, the relative size of these slopes are not easy to characterize. Consequently two phase diagrams (depending on the relative slope of $\dot{Q} = 0$ and $\dot{\lambda} = 0$) are given below. In all cases the $\dot{\lambda} = 0$ and $\dot{Q} = 0$ lines are shown as straight lines; they may in fact be curves.

Figure A.1



This phase diagram exhibits only two possible outcomes over time. Depending on the initial position in the phase space, Q will either rise without bound or approach zero. This indicates that the model allows the possibility of an unstable equilibrium. It could be argued that this case depicts what has historically occurred in certain regions of the United States.

Figure A.2



In this case, the motion in the phase space allows for potential equilibrium occurring at E. To achieve this equilibrium, however, it is necessary to have an initial position in sections I or III of the space. Any other initial position will result in instability. If, at the equilibrium position E, the associated air quality is acceptable, there may be no need for air quality standards.

Case B: One Effective Constraint

The matrix operations yield the following signs for this case:

$$\frac{dx_1}{dQ}, \frac{dx_2}{dQ} < 0, \frac{dK_1}{dQ}, \frac{dK_2}{dQ} < 0, \frac{dx_1}{d\lambda} = 0, \frac{dx_2}{d\lambda} < 0, \frac{dK_1}{d\lambda} = 0$$

$\frac{dK_2}{d\lambda} < 0$. Following the same method used in Case A, it was determined that $\left.\frac{d\lambda}{dQ}\right|_{\dot{Q} = 0} < 0$ and $\left.\frac{d\lambda}{dQ}\right|_{\dot{\lambda} = 0} < 0$.

By similar argument $\frac{d\dot{Q}}{dQ} = -D'(Q) > 0$ and $\frac{d\dot{\lambda}}{d\lambda} > 0$. Since some of the terms in the denominator of $\left.\frac{d\lambda}{dQ}\right|_{\dot{Q} = 0}$ are now zero, that line is expected to be more steeply sloped than in case A. The phase analysis and the interpretation remain identical to that given for Case A. However this does suggest an interesting policy implication. Selective use of effluent air standards may have little or no impact on air quality.

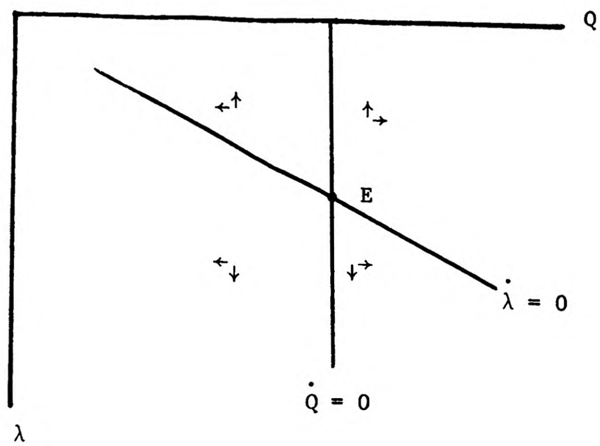
Case C: Both Constraints Effective

In this case the matrix operations provide the following signs:

$\frac{dx_1}{dQ}, \frac{dx_2}{dQ}, \frac{dK_1}{dQ}, \frac{dK_2}{dQ} < 0$ and all derivatives with respect to λ are zero. The methods described in case A allow the signing of slopes of $\dot{\lambda} = 0$ and $\dot{Q} = 0$. The curve $\dot{\lambda} = 0$ will again have a negative slope, and the $\dot{Q} = 0$ curve will be vertical. The $\dot{Q} = 0$ slope may be seen from the following argument. If $\dot{Q} = 0$, $Q_1 = \overline{Q}_1$ and $Q_2 = \overline{Q}_2$ then $g(Q_1, Q_2)$ must be constant and D(Q) must also be constant by $\dot{Q} = 0$. Hence Q must also be constant. The motion in the phase space is derived as before and exhibits behavior identical to cases A and B. There is only one phase diagran to consider.

In this case the equilibrium E is unstable. By law, positions to the right of $\dot{Q} = 0$ are not attainable. If the initial position is on $Q = 0$, Q will never change. If however Q ever falls to the left of $Q = 0$, Q will fall to zero. This must be so since the ability of the air to clean itself will dominate the maximum addition to pollution allowable under the law.

Figure C.1



2.2.2 Summary

In summary a model of a two firm economy with effluent standards for each firm has been given in this section. The model is from the perspective of a regional (or state) planner and is based on optimal control methods. The model exhibits various behavior. In the first two cases (no effluent constraints and one effluent constraint effective) both unstable and potentially stable behavior is observed. The attainment of the interior equilibrium depends upon the initial position and the relative strengths of the forces which move λ and Q. The third case (both effluent constraints effective) seems to be dominated by unstable movement toward zero pollution. The only alternative is maintenance of a constant level of pollution.

2.3 AMBIENT MODEL

In this section a model is given which essentially builds upon the efforts of the previous section. Again a two firm economy is assumed under the control of a planner who attempts to adjust the levels of output and effluent so that profit over time in the area is as large as possible. The difference is that rather than effluent standards, the planner is now confronted with a law which limits the stock of pollution in the air, i.e., an ambient air standard. Again the analysis makes use of optimal contral techniques. The variables given the last section retain their meaning and symbols. In addition the ambient air standard will be denoted by $\overline{Q}$, which is assumed to be constant.

The problem is to maximize

$$\int_0^T [P_1(t)\ x_1\ (t) - c_1(x_1,Q) - r(t)\ K_1(t) + P_2(t)\ x_2(t) - c_2(x_2,Q) - r(t)K_2(t)]dt \text{ subject to}$$

1. $\dot{Q} = g(h_1(x_1,K_1),\ h_2(x_2,K_2)) - D(Q)$
2. $\overline{Q} - Q \geq 0.$

The problem differs from the previous section in that the state variable, Q, is bounded. The Hamiltonian and first order condition take the following form.

$$H = P_1x_1 - c_1(x_1,Q) - rK_1 + P_2x_2 - c_2(x_2,Q) - rK_2 + \lambda\ [g(h_1(x_1,K_1),\ h_2(x_2,K_2)) - D(Q)] + \eta\ [\dot{\overline{Q}} - 2\ [g(h_1(x_1,K_1),\ h_2(x_2,K_2)) - D(Q)]]$$

$$(12)\quad \frac{\partial H}{\partial x_1} = P_1 - \frac{\partial c_1}{\partial x_1} + \lambda \frac{\partial g}{\partial Q_1}\frac{\partial h_1}{\partial x_1} - 2\eta \frac{\partial g}{\partial Q_1}\frac{\partial h_1}{\partial K_1} = 0$$

$$(13)\quad \frac{\partial H}{\partial K_1} = -r + \lambda \frac{\partial g}{\partial Q_1}\frac{\partial h_1}{\partial K_1} - 2\eta \frac{\partial g}{\partial Q_1}\frac{\partial h_1}{\partial K_1} = 0$$

$$(14)\quad \frac{\partial H}{\partial x_2} = P_2 - \frac{\partial c_2}{\partial x_2} + \lambda \frac{\partial g}{\partial Q_2}\frac{\partial h_2}{\partial x_2} - 2\eta \frac{\partial g}{\partial Q_2}\frac{\partial h_2}{\partial x_2} = 0$$

$$(15)\quad \frac{\partial H}{\partial K2} = -r + \lambda \frac{\partial g}{\partial Q2}\frac{\partial h2}{\partial K2} - 2\eta \frac{\partial g}{\partial Q2}\frac{\partial h2}{\partial K2} = 0$$

$$(16)\quad \dot{\lambda} = -\frac{\partial H}{\partial Q} = -\left[-\frac{\partial c_1}{\partial Q} - \frac{\partial c_2}{\partial Q} - \lambda D'(Q) + 2\eta\ D'(Q)\right]$$

$$(17)\quad \dot{Q} = g\{h_1(x_1,K_1),\ h_2(x_2,K_2)\} - D(Q)$$

(18) η is nonincreasing and constant if $\overline{Q} - Q > 0$.

The notation can be simplified by setting $\alpha = \lambda - 2\eta$. Observe that α will equal λ up to the point where the ambient air standard becomes effective. Given this transformation the first order conditions, 12-13 and 14-15, retain the economic interpretation of equations 1-2 and 3-4 in section 2.2.1.

The phase analysis is now considered. Using the matrix methods outlined in the appendix, the signs can be determined for the following derivatives:

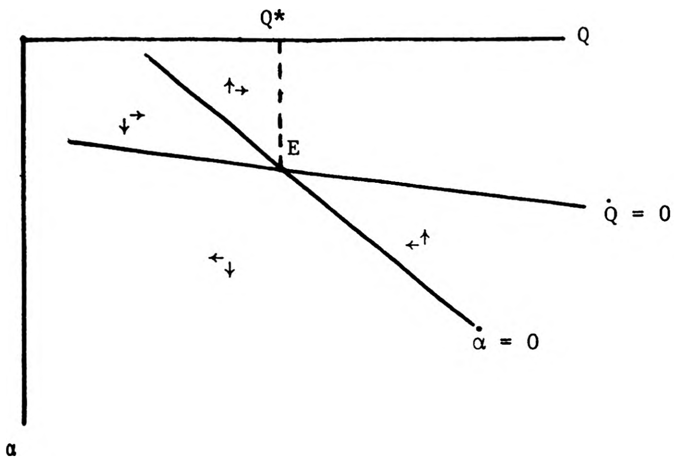
$$\frac{dx_1}{dQ},\ \frac{dK_2}{dQ},\ \frac{dK_1}{d\alpha},\ \frac{dK_2}{d\alpha} < 0 \text{ and } \frac{dK_1}{dQ},\ \frac{dK_2}{dQ},\ \frac{dK_1}{d\alpha},\ \frac{dx_2}{d\alpha} > 0.$$

These derivatives combined with the sign assumptions for the functions g, h, and c previously given yield $\frac{d\alpha}{dQ} < 0$ along $\dot{Q} = 0$ (provided $D'(Q)$ dominates) and $\frac{d\alpha}{dQ} < 0$ along $\dot{\alpha} = 0$ (provided $\frac{\partial^2 c}{\partial Q \alpha K}$ dominates). The motion in the phase space is given by $\frac{d\dot{Q}}{dQ} = -D'(Q) > 0$ and $\frac{d\dot{\alpha}}{d\alpha} = -D'(Q) > 0$.

This information can be summarized in the following phase diagram. There are two cases depending on the relative slopes of $\dot{Q} = 0$ and $\dot{\alpha} = 0$. In each case the behavior will be examined as if the constraint were not effective and then the constraint will be imposed.

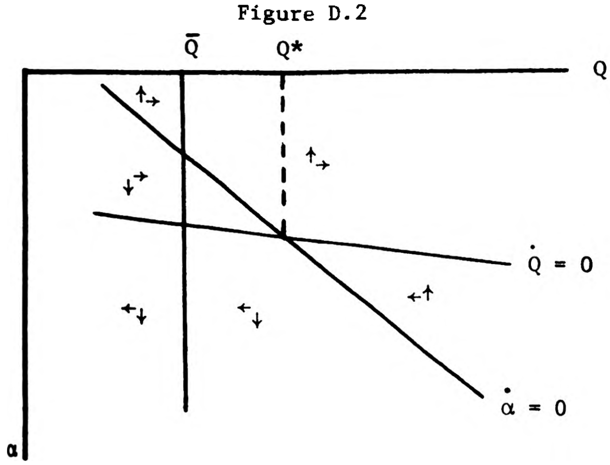
If $\dot{\alpha} = 0$ is relatively steeper than $\dot{Q} = 0$ the following phase space results.

Figure D.1



The position E is potentially stable, again depending on the initial position in the phase space and the relative strength of the forces moving α and Q. Suppose that the ambient air standard is now enforced at a level of Q less than Q*. The following phase diagram results.

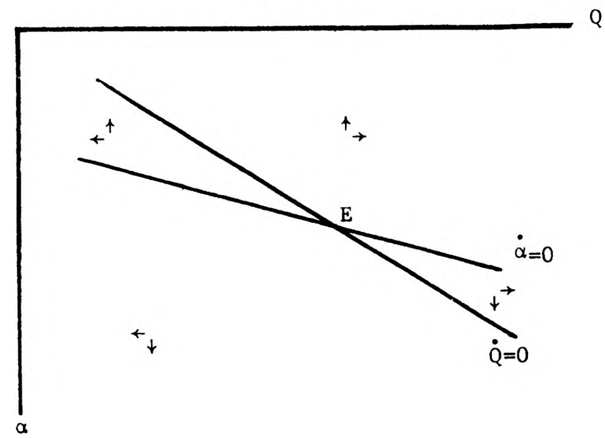
Figure D.2



In this case Q may never rise above $\bar{Q}$. If the path starts in an area below $\dot{\alpha} = 0$, the path will bounce off $\bar{Q}$ and Q will eventually go toward zero. If the path starts above $\dot{\alpha} = 0$, the path will bump up against $\bar{Q}$, and Q remain at that level.

The other possible case occurs when $\dot{Q} = 0$ is relatively steeper than $\dot{\alpha} = 0$. The phase analysis now appears thusly.

Figure D.3



This diagram (D.3) indicates that Q will either go to zero or infinity; the E position is unstable.

The imposition of the ambient air standard will result in paths which force Q toward zero (for any path starting below $\dot{\alpha} = 0$). Above $\dot{Q} = 0$, the path will reach $\bar{Q}$ and stay there. Paths starting between $\dot{\alpha} = 0$ and $\dot{Q} = 0$ will either go to zero or cross over $\dot{Q} = 0$ and go to $\bar{Q}$.*

In summary, this section has presented an economic model of pollution control via ambient air standards. The consequences of the model have been examined in the context of phase diagrams. In general, the ambient air standards will eventually cause pollution to either go to zero or $\bar{Q}$, the ambient air standard.

III. Summary and Conclusion

In this paper a model of the economic impact of air pollution standards has been given. The problem involves two firms and is considered from the point of view of a planner over a time horizon. The planner may be faced with either effluent or ambient air standards. The main analytical device has been optimal contral theory which allowed the analysis to develop in terms of phase diagrams.

The models suggest the following results. In the absence of constraint, there is some possibility that the optimal path may explode to either zero pollution or toward infinite pollution. There are also cases where finite levels may be potentially stable. In any case, when either type of constraint is imposed (i.e., ambient or effluent constraints on all firms) the impact is to limit the level of social pollution. In fact, under those conditions, the level of pollution may naturally fall toward zero since the additions to pollution are less than the native ability of the air to self clean. It is also noted that selective enforcement of effluent standards will not in general promote improved air quality.

*Figure D.4 appears at the end of the paper.

One last issue should be covered. What happens if the society decides to impose air standards (assume effluent for the time being) below current levels of pollution (assume no standards currently exist)? The full answer cannot be given by a phase analysis. What seems to be occurring is that a new initial position is being specified, and the certain results can be given. Unless the pollution regulation reduces $\dot{Q}$ to zero or less, the pollution will continue to grow. In the case where $\dot{Q}$ is reduced to zero, the new initial position will lie on $\dot{Q} = 0$ and (depending on λ) may be maintained or Q will fall to zero. That much can be reasoned within the phase analysis. Further understanding of that process (how the new initial position is generated) requires something more than phase analysis. Similar remarks hold for ambient air standards.

APPENDIX

This appendix attempts to set out some of the mathematical details which lie behind the analysis presented in Section II.

The first order conditions for the affluent problems* equations 1-4, 7, 8, provide implicitly relationships between the variables x_1, K_1, x_2, K_2, μ_1, μ_2, λ and Q. We wish to use the first six equations and solve for x_1, x_2, K_1, K_2, μ_1 and μ_2 in terms of λ and Q. The purpose of the solution is to obtain the relevant derivatives. In fact an explicit solution cannot be obtained for the variables in terms of λ and Q unless all functions are known; even then the functions may be so complex that a solution may not be attainable. However, we may be able to obtain the derivatives using the methods described as follows.

From the total derivatives of the six equations (1-4, 7, 8) we can obtain a system $Ax = b$ or

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{bmatrix} dx_1 \\ dK_1 \\ dx_2 \\ dK_2 \\ d\mu_1 \\ d\mu_2 \end{bmatrix} = \begin{bmatrix} b_{11}dQ + b_{12}d\lambda \\ b_{21}\, d\lambda \\ b_{31}dQ + b_{32}d\lambda \\ b_{41}\, d\lambda \\ b_{51}\, d\overline{Q}_1 \\ b_{61}\, d\overline{Q}_2 \end{bmatrix}$$

where the a_{ij} is the partial of the i^{th} first order condition with respect to the j^{th} variable. The matrix system can be manipulated to obtain (provided $|A| \neq 0$) $x = A^{-1}b$ which will yield the appropriate derivatives. The problem is that we do not know the magnitude of the a_{ij}'s; the sign of the a_{ij}'s will be known if suitable assumptions are made. Even then the sign of the elements of A^{-1} may not be determined. Enough assumptions will be made however to ensure that the sign pattern of A^{-1} can be determined.

We shall assume that A is negative definite. While there is nothing in the theory which assures that A will be negative definite, we are motivated to assume this condition since this condition would surely hold if the maximization were done in discrete time.** This information will be used to sign $|A|$ and the diagonal cofactors. The other elements of A^{-1} will be signed by calculating the appropriate cofactors and checking the signs. Unambiguous signs could not always be obtained, and when a conflicting sign pattern within the expansion of a cofactor arose, the cofactor was assigned the same sign as the majority of its components. Still in some cases signs could not be determined since the components were evenly split between plus and minus. In those cases it was frequently true that an alternative arrangement of terms would yield conclusive evidence as to the sign. In most cases where the signs were evenly split, a closer examination showed the term to be zero.

The exact sign pattern given to the inverse depends on the signs of the terms of the matrix A itself. In addition, the signs of the terms in the inverse will depend upon which constraints, if any, are effective. The signs of the terms in A will be given by the assumptions listed in the body of the paper.

*Similar statements apply to the ambient air standards as well.

**Note that A is not symmetric. The lack of symmetry is due to the existence of μ_i's in the last two rows of A which do not appear in last two columns of A. For the determinantal conditions, the μ_i's can be factored out. Thus we are assuming that a symmetric version of A is negative definite.

To reiterate, we assume $\frac{\partial^2 c_i}{\partial x_i^2} > 0$, $\frac{\partial c_i}{\partial x_i} > 0$, $\frac{\partial c_i}{\partial Q} > 0$, $\frac{\partial g}{\partial Q_i} > 0$, $\frac{\partial^2 g}{\partial Q_i^2} > 0$, $\frac{\partial^2 g}{\partial Q_1 \partial Q_2} < 0$, $\frac{\partial h_i}{\partial x_i} > 0$, $\frac{\partial h_i}{\partial K_i} < 0$, $\frac{\partial^2 h_i}{\partial x_i^2} > 0$, $\frac{\partial^2 h_i}{\partial x_i \partial K_i} > 0$ and $\frac{\partial^2 h_i}{\partial K_i^2} > 0$.

Further we assume $\left|(\lambda \frac{\partial g}{\partial Q_i} - \mu_i) \frac{\partial^2 h_i}{\partial x_i \partial K_i}\right| >$ $\left|\lambda \frac{\partial h_i}{\partial x_i} \frac{\partial^2 g}{\partial Q_i^2} \frac{\partial h_i}{\partial K_i}\right|$.

As we pointed out above, the sign convention for the partials of g is essentially arbitrary; other choices seemed to make the calculations more difficult.

In the case that both constraints are effective, A will have sign pattern (note $|A| > 0$ holds) as shown (the sign of A^{-1} is also shown).

$$A = \begin{bmatrix} - & - & + & - & - & 0 \\ - & - & - & + & + & 0 \\ + & - & - & - & 0 & - \\ - & + & - & - & 0 & + \\ - & + & 0 & 0 & 0 & 0 \\ 0 & 0 & - & + & 0 & 0 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} - & - & 0 & 0 & - & 0 \\ - & - & 0 & 0 & + & 0 \\ 0 & 0 & - & - & 0 & - \\ 0 & 0 & - & - & 0 & + \\ - & + & 0 & 0 & ? & - \\ 0 & 0 & - & + & - & ? \end{bmatrix}$$

We note that both constraints effective gives $\bar{Q}_1 - h_1(x_1, K_1) = 0$ and $\bar{Q} - h_2(x_2, K_2) = 0$; these are precisely a_{55} and a_{66}, respectively. Further, μ_1 and μ_2 will be (with no false corners) strictly positive.

If only one constraint is effective, say $\bar{Q}_2 - h_2(x_2, K_2) > 0$, $\bar{Q} - h_1(x_1, K_2) = 0$ then $\mu_2 = 0$ will hold, and A will have sign pattern as shown (the sign pattern of A^{-1} is also shown).***

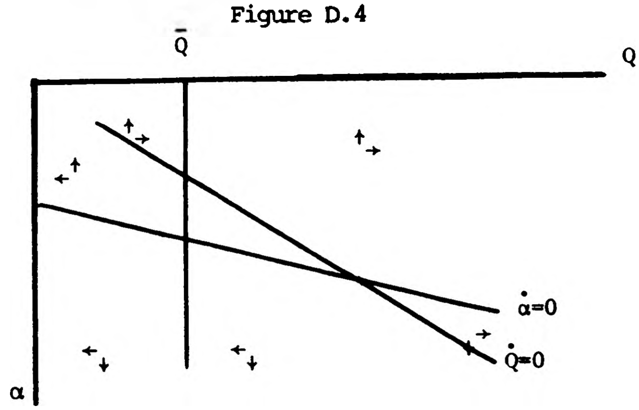
$$A = \begin{bmatrix} - & - & + & - & - & 0 \\ - & - & - & + & + & 0 \\ + & - & - & - & 0 & - \\ - & + & - & - & 0 & + \\ - & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} - & - & 0 & 0 & ? & 0 \\ - & - & 0 & 0 & ? & 0 \\ 0 & 0 & - & + & - & 0 \\ 0 & 0 & + & - & + & 0 \\ ? & ? & - & + & - & 0 \\ ? & ? & ? & ? & ? & ? \end{bmatrix}$$

If neither constraint is effective $a_{55} = \bar{Q}_1 - h_1(x_1, K_1) > 0$ and $a_{66} = \bar{Q}_2 - h_2(x_2, K_2) > 0$ hold; further $\mu_1 = \mu_2 = 0$. A will have the sign pattern as shown and the inverse can be calculated as well.

$$A = \begin{bmatrix} - & - & + & - & - & 0 \\ - & - & - & + & + & 0 \\ + & - & - & - & 0 & - \\ - & + & - & - & 0 & + \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} - & + & - & + & 0 & 0 \\ + & - & + & - & 0 & 0 \\ - & + & - & + & 0 & 0 \\ + & - & + & - & & \\ ? & ? & ? & ? & ? & 0 \\ ? & ? & ? & ? & 0 & ? \end{bmatrix}$$

From these various matrices the signs of the derivatives can be calculated and substituting into $\frac{d\lambda}{dQ}$ along $\dot{Q} = 0$ and along $\dot{\lambda} = 0$ to obtain the slope of the $\dot{Q} = 0$ and $\dot{\lambda} = 0$ lines.

Figure D.4



***We realize that the 6 x 6 version of both constraints ineffective (or one constraint effective) is a problem as $|A| = 0$. The relevant part of the inverse would be the first 4 x 4 (or 5 x 5) and the associated determinant of the 4 x 4 (5 x 5) would not be zero.