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Meeting the Challenges of Negotiated Mathematical Inquiry

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Set in the beginning of the year in one 4th grade classroom, this case study examined pedagogical practices that use children's observations and questions as a basis for mathematical investigations while providing experience with grade appropriate content. An initiating experience based on pentominoes, a geometry puzzle, became a springboard for students to pose problems for further inquiry. The researchers noted children's spontaneous comments as they worked, examined their written and sketched observations, and reframed the ideas as points of departure for later exploration. The follow-up investigations were co-created in unanticipated ways; although the researchers had identified several topics during their preplanning, several of the children's ideas suggested the most valuable mathematical ideas to pursue. On the other hand, the large number of potential inquiries posed an additional tension. Criteria were developed to guide the refining of choices for inquiry in a way that would honor children's ideas while keeping the organization manageable. As a result of this work, a framework for future investigations was developed.

"By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom" (National Council of Teachers of Mathematics [NCTM], 2000, p. 52).

I contemplated these recommendations one August as I formulated math plans for my incoming 4^{th} grade class. I wanted to begin the year with an experience that would pique children's curiosity and would encourage experimentation and informal conversation. Moreover, I wanted to use the children's observations and questions as a basis for investigation in order to validate them as mathematical inquirers.

At the same time, I was challenged by practical considerations. As much as I wanted to honor all of the children's questions and ideas, past experience had shown me that managing many individual inquiries can be overwhelming. The content of potential investigations was also a concern. As the teacher, I was responsible for addressing grade-level expectations in mathematics as well as providing experience with mathematical content that would most benefit the students' growth in conceptual understanding (NCTM, 2000). As I thought about these tensions of teaching, I realized that my interest in them extended well beyond the planning of one unit of study.

I shared my thinking with my husband, David, a professor of elementary mathematics education at our local university.[1] The year before, he had codirected a teacher-research grant that explored inquiry learning in mathematics, science, and language arts; I had been a teacher participant. Pursuing this inquiry together would give both of us an opportunity to further investigate this topic. We decided to design a systematic examination of the teaching of mathematics and of student learning; this article describes a portion of our collaborative work.

Following Hubbard and Power's (1993) advice to keep questions "open-ended enough to allow possibilities the researcher hasn't imagined" (p. 5), we developed the following to guide our research:

1. What are effective pedagogies for involving elementary children in mathematical inquiry?
2. What are features of the teacher's role in this process?
3. What happens to students' conceptual understanding, problem-solving strategies, and attitudes and dispositions during mathematical inquiry?

Together we decided to use the strategy of problem posing (Brown & Walter, 1990) to structure the unit. Since the first research question includes the process of making this choice, it is important first to describe the nature of the strategy. I then will outline the methodology of the study, describe how the research evolved during the implementation of instructional activities, and discuss the pedagogical framework that grew out of the final analysis.

Mathematical Inquiry and the Problem-Posing Strategy

Problem posing is a strategy that promotes the development of mathematical skills, strategies, attitudes, and dispositions. It involves identifying the attributes of a given problem and then changing one or more of them to create a new problem (Brown & Walter, 1990). A simple illustration could be the equation, $6 + 4 = 10$. One of the attributes of this problem is the equation has two addends. What if there were 3 addends? What new equations could be made (e.g., $2 + 4 + 4$, $3 + 4 + 3$)?

What are the benefits of engaging in this strategy? The National Council of Teachers of Mathematics (2000) notes, "People who see the world mathematically are said to have 'mathematical disposition.' Good problem solvers tend naturally to analyze situations carefully in mathematical terms and to pose problems based on situations they see" (p. 53). Problem posing makes this analytical process visible and, therefore, accessible to learners. It begins not merely with how to solve a problem, but with the question, "What is this problem all about?" (Moses, Bjork, & Goldenberg, 1990, p. 83). Identifying attributes develops observational skills. Experimenting with changing those attributes helps learners discover mathematical relationships. Brown and Walter (1990) assert, "[W]e understand something best in the context of changing it" (p. 139). In pursuing new problems, children also engage in planning solution strategies, carrying them out, and evaluating the results (NCTM, 2000).

Problem posing also fosters the development of positive attitudes and dispositions about mathematics because it encompasses not only answering a teacher's questions, but asking one's own (Brown & Walter, 1990). Any given problem implies a wide range of extensions, and generating those new problems can increase independence and risk-taking. Moses, Bjork, and Goldenberg (1990) note, "Ambiguity leaves room for curiosity, imagination, and generating one's own ideas" (p. 86). Furthermore, children of diverse experiences and abilities can find challenge and success in pursuing various investigations. Problem posing can potentially contribute to an environment in which children are "encouraged to explore, take risks, share failures and successes, and question one another" (NCTM, 2000, p. 53).

Despite the potential benefits, implementing the problem-posing strategy can bring challenges for the teacher. One involves the complex process of guiding children in developing problem-posing skills. Several researchers have addressed this potential problem. First, it is important for teachers to recognize that often children's comments imply, rather than directly state, possible avenues for further inquiry (Lindfors, 1999; Whitin, 2004). Thus, they must listen closely to these implied questions and make them accessible and public. Second, children need to be encouraged to take risks. Using phrases like "what if" and "make your own rules" (Brutlag, 1993, p. 134) can help children to entertain possibilities, take multiple perspectives, and imagine alternatives. Third, children whose experience in mathematics has consisted of following teacher-led procedures may need transitional experiences toward problem posing. For example, Gonzales (1998) suggests providing an activity, such as examining data, before asking specific questions. After some time for exploration, the teacher invites the class to orally brainstorm a variety of questions that could be posed. She also suggests using manipulatives for these introductory experiences. Moses, Bjork, and Goldenberg (1990) agree, especially in the context of an elementary classroom: "For young children, manipulatives often help make a mathematical territory feel familiar" (p. 84). This particular study began with two additional challenges in mind: ensuring worthwhile mathematical tasks and creating an efficient and manageable instructional structure.

The problem-posing strategy incorporates several key features of inquiry: observing carefully, adopting multiple points of view, raising questions, offering conjectures, making and carrying out plans, and reflecting on the results. As a process, it therefore has parallel examples in other subject areas. For example, a reader asks, "What if the story didn't end this way?" and creates an alternative ending. In writing workshop, a young author might wonder, "What if I change my story into a poem?" A child conducting a science experiment might ask, "What if we changed the height of the ramp?" The "what if" stance encourages curiosity, promotes ownership of ideas, and opens avenues for investigation. In short, problem posing in mathematics can play an important role in developing an inquiry stance across the curriculum.

Methodology

The study took place in a suburban school serving a middle and upper-middle income community located near a major southeastern city. The class was heterogeneous, including students receiving services for learning disabilities, gifted education, speech and language, and supplemental reading or mathematics instruction. About 84% of the students were White and 16% were African-American; roughly 16% qualified for free or reduced lunch.

As co-researchers, we decided to follow a case study design (Hitchcock & Hughes, 1991) in order to examine the relationships among our evolving pedagogies and the children's mathematical skills, attitudes, and dispositions. Data collection included records of instructional plans, observational notes taken during the hands-on activities, the children's written and drawn journal entries, transcripts of group conversations, and teachers' reflective memos throughout the study.

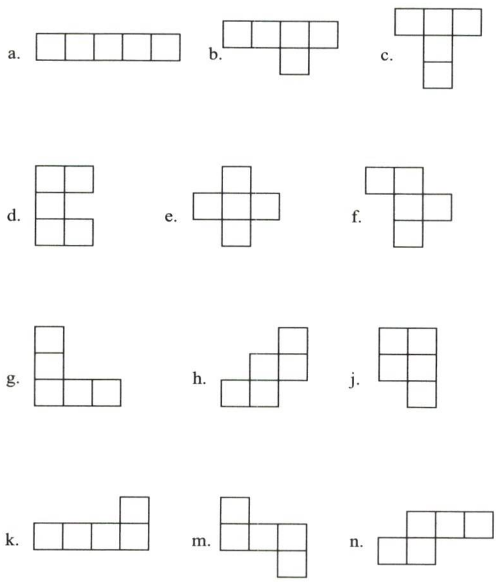
Data analysis followed the constant comparative method (Glaser & Straus, 1967; Goetz & LeCompte, 1984; Hubbard & Power, 1993). During the implementation of the project, we used themes and patterns that emerged from ongoing analysis to inform our evolving instructional plans. Categories for initial analysis included evidence of children's risk-taking, perseverance, collaboration, inventiveness, and problem-solving strategies as well as changes in our instructional plans. During this phase, it was important for us not to allow our preconceived notions about the nature of the problem-posing strategy to limit us from being open to unanticipated events and themes. As Eisner (1991) explains, "Knowing what to look for makes the search more efficient. At the same time, knowing what to look for can make us less likely to see things that were not part of our expectations" (p. 98). Indeed, unexpected themes did emerge and influence later events, especially in regard to the teacher's role. We, therefore, expanded our analysis to include a closer examination of the impact of the children's thinking on our plans. This early analysis, then, also became data that addressed the second and third research questions, in which we examined the role of the teacher and the nature of pedagogical decision-making. It was through the final analysis after the completion of the project that we developed an organizational framework for whole-class mathematical inquiry.

Planning and Preparation

Following Brown and Walter's (1990) recommendations, we first chose an appropriate problem, identified its attributes, and developed a tentative list of problem-posing extensions based upon those attributes. Choosing a problem involved several considerations. Having worked in the school district for several years, I was aware that most of the elementary mathematics instruction emphasized computation. Since this experience would open the year, we decided that a noncomputational task, such as geometry, would provide a more equitable challenge for all. We wanted to use familiar hands-on materials, as suggested by Moses, Bjork, and Goldenberg (1990). We also believed that using manipulatives would encourage the children to talk informally as they worked and to use drawing and writing to record and represent their findings. Finally, we could incorporate grade level curricular expectations involving geometric concepts. After some consideration, we chose a two-part puzzle involving geometric shapes with 5-square units, commonly called pentominoes (Hyde & Hyde, 1991; Walter, 1996). The first part asks: How many *different* shapes can be made from 5 squares, following these rules?

- Use all five squares for each figure;
- Sides must touch edge to edge. At least one complete edge of each square must touch another complete edge (vertex to vertex doesn't count); and
- Each shape must be different. If you flip, slide, or turn a shape so it matches another shape, then it does not count as a new shape.

The second part relates these twelve 2-dimensional shapes to solid geometry: Which of these shapes (copied on paper) can be folded to make a box without a top? Figure 1 shows the 12 possible configurations of the 5 tiles and indicates which of these can also be folded into a box.



Solutions b, c, e, f, h, k, m, and n can be folded into boxes without tops.

Figure 1. The Twelve Solutions for Pentominoes (5 square units)

This problem did meet our criteria. First, the notion of a puzzle suggests experimentation. The problem-solving process for this puzzle generally incorporates trial and error (e.g., as students move the tiles in various arrays to try to create other solutions). We thought that the ambiguity of trial and error could encourage attitudes such as risk-taking and perseverance. Both children who felt proficient with mathematics and those who were less confident could be successful and make valuable contributions for the whole class. Second, the problem also addresses key content in geometry:

- Transformational geometry (flips, slides, and turns),
- Congruence (Does this figure exactly match in size and shape another figure?), and
- Relationships between two- and three-dimensional shapes (How are squares and cubes similar/different?).

Third, since there are 12 possibilities for the first part of the puzzle, record keeping and representation play a natural and important role in the process. Finally, experimentation with the hands-on materials (square tiles) can generate a variety of observations. These observations are crucial because they can become the starting point for further investigations (Whitin, 2004).

We next examined the attributes of the first part of the problem and generated a list of potential problem-posing extensions for further inquiry:

Attribute	Extension
Five tiles for each shape	What if we used a different number of tiles?
Tiles are square	What if we used cubes instead of squares? What if we used triangles instead of squares?
Sides must touch edge to edge	What if the tiles could touch vertex to vertex?

As I will relate below, the children developed other problem-posing questions that were as mathematically rich (or even richer) as the ones we had posed. Despite the fact that we eventually revised our list substantially, it was beneficial to analyze the potential mathematical content embedded in the tasks (NCTM, 2000).

We reviewed our decision-making process with our research questions in mind. At this point our role was to provide children with experiences that incorporated key mathematical content and a structure that was carefully planned but not rigidly defined. During the two puzzle-solving sessions, we planned to observe the children with our list in mind, but we also intended to look for opportunities to build upon the children's interests for the follow-up inquiry investigations. We were ready to put our plan into action.

Early Negotiation: Building from Children's Observations

After giving the children time for free exploration with the tiles, we posed the challenge to find all possible shapes using 5 tiles. We instructed the children to keep a record of each shape on squared paper. We also asked them to respond to open-ended prompts in their math journals (e.g., "What do you notice? What do you find interesting? Surprising? What do your discoveries make you wonder?") (Whitin &

Whitin, 2000). Our intention for the prompts was twofold. We believed that through writing and drawing, the children would become more aware of their own problem-solving strategies (Gonzales, 1998). At the same time, analysis of their journals would inform our curricular decision-making. We encouraged the children to talk with their peers about their strategies and discoveries as they worked. We adults closely observed the children, taking notes about the ways they solved the problem, their spontaneous oral comments, and their interactions with peers and with us. By the end of the period, each child had discovered many of the shapes and, collectively, they found all 12.

After school, when we analyzed our notes and the children's journals, we identified aspects of the teacher's role that influenced our plans for the second day. In many instances children made spontaneous comments about their discoveries but did not capture their thinking in their journals. For example, when Julia noticed that she could flip one piece over and superimpose it on another piece to see if it matched (i.e., was congruent), David commented on her strategy and encouraged her to write it down. At the end of the period we made these kinds of discoveries public and then gave the children additional time to write and draw. We wanted to be sure to make children's thinking visible while it was still fresh in their minds and to encourage children to build off one another's ideas. During this post-sharing time, even children who had previously written very little added details and sketches. We suspected that these children used the examples to jump-start their own writing. Later, in examining the children's journals, we found that many of the entries were indeed related to the discoveries that we had shared aloud. We, therefore, wrote a reflective memo to be more intentional in making explicit the connections between the children's thinking and their record keeping and to build a climate of collaborative inquiry by systematically making individual discoveries public.

We also examined their journals in order to identify worthwhile problem-posing extensions suggested by their discoveries. We found a wide range of responses that included:

- While recording her solutions on the grid paper, Kaitlyn remarked that the resulting design looked like the computer game Tetris.

- Lisa was intrigued to find that two "L" shaped pieces could fit together to make a rectangle (see Figure 2).
- Rob commented, "If we used more tiles, we'd have more shapes."
- Dahlia noticed that the + shape could be rotated to look like an x.
- Anna observed, "If we could stand the tiles up on their edges, we could make a box."

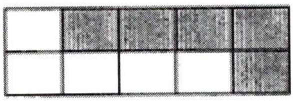


Figure 2. A Rectangle Composed of Two L-Shapes

When we studied these comments in light of our initial categories for analysis, we found evidence of risk-taking, inventiveness, and strategic thinking. Both Anna and Rob used the powerful word "if" to modify the attributes of the current problem and to envision the results of that change. We made a note to validate their willingness to change the problem and their strategic thinking (predicting and visualizing) and to make explicit the connection between their "if-statements" and problem-posing extensions (e.g., "What would happen if we used 6 tiles instead of 5?"). Dahlia and Lisa explored the properties of the shapes and discovered new relationships involving symmetry, while Kaitlyn made connections between the current problem and her previous experience. These discoveries likewise suggested further inquiry, but it would be our responsibility as teachers to make those possibilities evident for those individuals as well as their classmates.

Another observation we made about the first day's data surprised us in a different way. We humbly acknowledged that many of these comments implied mathematically-rich extensions not included on our preplanning list. In particular, it had not occurred to us that the grid paper could inspire problem posing. We now realized that Kaitlyn's spontaneous observation could be extended to an exploration involving tessellations. In fact, each of the 12 shapes does tessellate (Hyde &

Hyde, 1991). We had not anticipated that the children's discoveries would impact our preplanning after only one day. This experience afforded us an immediate opportunity to enhance not only Kaitlyn's, but also the other children's, confidence and curiosity by informing the class about the relationship between their discoveries and our plans. We added to our reflections to keep alert for other similar examples.

Drawing out Children's Strategic Thinking

With these new considerations in mind, we devoted another class period to the box-making experience. We used Anna's "what if" statement to introduce the lesson and to link her thinking to the work of mathematicians. We began, "Yesterday Anna wondered 'what if' we didn't keep the tiles flat? If we changed the rules for yesterday's puzzle, when we kept the tiles flat, we could make 3-dimensional shapes like boxes. Her idea is like a second puzzle that we will investigate today." We then described how mathematicians discover new patterns and relationships by asking "What if," making predictions, and developing strategies for investigation.

Next, we displayed the 12 possible pentomino solutions on the overhead and posed the problem, "Which of these shapes can be folded into a box without a top?" (Walter, 1996). Our instructions included:

- Copy each of the shapes on 1" grid paper and cut them out.
- Select a shape that you predict could become a box and draw it in your math journal. Explain your reasoning for the prediction.
- Try making the box, and record and explain the results.
- Continue with the other shapes. Describe what you notice. What about your results do you find interesting and/or surprising?

These prompts built upon those from the previous day. Open-ended invitations to "notice," "find interesting and/or surprising" could elicit observations that suggested further problem posing. We included directions to "predict" and "explain your reasoning" so that the children would make visible their strategic thinking (Whitin & Whitin, 2000).

This task was potentially more frustrating than the first, in which all of the children could be immediately successful in finding a variety of shapes with 5 tiles. In this task, 8 of the 12 shapes can be folded into a box (see Figure 1). We therefore closely observed the children to see how they tolerated the potential frustration. We were surprised and pleased to see how perseverant they were. One child, Andrea, decided to experiment with a shape that "didn't work" (see Figure 1-g), and found that she could fold it into "an envelope" rather than a box. To us, her willingness to change the rules and make a new discovery demonstrated a problem-posing spirit. Her classmate Ron again used the "what if" phrase to suggest an extension, commenting, "If we had 6 squares, we could make boxes with tops."

Other children discovered similarities between shapes that did "work" and strategically used that information. For example, Sheldon wrote that one of the successful box-making shapes "kind of looks like a mirror" (see Figure 3). He then searched through the shapes to find another "kind of mirror," and it formed a box as well. (In our class discussion later, we noted that his first example shows rotational symmetry while the second is an example of bilateral symmetry, or mirror image.)

At the conclusion of the period, we again invited the children to share their discoveries. We encouraged Sheldon, among others, to describe his problem-solving strategy and the way he preserved his thinking in his writing and drawing. Children like Andrea and Ron explained their extensions of the problem. We expanded upon their comments with our observations, especially highlighting their "never give up" attitude and "what if" stance. After this public forum, we provided a few minutes for the children to add to their journals.

These observations prompted us to reflect further about the data we were collecting. At this point, we suspected that the trial and error process required in making the 12 shapes contributed to the children's positive attitudes toward the box-making challenge. We believed that it was important for us to provide multiple opportunities for the children to take risks. We also saw that we could build their confidence by using *their* words and ideas in the real work of the classroom (e.g., using Anna's "what if" statement to introduce the activity) (Johnston, 2004). Using the children's observations as the basis for the problem-posing inquiries would build upon this idea.

However, we faced a new challenge when we examined how our list of possible extensions had grown considerably during this second activity. We, therefore, needed to develop criteria to guide us in refining the choices to offer the children for further exploration.

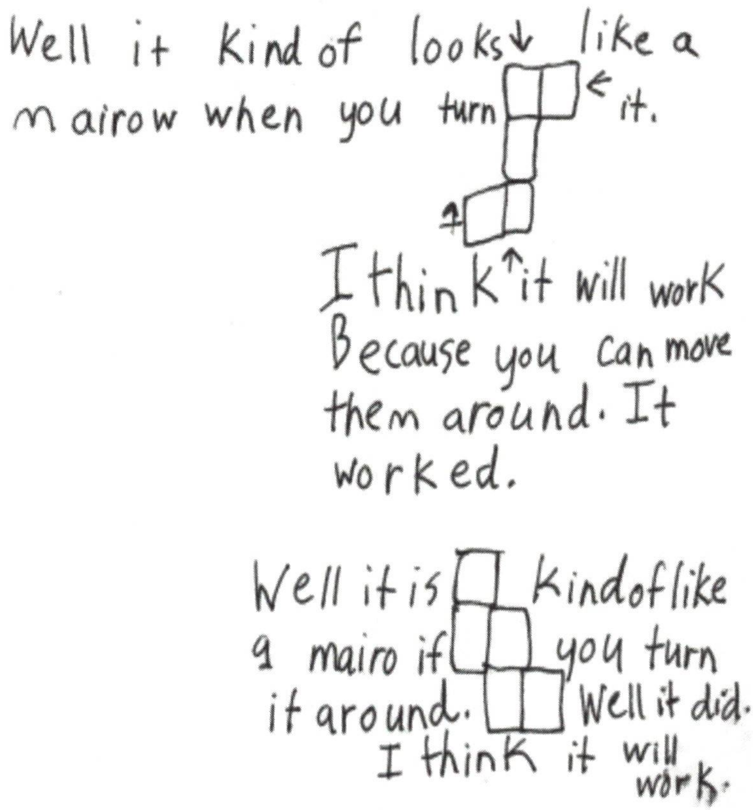


Figure 3. Sheldon's Discovery

Refining the Problem-Posing Choices

Based upon our notes from the two activities, we generated the following list of child-inspired problem-posing questions:

1. What if we used more tiles?
2. What if we tried to fill an area completely with all of the pentominoes? With 1, 2, or more shapes used again and again?
3. What if we tried to make a box with a top using 6 tiles?
4. What if we tried to make other shapes, such as envelopes?
5. What if we looked for a relationship between symmetrical shapes and their ability to be folded into boxes?

The children had generated many fascinating possibilities that we had not anticipated during our preplanning. Having so many potential investigations, however, raised a new dilemma. Although we wanted to include ideas from all the children, past experiences had taught us that offering too many choices becomes overwhelming for them and unmanageable for the teacher. In order to refine our list and limit the choice of investigations, we decided to follow two main guidelines:

1. Look for ideas with the greatest mathematical potential.
2. Incorporate as many of the children's ideas as possible.

The first criterion involved analyzing a task in terms of grade-level curricular expectations, opportunity for conceptual understanding and strategic thinking, and accessibility for children with a range of experience and expertise (NCTM, 2000). The second criterion addressed the children's growth in positive attitudes and dispositions (e.g., initiative, risk-taking). We wanted all of the children to recognize connections between their discoveries and the choices for further investigations. Children's observations that had less mathematical potential could be conceptually linked to our final choices. Part of our teacher role would be to make these connections visible, both to individual children and to the class as a group.

The challenge for filling a grid (inspired by Kaitlyn and Lisa and others) illustrates how we applied these criteria. We adults considered the range of mathematical ideas that would be addressed by investigations with area:

- Multiplication with an array model (e.g., Can we completely fill an 8"x10" grid with pentominoes?);
- Flips, slides, and turns (e.g., In what ways do pieces need to be flipped or turned to fill the grid?);
- Tessellations (e.g., Which pieces or combinations of pieces make repeating patterns with no spaces or overlaps?); and
- Symmetry (e.g., Do shapes with bilateral symmetry fill a grid most easily?).

Multiplication with an array model, transformations, and symmetry addressed grade-level expectations, and tessellations was an enrichment topic in the text. The interconnected ideas were consistent with NCTM's (2000) *Connections Standard*: "The notion that mathematical ideas are connected should permeate the school mathematical experience at all levels" (p. 64). Further, when we applied our second criterion, we noticed that most of the children's ideas were tied to one or more of these mathematical topics. Thus, filling an area seemed to be an appropriate problem-posing choice.

Using the criteria made it easy to discard several ideas, particularly two from our original adult-generated list. During the initial investigations, we had casually remarked to a few children, "I wonder what would happen if we tried triangles instead of squares?" Not one child pursued this suggestion during conversations or in their journals. With so many other fruitful suggestions to investigate, we decided to omit an exploration of triangles. We also eliminated the idea of using cubes instead of squares. Our underlying intention, to explore relationships between 2- and 3-dimensional shapes, was better addressed through the children's extensions of the box activity. After applying the criteria, the final choices for children included:

1. What if we tried to find different shapes with 6 tiles?
2. What if we tried to make boxes with tops using 6 tiles?
3. What if we tried to fill a grid with some pentomino shapes (1, 2, or more different shapes)?

Problem-Posing Investigations

The plan for the day of problem-posing investigations included having the children choose a problem from the refined list, make a plan, carry out the investigation, and share the results with the group. As before, we introduced the session by explaining how the children's comments, observations, and "what-if" statements related to the list we had developed. We also explained to them how we had discarded ideas from our original list based upon their ideas and discoveries. We wanted children to recognize how we as teachers could "grow and benefit from their interactions with students" (NCTM, 2000, p. 19). After this introduction, Sheldon raised his hand. Studying the list of choices, he offered a suggestion, "What if we used 4 tiles?" We wondered if our description of our changed roles had encouraged his response. To us, his comment was a healthy sign of the children's growing initiative and problem-posing spirit. In order to keep the choices manageable yet still honor his question, we suggested that he write down his idea to pursue after trying one of the listed choices.

Next we asked the children to choose one problem from the list, record it in their journals, and make a prediction or hypothesis about their solutions. We found that while the children worked, some of them modified their original ideas and pursued related tasks with our approval. In our later analysis we interpreted their modifications as further evidence of their willingness to take risks and pose their own problems by "making their own rules" (Brutlag, 1993, p. 134). Whether or not children changed their plans, however, we believe that having the children begin with a commitment on paper contributed to the overall success of the day.

Ron was one of the children to choose making boxes with tops. He wrote, "I want to use 6 tiles to make a whole box" in his journal. His procedure interested us. Instead of beginning by sketching a variety of arrangements of 6, he reviewed his entry from the initial box-making activity. We made a note to share his strategy with the class. We also thought it was likely that Sheldon's description of his "mirror" strategy from the previous day influenced Ron's strategic thinking.

Using his earlier journal did help Ron get started. He found that the + shaped pentomino was one of the easiest shapes to assemble into an open box. Building from this success, he predicted that a slightly modified shape would work as well, which it did (see Figure 4). Next, he developed a systematic plan to change this basic configuration in sequential steps. First, he moved one square unit down on one side (see arrow on right). The resulting shape also assembled into a cube. He then returned to his original design and moved a square on the left side up (see arrow on left). He continued to work in this systematic fashion to generate other solutions. Giving Ron additional time to explore the relationship between two- and three-dimensional figures afforded him the opportunity to develop this deductive problem-solving strategy and to invent an appropriate representation to record it. Later, during our final whole-class sharing, Ron demonstrated his strategy for the class. His sharing gave us the opportunity to discuss with the children a functional purpose of record-keeping: Writing and drawing allow us to preserve our thinking and use past experience to inform new learning (NCTM, 2000). We also affirmed his use of arrows as a valuable form of mathematical notation.

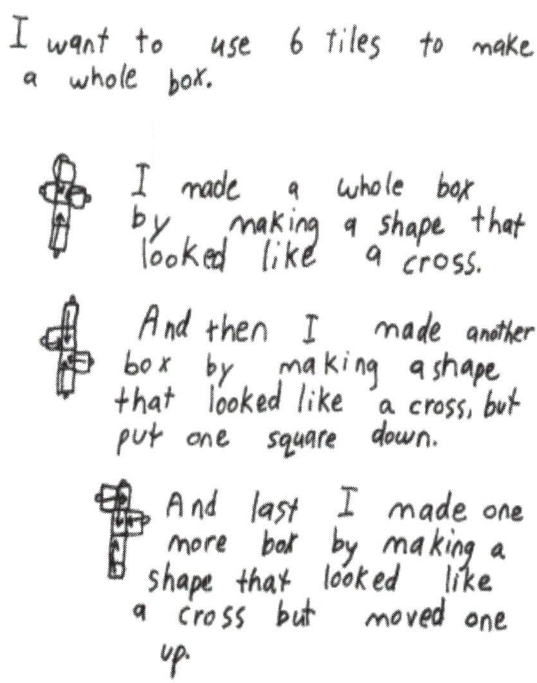


Figure 4. Ron's Prediction

The children who chose to fill an 8”x10” grid with areas containing 5 or 6 squares made other kinds of discoveries. They approached the task in various ways. Some chose one or two pentomino shapes and used them repeatedly to cover as much of the grid as possible, while others used all 12 shapes. By selecting the choice of filling an area, all these children gained valuable experience with transformational geometry. They constantly used flips, slides, and turns to help them fit the pieces together.

Several of these children also made numerical calculations as they filled the grid. Rather than use cutouts of the various shapes made from 6 tiles, Julia devised a more efficient strategy. She decided to color in 6 squares at a time, filling in as many squares as she could. She quickly verified the prediction that there would be more “ways” for arrangements of 6 tiles and, in fact, discovered many of the possible configurations. When the sheet was filled, she noticed that she had two unshaded squares left and commented, “I colored in 6 every time, and now I have 2 left over, so I colored 78 squares.” David asked her to explain her reasoning. She replied that 10 rows of 8 make 80, so 2 less is 78. He then challenged, “Is there another way to get the answer?” She counted 13 groups of 6, multiplied by 6, and verified the total, 78. This example again highlighted the teacher’s role to extend the children’s learning while following their lead. In this case, Julia both checked her work for accuracy and related two computational strategies.

With some adult encouragement, some of the children working with the grid showed increased levels of collaboration. When Brad found the two empty spaces left by 13 groups of 6, he predicted, “You could fill it up with 10-shapes” (imagining 8 shapes with 10 squares each). Dahlia overheard his remark and wondered aloud, “Could you fill it up with threes?” David suggested that they might try her idea, and the two collaboratively worked to find other factors for 80. Similarly, when Cody found the two remaining spaces, he predicted that shapes of five would work to fill the entire grid. “How do you know?” I (Phyllis) asked. Nate overhead this interchange and reasoned, “It would take 8 of those 5-shapes ... No ... 16, ’cause 8 x 5 = 40, and then times 2 (2 sets of 40) is 16.” Brad tried 6 squares and then predicted the results for 10, Cody tried 6 squares and then predicted 5, and their classmates extended or verified their ideas.

These examples suggest that the children were viewing problems as publicly owned. Dahlia felt comfortable extending Brad's observation, and Nate justified Cody's prediction. We suspected that our explicit oral connections and the daily summative forums played a major role in this trend. The children were beginning to establish a community of inquirers by collaborating based on a problem of mutual interest or by offering additional justification for a prediction. They also initiated their own additional problem-posing ventures. In the process they applied geometric principles and calculations involving multiplication to problem-solving contexts.

Reflection and Assessment

We concluded the mini-unit with a final strategy-sharing forum as well as individual reflection and assessment. By devoting time to a whole-group sharing session at the conclusion of these investigations, all of the children had the opportunity to learn alternative problem-solving strategies and ways to represent ideas through diagrams, such as Ron's. We also developed a rubric that incorporated several criteria: thinking flexibly; developing efficient, effective problem-solving strategies; taking risks; explaining mathematical processes clearly and accurately; and showing persistence (see Figure 5a). Part of our intent in designing the rubric was to raise the children's awareness about the important role that attitudes play in developing mathematical skills and strategies. In one column the children self-assessed their work; I added my evaluation from the teacher's perspective in the second. This rubric, along with a description of the project (see Figure 5b), initiated home-school communication.

Results of the Investigation

The goals of this research project were to explore the nuances of negotiated mathematical inquiry with particular attention to the teacher's role; to examine the development of students' attitudes, skills, and strategies in problem-solving contexts; and to identify routines and structures that facilitate these processes. The framework for negotiated inquiry (see Figure 6) grew out of the final analysis of the data.

Category	(Child's Name)'s Evaluation	Teacher's Evaluation
A: Thoughtful plans		
B: Persistence and industry		
C: Clear explanations		

Criteria:

A: Made thoughtful plans based on discoveries about shape; recorded plans in journal.

B: Showed persistence and worked industriously (e.g., trying even when the task was hard, looking for alternative solutions).

C: Wrote and drew to explain thinking clearly and accurately in mathematics journal.

Scoring:

3: Fully achieved the purpose of the task, including thoughtful, insightful interpretations and conjectures.

2: Completed most of the assignment and communicated ideas.

1: Did not accomplish the task; did not communicate ideas.

Figure 5a. Rubric for Final Assessment

The attached paper is an example of a kind of evaluation that we will use a lot this year. The children spent several days investigating patterns with geometric shapes, both two- and three-dimensional. They were required to record their findings in their journals. The chart, or rubric, names some of the important features of the task: making plans, persistence and industry, and explaining discoveries and reasoning in writing and diagrams. I asked the students to evaluate themselves on these criteria. (Of course, I described the categories in simpler terms.) I was very impressed with the seriousness and honesty with which they approached the task. I also gave my view of their work. There was a remarkable degree of agreement between me and individual children. When we differ, I am willing to discuss the matter more in depth with a child to further understand a child's perspective. The ultimate decision, however, does rest in my hands. I am excited about our first experience at self-reflection, and of course I wanted to share it with you.

I was particularly impressed with Ron's investigation of the three-dimensional cubes. He developed a sophisticated strategy to create different configurations of areas of six (six squares). He cut out his patterns and proved that certain arrangements could be folded successfully into perfect cubes. He was able to explain his thinking to the class.

Figure 5b. Sample Letter to Parents/Caregivers

The first part of the framework involves planning and preparation: choosing an appropriate problem, identifying attributes of the problem, and developing a tentative list of problem-posing extensions (Brown & Walter, 1990). Considerations in choosing a problem include grade-level curricular expectations, potential connections among mathematical concepts involved (NCTM, 2000), the prior experiences of the students, and their existing attitudes and dispositions. The incoming 4th graders in this study had limited experience talking and writing about their mathematical thinking, and their prior instruction had focused primarily on computation. We, therefore, chose a geometry puzzle using manipulatives (Gonzales, 1998; Moses, Bjork, & Goldenberg, 1990). Another part of the preparation process entails "anticipating the mathematical ideas that can be brought out by working on the problem" (NCTM, 2000, p. 53). After exploring the pentomino puzzle and identifying its attributes and parameters, we developed a tentative list of problem-posing extensions. Although this list served as a guide, we later learned how crucial it was to remain open-minded for new, more mathematically rich extensions to emerge from the children as they worked.

The second part of the framework includes one or more class sessions devoted to mathematical investigations. Key dimensions of the teacher's role emerged during these sessions. First, we needed to listen and observe closely as the children worked. Often their observations implied problem-posing extensions, but it was incumbent upon us to help them make their thinking visible (Gonzales, 1998; Lindfors, 1999; Whitin, 2004) by drawing out their voices. Providing open-ended response prompts gave structure yet flexibility to record observations. Even so, some children needed encouragement to bridge their oral comments to a written or drawn form. Second, the data suggested that the ways in which we responded to the children's ideas contributed to their positive attitudes and dispositions. For instance, we publicly legitimized Anna's willingness to modify the first-day's problem with her what-if statement. During subsequent investigations, many more children showed similar attitudes of risk-taking and inventiveness. Third, we found it valuable to make explicit the ways in which the children's thinking influenced our own. Our candid willingness to change perspectives, to abandon our first ideas, and to adopt others were all powerful demonstrations of the very kinds of attitudes we were trying to instill in the students. In addition, casting ourselves as learners as well as teachers served to empower the children as co-constructors of the problem-posing investigations. Finally, it was important for us to provide regular opportunities for the children to

A Framework for Negotiated Mathematical Inquiry

A. Planning and Preparation:
   1. Choose a mathematical investigation that has possibilities for a wide range of extensions. These experiences should embed several mathematical content strands and provide challenge for an array of mathematical learners.
   2. Prepare a tentative list of problem-posing questions that might arise from the initial experience.

B. Initial Experience(s) (one or more class sessions):

   1. Use open-ended prompts for children to use as they record the problems and their findings, and provide several minutes of concentrated writing time at the end of a work session. Suggested prompts include:
      a. What do you notice (about your pattern, construction, problem-solving strategy)?
      b. What do you find interesting? Surprising?
      c. What do your discoveries make you wonder about?
   2. During the work time, circulate and take notes about children's comments, surprises, anomalies, and discoveries. Add notes during the follow-up discussion of findings and as you review children's writing and drawings.

C. Refining Problem-Posing Choices

   1. Using the notes from class, revise your tentative problem-posing list, highlighting those that capitalize upon the children's interests. Add additional questions from children's unanticipated observations.
   2. Group similar questions, forming 3-5 clusters. Identify a question for each cluster, preserving as much of the children's own language as possible. In this way the children will see how their observations and musings can be turned into systematic investigations.
   3. List the questions on chart paper.

D. Problem-Posing Investigations (one or more class sessions):

   1. Post the chart of problem-posing questions. Demonstrate how various children's oral and written observations have led to these questions.
   2. After a brief discussion, ask that each child:
      a. Choose one of the problem-posing questions from the chart, record it on paper, and make a detailed plan if necessary.
      b. Make a prediction (conjecture, theory, hypothesis) about the findings.
      c. Carry out the investigation, comment upon the findings and add new questions.
   3. Following the investigations, allow time for whole-group sharing.
   4. Assess the children's work with an appropriate rubric. Problem-posing explorations are ideal for inclusion in students' portfolios and for publication on a school website.

Figure 6. A Framework for Negotiated Mathematical Inquiry

describe to their peers their discoveries and strategies and to follow these sharing sessions with time for others to put these ideas to immediate use. These forums enhanced the children's sense of ownership and promoted a collaborative spirit. It is likely that Ron's box-making strategy was influenced by Sheldon's sharing of his "mirror" strategy and that Dahlia and others felt welcome to spontaneously offer alternative solutions to their peers.

The third step of the framework returns the focus to mathematical content. Here, teachers review both their own tentative list of problem-posing questions and the implied questions generated from the children's work during the initiating experiences. By classifying the children's observations into conceptual categories, we found that we could reduce the number of choices for investigation and yet encompass most, if not all, of the children's ideas. We could then frame a problem that would address key content, develop conceptual understanding, and connect mathematical topics.

In the final part of the framework, the children choose a problem from the refined list, carry out the investigation, and share the results with the group. The relationship between the children's observations and the problems for investigation should be clear. At the conclusion of the inquiry the children present their results to their peers. They can also summarize in writing and visually represent their findings to share with a larger audience. Finally, self-assessment gives the children another important opportunity to reflect on their problem-solving strategies, their discoveries, and their attitudes and dispositions.

Conclusion

Problem-posing is a valuable strategy for encouraging children to generate questions for extended mathematical inquiry. In honoring student-generated extensions, however, teachers often face the dilemma, "What do I do with *all* of these questions? There simply isn't time to pursue every idea." It is also the responsibility of teachers to provide for mathematical content that will most benefit our students' growth in conceptual understanding and strategic thinking. The framework for negotiated inquiry described here can be used to deepen mathematical understanding while fostering attitudes and dispositions that are key to problem solving across the disciplines. The children demonstrated some of these attitudes in the final reflections that accompanied their rubrics:

Julia:	"What if" helped me notice how many things you can do with tiles and how much fun you can have.
Cody:	I learned about all the designs you could make. "What if" helped me think about all my wonders.
Wendell:	"What if" helped me make more ideas in my pattern. "What if" also helped my class. "What if" I hope will be the top #1 word in the state.

In these comments, the children cite important mathematical attitudes and dispositions, such as the ability to look closely, reflect on one's discoveries, and generate new insights for oneself and the class. As noted in the introductory quote from the National Council of Teachers of Mathematics (2000), these habits, skills, and perspectives extend well beyond the context of mathematics instruction as well. Problem posing, as a particular form of mathematical inquiry, supports children to experience the excitement of discoveries, to travel uncharted paths, and to participate in a community of inquirers.

References

Brown, S., & Walter, M. (1990). *The art of problem posing.* Hillsdale, NJ: Lawrence Erlbaum.

Brutlag, D. (1993). Making your own rules. In S. Brown & M. Walter (Eds.), *Problem posing: Reflections and applications* (pp. 130-133). Hillsdale, NJ: Lawrence Erlbaum.

Glaser, B., & Straus, A. (1967). *The discovery of grounded theory.* New York: Aldine.

Goetz, J., & LeCompte, J. (1984). *Ethnography and qualitative design in educational research.* Orlando, FL: Academic Press.

Gonzales, M. (1998). A blueprint for problem posing. *School Science and Mathematics, 98*(8), 448-452.

Hitchcock, G., & Hughes, D. (1991). *Research and the teacher: A qualitative introduction to school-based research.* New York: Routledge.

Hubbard, R., & Power, B. (1993). *The art of classroom inquiry.* Portsmouth, NH: Heinemann.

Hyde, A. A., & Hyde, P. R. (1991). *Mathwise.* Portsmouth, NH: Heinemann.

Johnston, P. H. (2004). *Choice words.* Portland, ME: Stenhouse.

Lindfors, J. (1999). *Children's inquiry.* New York: Teachers College Press.

Moses, B., Bjork, E., & Goldenberg, E. P. (1990). Beyond problem solving: Problem posing. In T. J. Cooney (Ed.), *Teaching and learning mathematics in the 1990s: 1990 NCTM yearbook* (pp. 82-91). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Walter, M. (1996). *Boxes, squares, and other things.* Reston, VA: National Council of Teachers of Mathematics.

Whitin, D. J. (2004). Building a mathematical community through problem posing. In R. Rubenstein (Ed.), *Perspectives on the teaching of mathematics*: 2004 NCTM Yearbook (pp. 129-140). Reston, VA: National Council of Teachers of Mathematics. Whitin, P., & Whitin, D. J. (2000). *Math is language too: Talking and writing in the mathematics classroom.* Urbana, IL: National Council of Teachers of English and Reston, VA: National Council of Teachers of Mathematics.

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Endnote

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