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Rainbow Perfect and Near-Perfect Matchings in Complete Graphs with Edges Colored by Circular Distance

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Abstract

Given an edge-colored complete graph K_n on n vertices, a perfect (respectively, near-perfect) matching M in K_n with an even (respectively, odd) number of vertices is rainbow if all edges have distinct colors. In this paper, we consider an edge coloring of K_n by circular distance, and we denote the resulting complete graph by $K_n^\bullet$. We show that when $K_n^\bullet$ has an even number of vertices, it contains a rainbow perfect matching if and only if $n = 8k$ or $n = 8k+2$, where k is a nonnegative integer. In the case of an odd number of vertices, Kirkman matching is known to be a rainbow near-perfect matching in $K_n^\bullet$. However, real-world applications sometimes require multiple rainbow near-perfect matchings. We propose a method for using a recursive algorithm to generate multiple rainbow near-perfect matchings in $K_n^\bullet$.

1 Introduction

Given an edge-colored undirected graph G, a *rainbow matching* (or *heterochromatic matching*) M in G is a matching (a set of edges without common vertices) such that all edges have distinct colors. While it is possible to find a maximum matching in G in polynomial time, computing a maximum rainbow matching is known to be an NP-hard problem. Indeed, the decision version of this problem is a classical example of NP-complete problems, even for edge-colored bipartite graphs [1].

If a graph G has an even number of vertices, a *rainbow perfect matching* in G is a rainbow matching that matches all vertices of the graph. If the number of vertices is odd, a *rainbow near-perfect matching* M in G is a rainbow matching in which exactly one vertex is unmatched. Note that in this paper, we use *RPM* to mean either a rainbow perfect matching or a rainbow near-perfect matching in the corresponding graph.

Over the past decade, finding RPMs in edge-colored graphs or hypergraphs has been studied for several classes of graphs, including complete bipartite graphs [2], r-partite graphs [3], Dirac bipartite graphs [4], random geometric graphs [5], and k-uniform, k-partite hypergraphs [6]. Most of the above studies assume that there are more colors used for coloring than there are colors among matchings. If we do not insist on perfect matchings, other studies have demonstrated the existence of large rainbow matchings in arbitrarily edge-colored graphs. Letting $\hat{\delta}(G)$ denote the minimum color degree of an edge-colored graph G, Wang and Li [7] first showed the existence of a rainbow matching whose size is $\left\lceil \frac{5\hat{\delta}(G)-3}{12} \right\rceil$, and conjectured that a tighter lower bound $\left\lceil \frac{\hat{\delta}(G)}{2} \right\rceil$ exists if $\hat{\delta}(G) \geq 4$ holds. LeSaulnier et al. [8] proved a weaker statement, that an edge-colored graph G always contains a rainbow matching of size at least $\left\lfloor \frac{\hat{\delta}(G)}{2} \right\rfloor$. Kostochka and Yancey [9] completed a proof of Wang and Li's conjecture. Letting n be the size of vertices of a graph G, Lo [10] showed that an edge-colored graph G contains a rainbow matching of size at least k, where $k = \min\left\{\hat{\delta}(G), \frac{2n-4}{7}\right\}$. If the graph is properly edge-colored (i.e., no two adjacent edges have the same color), several results [11, 12, 13, 14] have provided lower bounds for the size of a maximum rainbow matching. Other recent studies [15, 16, 17] have focused on finding large rainbow matchings under the stronger assumption that the given is strongly edge-colored. See Kano and Li [18] for a deeper analysis of rainbow subgraphs in an edge-colored graph.

To the best of our knowledge, there has been no research on finding RPMs in edge-colored complete graphs with exactly $\lfloor \frac{n}{2} \rfloor$ colors. If we consider any exact $\lfloor \frac{n}{2} \rfloor$-edge coloring of K_n where $n \geq 4$, the results in [19] show that the size of a maximum rainbow matching is bounded by $\mathcal{O}\left(\sqrt{\frac{n}{2}}\right)$. This indicates that there does not always exist an RPM in K_n by an arbitrary $\lfloor \frac{n}{2} \rfloor$-edge coloring. Thus, we consider RPMs in edge-colored complete graphs by a special $\lfloor \frac{n}{2} \rfloor$-edge coloring, called a circular-distance edge coloring. Such an edge-colored complete graph with n nodes is denoted as $K_n^{\bullet}$.

Scheduling for round robin tournaments is known as an important application for searching RPMs in $K_n^{\bullet}$. For generating a feasible round-robin tournament schedule, Kirkman [20] proposed a method based on a special RPM (Kirkman matching) in $K_n^{\bullet}$. Although Kirkman's method is popular for generating schedules for round-robin tournaments in European soccer leagues [21], it can produce unbalanced or unfair schedules [22, 23]. Other feasible solutions to the round-robin tournament scheduling problem may thus be needed for the next scheduling stage. Anderson [24] proposed an approach by enumerating all RPMs in $K_n^{\bullet}$ for balancing carry-over effects in tournaments with up to 24 teams. However, even though multiple RPMs in the graph $K_n^{\bullet}$ with an odd number of vertices are required in real-world applications, it has not been confirmed whether RPMs other than Kirkman matching exist for an arbitrary n.

The rest of the paper is organized as follows. In Section 2, we introduce basic notations, and we describe an application in tournament scheduling problem. In Section 3, we show results for the existence and non-existence of such RPMs in $K_n^{\bullet}$ when n is even. For $K_n^{\bullet}$ with an odd number of vertices, Kirkman showed there exists an RPM [20], which we call Kirkman matching. In Section 4, we show that there exists other RPMs different from Kirkman matching when n is an odd number larger than or equal to 7, and we then propose a method to generate multiple RPMs by using a recursive algorithm.

2 Preliminaries

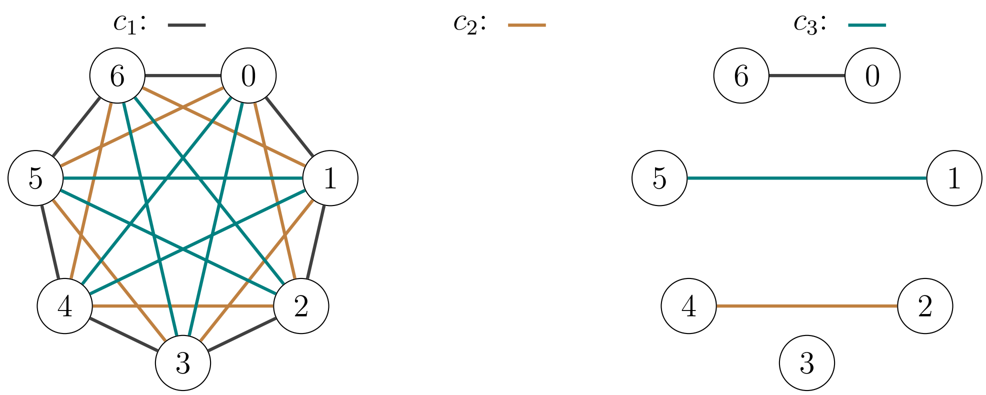
In this paper, all graphs are simple and undirected. A graph G is defined as $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We define a vertex set V containing n vertices as $V = \{0, 1, \ldots, n-1\}$, unless indicated otherwise. Given a graph G and a set of distinct colors $C = \{c_1, c_2, \ldots\}$, the *circular-distance edge coloring* of a graph G is a mapping $h : E(G) \to C$, where an edge $\{i, j\} \in E(G)$ is colored as

$$h(\{i, j\}) = c_{\min\{|i-j|, n-|i-j|\}}. \tag{1}$$

We denote by $K_n^{\bullet}$ the edge-colored complete graph K_n with each edge e colored by $h(e)$. According to coloring (1), $K_n^{\bullet}$ is colored in exactly $\lfloor \frac{n}{2} \rfloor$ colors, and it is not properly colored if $n \geq 3$. Here, we aim to find an RPM M in $K_n^{\bullet}$, that is, a rainbow matching M where $|M| = \lfloor \frac{n}{2} \rfloor$. Figures 1 and 2 respectively show examples of $K_7^{\bullet}$ and $K_8^{\bullet}$ with their RPMs.

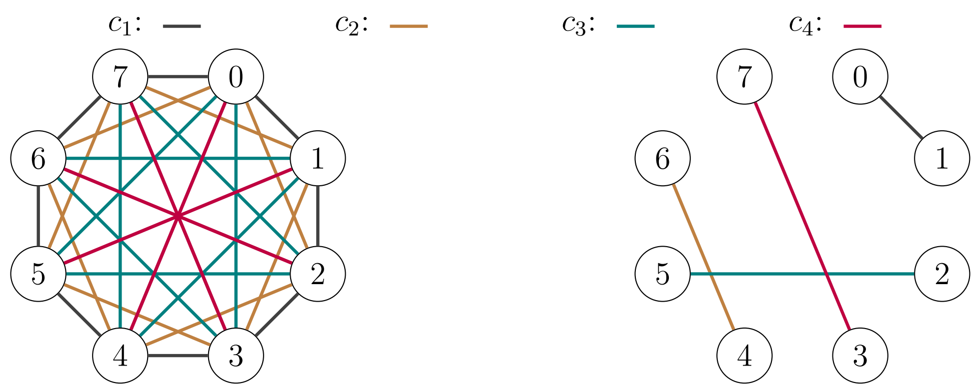
Given an RPM M in $K_n^{\bullet}$, an *α-rotated matching* of M denoted by $\mathrm{rot}(M, \alpha)$ is defined as

$$\mathrm{rot}(M, \alpha) = \{\{(i + \alpha) \bmod n, (j + \alpha) \bmod n\} \mid \forall \{i, j\} \in M\}. \tag{2}$$



(a) The edge-colored graph $K_7^\bullet$ (b) A rainbow near-perfect matching in $K_7^\bullet$

Figure 1: Examples of $K_7^\bullet$



(a) The edge-colored graph $K_8^\bullet$ (b) A rainbow perfect matching in $K_8^\bullet$

Figure 2: Examples of $K_8^\bullet$

Figure 3 shows the 1-rotated matching $\mathrm{rot}(M, 1)$ for the RPM M in Figure 1b. This example shows that if we arrange vertices $0, 1, \ldots, n-1$ around a cycle in clockwise order, $\mathrm{rot}(M, \alpha)$ matches by rotating matching M by $\frac{2\alpha}{n}\pi$ in the clockwise direction.

If M is an RPM in the graph $K_n^\bullet$, we observe that the following property holds from definition (2):

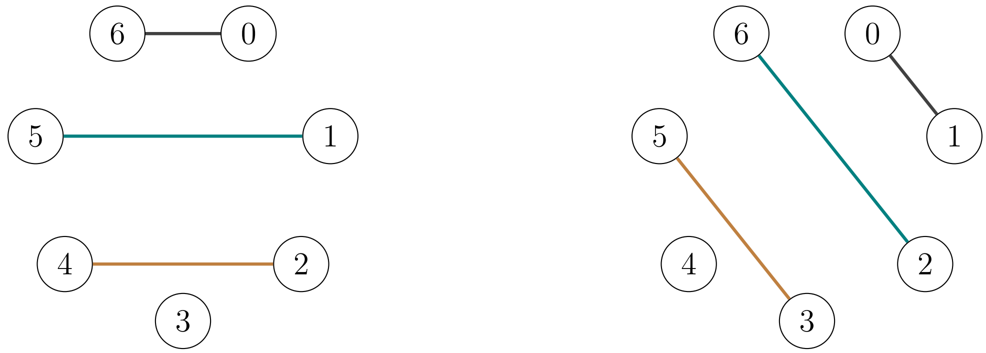
Property 2.1. $\forall \alpha \in \mathbb{Z}, \mathrm{rot}(M, \alpha)$ *is an RPM.*

Next, we define the reversed matching of M as $\mathrm{rev}(M)$:

$$\mathrm{rev}(M) = \{\{n-1-i, n-1-j\} \mid \forall \{i, j\} \in M\}. \tag{3}$$

Figure 4 shows the reversed matching $\mathrm{rev}(M)$ for the RPM M in Figure 2b. Given an RPM M in the graph $K_n^\bullet$, the following property holds by definition (3):

Property 2.2. *The matching* $\mathrm{rev}(M)$ *is also an RPM in the graph* $K_n^\bullet$.



(a) A rainbow near-perfect matching M for $K_7^\bullet$ (b) The 1-rotated matching $\mathrm{rot}(M, 1)$

Figure 3: Example 1-rotated RPM

(a) A rainbow perfect matching M in $K_8^\bullet$ (b) The reversed matching $\mathrm{rev}(M)$

Figure 4: Example reversed RPM

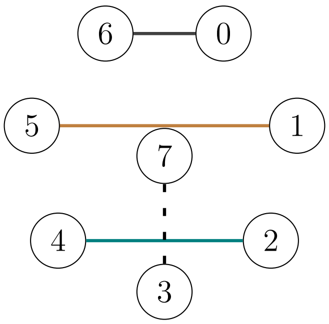
2.1 Round-robin tournament scheduling problem

In real-world applications, some combinatorial problems and their sub-problems are related to searches for an RPM in $K_n^\bullet$. A well-known example is scheduling for round-robin tournaments.

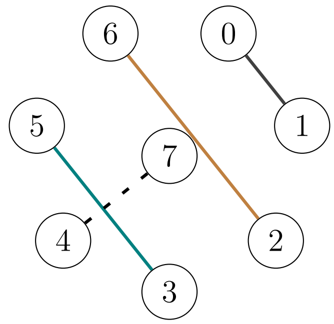
Given n teams (where n is even), a tournament organizer must decide which games take place in which rounds. The *round-robin tournament scheduling problem* (RTSP) aims to generate a schedule with $n-1$ rounds such that each pair of teams is matched exactly once, and each team plays exactly one game in each round.

Translating this problem into the language of graph theory, let each team be a vertex, and let an edge connecting vertices i and j represent a game between teams i and j. A perfect matching in the complete graph K_n thus describes a round of $n/2$ games. Therefore, the RTSP with n teams is the problem of decomposing the complete graph K_n into $n-1$ perfect matchings [25, 26, 27].

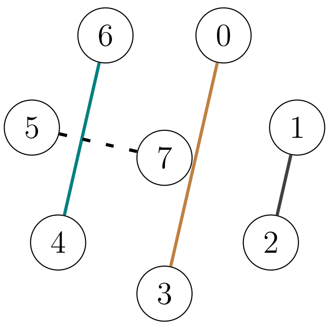
Next, we show that a decomposition of K_n can be formed based on any rainbow near-perfect matching M in $K_{n-1}^\bullet$.



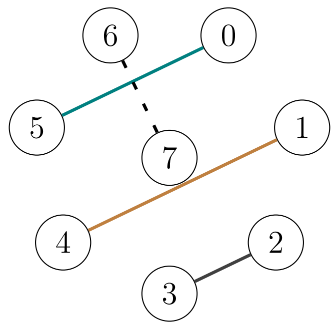
(a) Games in round 1



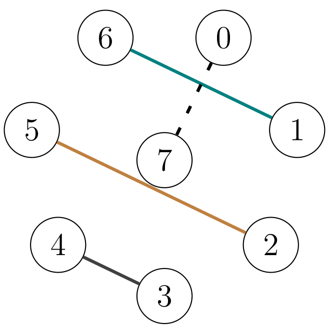
(b) Games in round 2



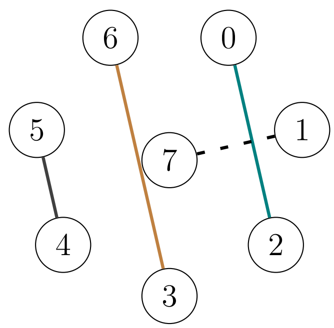
(c) Games in round 3



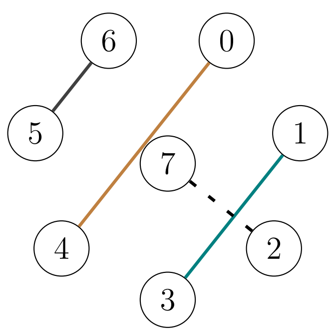
(d) Games in round 4



(e) Games in round 5



(f) Games in round 6



(g) Games in round 7

Figure 5: Feasible schedule of 8 teams based on a perfect rainbow matching of $K_7^\bullet$

Letting vertex $i' \in V(K^\bullet_{n-1})$ be a vertex not matched by M,

$$\mathrm{rot}(M, i) \cup \{\{(i' + i) \bmod (n-1), n-1\}\}, \qquad \forall i \in \{0, 1, \ldots, n-2\} \tag{4}$$

is a feasible decomposition of K_n. Another decomposition using the reversed RPM $\mathrm{rev}(M)$ can be similarly constructed as

$$\mathrm{rot}(\mathrm{rev}(M), i) \cup \{\{(n-1-i'+i) \bmod (n-1), n-1\}\}, \quad \forall i \in \{0, 1, \ldots, n-2\}. \tag{5}$$

Figure 5 shows a feasible schedule for $n = 8$ teams (a decomposition of K_8) using method (4) based on the RPM in Figure 1b. The matchings, generated by method (4) or method (5), partition the edge set of K_8 into 7 perfect matchings. Kirkman [20] first proposed this framework for scheduling round-robin tournaments. This approach uses a *Kirkman matching* M^{kir}_n in $K^\bullet_n$ as

$$M^{\mathrm{kir}}_n = \left\{ \{i, n-1-i\} \;\middle|\; \forall i \in \left\{0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right\} \right\}. \tag{6}$$

We call a schedule generated from Kirkman matching a *Kirkman schedule*. Figure 5 shows a Kirkman schedule with 8 teams. A Kirkman matching M^{kir}_n has the following special properties:

Property 2.3. *A Kirkman matching M^{kir}_n in $K^\bullet_n$ is an RPM if and only if $n = 2$ or n is odd.*

Property 2.4. *For an RPM M in the graph $K^\bullet_n$, if there exists α such that $\mathrm{rot}(M, \alpha) = \mathrm{rev}(M)$, then there exists β such that $M = \mathrm{rot}(M^{\mathrm{kir}}_n, \beta)$.*

Property 2.4 indicates that Kirkman matching M^{kir}_n and its α-rotated matchings are the only matchings that make decomposition (4) and decomposition (5) the same.

2.2 Normalization

To obtain different solutions by using method (4) and method (5), we first design an algorithm $\mathsf{norm}(M)$ that normalizes an RPM M using reverse and rotate operations. Algorithm 1 shows details of this normalization. We first clockwise rotate the matching M until edge $\{0, n-1\} \in M$ meets (lines 1–4). We then check each edge $\{i, j\}$ in M in ascending order of their color indexes. For the current edge $\{i, j\}$, if $i + j > n - 1$, the normalization finishes and returns $\mathrm{rev}(M)$; if $i + j < n - 1$, the normalization ends with the current M (lines 5–12). The normalization procedure returns the current M when all edges have been checked (line 13).

Properties 2.1 and 2.2 ensure that if M is an RPM in the graph $K^\bullet_n$, the normalization $\mathsf{norm}(M)$ is still an RPM. We call an RPM M a *normalized PRM* (N-RPM) if $\mathsf{norm}(M) = M$.

Regarding N-RPMs in the graph $K^\bullet_n$, the following properties hold:

Property 2.5. *If two N-RPMs M and M' are different, M' cannot be obtained from M by using α-rotate operators and reverse operators,*

Property 2.6. *A Kirkman matching M^{kir}_n is an N-RPM.*

Algorithm 1 Normalize a rainbow perfect matching: $\mathsf{norm}(M)$

Require: an RPM M.
Let $\{i, j\}$ be the edge colored in c_1 in M and $j > i$.
if $\{i, j\} \neq \{0, n-1\}$ **then**
  $M \leftarrow \mathrm{rot}(M, -j)$.
end if
for $k = 2$ **to** $k = \left\lfloor \frac{n}{2} \right\rfloor$ **do**
  Let $\{i, j\}$ be the edge colored in c_k in M.
  if $i + j < n - 1$ **then**
    return M.
  else if $i + j > n - 1$ **then**
    return $\mathrm{rev}(M)$.
  end if
end for
return M.

3 Rainbow perfect matchings in $K_n^\bullet$ with an even number of vertices

Letting k be a nonnegative integer, we show the results of considering the following two cases:

- $n = 8k$ or $n = 8k + 2$;
- $n = 8k + 4$ or $n = 8k + 6$.

3.1 Non-existence of an RPM when $n = 8k + 4$ or $n = 8k + 6$

We show the non-existence of RPM in the graph $K_n^\bullet$ for any $n \in \{8k + 4, 8k + 6\}$.

Theorem 3.1. *For any $k \in \mathbb{Z}_{\geq 0}$, no RPM exists in the graph $K_n^\bullet$ if $n \in \{8k + 4, 8k + 6\}$.*

Proof. We color all vertices in $K_n^\bullet$ using a function $\chi\colon V(K_n^\bullet) \to \{black, white\}$:

$$\chi(v) = \begin{cases} black & \text{if the label of } v \text{ is odd,} \\ white & \text{if the label of } v \text{ is even.} \end{cases} \tag{7}$$

First consider the case where $n = 8k + 4$. Function χ obtains $4k + 2$ black vertices and $4k + 2$ white ones in $K_{8k+4}^\bullet$. Assume there exists an RPM M in $K_{8k+4}^\bullet$; that is, M is a matching in $K_{8k+4}^\bullet$ containing colors $c_1, c_2, \ldots, c_{4k+2}$. According to the edge coloring in (1), edges with colors $c_1, c_3, \ldots, c_{4k+1}$ in M occupy $2k + 1$ black vertices and $2k + 1$ white ones, because the endpoints of each edge are colored in different colors. The remaining edges in M with colors $c_2, c_4, \ldots, c_{4k+2}$ consume an even number of vertices in both white and black, because the endpoints of each edge are same-colored. This contradicts the fact that numbers of white and black vertices in $K_{8k+4}^\bullet$ are both even.

A similar result holds for the case of $n = 8k + 6$, where using function χ in (7) obtains $4k + 3$ black vertices and $4k + 3$ white ones. Assume there exists an RPM M in $K^{\bullet}_{8k+6}$ containing colors $c_1, c_2, \ldots, c_{4k+3}$. The edges in M with colors $c_1, c_3, \ldots, c_{4k+3}$ account for $2k+2$ black vertices and $2k+2$ white ones because the endpoints of each edge are differently colored. Edges colored $c_2, c_4, \ldots, c_{4k+2}$ in M require an even number of both black and white vertices, because the endpoints of each edge are same-colored. This contradicts the fact that the numbers of white and black vertices in $K^{\bullet}_{8k+6}$ are both odd. □

3.2 Existence of an RPM when $n = 8k$ or $n = 8k + 2$

First consider the case where $k = 0$. The existence of an RPM is obvious because $\emptyset$ and $\{\{0, 1\}\}$ are the RPMs in $K^{\bullet}_0$ and $K^{\bullet}_2$, respectively. We then focus on $n = 8k$ and $n = 8k+2$ with $k \geq 1$. For such cases, we design the following matching T_n in the graph $K^{\bullet}_n$:

$$T_n = T'_n \cup T''_n \cup T'''_n \cup \bar{T}_n \tag{8}$$

where

$$T'_n = \begin{cases} \{\{1+i, 8k-2-i\} \mid i = 0, 1, \ldots, 2k-3\} & \text{if } n = 8k, \\ \{\{1+i, 8k-i\} \mid i = 0, 1, \ldots, 2k-2\} & \text{if } n = 8k+2; \end{cases} \tag{9}$$

$$T''_n = \begin{cases} \{\{2k+i, 6k-i\} \mid i = 0, 1, \ldots, k-1\} & \text{if } n = 8k, \\ \{\{2k+1+i, 6k+1-i\} \mid i = 0, 1, \ldots, k-1\} & \text{if } n = 8k+2; \end{cases} \tag{10}$$

$$T'''_n = \begin{cases} \{\{3k+i, 5k-2-i\} \mid i = 0, 1, \ldots, k-2\} & \text{if } n = 8k, \\ \{\{3k+1+i, 5k-1-i\} \mid i = 0, 1, \ldots, k-2\} & \text{if } n = 8k+2; \end{cases} \tag{11}$$

$$\bar{T}_n = \begin{cases} \{\{0, 4k-1\}, \{2k-1, 8k-1\}, \{5k-1, 5k\}\} & \text{if } n = 8k, \\ \{\{0, 2k\}, \{4k, 8k+1\}, \{5k, 5k+1\}\} & \text{if } n = 8k+2. \end{cases} \tag{12}$$

Theorem 3.2. *For any $n = 8k$ or $8k + 2$ with $k \in \mathbb{Z}_{\geq 1}$, T_n is an RPM in the graph $K^{\bullet}_n$.*

Proof. Tables 1 and 2 summarize the features of T'_n, T''_n, and T'''_n from (9)–(11). The columns "vertices" and "colors" show the covered vertices and colors, respectively.

For the case where $n = 8k$, Table 1 shows that T'_n, T''_n, and T'''_n share neither vertices nor colors, and that vertices

$$0, 2k-1, 4k-1, 5k-1, 5k, 8k-1$$

and colors

$$c_1, c_{2k}, c_{4k-1}$$

remain unmatched. The edge set $\bar{T}_{8k} = \{\{0, 4k-1\}, \{2k-1, 8k-1\}, \{5k-1, 5k\}\}$ satisfies all the remaining requirements, so T_n is an RPM.

For the case where $n = 8k + 2$, the same result that T_n is an RPM can be obtained from Table 2 and the definition of $\bar{T}_n$. □

Figure 6 shows the examples of T_{16} and T_{18}.

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Table 1: Features of T'_n, T''_n, and T'''_n when $n = 8k$

	vertices	colors	size
T'_n	$\{1, 2, \ldots, 2k-2\} \cup \{6k+1, 6k+2, \ldots, 8k-2\}$	$\{c_3, c_5, \ldots, c_{4k-3}\}$	$2k-2$
T''_n	$\{2k, 2k+1, \ldots, 3k-1\} \cup \{5k+1, 5k+2, \ldots, 6k\}$	$\{c_{2k+2}, c_{2k+4}, \ldots, c_{4k}\}$	k
T'''_n	$\{3k, 3k+1, \ldots, 4k-2\} \cup \{4k, 4k+1, \ldots, 5k-2\}$	$\{c_2, c_4, \ldots, c_{2k-2}\}$	$k-1$

Table 2: Features of T'_n, T''_n and T'''_n when $n = 8k+2$

	vertices	colors	size
T'_n	$\{1, 2, \ldots, 2k-1\} \cup \{6k+2, 6k+3, \ldots, 8k\}$	$\{c_3, c_5, \ldots, c_{4k-1}\}$	$2k-1$
T''_n	$\{2k+1, 2k+2, \ldots, 3k\} \cup \{5k+2, 5k+3, \ldots, 6k+1\}$	$\{c_{2k+2}, c_{2k+4}, \ldots, c_{4k}\}$	k
T'''_n	$\{3k+1, 3k+2, \ldots, 4k-1\} \cup \{4k+1, 4k+2, \ldots, 5k-1\}$	$\{c_2, c_4, \ldots, c_{2k-2}\}$	$k-1$

4 Rainbow near-perfect matchings in $K_n^\bullet$ with an odd number of vertices

Kirkman proposed Kirkman matching nearly 180 years ago. Property 2.3 indicates that an N-RPM exists in the graph $K_n^\bullet$ with an odd number of vertices. In this section, we first show the existence of an RPM through what we call *arch-recursive-slide* (ARS) matching, whose N-RPM is different from the Kirkman matching when $n \in \{7, 9, 11, \ldots\}$. We then propose an algorithm for generating multiple N-RPMs based on ARS matching.

4.1 Arch-recursive-slide (ARS) matching

In general, any odd number n can be expressed as

$$8k+1, 8k+3, 8k+5, \text{or } 8k+7 \qquad k \in \mathbb{Z}_{\geq 0}. \tag{13}$$

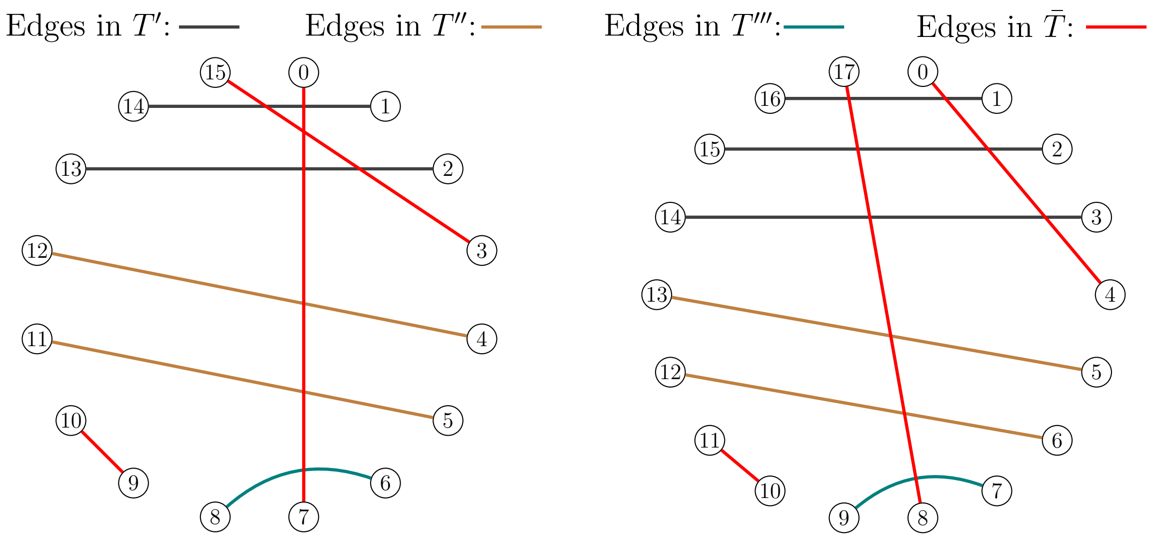
For the graph $K_n^\bullet$ with an odd number of vertices, we define the ARS matching Ξ_n as

$$\Xi_n = \Xi'_n \cup \Xi''_n \cup \Xi'''_n \tag{14}$$

where

$$\Xi'_n = \begin{cases} \emptyset & \text{if } n = 1, \\ \{\{i, 2k-1-i\} \mid i = 0, 1, \ldots, k-1\} & \text{if } n = 8k+1 \text{ or } n = 8k+3, \\ \{\{i, 2k+1-i\} \mid i = 0, 1, \ldots, k\} & \text{if } n = 8k+5 \text{ or } n = 8k+7; \end{cases} \tag{15}$$

$$\Xi''_n = \begin{cases} \emptyset & \text{if } n = 1, \\ \{\{2k+i, 4k+1+2i\} \mid i = 0, 1, \ldots, 2k-1\} & \text{if } n = 8k+1, \\ \{\{2k+i, 4k+1+2i\} \mid i = 0, 1, \ldots, 2k\} & \text{if } n = 8k+3, \\ \{\{2k+2+i, 4k+4+2i\} \mid i = 0, 1, \ldots, 2k\} & \text{if } n = 8k+5, \\ \{\{2k+2+i, 4k+4+2i\} \mid i = 0, 1, \ldots, 2k+1\} & \text{if } n = 8k+7; \end{cases} \tag{16}$$



(a) T_{16} (b) T_{18}

Figure 6: Examples of RPM matchings T_{16} and T_{18}

$$\Xi'''_n = \begin{cases} \emptyset & \text{if } n = 1, \\ \{\{4k+2i, 4k+2j\} \mid \{i,j\} \in \Xi_{2k+1}\} & \text{if } n = 8k+1, \\ \{\{4k+2+2i, 4k+2+2j\} \mid \{i,j\} \in \Xi_{2k+1}\} & \text{if } n = 8k+3, \\ \{\{4k+3+2i, 4k+3+2j\} \mid \{i,j\} \in \Xi_{2k+1}\} & \text{if } n = 8k+5, \\ \{\{4k+5+2i, 4k+5+2j\} \mid \{i,j\} \in \Xi_{2k+1}\} & \text{if } n = 8k+7. \end{cases} \tag{17}$$

We show Ξ_{33} as an example. According to (15)–(16), we obtain

$$\Xi'_{33} = \{\{0,7\}, \{1,6\}, \{2,5\}, \{3,4\}\}, \tag{18}$$

$$\Xi''_{33} = \{\{8,17\}, \{9,19\}, \{10,21\}, \{11,23\}, \{12,25\}, \{13,27\}, \{14,29\}, \{15,31\}\}. \tag{19}$$

From recursive formulation (17), to obtain Ξ'''_{33} we must compute Ξ_3 and Ξ_9 beforehand, as

$$\begin{aligned} \Xi_3 &= \Xi'_3 \cup \Xi''_3 \cup \Xi'''_3 \\ &= \emptyset \cup \{\{0,1\}\} \cup \emptyset \\ &= \{\{0,1\}\}, \\ \Xi_9 &= \Xi'_9 \cup \Xi''_9 \cup \Xi'''_9 \\ &= \{\{0,1\}\} \cup \{\{2,5\}, \{3,7\}\} \cup \{\{4+2i, 4+2j\} \mid \{i,j\} \in \Xi_3\} \\ &= \{\{0,1\}\} \cup \{\{2,5\}, \{3,7\}\} \cup \{\{4,6\}\} \\ &= \{\{0,1\}, \{2,5\}, \{3,7\}, \{4,6\}\} \end{aligned}$$

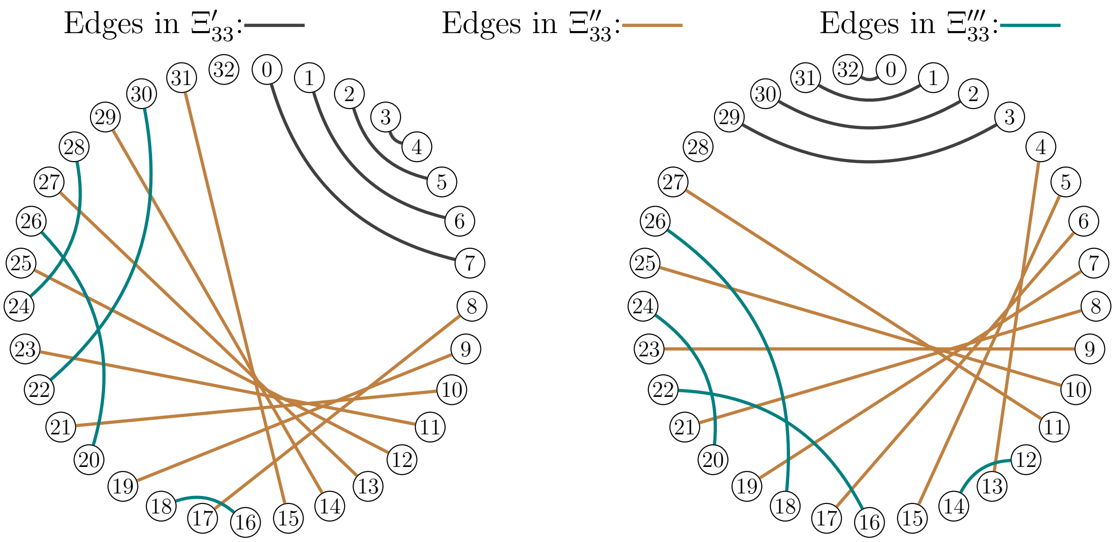
Thus, by using (17), we obtain Ξ'''_{33} as

$$\begin{aligned} \Xi'''_{33} &= \{\{16+2i, 16+2j\} \mid \{i,j\} \in \Xi_9\} \\ &= \{\{16,18\}, \{20,26\}, \{22,30\}, \{24,28\}\}. \end{aligned} \tag{20}$$

Finally, Ξ_{33} is formed by (18)–(20):

$$\begin{aligned}\Xi_{33} &= \Xi'_{33} \cup \Xi''_{33} \cup \Xi'''_{33} \\ &= \{\{0,7\},\{1,6\},\{2,5\},\{3,4\},\{8,17\},\{9,19\},\{10,21\},\{11,23\},\{12,25\},\{13,27\}, \\ &\quad \{14,29\},\{15,31\},\{16,18\},\{20,26\},\{22,30\},\{24,28\}\}\,. \end{aligned} \tag{21}$$

The ARS matching Ξ_{33} is an RPM for the graph $K^{\bullet}_{33}$ because the edges in (21) share no common vertices and cover all colors. Figure 7 shows Ξ_{33} and its corresponding N-RPM.



(a) ARS matching Ξ_{33}

(b) The N-RPM of Ξ_{33}

Figure 7: ARS matching Ξ_{33} and its N-RPM

In this paper, we call an RPM M in $K^{\bullet}_{n}$ a *cuttable RPM* if

$$\forall \{i,j\} \in M, |i-j| \leq n - |i-j| \tag{22}$$

holds. A necessary and sufficient condition for (22) is

$$\forall \{i,j\} \in M, |i-j| \leq \frac{n}{2}. \tag{23}$$

Note that the Kirkman matching M^{kir}_{n} is not cuttable where $n \geq 3$, but for any $n \in \mathbb{Z}_{\geq 0}$, matchings

$$\mathrm{rot}\left(M^{\mathrm{kir}}_{n}, \left\lfloor \frac{n+1}{4} \right\rfloor\right), \mathrm{rot}\left(M^{\mathrm{kir}}_{n}, -\left\lfloor \frac{n+1}{4} \right\rfloor\right) \tag{24}$$

are cuttable RPMs. Using this definition, we can prove that Ξ_n is an RPM:

Theorem 4.1. *For any odd number n, ARS matching Ξ_n is an RPM in the graph $K^{\bullet}_{n}$.*

Proof. The proof is by mathematical induction. For any $t \in \mathbb{Z}_{\geq 1}$, let $P(t)$ denote the statement that Ξ_n is a cuttable RPM for each $n \in \{1, 3, 5, \ldots, 8t-1\}$.
Base step ($t = 1$): $P(1)$ is true because according to (15)–(17)

$$\begin{aligned}
&\Xi_1 = \emptyset,\\
&\Xi_3 = \{\{0, 1\}\},\\
&\Xi_5 = \{\{0, 1\}, \{2, 4\}\},\\
&\Xi_7 = \{\{0, 1\}, \{2, 4\}, \{3, 6\}\}.
\end{aligned}$$

Each of these is an RPM in the corresponding graph $K_n^\bullet$, all involved edges $\{i, j\}$, satisfy $|i - j| \leq n/2$.
Inductive step $P(t) \to P(t + 1)$: Fix some $t \geq 1$, assuming that

Assumption 4.1. *for any $n \in \{1, 3, 5, \ldots, 8t-1\}$, Ξ_n is a cuttable RPM.*

To prove $P(t + 1)$, we must show that for any $n \in \{8t + 1, 8t + 3, 8t + 5, 8t + 7\}$, Ξ_n is a cuttable RPM in the graph $K_n^\bullet$. Table 3 summarizes the features of Ξ'_n and Ξ''_n according to (15)–(16), such as covered vertices (in the column "vertices"), covered colors (in the column "colors"), and size.

Table 3 shows that Ξ'_n and Ξ''_n share neither common vertices nor colors. The values in the last column indicate that statement $|i - j| \leq n/2$ holds for all edges $\{i, j\} \in \Xi'_n \cup \Xi''_n$. Table 4 lists the unmatched vertices and colors to be considered in $\Xi'''_{8t+1}, \Xi'''_{8t+3}, \Xi'''_{8t+5}$, and Ξ'''_{8t+7}.

From $t \geq 1$ and Assumption 4.1, Ξ_{2t+1} is a cuttable RPM for the graph $K_{2t+1}^\bullet$, using $2t$ distinct vertices in $\{0, 1, \ldots, 2t\}$. Thus, Ξ'''_{8t+1}, Ξ'''_{8t+3}, Ξ'''_{8t+5}, and Ξ'''_{8t+7} constructed by (17) cover $2t$ distinct vertices in the column "vertices" in Table 4.

Table 3: Features of Ξ'_n and Ξ''_n

	Ξ'_n			
n	vertices	colors	$\lvert\Xi'_n\rvert$	$\max_{\{i,j\}\in\Xi'_n} \lvert i-j\rvert$
$8k + 1$	$\{0, 1, \ldots, 2k - 1\}$	$\{c_1, c_3, \ldots, c_{2k-1}\}$	k	$2k - 1$
$8k + 3$	$\{0, 1, \ldots, 2k - 1\}$	$\{c_1, c_3, \ldots, c_{2k-1}\}$	k	$2k - 1$
$8k + 5$	$\{0, 1, \ldots, 2k + 1\}$	$\{c_1, c_3, \ldots, c_{2k+1}\}$	$k + 1$	$2k + 1$
$8k + 7$	$\{0, 1, \ldots, 2k + 1\}$	$\{c_1, c_3, \ldots, c_{2k+1}\}$	$k + 1$	$2k + 1$
	Ξ''_n			
n	vertices	colors	$\lvert\Xi''_n\rvert$	$\max_{\{i,j\}\in\Xi''_n} \lvert i-j\rvert$
$8k + 1$	$\{2k, 2k + 1, \ldots, 4k - 1\} \cup \{4k + 1, 4k + 3, \ldots, 8k - 1\}$	$\{c_{2k+1}, c_{2k+2}, \ldots, c_{4k}\}$	$2k$	$4k$
$8k + 3$	$\{2k, 2k + 1, \ldots, 4k\} \cup \{4k + 1, 4k + 3, \ldots, 8k + 1\}$	$\{c_{2k+1}, c_{2k+2}, \ldots, c_{4k+1}\}$	$2k + 1$	$4k + 1$
$8k + 5$	$\{2k + 2, 2k + 3, \ldots, 4k + 2\} \cup \{4k + 4, 4k + 6, \ldots, 8k + 4\}$	$\{c_{2k+2}, c_{2k+3}, \ldots, c_{4k+2}\}$	$2k + 1$	$4k + 2$
$8k + 7$	$\{2k + 2, 2k + 3, \ldots, 4k + 3\} \cup \{4k + 4, 4k + 6, \ldots, 8k + 6\}$	$\{c_{2k+2}, c_{2k+3}, \ldots, c_{4k+3}\}$	$2k + 2$	$4k + 3$

Table 4: Unmatched vertices and colors

n	vertices	colors
$8t + 1$	$\{4t, 4t + 2, \ldots, 8t\}$	$\{c_2, c_4, \ldots, c_{2t}\}$
$8t + 3$	$\{4t + 2, 4t + 4, \ldots, 8t + 2\}$	$\{c_2, c_4, \ldots, c_{2t}\}$
$8t + 5$	$\{4t + 3, 4t + 5, \ldots, 8t + 3\}$	$\{c_2, c_4, \ldots, c_{2t}\}$
$8t + 7$	$\{4t + 5, 4t + 7, \ldots, 8t + 5\}$	$\{c_2, c_4, \ldots, c_{2t}\}$

We next show that all colors in the column "colors" are covered. Assumption 4.1 guarantees that

$$\{|i-j| \mid \{i,j\} \in \Xi_{2t+1}\} = \{1,2,\ldots,t\}, \tag{25}$$

so Ξ'''_{8t+1}, Ξ'''_{8t+3}, Ξ'''_{8t+5}, and Ξ'''_{8t+7} constructed by (17) cover all colors in the column "colors" in Table 4. Note that (25) also indicates that

$$|i-j| \le 2t \le n/2$$

holds for each $\{i,j\}$ in Ξ'''_{8t+1}, Ξ'''_{8t+3}, Ξ'''_{8t+5}, and Ξ'''_{8t+7}.

Therefore, for each $n \in \{8t+1, 8t+3, 8t+5, 8t+7\}$, Ξ_n is an RPM in $K_n^\bullet$, and $|i-j| \le n/2$ holds for all edges $\{i,j\}$ in Ξ_n. □

For $n \in \{1,3,5\}$, all RPMs in the graph $K_n^\bullet$ are derived from the same N-RPM. However, if $n \in \{7,9,\ldots\}$, the N-RPM of ARS matching is different from Kirkman matching M_n^{kir} in the graph $K_n^\bullet$:

Property 4.1. *For any $n \in \{7,9,11,\ldots\}$, norm$(\Xi_n) \neq M_n^{\mathrm{kir}}$.*

Proof. From the definition of Ξ''_n in (16), $|\Xi''_n| \ge 2$ holds when $n \in \{7,9,\ldots\}$. When $n = 8k+1$ or $n = 8k+3$, edges $\{2k, 4k+1\}$ and $\{2k+1, 4k+3\}$ belong to Ξ''_n. If we arrange vertices $0,1,\ldots,n-1$ around a cycle in clockwise order, these two edges cross each other, and this cannot be changed by applying reverse and rotation operators. However, all edges in Kirkman matching M_n^{kir} are parallel, so the N-RPM of ARS matching is different from M_n^{kir} if $n = 8k+1$ or $n = 8k+3$. For $n = 8k+5$ or $n = 8k+7$, the same holds for edges $\{2k+2, 4k+4\}$ and $\{2k+3, 4k+6\}$ in Ξ''_n. □

4.2 Generating many N-RPMs based on ARS matching

We showed in the previous subsection that the ARS matching $\Xi_n = \Xi'_n \cup \Xi''_n \cup \Xi'''_n$ is an RPM in $K_n^\bullet$ if n is odd. In this subsection, we propose a method for generating many N-RPMs, based on the idea that valid variants of Ξ'''_{2k+1} lead to RPMs whose N-RPMs are different.

Given an RPM M in $K_n^\bullet$ where n is odd, we consider the following functions $f(M)$ and $g(M)$:

$$f(M) = \begin{cases} \mathrm{rot}(M, -2k) & \text{if } n = 8k+1 \text{ or } n = 8k+3, \\ \mathrm{rot}(M, -2k-2) & \text{if } n = 8k+5 \text{ or } n = 8k+7; \end{cases} \tag{26}$$

$$g(M) = \begin{cases} \mathrm{rot}(\mathrm{rev}(M), 2k) & \text{if } n = 8k+1 \text{ or } n = 8k+3, \\ \mathrm{rot}(\mathrm{rev}(M), 2k+2) & \text{if } n = 8k+5 \text{ or } n = 8k+7. \end{cases} \tag{27}$$

From Property 2.1, Property 2.2, and Theorem 4.1, $f(\Xi_n)$, $\mathrm{rev}(\Xi_n)$, and $g(\Xi_n)$ are RPMs in $K_n^\bullet$. Note that the four RPMs Ξ_n, $f(\Xi_n)$, $\mathrm{rev}(\Xi_n)$, and $g(\Xi_n)$ are all cuttable, and each is derived from the same N-RPM by Property 2.5.

We demonstrate how to generate many N-RPMs by example. In (21), we showed that $\Xi_{33} = \Xi'_{33} \cup \Xi''_{33} \cup \Xi'''_{33}$ is an RPM in $K_{33}^\bullet$. We used Ξ_9 in constructing Ξ'''_{33}. If we use

$f(\Xi_9)$, $\mathrm{rev}(\Xi_9)$ and $g(\Xi_9)$ instead of Ξ_9 when constructing Ξ'''_{33}, RPMs whose N-RPMs are different can be obtained for $K^\bullet_{33}$. Note that other cuttable RPMs such as $\mathrm{rot}(M_9^{\mathrm{kir}}, 2)$ and $\mathrm{rot}(M_9^{\mathrm{kir}}, -2)$ in (24) are also valid alternatives to Ξ_9.

Let $\Xi'''_n(M)$ denote

$$\Xi'''_n(M) = \begin{cases} \{\{4k+2i, 4k+2j\} \mid \{i,j\} \in M\} & \text{if } n = 8k+1, \\ \{\{4k+2+2i, 4k+2+2j\} \mid \{i,j\} \in M\} & \text{if } n = 8k+3, \\ \{\{4k+3+2i, 4k+3+2j\} \mid \{i,j\} \in M\} & \text{if } n = 8k+5, \\ \{\{4k+5+2i, 4k+5+2j\} \mid \{i,j\} \in M\} & \text{if } n = 8k+7. \end{cases} \tag{28}$$

We design a collection of RPMs F_n in the graph $K^\bullet_n$ as

$$F_n = \begin{cases} \{\emptyset\} & \text{if } n = 1, \\ \{\Xi'_n \cup \Xi''_n \cup \Xi'''_n(M) \mid M \in F_{2k+1}\} & \\ \quad \cup \{\Xi'_n \cup \Xi''_n \cup \Xi'''_n(f(M)) \mid M \in F_{2k+1}\} & \\ \quad \cup \{\Xi'_n \cup \Xi''_n \cup \Xi'''_n(\mathrm{rev}(M)) \mid M \in F_{2k+1}\} & \\ \quad \cup \{\Xi'_n \cup \Xi''_n \cup \Xi'''_n(g(M)) \mid M \in F_{2k+1}\} & \text{if } n = 8k+1, 8k+3, 8k+5, 8k+7. \end{cases} \tag{29}$$

Using this method, we obtain $\theta(n)$ different N-RPMs in the graph $K^\bullet_n$, where n is odd.

The following confirms the size of F_n in detail. For $n = 1$, we obtain $|F_n| = 1$ from (29); For $n = 3, 5, 7$, we construct F_n based on $F_1 = \{\emptyset\}$. Since $\emptyset = \mathrm{rev}(\emptyset) = f(\emptyset) = g(\emptyset)$, $F_n = \{\Xi_n\}$ holds for $n = 3, 5, 7$. When $M = \Xi_3$ or $M = \Xi_5$, since $M = g(M)$ and $\mathrm{rev}(M) = f(M)$ hold for $n = 9, 11, \ldots, 23$, F_n based on Ξ_3, Ξ_5 has 2 elements. For the other $n = 8k+i, i \in \{1, 3, 5, 7\}$, $|F_n| = 4|F_{2k+1}|$ holds. From the observations above, we can summarize that the size of F_n is bounded as:

$$\frac{1}{12}n < |F_n| \le n.$$

5 Conclusion

We studied a problem that searches for rainbow perfect/near-perfect matchings (RPM) in edge-colored complete graphs by a circular-distance edge coloring. Applications such as scheduling for round-robin tournaments are related to this problem. Letting k be a nonnegative integer, for the case when the number of the vertices n is even, we showed the existence of an RPM when $n = 8k$ or $n = 8k+2$, and the non-existence when $n = 8k+4$ or $n = 8k+6$. For the case when n is an odd number larger than 7, we showed there always exists an RPM different from the RPM discovered by Kirkman, Based on a special property of this new RPM, we proposed a recursive algorithm to generate $\theta(n)$ different RPMs.

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