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Characterizing Edge Betweenness-Uniform graphs

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Abstract

The *betweenness centality* of an edge e is, summed over all $u, v \in V(G)$, the ratio of the number of shortest u, v-paths in G containing e to the number of shortest u, v-paths in G. Graphs whose vertices all have the same edge betweenness centrality are called *edge betweeness-uniform*. It was recently shown by Madaras, Hurajová, Newman, Miranda, Flórez, and Narayan that of the over 11.7 million graphs with ten vertices or fewer, only four graphs are edge betweenness-uniform but not edge-transitive. In this paper we present new results involving properties of betweenness-uniform graphs.

1 Introduction

In social or complex network analysis, great attention is paid to determine the most important vertices or edges of a network. Vertices and edges that are central to a network lie on intersecting geodesics between pairs of vertices in the network. This idea was introduced by Anthonisse [1] and Freeman [6] in the context of social networks, where they defined vertex and edge betweenness centrality.

The *vertex betweenness* of a vertex x of a graph G is defined as

$$B(x) = \sum_{u,v \in V(G)} \frac{\sigma_{u,v}(x)}{\sigma_{u,v}}$$

where $\sigma_{u,v}$ is the number of shortest u, v-paths in G, and $\sigma_{u,v}(x)$ is the number of those shortest u, v-paths in G having x as an internal vertex. A survey of graph-theoretical properties of this graph invariant is given in [8].

This paper deals with *edge betweenness*, which is defined for an edge e of a graph G as the sum

$$B(e) = \sum_{u,v \in V(G)} \frac{\sigma_{u,v}(e)}{\sigma_{u,v}}$$

where $\sigma_{u,v}(e)$ is the number of shortest u, v paths in G that contain e. These metrics have appeared frequently in both social network and neuroscience literature [7, 18, 3, 17, 4, 11, 19].

Applications to network partitioning were given by Girvan and Newman in [9] in connection with their algorithm for determining the community structure of a network. Along with these applications, graph-theoretical properties of edge betweenness were studied, to some extent, in [5, 8, 13].

An *edge betweenness-uniform graph* is a graph where each edge in the graph has the same edge betweenness value. In [8] it was shown that distance-regular graphs are edge betweenness-uniform. Clearly, if a graph G is edge-transitive, then it is edge betweenness-uniform. However the converse is not true and we will further investigate graphs that are not edge-transitive but are edge betweenness-uniform. Graphs of this type appear to be exceedingly rare. It was shown independently by Hurajová and Madaras [12] and Newman, Miranda, Flórez, and Narayan [15] that of the over 11.7 million graphs with ten vertices or fewer, there are only four graphs that are edge betweenness-uniform but not edge-transitive. These are shown in Figure 1.

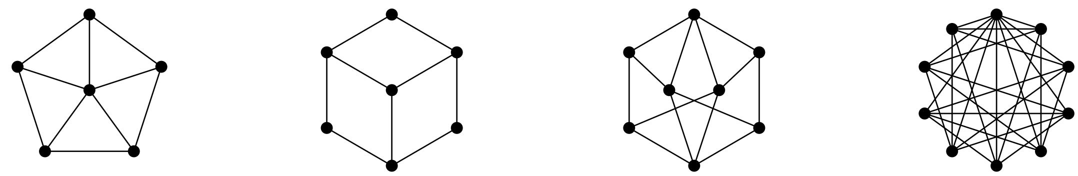

Figure 1: The four graphs on ten vertices or less that have uniform edge betweenness centrality but are not edge transitive.

They also proved that for each positive integer n the circulant graphs $C_{18n-3}(1, 6n)$ and $C_{18n+3}(1, 6n)$ (arisen from cycles on $18n \pm 3$ vertices by putting additional edges between vertices in circular distance $6n-1$) are edge betweenness-uniform and neither edge-transitive nor distance regular. Several other results and conjectures on edge-connectivity of edge betweenness-uniform graphs are found in [13]. Variants where graphs have only two or three different betweenness centrality values were investigated in [2].

Throughout this paper we consider connected graphs without loops or multiple edges. A vertex v in a graph G is called a *universal vertex* if v is adjacent to all other vertices in G. We will refer to the complete bipartite graph $K_{1,n-1}$ as a *star*. An edge is *pendant* if it is incident to a vertex of degree 1. The *join* $G \vee H$ of graphs G, H is the graph whose vertex set is the union of vertex sets of G and H, and the edge set is the union of edge sets of G, H and the set of all edges connecting vertices of G with the vertices of H.

The aim of this paper present new families of graphs that are not edge-transitive but are edge betweenness-uniform, and to obtain new results on their structure in general. In Section 2 we present properties of edge betweenness-uniform graphs.

While the graphs in Figure 1 do not readily extend to larger classes of graphs, they may posess properties that are present in other graphs that are not edge-transitive but are edge-betweenness-uniform. We note that the first and the fourth graphs have a universal vertex. This suggests the problem of determining when an edge betweenness-uniform graph contains a universal vertex. We investigate this problem in Subsection 3.1.

In Subsection 2.1 we investigate edge betweenness-uniform graphs that contain a universal vertex. We identify two graphs, the wheel on six vertices, and graph with ten vertices appearing in Figure 1. In these two examples the non-universal vertices have a degree equal to half the number of vertices. However this does not have to be the case. An example arises by taking the Schläfli graph (which has 27 vertices, is 16-regular and strongly regular) and adding to it a new universal vertex, then, according to Theorem 9, we obtain the edge betweenness-uniform graph on 28 vertices. It can be also checked using Wolfram Mathematica where non-universal vertices have degree 17.

2 Structure of edge betweenness-uniform graphs

We start with general results on the structure of edge betweenness-uniform graphs. In [13] it was conjectured that each connected edge betweenness-uniform graph is either a star or has

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edge connectivity at least 2. It was was proved that each cut-edge of an edge betweenness-uniform graph is a pendant edge incident with a vertex of degree at least 3. With the following theorem we confirm the mentioned conjecture:

Proposition 2.1. *Let G be a connected edge betweenness-uniform graph with a pendant edge. Then G is a star.*

Proof. It is easy to see that the only edge betweenness-uniform graphs of order at most four that contain at least one cut-edge are the stars. Assume G is an edge betweenness-uniform graph on $n \geq 5$ vertices with a pendant edge $e = uv$ where $\deg(u) = k + 1$ and $\deg(v) = 1$. Note that $3 \leq k \leq n - 2$.

Let C consist of all of the edges except for e that are incident to u. Note that $\sum_{f \in C} B(f) \leq 2(n-2) + k(k-1)$. Since G is edge betweenness-uniform and $B(e) = n - 1$, we have that $\sum_{f \in C} B(f) = k(n-1)$. Hence $k(n-1) = \sum_{f \in C} B(f) \leq 2(n-2) + k(k-1)$ which is equivalent to $k(n-1) \leq 2(n-2) + k(k-1)$, further adjusted to $n(k-2) \leq (k-2)(k+2)$. Thus $n \leq k+2$ and, combining this with the fact that $k \leq n-2$, we obtain that $k = n-2$, meaning that G is a star.

Next, we discuss edge betweenness-uniform graphs having vertices of degree 2. We will prove in the next theorem that it is impossible to have two vertices each of degree 2 where one vertex lies in a triangle and other not; in fact, this property can be extended to consider the closed neighbourhood of vertices with degree $k \geq 2$. □

Proposition 2.2. *If G is edge betweenness-uniform, then it cannot contain two vertices u and v each of degree $k \geq 2$ where the closed neighbourhood of u is a complete graph while the closed neighbourhood of v is not.*

Proof. Assume G is an edge betweenness-uniform graph. Let u be a vertex of degree k which, together with its neighbours, induces a complete subgraph of G. As a result, no shortest path between the neighbours of u will use an edge incident to u. Hence the only shortest paths containing edges incident to u will be shortest paths between u and vertices in $G - u$, yielding the sum of the edge betweenness over all edges incident to u being equal to $k(n-1)$. Suppose G also contains a vertex v of degree k with two non-adjacent neighbours x and y. Then at least one of the shortest x,y-paths in G must pass through v. Hence the sum of the edge betweenness over all edges incident to u will be strictly greater than $k(n-1)$, contradicting the fact that G is edge betweenness-uniform. □

Proposition 2.3. *Let G be a k-regular edge betweenness-uniform graph on n vertices. Then, for each edge $e \in E(G)$, $B(e) \geq \frac{n}{k}$.*

Proof. By [14], we have that, for each vertex u of G,

$$B(u) = \frac{1}{2}\left(\sum_{v \in N(u)} B(uv) - n + 1\right).$$

Since G is edge betweenness-uniform, $B(e) = b$ for each edge e of G. Moreover, G is k-regular so we get that

$$B(u) = \frac{1}{2}\left(kb - n + 1\right).$$

Note that G is also vertex betweenness-uniform because $B(u)$ does not depend on the selected vertex. In [16] it was proved that for betweenness-uniform graphs, $B(u) = 0$ if G is a complete graph or $B(u) \geq \frac{1}{2}$. Therefore, if G is not complete, then we get

$$B(u) = \frac{1}{2}\left(kb - n + 1\right) \geq \frac{1}{2}$$

which gives $kb - n + 1 \geq 1$ and thus $b \geq \frac{n}{k}$. □

3 Results

3.1 Edge-betweenness graphs with a universal vertex

In this section we investigate which edge-betweenness graphs have a universal vertex.

Theorem 3.1. *Let G be an edge betweenness-uniform graph on n vertices. Then*

1. *$G \cong K_n$ or G contains at most one universal vertex,*
2. *There is no edge $e = uv$ in G such that* $\deg(u) = n - 1$ *and* $\deg(v) = n - 2$,
3. *$G \cong K_3$ or there is no edge $e = uv$ in G such that* $\deg(u) = n - 1$ *and* $\deg(v) = 2$,
4. *$G \cong K_4$ or $G \cong W_6$ or there is no edge $e = uv$ in G such that* $\deg(u) = n - 1$ *and* $\deg(v) = 3$.

Proof. We prove each of the four cases individually.

1. Let G contain at least two vertices u, v of degree $n-1$. Then the edge uv has betweenness centrality equal to 1. If G is not a complete graph, then there exist the vertices w, z such that $wz \notin E(G)$ and $B(uw) \geq 1 + \frac{1}{\sigma_{wz}} > 1 = B(uv)$.
2. Let G contain a universal vertex u and a vertex v of degree $n - 2$. By Proposition 2.1 and part 1 above, we have $2 \leq \deg(y) \leq n - 2$ for each vertex y of G except of u. Since $\deg(v) = n - 2$, there is exactly one vertex w not adjacent to v; note that w has at least two neighbours u and z. We have $\deg(z) \leq n - 2$, but z is adjacent to each of u, v, w so there is another vertex x such that $zx \notin E(G)$. Then

   $$B(vz) \geq 1 + \frac{1}{\sigma_{vw}} + \frac{1}{\sigma_{zx}} > 1 + \frac{1}{\sigma_{vw}} = B(uv),$$

   yields a contradiction.

3. If $n-1=2$, then $G \cong K_3$. So let $n-1>2$ and let G contain a universal vertex u and a vertex v of degree 2. By the previous claim, each vertex y of $V(G)$ except of u, $2 \le \deg(y) < n-2$ holds. The vertex v has exactly two neighbours u, w where the vertex w is either of degree 2 or w has at least 3 neighbours. Let $V(G) \setminus \{u,v,w\} = \{w_1, w_2, \dots, w_{n-3}\}$. Then

$$B(uv) = 1 + \sum_{i=1}^{n-3} \frac{1}{\sigma_{vw_i}} > 1.$$

For $\deg(w)=2$ we get $B(vw) = 1 < B(uv)$.

4. If $n-1=3$, then $G \cong K_4$. Now let $n-1>3$ and let G contain a universal vertex u and a vertex v of degree 3. We denote $N(v) = \{u,w,z\}$ and $V(G) \setminus \{u,v,w,z\} = \{w_1, w_2, \dots, w_{n-4}\}$. By parts 1,2, and 3, for each vertex y of $V(G)$ except of u, $3 \le \deg(y) < n-2$ holds.

If $wz \in E(G)$, then

$$B(uv) = 1 + \sum_{i=1}^{n-4} \frac{1}{\sigma_{vw_i}}$$

and

$$B(vw) = 1 + \sum_{w_i \in N(w)} \frac{1}{\sigma_{vw_i}} < 1 + \sum_{i=1}^{n-4} \frac{1}{\sigma_{vw_i}} = B(uv).$$

Now let $wz \notin E(G)$. Since $\deg(w) \le n-3$, there exists a vertex w_k, $k \in \{1, 2, \dots, n-4\}$ such that $ww_k \notin E(G)$. We have

$$B(uv) = 1 + \frac{1}{\sigma_{vw_k}} + \sum_{i=1, i\neq k}^{n-4} \frac{1}{\sigma_{vw_i}}$$

and

$$B(vw) = 1 + \frac{1}{\sigma_{wz}} + \sum_{w_i \in N(w)} \frac{1}{\sigma_{vw_i}} \le 1 + \frac{1}{\sigma_{wz}} + \sum_{i=1, i\neq k}^{n-4} \frac{1}{\sigma_{vw_i}} = B(uv).$$

The equality $B(uv) = B(vw)$ holds if and only if $w_i \in N(w)$ for all $i \neq k$ and $\sigma_{vw_k} = \sigma_{wz}$. For w, z there exist at least two shortest w,z-paths in G, wvz and wuz. On the other hand, there are at most two shortest v, w_k-paths in G. Now either $zw_k \in E(G)$ and $\sigma_{vw_k} = 2$, or $zw_k \notin E(G)$ and $\sigma_{vw_k} = 1$. To achieve the equality $B(uv) = B(vw)$, we get that $zw_k \in E(G)$ and $\sigma_{vw_k} = \sigma_{wz} = 2$. One can see that both vertices w, z are adjacent to all other vertices of $\{w_1, w_2, \dots, w_{n-4}\}$ except one, w is not adjacent to w_k and there exists a vertex $w_j \in \{w_1, w_2, \dots, w_{n-4}\}$ such that $zw_j \notin E(G)$. It easy to see that $j \neq k$. Moreover, if there is a vertex $w_s \in \{w_1, w_2, \dots, w_{n-4}\} \setminus \{w_j, w_k\}$ then $\sigma_{wz} \ge 3$. This situation cannot occur. So $V(G) = \{u, v, w, z, w_j, w_k\}$, $\deg(u) = 5$ and $\deg(x) = 3$, $x \in V(G) \setminus \{u\}$.

There is only one graph on six vertices that satisfies all these conditions, namely, the wheel W_6.

□

3.2 Neighbourhood properties of edge betweenness-uniform graphs

The next theorems deal with vertex neighbourhood properties of edge betweenness-uniform graphs.

Theorem 3.2. *Let G be an edge betweenness-uniform graph on n vertices with $u, v \in V(G)$ and $uv \in E(G)$. If $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, then $G \cong K_n$.*

Proof. The vertices u, v have the same set of non-u,v neighbours $N = N(u) \setminus \{v\} = N(v) \setminus \{u\}$, therefore, the edge $e = uv$ belongs to exactly one shortest path (namely uv) and $B(uv) = 1$. We show that $N \cup \{u, v\} = V(G)$. In the opposite case there exist two vertices x, y such that $x \in N$, $y \in V \setminus (N \cup \{u, v\})$ and $xy \in E(G)$. Thus

$$B(ux) \geq 1 + \frac{1}{\sigma_{uy}} > 1 = B(uv).$$

We have a contradiction, which means that $N \cup \{u, v\} = V(G)$ and $\deg(u) = \deg(v) = n - 1$. Thus G has two universal vertices so $G \cong K_n$ (Theorem 3.1, Claim 1.) □

Theorem 3.3. *Let G be a graph on n vertices with a cut vertex u. Let $G \setminus \{u\} = G_1 \cup G_2$, $|V(G_1)| = n_1 > 4$, $|V(G_2) \cup \{u\}| = n_2 > 3$. If $|V(G_1) \cap N(u)| = 2$, then G is not edge betweenness-uniform.*

Proof. Let $v, w \in V(G_1) \cap N(u)$, $e_1 = uv$ and $e_2 = uw$. If we take one vertex from $V(G_1)$ and one from $V(G_2) \cup \{u\}$, then every shortest path connecting these two vertices must pass through e_1 or e_2, so

$$B(e_1) + B(e_2) \geq n_1 n_2.$$

If G is edge betweenness-uniform, then for any three edges f_1, f_2, f_3, it holds $B(f_1) + B(f_2) = 2B(f_3)$.
We consider two cases:

- Case 1: Let $n_1 \leq n_2$. Then we take a vertex $x \in V(G_1)$ furthest from u and an edge f incident to x.

$$2B(f) = 2 \sum_{y,z \in V(G)} \frac{\sigma_{y,z}(f)}{\sigma_{y,z}} = 2 \left(\sum_{y,z \in V(G_1)} \frac{\sigma_{y,z}(f)}{\sigma_{y,z}} + \sum_{y \in V(G_2)} \frac{\sigma_{x,y}(f)}{\sigma_{x,y}} \right) \leq 2 \left(\frac{n_1^2}{4} + n_2 \right) \leq$$

$$2 \frac{n_1 n_2}{4} + 2n_2 = n_2 \left(\frac{n_1 + 4}{2} \right) < n_2 \left(\frac{n_1 + n_1}{2} \right) = n_1 n_2 \leq B(e_1) + B(e_2). \qquad (1)$$

- Case 2: Let $n_1 > n_2$. Then we take a vertex $x \in V(G_2)$ that is furthest from u and an edge f incident to x.

$$2B(f) = 2\sum_{y,z\in V(G)} \frac{\sigma_{y,z}(f)}{\sigma_{y,z}} = 2\left(\sum_{y,z\in V(G_2)} \frac{\sigma_{y,z}(f)}{\sigma_{y,z}} + \sum_{y\in V(G_1)} \frac{\sigma_{x,y}(f)}{\sigma_{x,y}}\right) \le 2\left(\frac{n_2^2}{4} + n_1\right)$$

$< 2\frac{n_1 n_2}{4} + 2n_1 = n_1\left(\frac{n_2+4}{2}\right) \le n_1\left(\frac{n_2+n_2}{2}\right) = n_1 n_2 \le B(e_1) + B(e_2)$.

In both cases we get $2B(f) < B(e_1)+B(e_2)$ which means that G is not edge betweenness-uniform.

$\square$

3.3 Graphs with a uniform count of shortest paths

Theorem 3.4. *Let G be a graph on n vertices with $\sigma_{x,y}(G) = k$ for every non-adjacent $x, y \in V(G)$. Then G is edge betweenness-uniform if and only if every edge belongs to the same number of the shortest paths of G.*

Proof. For every two nonadjacent vertices there are exactly k shortest paths so $\sigma_{x,y} = k$ for every $x, y \in V(G)$, $xy \notin E(G)$. Let $e, f \in E(G)$. We have $\square$

$$B(e) = \sum_{x,y\in V(G)} \frac{\sigma_{x,y}(e)}{\sigma_{x,y}} = 1 + \sum_{xy\notin E(G)} \frac{\sigma_{x,y}(e)}{k} = 1 + \frac{1}{k}\sum_{xy\notin E(G)} \sigma_{x,y}(e),$$

$$B(f) = \sum_{x,y\in V(G)} \frac{\sigma_{x,y}(f)}{\sigma_{x,y}} = 1 + \sum_{xy\notin E(G)} \frac{\sigma_{x,y}(f)}{k} = 1 + \frac{1}{k}\sum_{xy\notin E(G)} \sigma_{x,y}(f).$$

If G is edge betweenness-uniform, then $B(e) = B(f)$ and $\sum_{xy\notin E(G)} \sigma_{x,y}(e) = \sum_{xy\notin E(G)} \sigma_{x,y}(f)$, which means that e as well as f belongs to the same number of shortest paths of G. On the other hand, if every edge e belongs to exactly $(1 + t)$ shortest paths (one of the paths being the edge e itself and $\sum\limits_{xy\notin E(G)} \sigma_{x,y}(e) = t$) then

$$B(e) = \sum_{x,y\in V(G)} \frac{\sigma_{x,y}(e)}{\sigma_{x,y}} = 1 + \sum_{xy\notin E(G)} \frac{\sigma_{x,y}(e)}{k} = 1 + \frac{1}{k}\sum_{xy\notin E(G)} \sigma_{x,y}(e) = 1 + \frac{t}{k}.$$

We see that $B(e)$ does not depend on the choice of an edge e of G, so G is edge betweenness-uniform.

Theorem 3.5. *Let G be an edge betweenness-uniform graph on n vertices such that, for every two nonadjacent vertices, there are exactly k shortest paths between them. If G has a universal vertex u, then every vertex of G except u has the same degree.*

Proof. Let G be an edge betweenness-uniform graph of order n and let $e = uv$, $f = uw$, and $e, f \in E(G)$. Since u is the universal vertex we get $\sigma_{v,x}(e) = 1$ and $\sigma_{w,y}(f) = 1$ for every $x, y \in V(G)$, $x \notin N(v)$, $y \notin N(w)$. Now

$$B(e) = \sum_{x,y \in V(G)} \frac{\sigma_{x,y}(e)}{\sigma_{x,y}} = 1 + \sum_{x \notin V(G) \setminus (N(v) \cup \{v\})} \frac{1}{k} = 1 + \frac{1}{k} (n - 1 - \deg(v))$$

and

$$B(f) = \sum_{x,y \in V(G)} \frac{\sigma_{x,y}(f)}{\sigma_{x,y}} = 1 + \sum_{x \notin V(G) \setminus (N(w) \cup \{w\})} \frac{1}{k} = 1 + \frac{1}{k} (n - 1 - \deg(w)) .$$

If $B(e) = B(f)$, then

$$1 + \frac{1}{k} (n - 1 - \deg(v)) = 1 + \frac{1}{k} (n - 1 - \deg(w))$$

and $\deg(v) = \deg(w)$. □

The next theorem presents graphs with the uniform count of shortest paths between nonadjacent vertices and the conditions for their edge betweenness-uniformity. An important family of such graphs is the family of *strongly regular graphs* determined by four parameters (n, k, λ, μ) as follows: the number of vertices is n, each vertex has degree k, and the number of common neighbours for each pair of adjacent (or nonadjacent) vertices is λ (or μ, respectively); other graphs with this property are trees and, in general, geodetic graphs.

Theorem 3.6. *Let $G \neq K_n$ be a strongly regular graph with parameters (n, k, λ, μ). Then $G \vee K_1$ is edge betweenness-uniform if and only if $k = 2\mu$.*

Proof. Let y be the universal vertex of $G \vee K_1$. Consider first the edge betweenness of an edge xy. Since G and $G \vee K_1$ have diameter 2, every pair of vertices with a positive contribution to $B(xy)$ shall contain x or y. We then have

$$B(xy) = 1 + \sum_{u \notin N_G(y)} \frac{\sigma_{u,y}(xy)}{\sigma_{u,y}} + \sum_{u \notin N_G(x)} \frac{\sigma_{u,x}(xy)}{\sigma_{u,x}} = 1 + 0 + \sum_{u \notin N_G(x)} \frac{1}{1 + \mu} =$$

$$1 + \frac{1}{1 + \mu}(n - k + 1)$$

as there is only one x, u-path passing through y and μ shortest x, u-paths in G. Next, let zw be an edge of G. We have

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$$B(zw) = 1 + \sum_{u \notin N_G(w)} \frac{\sigma_{u,w}(zw)}{\sigma_{u,w}} + \sum_{u \notin N_G(z)} \frac{\sigma_{u,z}(zw)}{\sigma_{u,z}} = 1 + \sum_{u \notin N_G(w)} \frac{\sigma_{u,w}(zw)}{1+\mu} +$$

$$\sum_{u \notin N_G(w)} \frac{\sigma_{u,s}(zw)}{1+\mu} = 1 + \frac{k-1-\lambda}{1+\mu} + \frac{k-1-\lambda}{1+\mu} = 1 + \frac{2(k-1-\lambda)}{1+\mu} \qquad (2)$$

since, for an endvertex of zw, say z, there are $k-1$ neighbours of z which are distinct from z, but λ of them are also the neighbours of w.

Now, $G \vee K_1$ is edge betweenness-uniform if and only if

$$1 + \frac{n-k-1}{1+\mu} = 1 + \frac{2(k-1-\lambda)}{1+\mu}$$

that is, if $2(k-1-\lambda) = n-k-1$. Using the fact $(n-k-1)\mu = k(k-\lambda-1)$, that holds for parameters of a strongly regular graph, we obtain

$$2\frac{(n-k-1)\mu}{k} = n-k-1$$

and, since $G \neq K_n$, the last equality holds if and only if $k = 2\mu$.

□

Examples of strongly regular graphs with $k = 2\mu$ are conference or Paley graphs, or the Schläfli graph (see [10] for details on these graphs). Another construction of edge betweenness-uniform graphs involves arc-transitive graphs (see again [10] p. 35).

Theorem 3.7. *Let G be an arc-transitive graph and let H be the graph obtained from G by subdividing each its edge by a vertex of degree 2. Then H is edge betweenness-uniform.*

Proof. By the definition of arc-transitivity for each pair of edges xy, uv of G (note that they may be the same) there exists an automorphism φ of G such that $\varphi([x,y]) = [v,u]$. Now let xt, wv be edges of H, t, and w have degree 2, and let y and u be the other neighbours of t and w, respectively. Then φ induces an automorphism φ' such that $\varphi'(x) = v$ and $\varphi'(t) = w$; thus, H is edge-transitive and therefore, edge betweenness-uniform. □

We state an extension of our work in the following conjecture.

Conjecture 3.1. If G is an edge betweenness-uniform graph with n vertices and G contains a universal vertex, then all other vertices in the graph have the same degree.

In addition, it appears that complete characterization of graphs that are edge betweenness-uniform but are not edge-transitive would be an ambitious problem. However most certainly other graphs in this family must surely exist.

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