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An Even 2-Factor in the Line Graph of a Cubic Graph

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Abstract

An *even* 2-factor is one such that each cycle is of even length. A 4-regular graph G is 4-edge-colorable if and only if G has two edge-disjoint even 2-factors whose union contains all edges in G. It is known that the line graph of a cubic graph without 3-edge-coloring is not 4-edge-colorable. Hence, we are interested in whether those graphs have an even 2-factor. Bonisoli and Bonvicini proved that the line graph of a connected cubic graph G with an even number of edges has an even 2-factor, if G has a perfect matching [Even cycles and even 2-factors in the line graph of a simple graph, Electron. J. Combin. **24** (2017), P4.15]. In this paper, we give a sufficient condition for the existence of an even 2-factor in the line graph of a connected cubic graph G that can be applied to cubic graphs having no perfect matching and satisfying certain conditions.

1 Introduction

In this paper, we deal with only simple graphs, unless otherwise stated. For a graph G, the *chromatic index* is the least integer k such that G has a k-edge-coloring, that is, an assignment of a color to each edge by at most k colors such that no two edges of the same color are adjacent. For a positive integer k, a graph is said to be *k-regular* if every vertex has degree exactly k. Vizing [7, 8] proved that every k-regular graph has the chromatic index either k or $k + 1$. A k-regular graph with chromatic index k is said to be *Class* 1, while one with chromatic index $k + 1$ is said to be *Class* 2. This means that the family of k-regular graphs can be partitioned into two classes by the chromatic index.

In this paper, we are interested in the case $k = 4$ and study the further partition of the 4-regular graphs of Class 2, in the view of a 2-factor. Recall that a 2-*factor* of a graph G is a spanning subgraph in which every vertex has degree 2. For spanning subgraphs $F_1, F_2, \ldots$ of a graph G, if $E(F_i) \cap E(F_j) = \emptyset$ for $i \neq j$ and $\bigcup_{i \geq 1} E(F_i) = E(G)$, then G is said to be *decomposed* into $F_1, F_2, \ldots$. The following theorem is classical but important.

Theorem 1.1 (Petersen [6])**.** Every 4-regular graph is decomposed into two 2-factors.

A 2-factor F of a graph G is said to be *even* if the length of each cycle in F is even. The following relation between the chromatic index of a 4-regular graph and its even 2-factors was mentioned in [1]. Since the proof is easy, we show that for self-containedness.

Proposition 1.2 (Bonisoli and Bonvicini, [1])**.** Let G be a 4-regular graph. Then G has the chromatic index 4 if and only if G can be decomposed into two even 2-factors.

Proof. Suppose that G has the chromatic index 4, that is, G has a 4-edge-coloring, say $1, 2, 3, 4$ for its colors. Then the edges of color 1 or 2 induce a 2-factor of G, and so do the edges of color 3 or 4. Since all cycles in the 2-factors contain two colors alternately, they are all even cycles. Thus, G can be decomposed into two even 2-factors.

Conversely, if G can be decomposed into two even 2-factors, then by coloring each cycle of one 2-factor by alternately color 1 and 2 and each cycle of the other 2-factor by alternately color 3 and 4, we obtain a 4-edge-coloring of G. $\square$

Therefore, if a 4-regular graph G is of Class 2, then G cannot be decomposed into two even 2-factors. With this proposition in mind, 4-regular graphs of Class 2 with an even 2-factor can be regarded to be closer to Class 1 than those without an even 2-factor. Thus, the existence of an even 2-factor gives a further hierarchy to the family of 4-regular graphs of Class 2. This fact motivates us to study sufficient conditions for the existence of an even 2-factor, which will be discussed in this paper.

The following is an easy observation, but useful.

Proposition 1.3. A 4-regular graph of odd order has no even 2-factors.

Therefore, we are interested in 4-regular graphs of even order and of Class 2. The line graph of a 3-regular graph of Class 2 with even number of edges is in fact such a graph, see the next section, and we focus on those graphs.

2   Line graph of a cubic graph

The *line graph* of a graph G is the graph $L(G)$ with the edges of G as its vertices, where two edges of G are adjacent in $L(G)$ if and only if they are adjacent in G. A 3-regular graph is also called a *cubic* graph, and the line graph of a cubic graph G is 4-regular, since each edge in G is adjacent with exactly four edges. Note that a graph has an even number of edges if and only if its line graph has an even order.

Cubic graphs of Class 2 with certain connectivity condition, such as the Petersen graph, have attracted by researchers as the name of *snark*. Jaeger proved that such graphs are related to 4-regular graphs of Class 2, through the line graph operation.

Theorem 2.1 (Jaeger [4])**.** The line graph of any cubic graph of Class 2 has no 4-edge-coloring.

By this theorem, one thinks the line graph of a cubic graph of Class 2 is a typical 4-regular graph of Class 2. By Proposition 1.2, such a graph cannot be decomposed into two even 2-factors. For the existence of an even 2-factor, Bonisoli and Bonvicini [1] proved the following theorem.

Theorem 2.2 (Bonisoli and Bonvicini, [1])**.** Let G be a connected cubic graph with an even number of edges. If G has a perfect matching, then the line graph $L(G)$ has an even 2-factor.

As mentioned in [1], there are connected cubic graphs G with an even number of edges such that G does not have a perfect matching, but the line graph $L(G)$ has an even 2-factor. The graph G depicted in Figure 1 is one of such graphs, where one can easily check that G does not have a perfect matching. (In fact, the graph G is known as the smallest simple cubic graph without perfect matching.) However, the line graph $L(G)$ of G has an even 2-factor, which can be seen by our main theorem.

In this paper, in order to cover those graphs, we show the following Theorem. An edge e in a graph G is called a *cut edge* if $G - e$ has more components than G. Recall that any cubic graph without perfect matching must contain a cut edge by Petersen's theorem [6]. Thus, to study cubic graphs without perfect matching, we consider conditions on cut edges. A vertex of degree 0 in a graph is said to be *isolated*.
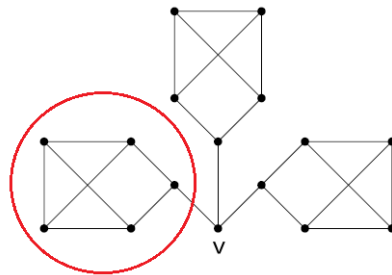
Figure 1: A connected cubic graph G with 24 edges. This graph G does not have a perfect matching, but satisfies Conditions (S1) and (S2).

Theorem 2.3. Let G be a connected cubic graph with an even number of edges, and let S be the set of cut edges in G. If G satisfies both of the following conditions, then the line graph $L(G)$ has an even 2-factor.

(S1) For each isolated vertex v in $G - S$, every component in $G - v$ has an odd number of edges.

(S2) Each component in $G - S$ that is not an isolated vertex is incident with at most two cut edges.

In Section 5, we discuss the best possibility of Conditions (S1) and (S2).

We show that the graph G depicted in Figure 1 satisfies the assumptions of Theorem 2.3. As remarked, it does not satisfy Theorem 2.2 since it has no perfect matching but Theorem 2.3, that will be proved later, show that the corresponding line graph has an even 2-factor. There are three cut edges in G, all of which are incident with the center vertex v. After removing the cut edges from G, exactly one isolated vertex v and three components that are not isolated vertices remain. Note that $G - v$ has three components each of which has seven edges, and hence Condition (S1) is satisfied. Each of the three components that are not isolated vertices is incident with exactly one cut edge, and hence Condition (S2) is also satisfied. Thus, by Theorem 2.3, we see that the line graph $L(G)$ of G has an even 2-factor.

If we take 2-edge-connected cubic graphs H of order 0 modulo 4 with one edge subdivided and replace one component of $G - v$ (indicated by a red circle in Figure 1) with H by joining v with the vertex obtained by subdivision, then we obtain cubic graphs to which we can apply Theorem 2.3 but not Theorem 2.2. (We leave the readers to check that the obtained graph has an even number of edges.) Therefore, there are infinitely many connected cubic graphs G with an even number of edges such that G does not have a perfect matching, but satisfies Conditions (S1) and (S2).

Remark 1: Let G be a connected cubic graph satisfying the assumptions in Theorem 2.3. Suppose that there does not exist an isolated vertex in $G - S$. By Condition (S2), we see that all cut edges are contained in a path in G. It follows from the Petersen's theorem (see Theorem 3.3 below) that G has a perfect matching, and hence Theorem 2.2 implies that the line graph $L(G)$ of G has an even 2-factor.

On the other hand, if there exists an isolated vertex in $G - S$, then we can easily see that G does not have a perfect matching, and hence we cannot apply Theorem 2.2 to such a graph G.

Therefore, the family of cubic graphs to which we can apply Theorem 2.3 but not Theorem 2.2 is indeed equivalent to the family of connected cubic graphs G with even number of edges such that Conditions (S1) and (S2) are satisfied and there is an isolated vertex in $G - S$.

Remark 2: Our proof of Theorem 2.3 uses different ideas from the proof of Theorem 2.2 by Bonisoli and Bonvicini [1]. In [1], they consider a pair of P_2-decompositions, where a P_2-*decomposition* of a graph G is a decomposition into subgraphs of G all of which are isomorphic to the path of order three (of length two). They gave a necessary and sufficient condition for the line graph to have an even 2-factor in terms of a pair of *disjoint* P_2-decompositions, that is, two P_2-decompositions without common members, see [1, Proposition 5]. The existence of a pair of disjoint P_2-decompositions is a refinement of a result by Kotzig [5]. We give an alternative proof of the existence of an even 2-factor in the line graph of a cubic graph through a decomposition into extended circuits with an even number of edges. Our approach relies on the results in [2] (see Proposition 3.1) and then we adjust the parity of the length of each cycle.

3 Preliminary

In this section, we give some terminology and propositions that are used in our proof. A *circuit* is a connected graph in which every vertex has positive even degree. An *extended circuit* is a circuit together with edges that are incident with vertices in the circuit, where the circuit is called the *base* of the extended circuit. A *star* is the complete bipartite graph $K_{1,r}$ for some integer $r \geq 1$. The following is essentially shown by Harary and Nash-Williams [3].

Proposition 3.1 (c.f. [3]). Let G be a graph of order at least three. Then the line graph of G contains a Hamiltonian cycle if and only if G is either an extended circuit or a star.

Using this idea, we can show that the line graph $L(G)$ of a graph G has a 2-factor if and only if G can be decomposed into extended circuits and stars with at least three edges. This was essentially shown by Gould and Hynds [2]. When we restrict ourselves to an even 2-factor in the line graph of a cubic graph, we obtain the following.

Proposition 3.2. Let G be a connected cubic graph. Then the line graph $L(G)$ of G has an even 2-factor if and only if G can be decomposed into extended circuits with even number of edges.

Proof. Since any subgraph isomorphic to a star in a cubic graph can have at most three edges, it follows from Proposition 3.1 that each even cycle in $L(G)$ corresponds to an extended circuit in G. This observation, together with the criteria by Gould and Hynds [2] shows the statement. $\square$

We also use the following theorem.

Theorem 3.3 (Petersen, [6]). For a connected cubic graph G, if all cut edges are contained in a path in G, then G has a 2-factor.

4 Proof of the main theorem

Let G be a connected cubic graph with an even number of edges, and S be the set of cut edges. Suppose that G satisfies Conditions (S1) and (S2). By Proposition 3.2 it suffices to find a decomposition of G into extended circuits with even number of edges.

Let v be an isolated vertex in $G - S$, and let e_1, e_2, e_3 be the edges incident with v in G. Note that e_1, e_2, e_3 are all cut edges in G. We replace v with three new vertices v_1, v_2, v_3 so that the edge e_i is incident with v_i for $i = 1, 2, 3$, see Figure 2. This operation breaks G into three components each containing v_i for some $i = 1, 2, 3$.
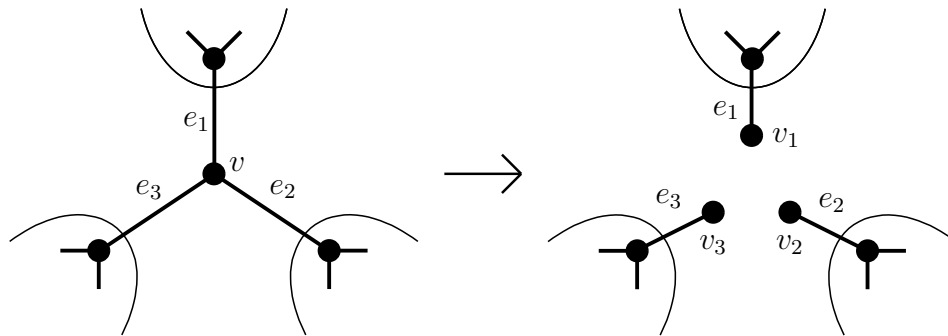

Figure 2: The operation for an isolated vertex v in $G - S$.

We perform this operation to all isolated vertices v in $G - S$ one by one, and let G' be the obtained graph. With abuse of notation, we may regard that G' has the edge set same as G, that is, $E(G') = E(G)$.

Claim 1. Each component of G' has an even number of edges, and in particular, at least three vertices.

Proof. Let $v^1, v^2, \ldots$ be the isolated vertices in $G - S$ in the order that we perform the operation to obtain G', and let G^i be the graph obtained by the operation at v^i. Let $G^0 = G$. By the induction on the number of steps, we prove the stronger statement that

- every component of G^i has an even number of edges and

- (S1') for every isolated vertex v^j in $G - S$ with $j \geq i + 1$, each of the three components in $G^i - v^j$ that have a vertex adjacent with v^j in G has an odd number of edges.

Note that the original graph $G = G^0$ has an even number of edges and satisfies Condition (S1') by the assumption and Condition (S1), respectively.

We assume that $i \geq 0$ and every component in G^i has an even number of edges and satisfies Condition (S1'). Suppose that there still remains the isolated vertex v^{i+1} in $G - S$ that has not performed the operation yet. Let C^i be the component of G^i containing v^{i+1}.

Note that C^i contains an even number of edges. If we perform the operation at v^{i+1} to obtain G^{i+1}, C^i breaks into three components. By Condition (S1') for G^i, each component of $C^i - v^{i+1}$ has an odd number of edges. Hence together with the edge incident with v^{i+1}, we see that every component of G^{i+1} has an even number of edges.

Therefore, it suffices to check Condition (S1') for G^{i+1}. Let v^j be an isolated vertex in $G - S$ with $j \geq i + 2$. If v^j is not contained in C^i, then we have nothing to show since each component of $G^{i+1} - v^j$ that has a vertex adjacent with v^j in G is also a component of $G^i - v^j$. Thus, we may assume that v^j is contained in C^i. Let C^{i+1} be the component of G^{i+1} containing v^j, which is obtained by the operation at v^{i+1}. Note that C^{i+1} has an even number of edges. Let D_1, D_2, D_3 be the three components of $G^{i+1} - v^j$ that have a vertex adjacent with v^j in G. Note that they are also the three components of $C^{i+1} - v^j$. By symmetry, we may assume that D_1 contains the vertex obtained by the operation at v^{i+1}. In this case, D_2 and D_3 are also components of $G^i - v^j$. By Condition (S1') for G^i, both $|E(D_2)|$ and $|E(D_3)|$ are odd. Since $|E(D_1)| + |E(D_2)| + |E(D_3)| = |E(C^{i+1})| - 3$ (where 3 corresponds to the three edges incident with v^j) and C^{i+1} has an even number of edges, we have that $|E(D_1)|$ is also odd. This proves that G^{i+1} satisfies Condition (S1'), and completes the proof of Claim 1. □

Since we regard $E(G') = E(G)$, it follows from Proposition 3.2 that it suffices to find a decomposition of G' into extended circuits with even number of edges. To do that, we focus on each component of G'. Let C be a component of G'. The next claim shows that C has a certain set of disjoint cycles, each of which will be a base circuit of an extended circuit that we will find.

Claim 2. C has pairwise disjoint cycles whose union contains all vertices of degree three in C.

Proof. Note that each vertex of C has degree either one or three. To each vertex of degree one, we add the subdivided-K_4, which is the graph obtained from K_4 by subdividing an edge once, as in Figure 3. We denote by C^+ the resultant graph, which is a cubic graph.
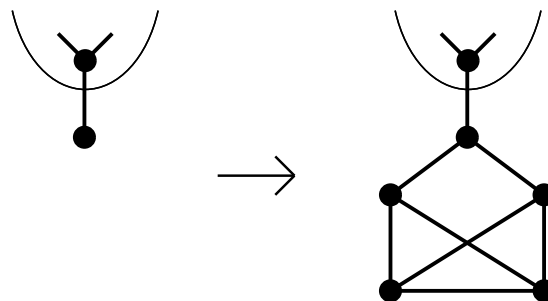

Figure 3: The addition of subdivided-K_4 to a vertex of degree one in C.

Consider the block-cut tree T_C of C^+, and suppose that T_C has three leaves. This means that there is a cut vertex that is incident with three blocks or there is a block that has three cut vertices. However, this contradicts the construction of G' and Condition (S2), respectively. Thus, T_C has at most two leaves. This implies that all cut edges are contained

in a path in C^+. By Theorem 3.3, C^+ has a 2-factor. By deleting a cycle in the added subdivided-K_4's, we obtain the desired cycles. □

Let $X_1, X_2, \ldots, X_t$ be the pairwise disjoint cycles whose union contains all vertices of degree three in C, which are obtained by Claim 2. Now, we will find an assignment f that assigns each edge e in $E(C) - \bigcup_{i=1}^{t} E(X_i)$ to a cycle $f(e) = X_i$ for some $1 \leq i \leq t$ so that $f(e)$ contains at least one of the end vertices of e and for $1 \leq i \leq t$,

$$E(X_i) \cup \{e : f(e) = X_i\}$$

has an even number of edges. By the conditions on the assignment f, we see that $E(X_i) \cup \{e : f(e) = X_i\}$ induces an extended circuit with even number of edges, and those extended circuits form a decomposition of C with properties mentioned in Proposition 3.2.

To find such an assignment f, we delete all vertices of degree one and contract each X_i into one vertex for $1 \leq i \leq t$, keeping all edges in $E(C) - \bigcup_{i=1}^{t} E(X_i)$, even in the case when multiples edges or loops are created. Let $\widetilde{T}$ be a spanning tree of the resultant graph. Note that all edges in $\widetilde{T}$ connect two vertices of degree three in C that belong to distinct cycles in $X_1, X_2, \ldots, X_t$.

To each edge e in $E(C) - \bigcup_{i=1}^{t} E(X_i) - E(\widetilde{T})$, we first assign X_i as $f(e)$ so that X_i contains at least one of the end vertices of e. Since C contains at least three vertices (by Claim 1) and each vertex of degree three in C is contained in one of $X_1, X_2, \ldots, X_t$, such a cycle X_i must exist.

We regard $\widetilde{T}$ as a rooted tree with root X_1 such that all edges are oriented from the leaves to the root. From the leaves of $\widetilde{T}$ we will find the assignment of the edges in $E(\widetilde{T})$ along the tree $\widetilde{T}$ as follows. Suppose that for a cycle X_i with $X_i \neq X_1$, all edges in $\widetilde{T}$ incoming to X_i are already assigned by f, including the case that X_i corresponds to a leaf of $\widetilde{T}$. Let e be an edge in $\widetilde{T}$ that outgoes to the parent, say X_j, of X_i in $\widetilde{T}$. In this case, we assign X_i or X_j to e as $f(e)$ so that $E(X_i) \cup \{e : f(e) = X_i\}$ has an even number of edges. By this procedure, we can assign all edges in $\widetilde{T}$ one by one so that $E(X_i) \cup \{e : f(e) = X_i\}$ has an even number of edges for any i with $i \neq 1$. It follows from Claim 1 that C has an even number of edges, which is the union of $E(X_i) \cup \{e : f(e) = X_i\}$ over all $1 \leq i \leq t$. Hence, $E(X_1) \cup \{e : f(e) = X_1\}$ also has an even number of edges.

Therefore, each component C of G' has a decomposition of G' into extended circuits with even number of edges, and so G does. This completes the proof of Theorem 2.3. □

5 Conclusion

In this paper, we give a new sufficient condition for the line graph of a connected cubic graph to have an even 2-factor. Before closing this paper, we discuss the best possibility of Conditions (S1) and (S2).

One may think that Condition (S1) is technical, but we emphasize that it is indeed a necessary condition. Let G be a connected cubic graph with an even number of edges, and S be the set of cut edges in G. Suppose that for an isolated vertex v in $G - S$, there are some components in $G - v$ that have an even number of edges. If we perform the operation to v

as in Figure 2, we obtain the three components, say D_1, D_2, D_3, some of which have an odd number of edges. By symmetry, we may assume that D_1 has an odd number of edges. Thus, D_1 cannot have a decomposition into extended circuits with even number of edges. Since no extended circuit in G can contain edges in two of D_1, D_2, D_3, this shows that G does not have a decomposition into extended circuits with even number of edges. By Proposition 3.2, the line graph $L(G)$ of G does not have an even 2-factor.

On the other hand, we assume Condition (S2) by a technical reason. For example, consider the connected cubic graph depicted in the left of Figure 4, which has three cut edges represented by double lines. Since the center component of $G - S$ is incident with three cut edges, this does not satisfy Condition (S2). However, we can check that the line graph of G has an even 2-factor.

On the other hand, the connected cubic graph G' depicted in the right Figure 4, does not satisfy Condition (S2) and the line graph of G' has no even 2-factor. Thus, in order to give a further extension of Theorem 2.3, we need to distinguish those graphs in Figure 4, which makes the statement more complicated. Therefore, we decide to state our main theorem as in Theorem 2.3.
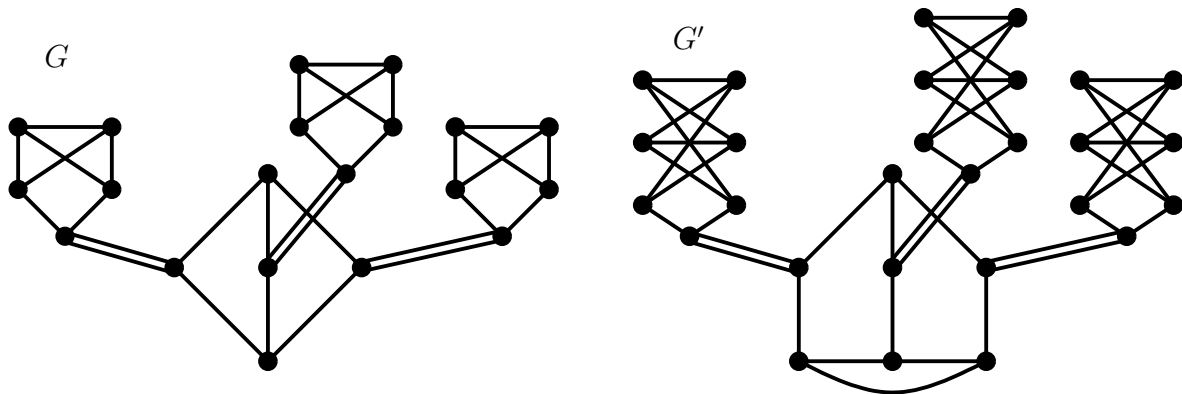

Figure 4: Connected cubic graphs G (left) and G' (right) with three cut edges, which are indicated by double lines. Both of the graphs G and G' do not satisfy Condition (S2). The line graph $L(G)$ of G has an even 2-factor, while the line graph $L(G')$ of G' has no even 2-factor.

Acknowledgment

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