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Facial Achromatic Number of Triangulations with Given Guarding Number

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Cover Page Footnote

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Abstract

A (not necessarily proper) k-coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of a graph G on a surface is a *facial t-complete k-coloring* if every t-tuple of colors appears on the boundary of some face of G. The maximum number k such that G has a facial t-complete k-coloring is called a *facial t-achromatic number* of G, denoted by $\psi_t(G)$. In this paper, we investigate the relation between the facial 3-achromatic number and guarding number of triangulations on a surface, where a *guarding number* of a graph G embedded on a surface, denoted by $\mathrm{guard}(G)$, is the smallest size of its *guarding set* which is a generalized concept of guards in the art gallery problem. We show that for any graph G embedded on a surface, $\psi_{\Delta(G^*)}(G) \leq \mathrm{guard}(G) + \Delta(G^*) - 1$, where $\Delta(G^*)$ is the largest face size of G. Furthermore, we investigate sufficient conditions for a triangulation G on a surface to satisfy $\psi_3(G) = \mathrm{guard}(G) + 2$. In particular, we prove that every triangulation G on the sphere with $\mathrm{guard}(G) = 2$ satisfies the above equality and that for one with guarding number 3, it also satisfies the above equality with sufficiently large number of vertices.

1 Introduction

All graphs considered in this paper are finite, undirected and simple unless otherwise mentioned. For a graph G, let $V(G)$ be the set of vertices and let $\deg_G(v)$ be the *degree* of a vertex v in G which is the number of edges incident to v. A *complete graph* K_n is a graph with n vertices in which every distinct two vertices are adjacent. A graph G is said to be *embedded* on a surface F^2 if G is drawn on F^2 without edge crossings. The *face size* of a face f is the length of the boundary of f and $\Delta(G^*)$ means the maximum face size of faces in G, where G^* is the dual graph of G. A *triangulation* on a surface F^2 is a graph embedded on F^2 such that every face is triangular. For basic terms and notations not defined here, we refer to [13].

1.1 Background of our study and a general relationship

In the art gallery problem, one is given a simple polygon P in the plane and asked to find the smallest subset S of points of P, called *guards*, so that every point in P is seen by at least one point in S. The results on this problem address the extremal question; how many guards are needed to guard a simple polygon with n points? Chvátal [10] gave the first solution to the question by proving that $\lfloor \frac{n}{3} \rfloor$ guards are sufficient and sometimes necessary to guard a polygon with n points. Subsequently, Fisk [15] presented an elegant proof to this theorem: First, every simple polygon in the plane can be triangulated by adding only edges. Since the resulting configuration can be regarded as a maximal outerplanar graph, such a graph has a proper 3-coloring of points, and hence each color class in its 3-coloring becomes the desired set of guards. In this context, for a graph G embedded on a surface, a generalized concept, called a *polychromatic k-coloring* of G, is well studied, which is a (not necessarily proper) coloring of vertices of G such that all k colors appear on the boundary of each face of G. The *polychromatic number* of G, denoted by $p(G)$, is the maximum integer k such that G has a polychromatic k-coloring. In general, for a graph G on the sphere, it is NP-complete to determine if $p(G) \geq 3$ [2]. In the same paper, many results are provided for polychromatic coloring of planar graphs, and for that of graphs embedded on other surfaces, see [20, 22]. For more results and other topics, see [6, 14, 19, 21] and a survey [12, Section 6].

The guarding set of graphs on surfaces is a generalization of guards in the art gallery problem. (Note that the polygon P in the art gallery problem is a geometric graph but in the generalization of guards on surfaces is not.) Let G be a graph embedded on a surface. A *guarding set* of G is a subset $S \subseteq V(G)$ such that each face of G is incident to a vertex in S. The *guarding number* of G, denoted by $\mathrm{guard}(G)$, is the number of vertices in a smallest guarding set of G. Similarly to the polychromatic number, it is NP-hard to find the guarding number of a given graph on the sphere [11]. By the definition, for any graph G on a surface with n vertices, $\mathrm{guard}(G) \leq \frac{n}{p(G)}$. Therefore, many results for upper bounds of the guarding number can be obtained from known results on the polychromatic number. For example, since every triangulation on the sphere with $n \geq 4$ vertices has polychromatic number at least 2 [6], its guarding number is at most $\frac{n}{2}$. Note that the above bound is sharp since there exists a triangulation G on the sphere with $\mathrm{guard}(G) = \lfloor \frac{n}{2} \rfloor$ [6]. The guarding set is also considered in several variant settings; for example, see [5, 7].

In this paper, we investigate the relation between the guarding number and facial achromatic number of graphs on surfaces, where the latter is recently introduced by the authors in [23]. A (not necessarily proper) k-coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of a graph G on a surface is a *facial t-complete k-coloring* if every t-tuple of colors appears on the boundary of some face of G. The maximum number k such that G has a facial t-complete k-coloring is called a *facial t-achromatic number* of G, denoted by $\psi_t(G)$. (If we do not need to care about the value t, then we simply call it a facial achromatic number.) It is clear that $t \leq \Delta(G^*)$. We instinctively see that if the number of vertex disjoint faces of G becomes larger, then so do $\mathrm{guard}(G)$ and $\psi_t(G)$. It is proved in [23] that if the number of vertex disjoint faces of a triangulation G is at least $\binom{n}{3}$, then $\psi_3(G) \geq n$ and that the cubic order with respect to n is sharp. On the other hand, it is trivial that for any graph G, $\mathrm{guard}(G)$ is at least the number of vertex disjoint faces. Thus, we can expect that $\psi_t(G)$ can be bounded by a function of $\mathrm{guard}(G)$ for any graph G, and in fact, we have the following general relationship.

Theorem 1.1. *Let G be a graph on a surface. Then*

$$\psi_{\Delta(G^*)}(G) \leq \mathrm{guard}(G) + \Delta(G^*) - 1.$$

Corollary 1.2. *Let G be a graph on a surface with n vertices. Then*

$$\psi_{\Delta(G^*)}(G) \leq \frac{n}{p(G)} + \Delta(G^*) - 1.$$

The bound in Theorem 1.1 is best possible in general. For example, a cycle C_n of order n embedded on the sphere has $\Delta(C_n^*) = n$, $\mathrm{guard}(C_n) = 1$ and $\psi_n(C_n) = n$.

Regarding a face of a triangulation G as a hyperedge, we can obtain a 3-uniform hypergraph H_G from G. Observe that the matching number and covering number of H_G corresponds to the number of vertex disjoint faces and guarding number of G, respectively. Thus, Theorem 1.1 together with Aharoni's result [1] leads to the following.

Corollary 1.3. *For any triangulation G on a surface,*

$$\psi_3(G) \leq 2f(G) + 2,$$

where $f(G)$ is the number of vertex disjoint faces of G.

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1.2 Sufficient condition for $\psi_3(G) = \mathrm{guard}(G) + 2$

It seems to be difficult to determine the facial achromatic number of graphs on surfaces since to find the guarding number is NP-hard in general even if those given graphs are triangulations. Thus, we focus on triangulations G satisfying $\psi_3(G) = \mathrm{guard}(G) + 2$.

We first establish a sufficient condition for a triangulation on a surface to satisfy the above equality, as follows. (Note that no trivial triangulation has guarding number 1.)

Theorem 1.4. *Let G be a triangulation on a surface and let $S = \{v_1, v_2, \ldots, v_k\}$ be a guarding set of G with $|S| = \mathrm{guard}(G) = k \geq 2$. If $\deg_G(v_i) \geq \frac{1}{3}k^3 + k^2 + \frac{5}{3}k - 1$ for each $i \in \{1, \ldots, k\}$, then $\psi_3(G) = k + 2$.*

The condition of Theorem 1.4 seems to be very strict and so one may think that the theorem is trivial. However, Theorem 1.4 gives us an important observation for the relation between the facial achromatic number and the number of vertex disjoint faces: Recall that for any triangulation G on a surface, if G has at least $\binom{m}{3}$ vertex disjoint faces, then $\psi_3(G) \geq m$, and to improve its coefficient of the sufficient condition is proposed as an open problem [23]. Theorem 1.4 means that by increasing the degree of several specific vertices, we can make the facial achromatic number of a given triangulation as large as possible even if the number of vertex disjoint faces is fixed. (Note that the number of vertex disjoint faces of a graph G is at most $\mathrm{guard}(G)$.) Hence the theorem asserts that the degree condition may improve the coefficient of the evaluation of the number of vertex disjoint faces.

Moreover, we cannot relax the sufficient condition of Theorem 1.4 to the degree sum condition. Let G be a triangulation on the sphere shown in Figure 1. Graphs inside of xay and xyb are isomorphic to a standard form with guarding set $\{x, y\}$, where a *standard form* is shown in Figure 2. Observe that $\{x, y, z, w\}$ is a guarding set which forms a copy of K_4. By Theorem 1.1, we have that $4 \leq \psi_3(G) \leq 6$ and we suppose to the contrary that $\psi_3(G) = 6$. In this case, vertices in the guarding set are colored by different colors mutually. (Otherwise, a tuple of three colors which are not used for vertices in the guarding set cannot appear.) Thus, we may assume that x, y, z and w are colored by colors $1, 2, 3$ and 4, respectively. Since $\psi_3(G) = 6$, at least four faces colored by colors other than 1 and 2 must appear. However, there are exactly two faces zuu' and wvv' which are guarded by neither x nor y, a contradiction. Therefore, $\psi_3(G)$ is at most 5. On the other hand, if we put standard forms of sufficiently large order in the regions surrounded by xay and xyb in G, then the sum of degree of x and y becomes sufficiently large (e.g., larger than $\frac{1}{3} \times 4^3 + 4^2 + \frac{5}{3} \times 4 - 1 = 43$). However, the number of faces which are guarded by neither x nor y is not increased. Thus, no matter how large the sum of degree is, $\psi_3(G)$ is still at most 5. Therefore, we cannot relax the sufficient condition of Theorem 1.4 to the degree sum condition.

1.3 Triangulations with small guarding number

In general, it seems to be not easy to characterize all triangulations G with $\mathrm{guard}(G) \geq 4$ and $\psi_3(G) = \mathrm{guard}(G)+2$, since the structure of such triangulations is rather flexible. In fact, for any fixed $k \geq 5$, we can construct infinitely many triangulations G with $\mathrm{guard}(G) = k$ and $\psi_3(G) < k+2$ similarly to Figure 1: Let H be a standard form with k' vertices for $k' \geq 4$ and let $v_1, \cdots, v_{k'}$ be vertices of H as shown in Figure 2. Let G be a triangulation on the sphere

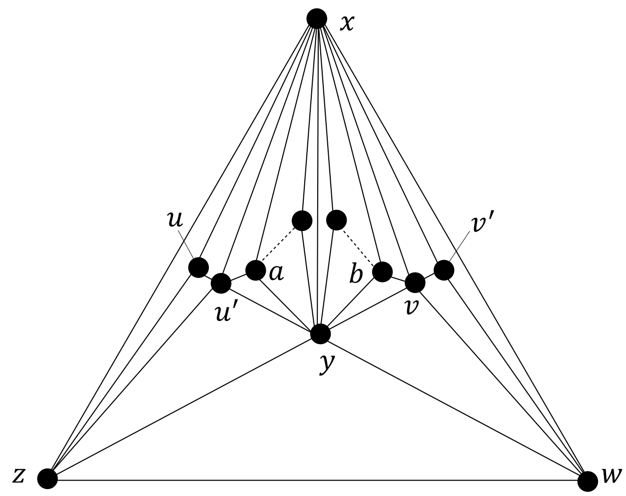

Figure 1: A triangulation G on the sphere with guarding number 4 and $\psi_3(G) < 6$

obtained from H by identyfying $v_{k'-2}v_{k'-1}v_{k'}$ with xzw of Figure 1 and adding vertices inside of $v_iv_{i+1}v_{k'-1}$ for $i = 1, \dots, k'-3$ as the subgraph surrounded by $v_iv_{i+1}v_{k'-1}$ is isomorphic to a standard form such that v_i and v_{i+1} are vertices in the guarding set of the standard form as shown in Figure 3. Thus, $v_1, \dots, v_{k'-2}$ are in the guarding set of G. Moreover, since the subgraph surrounded by $v_{k'-2}v_{k'-1}v_{k'}$ is isomorphic to Figure 1, $v_{k'-1}, v_{k'}$ and y of Figure 3 are in the guarding set of G. Thus, $\mathrm{guard}(G) = k'+1$. Put $k = k'+1$ and consider whether G has a facial 3-complete $(k+2)$-coloring. Suppose that G has a facial 3-complete $(k+2)$-coloring and we may assume that v_i is assigned with color i for $i = 1, \cdots, k'$ and y is assigned with color k. Since $\psi_3(G) = k+2$, four faces colored by $k'-1, k', k+1$ and $k+2$ must appear. However, faces other than $uu'v_{k'-1}$ and $v'vv'_k$ have at least one vertex colored by other than the above four colors. Thus, G has no facial 3-complete $(k+2)$-coloring.

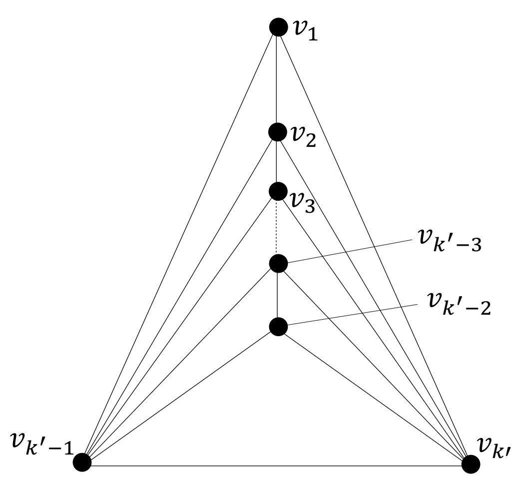

Figure 2: A standard form H with k' vertices

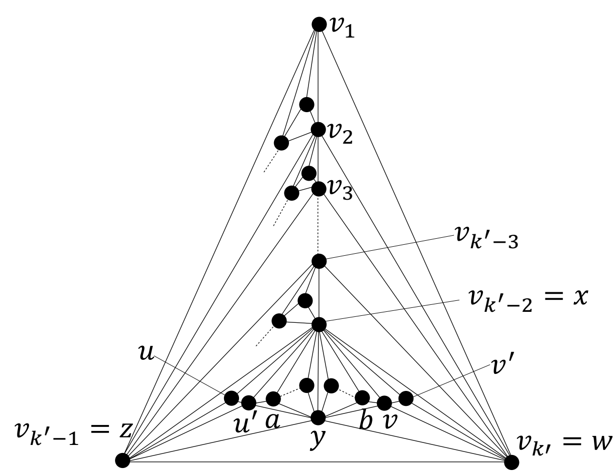

Figure 3: A graph G obtained from H

By the above examples, we focus on a triangulation with guarding number at most 3. If the minimum guarding set S is of size 2, then there are only two cases; S is an independent set or S induces a K_2. Thus we can completely characterize triangulations with guarding number 2 (Proposition 3.1), and then we have the following using the characterization.

Theorem 1.5. *Let G be a triangulation on the sphere with* $\mathrm{guard}(G) = 2$. *Then* $\psi_3(G) = 4$.

On the other hand, a triangulation G with $\mathrm{guard}(G) = 3$ satisfies the equality if G has sufficiently large number of vertices.

Theorem 1.6. *Let G be a triangulation on the sphere with* $|V(G)| \geq 11$ *and* $\mathrm{guard}(G) = 3$. *Then* $\psi_3(G) = 5$.

In particular, when the minimum guarding set of a triangulation G on the sphere with $\mathrm{guard}(G) = 3$ induces a copy of K_3, the lower bound of the order can be reduced by one.

Theorem 1.7. *Let G be a triangulation on the sphere with* $|V(G)| \geq 10$ *and* $\mathrm{guard}(G) = 3$. *If G has a minimum guarding set inducing a K_3, then* $\psi_3(G) = 5$.

We show Theorem 1.6 by a computer-assisted proof. The program "plantri" (`http://users.cecs.anu.edu.au/~bdm/plantri/`) is one of the fastest C program which generates certain type of graphs on the sphere. We generate triangulations on the sphere by plantri, and implement a Python program which determines the guarding set of the graph and whether it has a facial 3-complete 5-coloring. The program and the data generated by plantri are shown in the first author's webpage `https://sites.google.com/view/naokimatsumoto/data?authuser=0`.

2 Remarks on Theorems 1.6 and 1.7

We first show that the evaluation of the number of vertices in Theorem 1.6 is best possible unless the minimum guarding set induces a copy of K_3. By the computer program, we found some triangulations G on the sphere with $|V(G)| = 10$ and $\mathrm{guard}(G) = 3$ which have no facial 3-complete 5-coloring (see Figure 4). The triangulation G_1 in the left hand of Figure 4 has the minimum guarding set $\{a, c, d\}$ which induces a path of length 2, and $\{a, b, d\}$ which induces a path of length 1 and one isolated vertex. Moreover, the triangulation G_2 in the right hand of Figure 4 has the minimum guarding set $\{a, b, c\}$ which induces three isolated vertices.

Next, we show that the evaluation of the number of vertices in Theorem 1.7 is best possible. A *pseudocomplete k-coloring* of G is a (not necessarily proper) k-coloring such that each pair of colors appears on at least one edge (cf. [3, 16]). The *pseudoachromatic number* of G denoted by $\psi(G)$ is the maximum number k for which G has a pseudocomplete k-coloring.

Theorem 2.1 (Yegnanaranayan [25])**.** *Let C_n be the cycle on n vertices for $n \geq 3$. Then*

$$\begin{cases} \psi(C_n) = 2k & (2k^2 \leq n \leq k(2k+1) - 1). \\ \psi(C_n) = 2k+1 & (k(2k+1) \leq n \leq 2k^2 + 4k + 1). \end{cases}$$

Lemma 2.2. *Let G be a triangulation on the sphere with* $|V(G)| \leq 9$ *and* $\mathrm{guard}(G) = 3$. *If* $\psi_3(G) = 5$, *then there exists a vertex v such that the degree of v is 8.*

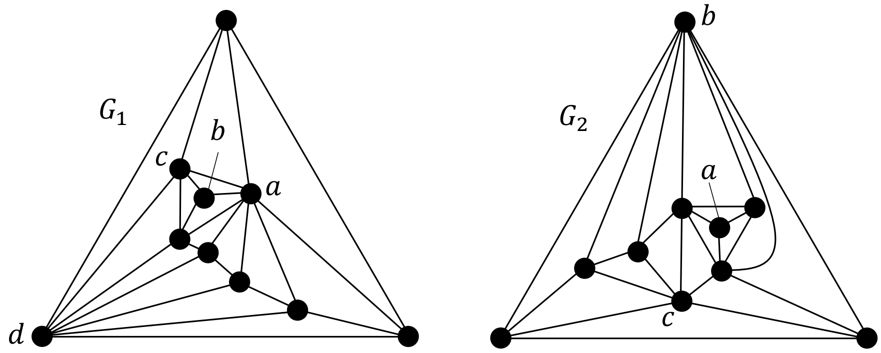

Figure 4: Triangulations G_1 and G_2 on the sphere with guarding number 3 which have no facial 3-complete 5-coloring

Proof. Since $|V(G)| \leq 9$, there exists a color which is assigned to only one vertex in G for a facial 3-complete 5-coloring c of G. Let v be a vertex in G such that $c(v) = 1$ and suppose that $c(u) \in \{2, 3, 4, 5\}$ for any vertex u in G other than v. Since color 1 appears on only v, triads including color 1 must appear around v. Thus, all of the pair of four colors other than color 1 appear in the link of v. By Theorem 2.1, $\psi(C_7) < 4$ and hence, the degree of v is 8. □

Let G be a triangulation on the sphere shown in Figure 5. Since the maximum degree of G is seven, G has no facial 3-complete 5-coloring by Lemma 2.2. Thus, the evaluation of the number of vertices in Theorem 1.7 is the best since the minimum guarding set $\{x, y, z\}$ forms K_3.

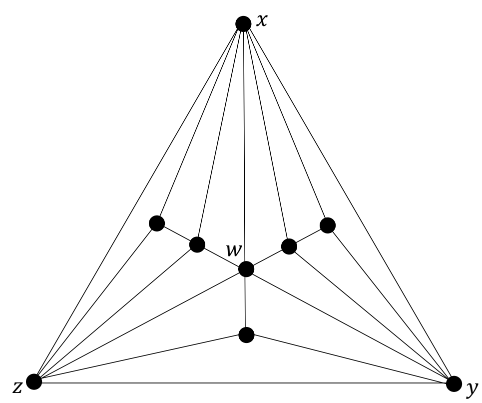

Figure 5: A triangulation G on the sphere with guard$(G) = 3$ and $|V(G)| = 9$ whose the minimum guarding set induces a copy of K_3

Finally we notice triangulations G with $\psi_3(G) = \mathrm{guard}(G)+2$ for large guarding number. Recall that the guarding number of a triangulation on the sphere can be a half of the number of vertices in general. On the other hand, for a triangulation G on a surface F^2 with equality in Theorem 1.1, the guarding number guard(G) is bounded by $O(\sqrt[3]{|V(G)|})$, since $\binom{\mathrm{guard}(G)+2}{3} \leq 2|V(G)| - 2\epsilon(F^2)$ by Euler's formula, where $\epsilon(F^2)$ is the Euler characteristic of F^2. Moreover, we obtain the following proposition which implies that this order is the best in general.

Proposition 2.1. *There exist infinitely many triangulations G on the sphere whose order of the guarding number is $\Omega(\sqrt[3]{|V(G)|})$.*

Proof. Let G be a double wheel with $|V(G)| = m + 2$ for $m \geq 2$, where a *double wheel* of order $m+2$ is a triangulation with degree sequence $(m, m, 4, 4, \ldots, 4)$. Let $w_1, w_2, \cdots, w_m$ be vertices of degree 4 in G for $m \geq 2$ and u and v be other vertices of G. Put $2\binom{m+1}{2}$ vertices into the inside of $w_i w_{i+1} u$ for $i = 1, \cdots, m$ such that a graph whose boundary is $w_i w_{i+1} u$ is a standard form, where w_i and w_{i+1} guard its vertices as shown in Figure 6 and we call it G'. The guarding number of G' is m and $|V(G')| = m + 2 + m \times 2\binom{m+1}{2} = m^3 + m^2 + m + 2$. Thus, the order of the guarding number of G' is $\Omega(\sqrt[3]{|V(G')|})$. □

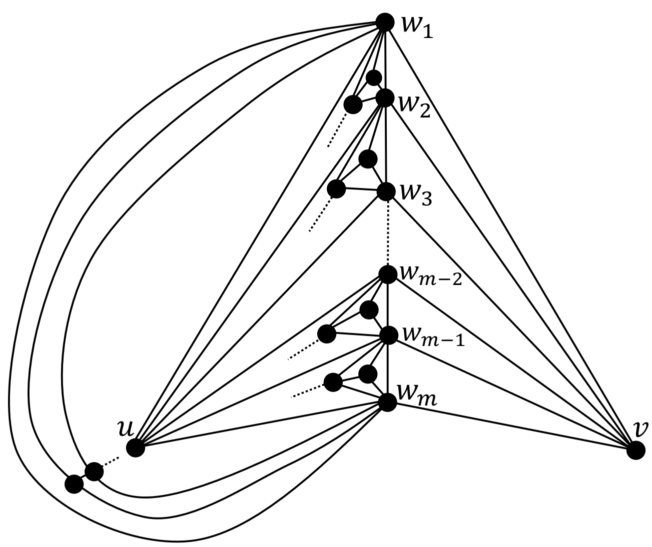

Figure 6: A triangulation G' on the sphere such that the order of guard(G') is $\Omega(\sqrt[3]{|V(G')|})$

3 Proofs of Theorems 1.1, 1.4 and 1.5

Proof of Theorem 1.1. Let G be a graph on a surface and let S be a minimum guarding set of G. Since every face is incident to a vertex in S, for any facial $\Delta(G^*)$-complete coloring, every $\Delta(G^*)$-tuple of colors appearing on a face contains a color used for a vertex in S. Thus, if $\psi_{\Delta(G^*)}(G) \geq \mathrm{guard}(G) + \Delta(G^*)$, then there must exist a face which is not incident to any vertex in S, contrary to that S is a guarding set of G. □

Proof of Theorem 1.4. Suppose that each vertex v_i in the guarding set S has been assigned with color i for each $i = 1, \ldots, k$ and other vertices in G has not been assigned with colors. We consider to assign colors to vertices of the link of v_1 to v_k in order. If there exist $\binom{k+1}{2}$ independent edges on the link of v_1 such that endvertices of the edges have not been assigned with colors, then each pair of $k+1$ colors can appear on those edges. Since there are at most $k-1$ vertices which have been assigned with colors in the link of v_1, we may assume that there are $k-1$ vertices in the guarding set on the link of v_1. Let $v'_1, \cdots, v'_{k-1}$ be vertices in S lying on the link of v_1 in clockwise and let V_i be the set of vertices on the link of v_1 between v'_i and v'_{i+1} for $i = 1, \cdots, k-1$ where $v'_k = v'_1$. Note that each V_i contains no vertex in S. If $\left\lfloor \frac{|V_1|}{2} \right\rfloor + \cdots + \left\lfloor \frac{|V_{k-1}|}{2} \right\rfloor \geq \binom{k+1}{2}$, then all triads containing color 1 can appear around v_1. Thus, if $\deg(v_1) \geq 2\binom{k+1}{2} + 2(k-1) = k^2 + 3k - 2$, then we can assign colors to exactly $2\binom{k+1}{2}$ vertices among the neighborhoods of v_1, so that all triads containing color 1 appears around v_1.

Next, we consider to assign colors to vertices on the link of v_2. In this case, there are at most $(k-1)+2\binom{k+1}{2}$ vertices which have been assigned with colors. Since all triads containing color 1 has appeared, we would like to make triads containing color 2 around v_2 which does not have color 1. Similarly to the previous case, if there are $2\binom{k}{2} + 2\{(k-1) + 2\binom{k+1}{2}\} = 3k^2 + 3k - 2$ vertices, then all triads containing color 2 can appear around v_2 and hence, we color exactly $2\binom{k}{2}$ vertices on the link of v_2.

Repeating such a consideration, for the link of v_k, there are at most $(k-1)+\sum_{i=1}^{k-1} 2\binom{k+2-i}{2}$ vertices which have been assigned with colors. Thus, if there are $2\binom{k+2-k}{2} + 2\{(k-1) + \sum_{i=1}^{k-1} 2\binom{k+2-i}{2}\} = \frac{1}{3}k^3 + k^2 + \frac{5}{3}k - 1$ vertices, then all triads containing color k can appear around v_k. This completes the proof of Theorem 1.4. □

Before we prove Theorem 1.5, we provide the following characterization.

Proposition 3.1. *Let G be a triangulation on the sphere with* $\mathrm{guard}(G) = 2$. *Then G is isomorphic to either a double wheel or a standard form.*

Proof. Let $S = \{x, y\}$ be a guarding set of a triangulation G on the sphere and let $L(x) = u_1, \ldots, u_k$ and $L(y) = v_1, \ldots, v_\ell$ be the links of x and y in clockwise and anti-clockwise orders, respectively. Since S is a guarding set, there exists no face which contains neither x nor y in its boundary. Thus, $k = \ell$.

We suppose that $xy \in E(G)$, and without loss of generality, we may suppose that $u_1 = y$, $v_1 = x$, $u_2 = v_2$ and $v_k = u_k$. Then we also have $u_i = v_i$ for each $i \in \{3, \ldots, k-1\}$ by the planarity of G, since otherwise, say $u_j \neq v_j$ for some $j \in \{3, \ldots, k-1\}$, there exists a face which is not guarded by S in the region containing both u_j and v_j in its boundary (but not containing x and y in its interior). Thus, G is isomorphic to the standard form. Similarly, if $xy \notin E(G)$, then we have $u_i = v_i$ for each $i \in \{3, \ldots, k-1\}$, and hence, G is isomorphic to a double wheel. □

Proof of Theorem 1.5. Let G be a triangulation on the sphere with $\mathrm{guard}(G) = 2$ and $n \geq 4$ vertices. By Proposition 3.1, G is isomorphic to either a double wheel or a standard form. In the former case, we assign three vertices on the boundary of a face containing a vertex of degree $n-2$ with colors 2, 3 and 4. Then we assign color 1 to all other vertices. In the latter case, we assign two vertices of degree $n-2$ with colors 1 and 2 and two vertices of degree 3 with colors 3 and 4. Then we assign color 3 to all other vertices of degree 4 (if exist). It is easy to see that the above two colorings are both facial 3-complete 4-colorings. Thus, the theorem holds. □

4 Proof of Theorems 1.6 and 1.7

Proof of Theorem 1.6. We prove the theorem by induction on $|V(G)|$. We can first confirm that every triangulation G on the sphere with $|V(G)| = 11$ has guarding number 3 and a facial 3-complete 5-coloring by running the computer program.

Let G be a triangulation on the sphere with $\mathrm{guard}(G) = 3$ and $|V(G)| = n \geq 12$, and suppose that every triangulation on the sphere with guarding number 3 and at most $n-1$ vertices has a facial 3-complete 5-coloring. If G is neither a standard form nor a double wheel, then there exists an edge xy of G such that G is transformed to neither a standard form nor a double wheel by contracting xy (and deleting resultant multiple edges). Thus, we can transform G into a triangulation on the sphere G' with $|V(G')| = n-1$ and $\mathrm{guard}(G') = 3$ by the contraction of xy. Since G' has a facial 3-complete 5-coloring by induction hypothesis, we color vertices of G other than x and y by the same color as ones of G', and x and y by the same color as $x = y$ in G'. Therefore, G has a facial 3-complete 5-coloring. □

Next we give a non-computer-assisted proof to Theorem 1.7. We first prepare the following lemma.

Lemma 4.1. *Let G be a triangulation on the sphere with* $\mathrm{guard}(G) = 3$ *and $S = \{x, y, z\}$ be a minimum guarding set of G. If S induces a K_3, then there exist one or two vertices w and w' which are adjacent to all of x, y and z, and each graph whose boundary is uvv', where $u \in \{w, w'\}$ and $v, v' \in \{x, y, z\}$ with $v \neq v'$, is isomorphic to a standard form or a triangle.*

Proof. First, we shall show the existence of a vertex w which is adjacent to all of x, y and z. Let R be a region surrounded by a cycle xyz. If there exists no vertex in G other than x, y and z, then G is not a triangulation on the sphere by the definition. Thus, we may assume that there exists at least one vertex in R other than x, y and z.

Suppose to the contrary that all vertices in R are adjacent to at most two of x, y and z. Since G is a triangulation on the sphere, there exists a vertex a in R such that xya is a boundary of a face. Similarly, there exists a vertex b in R such that xbz is a boundary of a face. If $a = b$, then the vertex is adjacent to all of x, y and z, a contradiction. Thus, we may assume that $a \neq b$.

Let $v_1 v_2 \cdots v_k$ be a part of the link of x in R for $k \geq 2$, where $v_1 = a$ and $v_k = b$. Since G is a triangulation on the sphere, there exist two faces which contain an edge $v_1 v_2$. One of such faces is $x v_1 v_2$ and another one is $v_1 v_2 y$ or $v_1 v_2 z$ since $\{x, y, z\}$ is a guarding set. If the face is $v_1 v_2 z$, then it implies that a is the desired vertex. Thus, we may assume that the face is $v_1 v_2 y$. By repeating this argument, there exists no face $v_i v_{i+1} z$ for $i = 1, \cdots, k-2$. However, for an edge $v_{k-1} v_k$, at least one of faces $v_{k-1} v_k y$ and $v_{k-1} v_k z$, that is, either v_{k-1} or v_k is the desired vertex, a contradiction. Therefore, there exists a vertex which is adjacent to all of x, y and z.

For the other region R' surrounded by xyz, similarly to the above, if the number of vertices inside of R' is at least one, then there exists a vertex which is adjacent to all of x, y and z. If there exist vertices inside R and R' other than such vertices, then they cannot be adjacent to all of x, y and z by planarity. Therefore, the number of the desired vertices is one or two. Moreover, it is easy to see that the second claim in the lemma follows from Proposition 3.1. □

Proof of Theorem 1.7. Let $S = \{x, y, z\}$ be a minimum guarding set of G and suppose that S induces a copy of K_3. Let R and R' be two regions surrounded by xyz. By Lemma 4.1, there exist one or two vertices w and w' which are adjacent to all of x, y and z in R and R', respectively. We prove the theorem by induction on the number of vertices in the following two cases.

Case 1. G has either w or w', say w.

Let R_1, R_2 and R_3 be the regions whose boundaries are xyw, xwz and wyz, respectively; see Figure 7. We first consider the base case, where $|V(G)| = 10$. In this case, there are six vertices other than x, y, z and w. If exactly one of R_1, R_2 and R_3 has vertices inside of it, then G is isomorphic to a standard form by Lemma 4.1 and hence, guard$(G) = 2$ by Proposition 3.1, a contradiction. Thus, at least two of R_1, R_2 and R_3 have vertices inside of it.

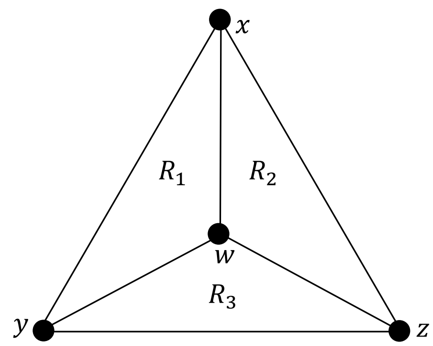

Figure 7: The structure of G which has only w

Let (i, j, k) be a sequence representing that there are i, j and k vertices inside of R_1, R_2 and R_3, respectively. By symmetry, it suffices to check the cases $(1, 5, 0)$, $(2, 4, 0)$, $(3, 3, 0)$, $(1, 1, 4)$, $(1, 2, 3)$ and $(2, 2, 2)$. As shown in Figure 8, G has a facial 3-complete 5-coloring in each case.

Suppose that $|V(G)| = n \geq 11$ and for every triangulation with at most $n - 1 (\geq 10)$ vertices satisfying the same condition of G, the theorem holds. Since guard$(G) = 3$, there exist at least two regions among R_1, R_2 and R_3 which have vertices. Moreover, there exists a region which have at least two vertices, say R_1, and hence, there exists a vertex v of degree 3 inside of R_1. Let v' be a vertex which is adjacent to v other than x and y. Let H be the graph obtained from G by removing v. By induction hypothesis, H has a facial 3-complete 5-coloring c by the assumption. We define a 5-coloring c' of G as $c'(a) = c(a)$ for any $a \in V(H)$ and $c'(v) = c(v')$. Observe that the triad appearing on a face xyv' in H appears a face xyv in G. Therefore, G has a facial 3-complete 5-coloring.

Case 2. G has both w and w'.

Let $R_1, R_2, R_3, R'_1, R'_2$ and R'_3 be the regions of G surrounded by a cycle of length 3 as shown in Figure 9. We first consider the base case, where $|V(G)| = 10$. If at most one of $R_1, R_2, R_3, R'_1, R'_2$ and R'_3 has vertices inside of it, then G is isomorphic to a standard form by Lemma 4.1 and hence, guard$(G) = 2$ by Proposition 3.1, a contradiction. Moreover,

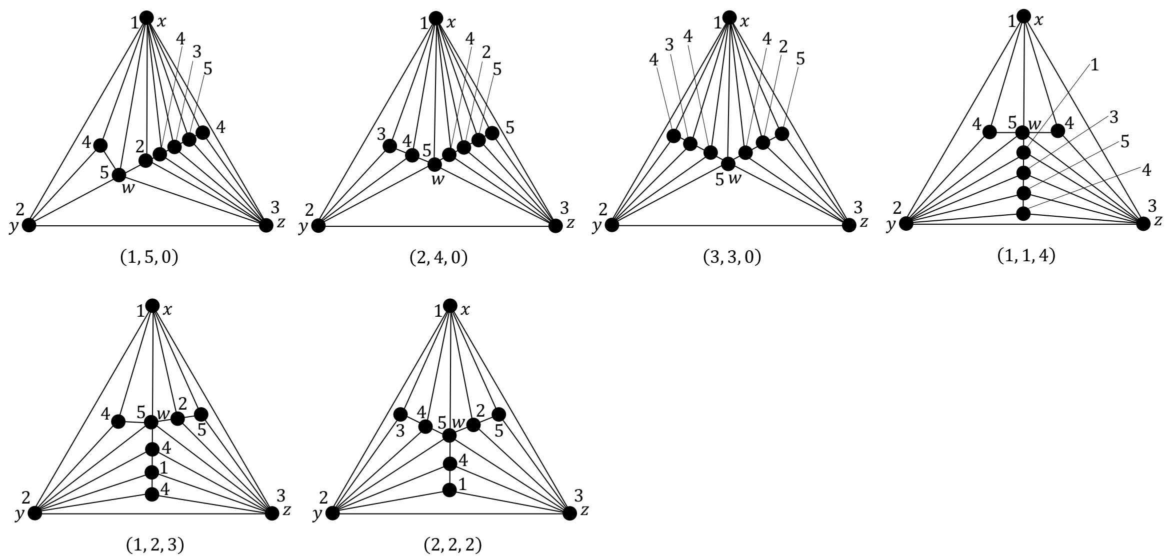

Figure 8: Facial 3-complete 5-colorings of G when G has only w

if only R_i and R'_i for $i = 1, 2, 3$ have vertices inside of them, then G is isomorphic to a standard form, a contradiction. Thus, we may assume that at least two of the regions of G have vertices inside of them other than the above case.

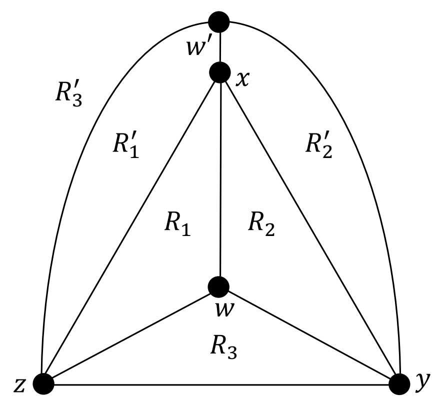

Figure 9: The structure of G which has w and w'

Let (i, j, k, l, m, n) be a sequence representing that there are i, j, k, l, m and n vertices inside of $R_1, R_2, R_3, R'_1, R'_2$ and R'_3, respectively. By symmetry, it suffices to consider the cases $(1, 4, 0, 0, 0, 0)$, $(2, 3, 0, 0, 0, 0)$, $(1, 0, 0, 0, 4, 0)$, $(2, 0, 0, 0, 3, 0)$, $(1, 1, 3, 0, 0, 0)$, $(1, 2, 2, 0, 0, 0)$, $(1, 1, 0, 3, 0, 0)$, $(1, 3, 0, 1, 0, 0)$, $(3, 1, 0, 1, 0, 0)$, $(1, 2, 0, 2, 0, 0)$, $(2, 1, 0, 2, 0, 0)$, $(2, 2, 0, 1, 0, 0)$, $(1, 1, 0, 0, 0, 3)$, $(3, 1, 0, 0, 0, 1)$, $(1, 2, 0, 0, 0, 2)$, $(2, 2, 0, 0, 0, 1)$, $(1, 1, 1, 2, 0, 0)$, $(2, 1, 1, 1, 0, 0)$, $(1, 2, 1, 1, 0, 0)$, $(1, 1, 0, 1, 2, 0)$, $(1, 1, 0, 0, 1, 2)$, $(1, 2, 0, 0, 1, 1)$ and $(1, 1, 1, 1, 1, 0)$. As shown in Figure 10, G has a facial 3-complete 5-coloring in each case.

Thus, we may suppose that $|V(G)| \geq 11$. If a region contains at least two vertices inside of it, then the theorem holds by induction on $|V(G)|$ similarly to Case 1. Hence, each region contains exactly one inner vertex, that is, $|V(G)| = 11$. In this case, all vertices other than

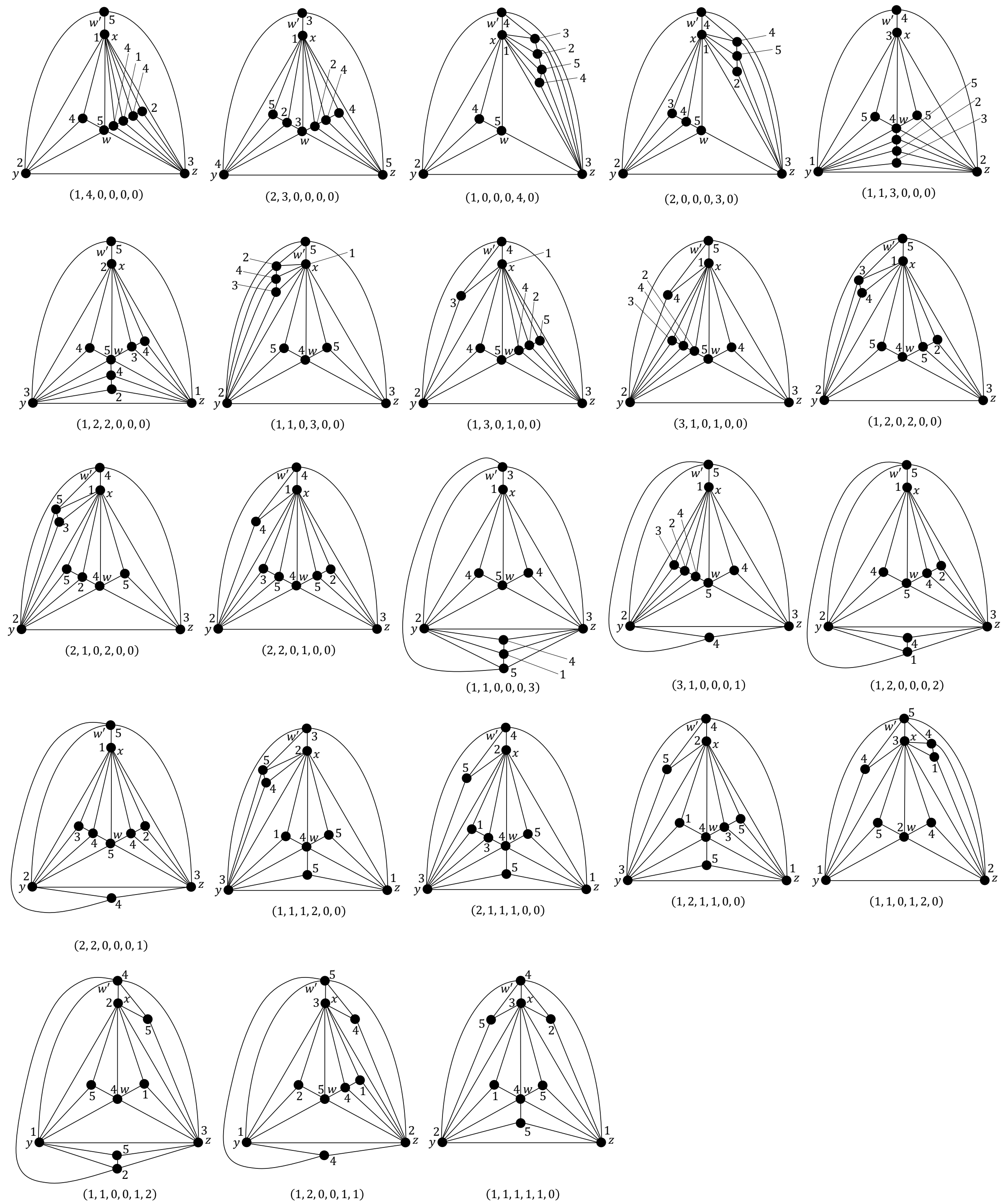

Figure 10: Facial 3-complete 5-colorings of G when G has w and w'

x, y, z, w and w' are of degree 3. By removing one of vertices of degree 3, say v, we have the last graph in Figure 10 with a facial 3-complete 5-coloring. Therefore, similarly to the above cases, by assigning a color used for a neighbor of v to v itself, we can obtain a facial

3-complete 5-coloring of G. This completes the proof of Theorem 1.7. □

5 Concluding remarks

In this section, we describe the relationship between the facial achromatic number and other two invariants, called a domination number and a vertex cover number. The *domination number* of a graph G, denoted by $\gamma(G)$, is the size of a minimum dominating set of G, where a *dominating set* S of G is a subset of $V(G)$ such that each vertex of G is adjacent to S or is in S. The *vertex cover number* of a graph G, denoted by $\tau(G)$, is the size of a minimum vertex cover of G, where a *vertex cover* U of G is a subset of $V(G)$ such that each edge of G is incident to a vertex in U. (Since there is a large amount of literatures for domination number and vertex cover number, we refer the readers to survey several books and articles [4, 9, 17, 18, 24].)

For a triangulation G on a surface, the guarding number lies on between the domination number and vertex cover number, i.e., $\gamma(G) \leq \mathrm{guard}(G) \leq \tau(G)$ (cf. [8]). Thus, for any triangulation G on a surface, $\psi_3(G) \leq \tau(G) + 2$ holds by Theorem 1.1. However, this estimation is not best possible. In fact, we have the following sharp bound. (For example, the complete graph K_4 embedded on the sphere attains the equality of the following since $\psi_3(K_4) = 4$ and $\tau(K_4) = 3$.)

Proposition 5.1. *Let G be a triangulation on a surface. Then*

$$\psi_3(G) \leq \tau(G) + 1.$$

Proof. Let G be a graph on a surface and let U be a minimum vertex cover of G. Observe that every face is incident to at least two vertices in U (since at least two vertices are needed for covering three edges in a triangle). Let $c : V(G) \to \{1, 2, \ldots, k\}$ be a facial 3-complete k-coloring. Suppose to the contrary that $k \geq \tau(G) + 2$, and without loss of generality, all vertices in U are colored by distinct $\tau(G)$ colors, say $1, 2, \ldots, \tau(G)$. By the above observation, every face can use at most one color which is not used for any vertex in U, that is, there exists no face with a triple $\{\tau(G)+1, \tau(G)+2, r\}$, where $r \in \{1, 2, \ldots, \tau(G)\}$, a contradiction. □

One may mind whether $\psi_3(G) \leq \gamma(G) + 2$ holds since $\gamma(G) \leq \mathrm{guard}(G)$. However, the inequality does not hold in general: The triangulation H shown in Figure 11 is obtained from the standard form (of sufficiently large order) by adding a vertex of degree 3 to each of faces guarded by x. Note that $\gamma(H) = 1$ and H contains many vertex disjoint faces. It is easy to see that the facial achromatic number increases as the number of vertex disjoint faces gets bigger (cf. [23]). Thus $\psi_3(H) \leq \gamma(H) + 2 = 3$ does not hold.

Observe that for the triangulation H shown in Figure 11, $\mathrm{guard}(H)$ is about as large as the number of vertex disjoint faces. Thus, we intuitively guess that the family of triangulations G with $\psi_3(G) \leq \gamma(G) + 2$ is not so large. Moreover, it is difficult to completely characterize triangulations G with $\psi_3(G) = \mathrm{guard}(G) + 2$; nevertheless characterizing ones G with $\psi_3(G) = \gamma(G) + 2$ seems to be more easier, since the corresponding family of triangulations avoids the above example. Therefore, we conclude the paper with preparing the following problem.

Problem 5.1. *Characterize triangulations G on surfaces such that $\psi_3(G) = \gamma(G) + 2$.*

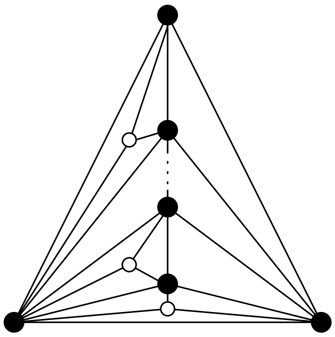

Figure 11: A triangulation H with domination number 1 but high facial achromatic number

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