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The Generalized Wiener-Hopf Equations for the elastic wave motion in angular regions

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In this work, we introduce a general method to deduce spectral functional equations in elasticity and thus, the Generalized Wiener-Hopf Equations (GWHEs), for the wave motion in angular regions filled by arbitrary linear homogeneous media and illuminated by sources localized at infinity. The work extends the methodology used in electromagnetic applications and proposes for the first time a complete theory to get the GWHEs in elasticity. In particular we introduce a vector differential equation of first order characterized by a matrix that depends on the medium filling the angular region. The functional equations are easily obtained by a projection of the reciprocal vectors of this matrix on the elastic field present on the faces of the angular region. The application of the boundary conditions to the functional equations yields GWHEs for practical problems. This paper extends and applies the general theory to the challenging canonical problem of elastic scattering in angular regions.

1. Introduction

In [1], we have applied a general theory to obtain spectral functional equations in electromagnetics and thus Generalized Wiener-Hopf Equations (GWHEs) for scattering problem in angular regions filled by arbitrarily linear media, inspired by [2] and described also in [3]. The monographs [4]- [5] show the efficacy of the generalization of the Wiener-Hopf (WH) technique in practical electromagnetic wave scattering problems in presence of geometries containing angular regions and/or stratified planar regions, see references therein.

In this paper we implement for the first time the methodology to the challenging canonical problem of elastic scattering in angular regions where some physical quantities are tensors. The technique consists of three steps: 1) the deduction of functional equations in spectral domain of sub-regions that constitute the whole geometry of the problem, 2) the imposition of boundary conditions to get the GWHEs and, 3) the solution of the system of the WH equations using exact or semianalytical approximate techniques of factorization as the Fredholm factorization technique [6]- [7].

This paper is focused on the first fundamental step and introduce the potentialities to develop the other two steps through validations. We follows the method to obtain the WH equations in spectral domain proposed by Jones [8]- [9], with the application of Fourier/Laplace transforms directly to the PDE formulation of the problem avoiding the tricky derivation of the Green's function in the natural domain. In this work we use a first order differential vector formulation for continuous components of the fields, inspired by Bresler and Marcuvitz in [10] for stratified media in electromagnetics. We note that some of theoretical aspects used in electromagnetics (see [1]) are not available in elasticity or are cumbersome to be extended. For this reason, the GWHEs derivation for scattering by angular regions in elasticity is more complicated and challenging, although following the same general theory. Indeed, the authors of this paper have preliminary introduced in [4]- [5] an abstract formulation for simplified elastic scattering problem concerning the semi-infinite crack and some initial aspects of wedge problems.

In this paper, we first extend the formulation presented in [1] to elastic problems in angular regions using oblique Cartesian coordinates. It yields a matrix differential problem of first order whose unknowns are the field components continuous to the faces of the angular regions. The application of Laplace transform along one face of the angular region and the assumption of problem invariance along the edge profile yield a matrix ordinary differential problem of first order. Following [1] based on [11], we develop a spectral solution before imposing boundary conditions based on the derivation of the dyadic Green's functions in terms of eigenvectors and eigenvalue of the algebraic matrix operator (of the first order differential formulation) .

The projection of the solution on reciprocal vectors allows to get a set of functional equations that relate the Laplace transforms of continuous field components along one face of the angular regions to the ones of the other face. The imposition of boundary conditions yields a set of GWHEs for practical angular region problems.

For the sake of simplicity, even if challenging, this work is focused on elastic wedge problem filled by an elastic isotropic solid and extendable to anisotropic media. This problem is considered a fundamental problem in the mathematical theory of elastic diffraction and, despite numerous attempts to solve it in closed form, no exact solution exists for arbitrary aperture angle of the wedge region. Three major semi-analytical approaches [12]- [14] have been proposed to solve this problem in the two-dimensional case (i.e. at normal incidence). The first method is presented by Budaev in his monograph [12] that is based on the Sommerfeld integral (SI) representation of the elastic potentials and extends the popular and effective Sommerfeld-Malyuzhinets (SM) method to wedge problems with two concurrent different propagation constants. The difference equations, that initially arise from this formulation, are reduced to singular integral equations that are treated with a regularization method. Further interesting aspects of this formulation are presented also in [15]. A second method to study elastic wedge problems is reported in [13], where the scattered field by the faces of the wedge is related to the Fourier transforms of the displacement field of the faces (the spectral functions). Applying the Fourier transforms to the differential formulation of the elastic field and taking into account the boundary conditions, the authors obtain singular integral equations in terms of the spectral functions, that are numerically solved by using the Galerkin collocation method. An important aspect of this work is the use of recursive equations that provide analytical continuation (propagation of the solution) of the approximate spectral functions obtained by the numerical solution in a certain strip. New development of this method are reported in [16], where double Fourier transforms are introduced to obtain the kernels of the singular integral equations. In [17] the method is extended to 3D problems, however the proposed functional equations in spectral domain are again written in terms of singular integral operators and not in an algebraic form. The concept of spectral representation of the displacements on the wedge faces is applied also by Gautesen's group works [18]- [20], [14] that, according to our opinion, have produced the best practical results in the solution of the two dimensional elastic isotropic wedge problem [14]. The difference with respect to [13] is the use of an integral representation in terms of the displacements in the natural domain. Substantially, the integral representations of this method are those that in electromagnetism are called Kirchhoff's representations. The kernel of the integral representations are suitable Green functions of the free space and the integral does not contain components of the stress tensors. The traction-free boundary conditions on the faces of the wedge impose this property. Another important aspect in these works is to resort to an extinction theorem that allows to impose the vanishing of the displacement outside the elastic wedge. The application of the theorem allows to use unilateral Fourier transform (or Laplace transform) on the Gautesen (Kirchhoff) integral representations and it yield functional equations which are algebraic with respect to the Laplace transforms of the displacements on the two faces of the wedge. We note that the arguments of the Laplace transforms of the displacements on the two faces are different. Substantially, the functional equations obtained in [14] are GWHEs[1], although not defined in this way.

In this paper we derive with a systematic and efficient method spectral functional equations in algebraic form useful to derive GWHEs in 3D elastic wedge problems. These equations are validated by comparison with the ones proposed in [14]. The proposed method has the following important characteristics:

(i) The functional equations are easily obtained in terms of eigenvectors and eigenvalues of a matrix that characterizes the medium filling the angular region.
(ii) These functional equations hold independently from the boundary conditions of the angular region.
(iii) The application of boundary conditions yields a system of GWHEs for a specific problem.
(iv) The deduction of the GWHEs is general, since the method can be applied to study wave motion in angular regions filled by arbitrary linear media.

We remark that property (i) avoids the introduction of Kirchhoff type representations that require the computation of the Green's function. This computation can be difficult in elasticity, see Gautesen's group works [14]. Property (ii) allows the possibility to study complex wave motion problems constituted of different angular sub-regions or angular regions connected to planar stratified media, see in electromagnetics [21]- [24]. The third and the fourth characteristics allow the derivation of GWHEs in isotropic elastic media with plane wave source at skew incidence and in the general case of an elastic wedge filled by anisotropic medium. Moreover, we note that it is possible to directly compute from the spectral solution of the GWHEs the field in every point of the angular regions, avoiding Kirchhoff's representations and Green's function in natural domain. In particular the diffracted field component can be asymptotically computed with the saddle point method. A last but not less important property of the GWHE formulations of wedge problems is constituted by the set of mathematical tools in complex analysis. The Wiener-Hopf technique provides powerful solution methods based on exact and approximate factorization methods. In their works, Gautesen et al. have proposed a possible original method to deal with GWHEs of elastic wedge problems, exploiting analytical properties of the unknowns, see [14] and references therein. We propose, alternatively, the Fredholm factorization method [6]- [7] which is an effective semi-analytical technique for the solution of arbitrary GWHEs and it is based on the reduction of the factorization problem to Fredholm integral equations of second kind. We expect, in a future work, to effectively apply the Fredholm factorization to solve the GWHEs of elastic wedge problem using the same methodology applied in electromagnetic scattering from dielectric wedge [5].

The paper is organized into eight sections and we assume plane wave sources and/or sources localized at infinity in time harmonic fields with a time dependence specified by $e^{j\omega t}$ (electrical engineering notation) which is suppressed. In Section 2, we introduce the first order vector differential formulation for continuous components of the elastic field in an indefinite homogeneous medium. Note that, while in electromagnetics the continuous components of field are the transversal ones, in elasticity we have a more complex definition in term of stress tensor

[1]The GWHEs differ from the Classical Wiener-Hopf equations (CWHEs) for the definitions of the unknowns in spectral domain. While CWHEs introduce plus and minus functions that are always defined in the same complex plane, the GWHEs present plus and minus functions that are defined in different complex planes but related together.

and velocity vector. The theory presented in Section 2 is also useful to study propagation in stratified media. Using oblique Cartesian coordinates and taking into account the results of Section 2, Section 3 describes the novel application of the method to angular regions, yielding the oblique first order vector differential formulation for continuous components of the elastic field. The application of Laplace transform along one face of the angular region and assumption of problem with invariance along the edge profile yield a vector ordinary differential problem of first order (oblique equations). The solution of these oblique equations, projected on the reciprocal vectors of an algebraic matrix defined in Section 2, provides the functional equations of an arbitrary angular region (Section 4). It is remarkable that we get functional equations independently from the materials and the sources that can be present outside of the considered angular region. Explicit expressions in algebraic form are reported in Section 5 for isotropic media and arbitrary boundary conditions. Section 6 shows the validation of functional equations in special simplified cases reported in literature by other authors for the planar problem; and Section 7 reports the validation of functional equations by evaluating the characteristic impedances of half spaces in planar problem. Finally, conclusions are reported in section 8 and a glossary of the symbols useful for the readability of the text is provided at the end. We remark that, according to our opinion, the functional equations for the non planar (3D) general case, are deduced and reported for the first time in literature in this paper at Section 5. We finally state that the scope of our paper is to present algebraic spectral functional equations for arbitrary boundary conditions for 3D wave motion problems in angular regions that are useful for the examination of practical problems by imposing specific boundary conditions yielding GWHE formulations.

2. First order differential equations for continuous components of the elastic field in an indefinite rectangular isotropic medium

In this section we study elastic wave propagation in stratified media along a direction (say y) and, consequently in Section 3, we use these results to develop the theory for angular regions.

The evaluation of the physical fields in an elastic linear medium can be generally described by a system of partial differential equation of first order. In absence of sources localized at finite or in presence of plane wave sources, the system is constituted of the translational equation of motion and the stress-displacement equation [25]- [26], i.e. considering dydadic notation and time harmonic regime we have

$$\nabla \cdot \underline{T} = -\rho\omega^2 \mathbf{u}, \tag{2.1}$$

$$\underline{S} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})'), \tag{2.2}$$

where $\underline{T}$, $\underline{S}$, $\mathbf{u}$ are respectively the stress tensor, the strain tensor and the displacement vector and, ρ is the mass density ($'$ stands for transpose). In a general media the stress and strain tensors have constitutive relation given by the Hooke's law

$$\underline{T} = \underline{\underline{C}} : \underline{S}, \tag{2.3}$$

where $\underline{\underline{C}}$ is a fourth order stiffness tensor that in isotropic media simplifies to

$$\underline{\underline{C}} = \lambda \underline{I}\, \underline{I} + 2\mu \underline{\underline{I}}^{sym}, \tag{2.4}$$

where λ and μ are the Lamé's constants of the elastic medium and, $\underline{I}$ and $\underline{\underline{I}}^{sym}$ are respectively the unit dyadic and the symmetric fourth order unit dyadic (tetradic).

Using vector (Voigt) representation for tensor quantities [25] we re-write (2.1) as

$$\nabla_T \mathbf{T} = j\omega\, \mathbf{p}, \tag{2.5}$$

$$\nabla_v \mathbf{v} = j\omega\, \mathbf{S}, \tag{2.6}$$

with

$$\nabla_T = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix}, \quad \nabla_v = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{pmatrix} = (\nabla_T)', \tag{2.7}$$

and where $\mathbf{T}$, $\mathbf{S}$, $\mathbf{p}$ and $\mathbf{v}$ are respectively the symmetric stress tensor in six-component vector form (2.8), the symmetric strain tensor in six-component vector form (2.8), the vector momentum density $\mathbf{p} = \rho \mathbf{v}$ and the vector particle velocity $\mathbf{v} = j\omega \mathbf{u}$:

$$\mathbf{T} = (T_{xx}, T_{yy}, T_{zz}, T_{yz}, T_{xz}, T_{xy})', \quad \mathbf{S} = (S_{xx}, S_{yy}, S_{zz}, 2S_{yz}, 2S_{xz}, 2S_{xy})'. \tag{2.8}$$

Inspired by [1] for electromagnetic applications, to effectively study wave motion problems in elasticity, it is convenient to introduce the concept of transverse equations using abstract notation.

The homogeneous abstract form of (2.5) and (2.6), see section 2.9 of [4], is

$$\Gamma_{\nabla}\, \boldsymbol{\psi} = j\omega \boldsymbol{\theta}, \tag{2.9}$$

where Γ_{∇} is a matrix differential operator of first order that relates the fields ψ and θ:

$$\boldsymbol{\psi} = \begin{pmatrix} \mathbf{T} \\ \mathbf{v} \end{pmatrix}, \quad \boldsymbol{\theta} = \begin{pmatrix} \mathbf{S} \\ \mathbf{p} \end{pmatrix}, \quad \Gamma_{\nabla} = \begin{pmatrix} 0 & \nabla_v \\ \nabla_T & 0 \end{pmatrix}. \tag{2.10}$$

The vectors $\boldsymbol{\psi}$ and $\boldsymbol{\theta}$ have constitutive relation defined by the equation

$$\boldsymbol{\theta} = \mathbb{W}\, \boldsymbol{\psi}, \tag{2.11}$$

where the matrix $\mathbb{W}$ depends on the medium that is considered.

In order to close the mathematical problem (2.9)-(2.11), we need to enforce the geometrical domain of the problem, its boundaries conditions and the radiation condition.

For simplicity, in the following, we consider isotropic loss-less material, however we claim that transversal elastic equations in a general anisotropic medium assume the same form. Considering the Hooke's law $\mathbf{T} = \mathbb{C}\, \mathbf{S}$ in lossless isotropic medium we obtain

$$\mathbb{W} = \begin{pmatrix} \mathbb{C}^{\mathbf{-1}} & \mathbb{O} \\ \mathbb{O} & \mathbb{R} \end{pmatrix}, \mathbb{C} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}, \mathbb{R} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}. \tag{2.12}$$

In the following we use also alternative parameters to define the medium characteristics with respect to the mass density ρ, and the Lamé's constants λ and μ:

$$k_p = \omega \sqrt{\frac{\rho}{\lambda + 2\mu}}, \quad k_s = \omega \sqrt{\frac{\rho}{\mu}}, \quad Z_o = \frac{k_s \mu}{\omega}, \tag{2.13}$$

where k_p is the propagation constant of the longitudinal/principal wave, k_s is the propagation constant of the transversal/secondary wave (vertical or horizontal) and the impedance Z_o is a quantity such that stress components have same dimensions of velocity components time Z_o.

Comparing the equations (2.9)-(2.12) to the ones reported in [1] for electromagnetic applications, we note that the stress $\mathbf{T}$, the particle velocity $\mathbf{v}$, the strain $\mathbf{S}$ and the momentum density $\mathbf{p}$ are analogous respectively to the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, the dielectric induction $\mathbf{D}$ and the magnetic induction $\mathbf{B}$ with constitutive relations $\mathbf{T} = \mathbb{C}\, \mathbf{S}$ and $\mathbf{p} = \rho \mathbf{v}$ analogous respectively to $\mathbf{E} = \varepsilon^{-1} \mathbf{D}$ and $\mathbf{B} = \mu \mathbf{H}$ (where ε, μ can be either scalar or tensor). Moreover (2.5)-(2.6) are the elastic analogue of Maxwell's equations in electromagnetism.

Substituting (2.11) into (2.9) with (2.12)-(2.13) we get the nine equations that relate the stress $\mathbf{T}$ with the velocity $\mathbf{v}$ [4]:

$$\left(\Gamma_{\nabla} - j\omega\mathbb{W}\right)\boldsymbol{\psi} = \mathbf{0}, \tag{2.14}$$

whose explicit form is

$$\begin{cases} D_x T_{xx} + D_z T_{xz} + D_y T_{xy} = jk_s Z_o v_x \\ D_y T_{yy} + D_z T_{yz} + D_x T_{xy} = jk_s Z_o v_y \\ D_z T_{zz} + D_y T_{yz} + D_x T_{xz} = jk_s Z_o v_z \\ D_x v_x = \frac{jk_s[2k_p^2(T_{xx}-T_{yy}-T_{zz})+k_s^2(-2T_{xx}+T_{yy}+T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o} \\ D_y v_y = \frac{jk_s[k_s^2(T_{xx}-2T_{yy}+T_{zz})-2k_p^2(T_{xx}-T_{yy}+T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o} \\ D_z v_z = \frac{jk_s[k_s^2(T_{xx}+T_{yy}-2T_{zz})-2k_p^2(T_{xx}+T_{yy}-T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o} \\ D_z v_y + D_y v_z = \frac{jk_s T_{yz}}{Z_o} \\ D_z v_x + D_x v_z = \frac{jk_s T_{xz}}{Z_o} \\ D_y v_x + D_x v_y = \frac{jk_s T_{xy}}{Z_o} \end{cases}, \tag{2.15}$$

where $D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}, D_z = \frac{\partial}{\partial z}$.

While the constitutive parameters change only in one direction, say y, using the divergence theorem [25], it is possible to demonstrate that the continuous components of $\boldsymbol{\psi}$ at interfaces are the ones of $\boldsymbol{v}$ and $\boldsymbol{n} \cdot \underline{T}$, where $\boldsymbol{n}$ is the unit normal at the interface, i.e.

$$\boldsymbol{\psi}_t = (T_{yy}, T_{yz}, T_{xy}, v_x, v_y, v_z)'. \tag{2.16}$$

The transverse equations of a field are equations that involve only the components that remain continuous along the stratification according to the boundary conditions on the interfaces and, starting from (2.15), in general they assume the following form

$$\boxed{-\frac{\partial}{\partial y}\boldsymbol{\psi}_t = \mathcal{M}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right)\boldsymbol{\psi}_t} \tag{2.17}$$

where we have a first order derivative along y and a matrix differential operator in x and z.

The reduction of the elastic differential problems to the transverse equations starts from deriving expressions of the discontinuous components (along y) direction (T_{xx}, T_{zz}, T_{xz}) from the 4th, the 6th and the 8th of (2.15). We get:

$$\begin{cases} T_{xx} = \frac{k_p{}^2(-2k_s T_{yy}+4jZ_o(D_x v_x + D_z v_z))+k_s{}^2(k_s T_{yy}-2jZ_o(2D_x v_x + D_z v_z))}{k_s{}^3} \\ T_{zz} = \frac{k_p{}^2(-2k_s T_{yy}+4jZ_o(D_x v_x + D_z v_z))+k_s{}^2(k_s T_{yy}-2jZ_o(D_x v_x + 2D_z v_z))}{k_s{}^3} \\ T_{xz} = -\frac{j(D_z v_x + D_x v_z)Z_o}{k_s} \end{cases}. \tag{2.18}$$

By substituting (2.18) into the six non used equations of (2.15) (i.e. equations at line 1,2,3,5,7,9) we get the transverse equations (2.17) where

$$\mathcal{M}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right) = \begin{pmatrix} 0 & D_z & D_x & 0 & -jk_s Z_o & 0 \\ D_z - \frac{2D_z k_p{}^2}{k_s{}^2} & 0 & 0 & \frac{jD_x D_z(4k_p{}^2-3k_s{}^2)Z_o}{k_s{}^3} & 0 & M_{26}(D_z, D_x) \\ D_x - \frac{2D_x k_p{}^2}{k_s{}^2} & 0 & 0 & M_{34}(D_z, D_x) & 0 & \frac{jD_x D_z(4k_p{}^2-3k_s{}^2)Z_o}{k_s{}^3} \\ 0 & 0 & -\frac{jk_s}{Z_o} & 0 & D_x & 0 \\ -\frac{jk_p{}^2}{k_s Z_o} & 0 & 0 & D_x - \frac{2D_x k_p{}^2}{k_s{}^2} & 0 & D_z - \frac{2D_z k_p{}^2}{k_s{}^2} \\ 0 & -\frac{jk_s}{Z_o} & 0 & 0 & D_z & 0 \end{pmatrix}, \tag{2.19}$$

$$M_{34}(D_z, D_x) \quad = \quad -\frac{j\left(k_s{}^4+\left(4D_x{}^2+D_z{}^2\right)k_s{}^2-4D_x{}^2k_p{}^2\right)Z_o}{k_s{}^3}, \tag{2.20}$$

$$M_{26}(D_z, D_x) \quad = \quad -\frac{j\left(k_s{}^4+\left(D_x{}^2+4D_z{}^2\right)k_s{}^2-4D_z{}^2k_p{}^2\right)Z_o}{k_s{}^3}, \tag{2.21}$$

and where $D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}, D_z = \frac{\partial}{\partial z}$.

The transverse equations along y direction takes the form reported in (2.17) where $\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$ is matrix differential operator of arbitrary differential order and dimension that, in case of electromagnetic and elastic problems, have respectively dimension 4 and 6, both with differential order 2 in x and z. In the following, we assume that the geometry of the elastic wave-motion problem as well as the eventual boundary conditions are invariant along the z-direction, thus, without loss of generality, when a source depends on a $e^{-j\alpha_o z}$ factor, also the total field depends on the same factor, i.e. $\boldsymbol{\psi}_t = \boldsymbol{\psi}_t(x, y, z) = \mathbf{f}(x, y)e^{-j\alpha_o z}$, see for instance [17] before (2.8). Of course, the same behavior can be obtained by applying Fourier transform also along z direction and assuming an incident plane wave with a particular skew direction that yields $e^{-j\alpha_o z}$. However, for simplicity, we prefer to avoid the use of a double Fourier transform, recalling that in the present context an arbitrary source can be expanded in a summation of plane waves.

It yields $\frac{\partial}{\partial z}\boldsymbol{\psi}_t(x, y, z) = -j\alpha_o \boldsymbol{\psi}_t(x, y, z)$, i.e. $\frac{\partial}{\partial z} \to -j\alpha_o$, thus

$$\mathcal{M}(\frac{\partial}{\partial z}, \frac{\partial}{\partial x}) = \mathcal{M}(-j\alpha_o, \frac{\partial}{\partial x}) = \mathbb{M}_o + \mathbb{M}_1\frac{\partial}{\partial x} + \mathbb{M}_2\frac{\partial^2}{\partial x^2}, \tag{2.22}$$

where $\mathbb{M}_m$ with $m = 0, 1, 2$ are 6x6 matrices and do not depend on x, as they are easily derived from (2.19):

$$\mathbb{M}_o = \begin{pmatrix} 0 & -j\alpha_o & 0 & 0 & -jk_sZ_o & 0 \\ -j\alpha_o\left(1-\frac{2k_p{}^2}{k_s{}^2}\right) & 0 & 0 & 0 & 0 & -\frac{jZ_o\left(4\alpha_o{}^2k_p{}^2+k_s{}^4-4\alpha_o{}^2k_s{}^2\right)}{k_s{}^3} \\ 0 & 0 & 0 & -\frac{jZ_o\left(k_s{}^2-\alpha_o{}^2\right)}{k_s} & 0 & 0 \\ 0 & 0 & -\frac{jk_s}{Z_o} & 0 & 0 & 0 \\ -\frac{jk_p{}^2}{k_sZ_o} & 0 & 0 & 0 & 0 & -j\alpha_o\left(1-\frac{2k_p{}^2}{k_s{}^2}\right) \\ 0 & -\frac{jk_s}{Z_o} & 0 & 0 & -j\alpha_o & 0 \end{pmatrix}, \tag{2.23}$$

$$\mathbb{M}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha_oZ_o\left(4k_p{}^2-3k_s{}^2\right)}{k_s{}^3} & 0 & 0 \\ 1-\frac{2k_p{}^2}{k_s{}^2} & 0 & 0 & 0 & 0 & \frac{\alpha_oZ_o\left(4k_p{}^2-3k_s{}^2\right)}{k_s{}^3} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\frac{2k_p{}^2}{k_s{}^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.24}$$

$$\mathbb{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{jZ_o}{k_s} \\ 0 & 0 & 0 & \frac{4jZ_o\left(k_p{}^2-k_s{}^2\right)}{k_s{}^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.25}$$

(a) The eigenvalues and the eigenvectors of $\mathcal{M}$ in spectral domain

By applying Fourier transform along x direction to (2.17) with (2.22)-(2.25) ($\mathbb{M}_m = 0,\ m > 2$) in absence of source, we obtain an ordinary vector first order differential equation

$$-\frac{d}{dy}\boldsymbol{\Psi}_t(\eta) = \mathbb{M}(\eta)\,\boldsymbol{\Psi}_t(\eta), \tag{2.26}$$

where $\boldsymbol{\psi}_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{\Psi}_t(\eta)e^{-j\eta x}d\eta$ (notation with omission of y, z dependence) and

$$\mathbb{M}(\eta) = \mathcal{M}(-j\alpha_o, -j\eta) = \mathbb{M}_o - j\eta\mathbb{M}_1 - \eta^2\mathbb{M}_2, \tag{2.27}$$

where $\frac{\partial}{\partial z} \to -j\alpha_o$ for the field factor $e^{-j\alpha_o z}$ (see comment before (2.22)) and $\frac{\partial}{\partial x} \to -j\eta$ for the property of Fourier transforms.

Now, let us investigate the properties of the eigenvalue problem (2.28) associated to (2.26):

$$\mathbb{M}(\eta)\,\mathbf{u}_i(\eta) = \lambda_i(\eta)\mathbf{u}_i(\eta), \tag{2.28}$$

where $\mathbf{u}_i(\eta)$ and λ_i $(i = 1..n)$ are respectively the eigenvectors and the eigenvalues of the 6x6 matrix $\mathbb{M}(\eta)$ (2.27). In presence of a passive medium we observe that three eigenvalues (say $\lambda_1,\ \lambda_2,\ \lambda_3$) present non-negative real part and the other three eigenvalues (say $\lambda_4,\ \lambda_5,\ \lambda_6$) present non-positive real part. In the following we use also alternative expressions:

$$\lambda_1 = j\xi_p(\eta) = -\lambda_4, \quad \lambda_2 = \lambda_3 = j\xi_s(\eta) = -\lambda_5 = -\lambda_6. \tag{2.29}$$

The explicit form of (2.29) are expressed in terms of $\tau_{op} = \sqrt{k_p^2 - \alpha_o^2}$, $\tau_{os} = \sqrt{k_s^2 - \alpha_o^2}$

$$\xi_p(\eta) = \sqrt{\tau_{op}^2 - \eta^2}, \quad \xi_s(\eta) = \sqrt{\tau_{os}^2 - \eta^2}, \tag{2.30}$$

with $Im[k_{p,s}] < 0$, $Im[\tau_{op,os}] < 0$ in lossy media. Since $k_{p,s}^2 = k_x^2 + k_y^2 + k_z^2 = \eta^2 + \xi_{p,s}^2 + \alpha_o^2$, $\xi_{p,s}(\eta)$ are multivalued functions of η. In the following we assume as proper sheets of $\xi_{p,s}(\eta)$, the ones with $\xi_{p,s}(0) = \tau_{op,os}$ and as branch lines of $\xi_{p,s}(\eta)$ the classical line $Im[\xi_{p,s}(\eta)] = 0$ (see in practical engineering estimations Ch. 5.3b of [32]) or the vertical line $(Re[\eta] = Re[\tau_{os,op}],\ Im[\eta] < Im[\tau_{os,op}])$. In (2.29) we have that $\lambda_1,\ \lambda_2,\ \lambda_3$ $(\lambda_4,\ \lambda_5,\ \lambda_6)$ are related to progressive (regressive) waves and, $\xi_{p,s}$ are with non-positive imaginary part. In this framework we associate the direction of propagation to attenuation phenomena.

Since the matrix $\mathbb{M}(\eta)$ is diagonalizable, $\mathbb{M}(\eta)$ is semi-simple [2] [33], Ch. V.9. The semi-simple property is fundamental to develop the procedure as it yields a set of independent eigenvectors $\mathbf{u}_i(\eta)$ even with same eigenvalues. Although a theory about geometric and mathematical multiplicity of eigenvalues is available, in practice, we checked the diagonalizability of $\mathbb{M}(\eta)$ using Jordan decomposition algorithm that in our case yields $\mathbb{M}(\eta) = \mathbb{U}^{-1}\mathbb{D}\mathbb{U}$ where the matrix $\mathbb{U}$ is a matrix with column elements $\mathbf{u}_i(\eta)$ and $\mathbb{D}$ is a diagonal matrix with diagonal elements the eigenvalues λ_i. In relation to the eigenvectors $\mathbf{u}_i(\eta)$, we introduce the reciprocal vectors $\boldsymbol{\nu}_i(\eta)$ (see chapter 3.16 of [33]) that, in the general elastic case with $\alpha_o \neq 0$, can be computed by inversion of the matrix $\mathbb{U}$. The vectors $\boldsymbol{\nu}_i(\eta)$ satisfy the bi-orthogonal relations

$$\boldsymbol{\nu}_j \cdot \mathbf{u}_i = \delta_{ji}, \quad i.e. \quad \underline{1}_t = \sum_{i=1}^{6}\mathbf{u}_i\boldsymbol{\nu}_i, \tag{2.31}$$

where $\cdot$ is the vector scalar product, δ_{ij} is the Kronecker symbol and, $\underline{1}_t$ is the unit dyadic defined in terms of dyadic products and such that $\underline{1}_t \cdot \mathbf{a} = \mathbf{a} \cdot \underline{1}_t = \mathbf{a}$ for an arbitrary vector $\mathbf{a}$.

From a physics point of view, the eigenvalues $\lambda_1 = -\lambda_4$ are associated to longitudinal P (principal) waves, while $\lambda_2 = -\lambda_5$ and $\lambda_3 = -\lambda_6$ are relevant to the transversal S (secondary) waves of two types: secondary vertical (SV) and secondary horizontal (SH). The P, SV and SH waves are not decoupled when $\alpha_o \neq 0$, while if $\alpha_o = 0$ we have two decoupled problems: one related to P and SV waves (planar problem) and the other to SH waves (antiplanar problem).

[2] A square matrix of dimension n is called semi-simple iff it has a basis of eigenvectors in $\mathbb{R}^n$.

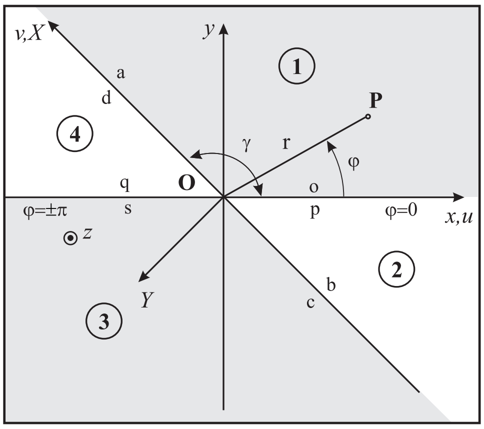

Figure 1. Angular regions and oblique Cartesian coordinates. The figure reports the x, y, z Cartesian coordinates and r, φ, z cylindrical coordinates useful to define the oblique Cartesian coordinate system u, v, z with reference to the angular region 1 $0 < \varphi < \gamma$ with $0 < \gamma < \pi$. In the figure, the space is divided into four angular regions delimited by $\varphi = \pm\gamma, 0, \pi$, and the face boundaries are labeled a, b, c, d, o, p, q, s. The figure reports also the local-to-face-a Cartesian coordinate system $X, Y, Z \equiv z$. Note that $x \equiv u$ and $v \equiv X$.

The computation of eigenvectors in (2.28), using Wolfram Mathematica ®, it yields in compact notation

$$\mathbb{U}=\begin{pmatrix} \frac{Z_o\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{k_s\alpha_o} & -\frac{2Z_o\xi_s}{k_s} & 0 & \frac{Z_o\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{k_s\alpha_o} & \frac{2Z_o\xi_s}{k_s} & 0 \\ -\frac{2Z_o\xi_p}{k_s} & -\frac{\alpha_o Z_o}{k_s} & -\frac{Z_o\xi_s}{k_s} & \frac{2Z_o\xi_p}{k_s} & -\frac{\alpha_o Z_o}{k_s} & \frac{Z_o\xi_s}{k_s} \\ -\frac{2\eta Z_o\xi_p}{k_s\alpha_o} & \frac{Z_o\left(\xi_s^2-\eta^2\right)}{k_s\eta} & \frac{\alpha_o Z_o\xi_s}{k_s\eta} & \frac{2\eta Z_o\xi_p}{k_s\alpha_o} & \frac{Z_o\left(\xi_s^2-\eta^2\right)}{k_s\eta} & -\frac{\alpha_o Z_o\xi_s}{k_s\eta} \\ \frac{\eta}{\alpha_o} & -\frac{\xi_s}{\eta} & -\frac{\alpha_o}{\eta} & \frac{\eta}{\alpha_o} & \frac{\xi_s}{\eta} & -\frac{\alpha_o}{\eta} \\ \frac{\xi_p}{\alpha_o} & 1 & 0 & -\frac{\xi_p}{\alpha_o} & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \tag{2.32}$$

whose columns are $\mathbf{u}_i(\eta)$ corresponding to the eigenvalues as defined and ordered in (2.29). The inverse of $\mathbb{U}$ yields in its rows the reciprocal vectors $\boldsymbol{\nu}_i(\eta)$:

$$\mathbb{V}=\begin{pmatrix} -\frac{\alpha_o}{2k_sZ_o} & -\frac{{\alpha_o}^2}{2k_sZ_o\xi_p} & -\frac{\alpha_o\eta}{2k_sZ_o\xi_p} & \frac{\alpha_o\eta}{{k_s}^2} & -\frac{\alpha_o\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{2{k_s}^2\xi_p} & \frac{{\alpha_o}^2}{{k_s}^2} \\ -\frac{{\alpha_o}^2+\eta^2}{2k_sZ_o\xi_s} & \frac{\alpha_o}{2k_sZ_o} & \frac{\eta}{2k_sZ_o} & \frac{\eta\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{2{k_s}^2\xi_s} & \frac{{\alpha_o}^2+\eta^2}{{k_s}^2} & \frac{\alpha_o\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{2{k_s}^2\xi_s} \\ \frac{\alpha_o}{2k_sZ_o} & -\frac{(k_s-\alpha_o)(k_s+\alpha_o)}{2k_sZ_o\xi_s} & \frac{\alpha_o\eta}{2k_sZ_o\xi_s} & -\frac{\alpha_o\eta}{{k_s}^2} & -\frac{\alpha_o\xi_s}{{k_s}^2} & \frac{1}{2}-\frac{{\alpha_o}^2}{{k_s}^2} \\ -\frac{\alpha_o}{2k_sZ_o} & \frac{{\alpha_o}^2}{2k_sZ_o\xi_p} & \frac{\alpha_o\eta}{2k_sZ_o\xi_p} & \frac{\alpha_o\eta}{{k_s}^2} & \frac{\alpha_o\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{2{k_s}^2\xi_p} & \frac{{\alpha_o}^2}{{k_s}^2} \\ \frac{{\alpha_o}^2+\eta^2}{2k_sZ_o\xi_s} & \frac{\alpha_o}{2k_sZ_o} & \frac{\eta}{2k_sZ_o} & -\frac{\eta\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{2{k_s}^2\xi_s} & \frac{{\alpha_o}^2+\eta^2}{{k_s}^2} & -\frac{\alpha_o\left({\alpha_o}^2+\eta^2-\xi_s^2\right)}{2{k_s}^2\xi_s} \\ \frac{\alpha_o}{2k_sZ_o} & \frac{(k_s-\alpha_o)(k_s+\alpha_o)}{2k_sZ_o\xi_s} & -\frac{\alpha_o\eta}{2k_sZ_o\xi_s} & -\frac{\alpha_o\eta}{{k_s}^2} & \frac{\alpha_o\xi_s}{{k_s}^2} & \frac{1}{2}-\frac{{\alpha_o}^2}{{k_s}^2} \end{pmatrix}. \tag{2.33}$$

In the following Sections 3-5, the eigenvectors $\mathbf{u}_i(\eta)$ and the reciprocal vectors $\boldsymbol{\nu}_i(\eta)$ will be used to obtain functional equations that relates spectral quantities in elastic wave motion problems between the two terminal faces of an angular region for an arbitrary α_o, i.e. non planar problems. We also note that $\mathbf{u}_i(\eta)$ and $\boldsymbol{\nu}_i(\eta)$ can be used to build the solution of the transverse equations (2.26) in Laplace domain for elastic wave motion problems in a rectangular stratified region [31].

3. First order differential oblique equations for continuous components of the elastic field in an angular region

In this section we introduce the oblique equations for continuous components of the elastic field in an angular region using an oblique system of Cartesian axes and applying the properties reported in Section 2 for rectangular regions. In the following sections, first, we deduce spectral functional equations then, by imposing boundary conditions, the GWHEs for angular shaped regions.

With reference to Fig. 1 where angular regions are defined thorough the angle γ $(0<\gamma<\pi)$, we introduce the oblique Cartesian coordinates u, v, z in terms of the Cartesian coordinates x, y, z:

$$u = x - y\,\cot\gamma,\; v = \frac{y}{\sin\gamma} \quad or \quad x = u + v\,\cos\gamma,\; y = v\sin\gamma, \tag{3.1}$$

with partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = -\cot\gamma\,\frac{\partial}{\partial u} + \frac{1}{\sin\gamma}\,\frac{\partial}{\partial v}, \\ \frac{\partial}{\partial u} &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} = \cos\gamma\,\frac{\partial}{\partial x} + \sin\gamma\,\frac{\partial}{\partial y}. \end{aligned} \tag{3.2}$$

Starting from (2.17) with (2.22) the transverse (with respect to y) equation of dimension $n=6$ for an elastic problem with invariant geometry along z-direction (i.e. $e^{-j\alpha_o z}$) is

$$-\frac{\partial}{\partial y}\boldsymbol{\psi}_t = \mathcal{M}(-j\alpha_o, \frac{\partial}{\partial x})\,\boldsymbol{\psi}_t = (\mathbb{M}_o + \mathbb{M}_1\frac{\partial}{\partial x} + \mathbb{M}_2\frac{\partial^2}{\partial x^2})\,\boldsymbol{\psi}_t. \tag{3.3}$$

Note that for elastic problems, we have second differential order in x. Substituting (3.2), in particular $\frac{\partial}{\partial x} = \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial y} = -\cot\gamma\,\frac{\partial}{\partial u} + \frac{1}{\sin\gamma}\,\frac{\partial}{\partial v}$, into (3.3), we obtain

$$-\frac{\partial}{\partial v}\boldsymbol{\psi}_t = \mathcal{M}_e(-j\alpha_o, \frac{\partial}{\partial u})\,\boldsymbol{\psi}_t = (\mathbb{M}_{eo} + \mathbb{M}_{e1}\frac{\partial}{\partial u} + \mathbb{M}_{e2}\frac{\partial^2}{\partial u^2})\,\boldsymbol{\psi}_t, \tag{3.4}$$

where

$$\mathbb{M}_{eo} = \mathbb{M}_o\sin\gamma, \quad \mathbb{M}_{e1} = \mathbb{M}_1\sin\gamma - \mathbb{I}\cos\gamma, \quad \mathbb{M}_{e2} = \mathbb{M}_2\sin\gamma. \tag{3.5}$$

For the sake of simplicity and in order to get simple explicit expressions, we consider homogeneous isotropic media filling the angular regions. In this case the explicit forms of $\mathbb{M}_{em}$, $m=0,1,2$ (3.5) are straightforwardly derived from (2.23)-(2.25). By applying the Fourier transform along $x=u$ direction to (3.4), i.e. $\boldsymbol{\psi}_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\boldsymbol{\Psi}_t(\eta)e^{-j\eta x}d\eta$ with notation omitting v, z dependence, we obtain the ordinary system of differential equations

$$-\frac{\partial}{\partial v}\boldsymbol{\Psi}_t = \mathbb{M}_e(\gamma,\eta)\,\boldsymbol{\Psi}_t \tag{3.6}$$

with

$$\mathbb{M}_e(\gamma,\eta) = \mathcal{M}_e(-j\alpha_o, -j\eta) = \mathbb{M}_{eo} - j\eta\mathbb{M}_{e1} - \eta^2\mathbb{M}_{e2} \tag{3.7}$$

since $\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \overset{FT}{\leftrightarrow} -j\eta$.

(a) Link between eigenvalues of $\mathbb{M}(\eta)$ and $\mathbb{M}_e(\gamma,\eta)$

In the oblique coordinate system, the solution of (3.6) is related to the eigenvalue problem

$$\mathbb{M}_e(\gamma,\eta)\,\mathbf{u}_{ei}(\gamma,\eta) = \lambda_{ei}(\gamma,\eta)\mathbf{u}_{ei}(\gamma,\eta), \tag{3.8}$$

where λ_{ei} and $\mathbf{u}_{ei}(\gamma,\eta)$ $(i=1..n)$ are respectively the eigenvalues and the eigenvectors of the 6x6 matrix $\mathbb{M}_e(\gamma,\eta)$. Using (3.6) and (3.7) equation (3.8) becomes

$$(\mathbb{M}_o\sin\gamma - j\eta\mathbb{M}_1\sin\gamma - \eta^2\mathbb{M}_2\sin\gamma)\,\mathbf{u}_{ei}(\gamma,\eta) = (\lambda_{ei}(\gamma,\eta) - j\eta\cos\gamma)\mathbf{u}_{ei}(\gamma,\eta) \tag{3.9}$$

and thus

$$\mathbb{M}(\eta)\,\mathbf{u}_{ei}(\gamma,\eta) = \left(\frac{\lambda_{ei}(\gamma,\eta) - j\eta\cos\gamma}{\sin\gamma}\right)\mathbf{u}_{ei}(\gamma,\eta). \tag{3.10}$$

Comparing (3.10) with (2.28) we observe the relation among the eigenvalues and the eigenvectors of the two problems. The two problems defined by the matrices $\mathbb{M}(\eta)$ and $\mathbb{M}_e(\gamma,\eta)$ have same eigenvectors

$$\mathbf{u}_{ei}(\gamma,\eta) = \mathbf{u}_i(\eta), \tag{3.11}$$

thus same reciprocal vectors and related eigenvalues

$$\frac{\lambda_{ei}(\gamma,\eta) - j\eta\cos\gamma}{\sin\gamma} = \lambda_i(\eta). \tag{3.12}$$

Since $\mathbb{M}_e(\gamma,\eta)$ and $\mathbb{M}(\eta)$ have same eigenvectors (3.11), i.e. $\mathbf{u}_i(\eta)$ reported in the columns of (2.32), we note the important property that the eigenvectors of $\mathbb{M}_e(\gamma,\eta)$ do not depends on the aperture angle γ of the angular region (Fig. 1). From (3.12), the eigenvalues λ_{ei} of $\mathbb{M}_e(\gamma,\eta)$ can be re-written using the notation (2.29)-(2.30):

$$\begin{aligned}
\lambda_{e1}(\gamma,\eta) &= j(\eta\cos\gamma + \sin\gamma\,\xi_p(\eta)),\\
\lambda_{e2,e3}(\gamma,\eta) &= j(\eta\cos\gamma + \sin\gamma\,\xi_s(\eta)),\\
\lambda_{e4}(\gamma,\eta) &= j(\eta\cos\gamma - \sin\gamma\,\xi_p(\eta)),\\
\lambda_{e5,e6}(\gamma,\eta) &= j(\eta\cos\gamma - \sin\gamma\,\xi_s(\eta)).
\end{aligned} \tag{3.13}$$

where the first three λ_{ei} are related to progressive waves and the last three to regressive waves according to the definitions reported in Section 2. The corresponding eigenvectors and reciprocal vectors corresponding to λ_{ei} are $\mathbf{u}_i$ and $\boldsymbol{\nu}_i$ reported in (2.32) and (2.33) according to (3.11).

As we will see in the next section, the bi-orthogonal basis $\mathbf{u}_i$ and $\boldsymbol{\nu}_i$ can be used to build the solution of the transverse equations (3.6) in Laplace domain for elastic wave motion problems in an angular region with arbitrary α_o, i.e. non planar problems.

4. Solution of the oblique equations for angular regions

With reference to Fig. 1, let us introduce the Laplace transforms of $\boldsymbol{\psi}_t(u,v)$ (2.16)

$$\tilde{\boldsymbol{\psi}}_t(\eta,v) = \int_0^\infty e^{j\eta\, u}\boldsymbol{\psi}_t(u,v)du \tag{4.1}$$

for regions 1,2 and $\tilde{\boldsymbol{\psi}}_t(\eta,v) = \int_{-\infty}^0 e^{j\eta\, u}\boldsymbol{\psi}_t(u,v)du$ for regions 3,4. The Laplace transforms applied to (3.4) yield:

$$-\frac{d}{dv}\tilde{\boldsymbol{\psi}}_t = \mathbb{M}_e(\gamma,\eta)\,\tilde{\boldsymbol{\psi}}_t + \boldsymbol{\psi}_s(v) \tag{4.2}$$

with

$$\mathbb{M}_e(\gamma,\eta) = \mathbb{M}_{eo} - j\eta\,\mathbb{M}_{e1} - \eta^2\mathbb{M}_{e2}. \tag{4.3}$$

Note that (4.3) and (3.7) share the same symbol and explicit mathematical expression, however the first is related to a Fourier transform while the second to a Laplace transform, thus obviously they have the same eigenvalues and eigenvectors.

The term $\boldsymbol{\psi}_s(v)$ is obtained from the derivative property of the Laplace transform and for each angular region we obtain a different expression. In particular, we indicate with $\boldsymbol{\psi}_{as}(v)$ the value of $\boldsymbol{\psi}_s(v)$ on the face a, see Fig. 1, $(0 \le v < +\infty, u = 0_+)$, with $\boldsymbol{\psi}_{bs}(v)$ the value of $\boldsymbol{\psi}_s(v)$ on the face b $(-\infty \le v < 0, u = 0_+)$, with $\boldsymbol{\psi}_{cs}(v)$ the value of $\boldsymbol{\psi}_s(v)$ on the face c $(-\infty \le v < 0, u = 0_-)$ and with $\boldsymbol{\psi}_{ds}(v)$ the value of $\boldsymbol{\psi}_s(v)$ on the face d $(0 \le v < +\infty, u = 0_-)$.

Since (4.2) is a system of six ordinary differential equations of first order with constant coefficients in a semi-infinite interval, we have mainly two methods for its solution: 1) to apply the dyadic Green's function procedure in v domain, 2) to apply the Laplace transform in v that yields a linear system of six algebraic equations from which one can write down the general solution in terms of eigenvalues and eigenfunctions. We note that both methods are effective and in particular the second method is more useful for representing the spectral solution in each point of the considered angular region. However, it initially introduces complex functions of two variables. As proposed in the following subsections, we prefer the first method because, by this way, we get the functional equations of the angular regions that involve directly complex functions of one variable.

Using the concept of non-standard Laplace transforms (see section 1.4 of [4]), the validity of (4.2) and (4.3) in absence of sources is extended to the total fields in presence of plane-wave sources or sources located at infinity from any direction yielding isolated poles in spectral domain.

With reference to Fig. 1, let us now focus the attention on the angular region 1 in details. The results for the other regions will follow a similar procedure. We observe that the selection of four angular regions as in Fig. 1 related to a unique aperture angle γ does not limit the applicability of the method. In fact all the equations (once derived) can be used with a different appropriate aperture angle just replacing γ with the proper value. The purpose of deriving the functional equations with a unique γ is related to the fact that we formulate and solve the angular region problems by analyzing once and for all the matrix $\mathbb{M}_e(\gamma,\eta)$ (4.3). We recall also that the imposition of boundary conditions and media for each region will be made only while examining a practical problem and it yields GWHEs from the functional equations.

(a) Region 1: $u>0, v>0$

Focusing the attention on region 1 (Fig. 1), i.e. $u>0, v>0$, (4.2) holds with

$$\boldsymbol{\psi}_s(v)=\boldsymbol{\psi}_{as}(v)=-\mathbb{M}_{e1}\,\boldsymbol{\psi}_t(0_+,v)+j\eta\,\mathbb{M}_{e2}\,\boldsymbol{\psi}_t(0_+,v)-\mathbb{M}_{e2}\,\frac{\partial}{\partial u}\boldsymbol{\psi}_t(0_+,v). \tag{4.4}$$

Equation (4.2) is a system of differential equations of first order of dimension six, whose solution $\tilde{\boldsymbol{\psi}}_t$ is obtainable as sum of a particular integral $\tilde{\boldsymbol{\psi}}_p$ with the general solution of the homogeneous equation $\tilde{\boldsymbol{\psi}}_o$ [11]:

$$\tilde{\boldsymbol{\psi}}_t=\tilde{\boldsymbol{\psi}}_o+\tilde{\boldsymbol{\psi}}_p. \tag{4.5}$$

The solution of the homogeneous equation must satisfy

$$-\frac{d}{dv}\tilde{\boldsymbol{\psi}}_o=\mathbb{M}_e(\gamma,\eta)\,\tilde{\boldsymbol{\psi}}_o. \tag{4.6}$$

Considering the solution form $\tilde{\boldsymbol{\psi}}_o=C\,e^{-\lambda(\gamma,\eta)v}\mathbf{u}(\eta)$, the most general solution is

$$\tilde{\boldsymbol{\psi}}_o(\gamma,v)=\sum_{i=1}^{6}C_i e^{-\lambda_{ei}(\gamma)\,v}\mathbf{u}_i(\eta), \tag{4.7}$$

where λ_{ei} and $\mathbf{u}_i$ (i=1..6) are the eigenvalues and the eigenvectors of the matrix $\mathbb{M}_e(\gamma,\eta)$ respectively reported at (3.13) and (2.32).

In presence of a passive medium, following the properties described in Section 2(a), we observe that the first three eigenvalues $\lambda_{ei}, i=1,2,3$ present non-negative real part and are related to progressive waves along positive v direction while the last three eigenvalues $\lambda_{ei}, i=4,5,6$ present non-positive real part and are related to regressive waves. The evaluation of the particular integral $\tilde{\boldsymbol{\psi}}_p(\eta,v)$ of (4.2) is easier if carried out in dyadic notation i.e.

$$-\frac{d}{dv}\tilde{\boldsymbol{\psi}}_t=\underline{M}_e(\gamma,\eta)\cdot\tilde{\boldsymbol{\psi}}_t+\boldsymbol{\psi}_s(v), \tag{4.8}$$

where $\underline{M}_e$ is the dyadic counterpart of the matrix $\mathbb{M}_e$ assuming canonical basis[3]. It yields

$$\tilde{\boldsymbol{\psi}}_p(\eta,v)=-\int_0^\infty \underline{G}(v,v')\cdot\boldsymbol{\psi}_s(v')dv', \tag{4.9}$$

where $\underline{G}(v,v')$ is the dyadic Green's function of (4.8), i.e. solution of

$$\frac{d}{dv}\underline{G}(v,v')+\underline{M}_e(\gamma,\eta)\cdot\underline{G}(v,v')=\delta(v-v')\underline{1}_t \tag{4.10}$$

with the unit dyadic $\underline{1}_t$ of dimension six.

[3] Any dyadic $\underline{A}=\sum_{ij}A_{ij}\mathbf{e_i}\mathbf{e_j}$ can be represented by a matrix $\mathbb{A}$ with elements A_{ij} where $\mathbf{e_i}$ are unit vectors and vice versa.

Based on the theory reported in [31] and [11], we apply the methodology reported in Section 4 and Appendix B of [1], where we build the dyadic Green's function for arbitrary boundary conditions by selecting progressive and regressive waves in indefinite half-space as homogeneous solutions of (4.10). It yields:

$$\underline{G}(v,v') = \begin{cases} \sum_{i=1}^{3} \mathbf{u}_i \boldsymbol{\nu}_i e^{-\lambda_{ei}(\gamma,\eta)(v-v')}, & v > v' \\ -\sum_{i=4}^{6} \mathbf{u}_i \boldsymbol{\nu}_i e^{-\lambda_{ei}(\gamma,\eta)(v-v')}, & v < v' \end{cases}. \tag{4.11}$$

In our framework, we avoid to impose the boundary condition at this step, since we want to find functional equations that are free of this constraint, as described in [1] based on [11]. Only, while investigating a practical problem, we will impose boundary condition to the functional equations (for instance in region 1 at face $\varphi = 0$ i.e. $u > 0$, $v = 0$ and face $\varphi = \gamma$ i.e. $u = 0$, $v > 0$) yielding GWHEs of the problem.

By substituting (4.7) and (4.9) with (4.11) into (4.5), it yields

$$\begin{aligned} \tilde{\boldsymbol{\psi}}_t(\eta,v) = \sum_{i=1}^{6} C_i e^{-\lambda_{ei}(\gamma)\,v}\mathbf{u}_i - \sum_{i=1}^{3} \mathbf{u}_i\boldsymbol{\nu}_i \cdot \int_0^v e^{-\lambda_{ei}(\gamma,\eta)(v-\,v')}\boldsymbol{\psi}_{as}(v')dv' + \\ + \sum_{i=4}^{6} \mathbf{u}_i\boldsymbol{\nu}_i \cdot \int_v^\infty e^{-\lambda_{ei}(\gamma,\eta)(v-\,v')}\boldsymbol{\psi}_{as}(v')dv'. \end{aligned} \tag{4.12}$$

Looking at the asymptotic behavior of (4.12) for $v \to +\infty$ we have that the divergent terms are the ones in $\sum_{i=4}^{6} C_i e^{-\lambda_{ei}(\gamma)\,v}\mathbf{u}_i$. For this reason we assume $C_i = 0$, $i = 4, 5, 6$. Note in particular the vanishing of the last three integral terms as $v \to +\infty$ (last sum in (4.12)).

Setting $v = 0$ in (4.12), we have

$$\tilde{\boldsymbol{\psi}}_t(\eta,0) = \sum_{i=1}^{3} C_i\mathbf{u}_i + \sum_{i=4}^{6} \mathbf{u}_i\boldsymbol{\nu}_i \cdot \int_0^\infty e^{\lambda_{ei}(\gamma,\eta)v'}\boldsymbol{\psi}_{as}(v')dv'. \tag{4.13}$$

Multiplying (4.13) by $\boldsymbol{\nu}_i(\eta)$ for $i = 1..6$, using bi-orthogonality, we obtain

$$\begin{cases} \boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_t(\eta,0) = C_i, & i = 1,2,3 \\ \boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_t(\eta,0) = \boldsymbol{\nu}_i \cdot \breve{\boldsymbol{\psi}}_{as}(-j\lambda_{ei}(\gamma,\eta)), & i = 4,5,6 \end{cases}, \tag{4.14}$$

where $\lambda_{ei}(\gamma,\eta)$ are reported in (3.13) and $\breve{\boldsymbol{\psi}}_{as}(\chi)$ is the Laplace transform in v along face a ($v = r$ in cylindrical coordinates)

$$\breve{\boldsymbol{\psi}}_{as}(\chi) = \int_0^\infty e^{j\chi v}\boldsymbol{\psi}_{as}(v)dv. \tag{4.15}$$

We note that in the first three equations of (4.14) we use progressive reciprocal vectors and we obtain C_i that are needed in the computation of the homogeneous portion of the solution $\tilde{\boldsymbol{\psi}}_t(\eta,v)$ (4.12) through the Green's function method. In particular, the unknowns C_i, $i = 1,2,3$ are related to the Laplace transform $\tilde{\boldsymbol{\psi}}_t(\eta,0)$ evaluated in the lower face of the angular region ($v = 0$). We now focus the attention on the last three equations of (4.14) obtained by using regressive reciprocal vectors that yield the three functional equations of the angular region. We re-write them as

$$\boldsymbol{\nu}_i \cdot \tilde{\boldsymbol{\psi}}_t(\eta,0) = \boldsymbol{\nu}_i \cdot \breve{\boldsymbol{\psi}}_{as}(-m_{ai}(\gamma,\eta)), \quad i = 4,5,6 \tag{4.16}$$

with

$$\begin{aligned} m_{a4}(\gamma,\eta) = m_p(\gamma,\eta) = j\lambda_{e4}(\gamma,\eta) = -\eta\cos\gamma + \xi_p\sin\gamma, \\ m_{a5,a6}(\gamma,\eta) = m_s(\gamma,\eta) = j\lambda_{e5,e6}(\gamma,\eta) = -\eta\cos\gamma + \xi_s\sin\gamma. \end{aligned} \tag{4.17}$$

In (4.16) the Laplace transforms of combinations of the field components defined on the boundaries of an angular region, i.e. $v = 0$ (face o) and $u = 0$ (face a) in Fig. 1, are related to each other. These functional equations are the starting point to define the GWHEs of region 1. They are valid for any linear isotropic elastic medium filling the region. Moreover, in (4.16), we note that the reciprocal vectors and eigenvectors do not appear in the definitions of the Laplace transforms of the field. Only the eigenvalues are used as argument of the Laplace transforms at the right hand side. In the following we apply the notation $+$ to $\tilde{\boldsymbol{\psi}}_t(\eta,0)$ and $\breve{\boldsymbol{\psi}}_{as}(-m_{ai}(\gamma,\eta))$, i.e. $\tilde{\boldsymbol{\psi}}_{t+}(\eta,0)$ and $\breve{\boldsymbol{\psi}}_{as+}(-m_{ai}(\gamma,\eta))$, to highlight that these Laplace transforms are plus functions respectively in η and $\chi=-m_{ai}(\gamma,\eta)$, i.e. they are regular in the upper half plane of the complex plane η and χ.

Note that the functional equations (4.16) contains spectral unknowns defined into two different complex planes (η and $\chi=-m_{ai}(\gamma,\eta)$) related together via (4.17) and thus, we impose the boundary conditions we get GWHEs and not CWHEs (except in the case $\gamma=\pi$).

Explicit form of functional equations (4.16) are obtained and reported in Section 5 for isotropic media, however the theory reported in this paper can be applied to more complex media.

(b) From Region 1 to the other angular regions

Now, let us repeat the procedure for region 2 (Fig. 1), i.e. $u>0, v<0$. The solution $\tilde{\boldsymbol{\psi}}_t(\eta,v)$ of the system of differential equations of first order of dimension six (4.2) is obtainable as sum (4.5) of the general homogeneous solution $\tilde{\boldsymbol{\psi}}_o$ with a particular integral $\tilde{\boldsymbol{\psi}}_p$ defined in terms of

$$\boldsymbol{\psi}_s(v)=\boldsymbol{\psi}_{bs}(v)=-\mathbb{M}_{e1}\,\boldsymbol{\psi}_t(0_+,v)+j\eta\,\mathbb{M}_{e2}\,\boldsymbol{\psi}_t(0_+,v)-\mathbb{M}_{e2}\,\frac{\partial}{\partial u}\boldsymbol{\psi}_t(0_+,v). \tag{4.18}$$

in region 2 ($v<0$). We note that (4.18) is equal to (4.4) but with different support in v. The homogeneous solution takes the form (4.7). In presence of a passive medium, we recall that the first three eigenvalues present non-negative real part and are related to progressive waves along positive v while the last three eigenvalues present non-positive real part and are related to regressive waves, thus looking at the asymptotic behavior of (4.7) for $v\to-\infty$ we have $C_i=0,\ i=1,2,3$. Once obtained the dyadic Green's function specialized for region 2, the solution is

$$\begin{array}{l}\tilde{\boldsymbol{\psi}}_t(\eta,v)=\sum\limits_{i=4}^{6}C_i\mathbf{u}_i e^{-\lambda_{ei}(\gamma,\eta)\,v}-\sum\limits_{i=1}^{3}\mathbf{u}_i\boldsymbol{\nu}_i\cdot\int_{-\infty}^{v}e^{-\lambda_{ei}(\gamma,\eta)(v-v')}\boldsymbol{\psi}_{bs}(v')dv'+\\ \qquad+\sum\limits_{i=4}^{6}\mathbf{u}_i\boldsymbol{\nu}_i\cdot\int_{v}^{0}e^{-\lambda_{ei}(\gamma,\eta)(v-v')}\boldsymbol{\psi}_{bs}(v')dv'\end{array}. \tag{4.19}$$

before imposing the boundary conditions. Setting $v=0$ in (4.19), we have

$$\tilde{\boldsymbol{\psi}}_t(\eta,0)=\sum_{i=4}^{6}C_i\mathbf{u}_i-\sum_{i=1}^{3}\mathbf{u}_i\boldsymbol{\nu}_i\cdot\int_{-\infty}^{0}e^{\lambda_{ei}(\gamma,\eta)v'}\boldsymbol{\psi}_{bs}(v')dv'\ . \tag{4.20}$$

Multiplying (4.20) by $\boldsymbol{\nu}_i(\eta)$ for $i=1..6$, using bi-orthogonality, we obtain

$$\left\{\begin{array}{l}\boldsymbol{\nu}_i\cdot\tilde{\boldsymbol{\psi}}_t(\eta,0)=C_i,\quad i=4,5,6\\ \boldsymbol{\nu}_i\cdot\tilde{\boldsymbol{\psi}}_t(\eta,0)=-\boldsymbol{\nu}_i\cdot\breve{\boldsymbol{\psi}}_{bs}(j\lambda_{ei}(\gamma,\eta)),\quad i=1,2,3\end{array}\right., \tag{4.21}$$

where $\lambda_{ei}(\gamma,\eta)$ are reported in (3.13) and where

$$\breve{\boldsymbol{\psi}}_{bs}(\chi)=\int_{-\infty}^{0}e^{-j\chi v}\boldsymbol{\psi}_{bs}(v)dv=\int_{0}^{\infty}e^{j\chi r}\boldsymbol{\psi}_{bs}(-r)dr \tag{4.22}$$

is the left Laplace transform of $\boldsymbol{\psi}_{bs}(v)$ in v along face b (Fig. 1) or the Laplace transform in r of $\boldsymbol{\psi}_{bs}(-r)$ in cylindrical coordinates (r,φ,z). The properties of (4.21) are the same as for region 1. In particular, we focus the attention on the last three equations obtained by using progressive reciprocal vectors that yield the functional equations of the angular region. We re-write them as

$$\boldsymbol{\nu}_i\cdot\tilde{\boldsymbol{\psi}}_t(\eta,0)=-\boldsymbol{\nu}_i\cdot\breve{\boldsymbol{\psi}}_{bs}(-m_{bi}(\gamma,\eta)),\quad i=1,2,3 \tag{4.23}$$

with

$$\begin{array}{l}m_{b1}(\gamma,\eta)=m_{pb}(\gamma,\eta)=-j\lambda_{e1}(\gamma,\eta)=\eta\cos\gamma+\xi_p\sin\gamma,\\ m_{b2,b3}(\gamma,\eta)=m_{sb}(\gamma,\eta)=-j\lambda_{e2,e3}(\gamma,\eta)=\eta\cos\gamma+\xi_s\sin\gamma.\end{array} \tag{4.24}$$

In (4.23) the Laplace transforms of combinations of the field components defined on the boundaries of an angular region, i.e. $v=0$ (face o) and $u=0$ (face b) in Fig. 1, are related together. These functional equations are the starting point to define the GWHEs of region 2 by imposing boundary conditions and in particular they can be coupled to the ones of region 1 to build a structure with two angular regions with different elastic properties.

Observing (4.23), we note that at the second members we have that, in general, $\breve{\boldsymbol{\psi}}_{bs}(-m_{bi}(\gamma,\eta))$ contains discontinuous field components at the boundary $u=0, v<0$ of the angular region, while $\tilde{\boldsymbol{\psi}}_t(\eta,0)$ (by definition 2.16) is continuous at the boundary $u>0, v=0$.

Similarly to what has been done in [1] for electromagnetic applications, we can repeat the procedure to obtain functional equations for regions 3 and 4 (Fig. 1).

5. Explicit form of the functional equations for non planar (3D) problems in angular regions

In this Section, according to our opinion, we deduce and report for the first time in literature explicit spectral functional equations in algebraic form for the non planar (3D) elastic scattering problem in isotropic angular regions with arbitrary boundary conditions.

(a) Explicit form for region 1

We remark that (4.16) are the functional equations of region 1 for an elastic wave motion problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). The functional equations for the 2D (planar and antiplanar) problems are a particular case of the general wave motion problem with $\alpha_o=0$. In the following we demonstrate for validation that the GWHEs obtained from the proposed functional equations enforcing the boundary conditions and the functional equations obtained in [14] using the Gautesen (Kirchhoff) integral representations in the natural domain are identical, although the applied notations are different from each other and not immediate in the comparison.

To explicitly represent (4.16) in region 1, we need $\boldsymbol{\nu}_i$ reported in the rows of V (2.33), the Laplace transform of the field $\tilde{\boldsymbol{\psi}}_t(\eta,0)$ along $x, u>0, v=0_+$ (face o, see Fig. 1) and the Laplace transform $\breve{\boldsymbol{\psi}}_{as}(-m_{ai}(\gamma,\eta))$ along $x, u=0_+, v>0$ (face a, see Fig. 1). An important property of functional equations is that they report combination of field components that are continuous on the two boundary of the angular region. This property is fundamental to enforce boundary conditions in particular while connecting the angular region to a different body. We observe that, while $\tilde{\boldsymbol{\psi}}_t(\eta,0)$ is continuous at face o by definition (2.16), we need some mathematical manipulations to demonstrate that $\breve{\boldsymbol{\psi}}_{as}(-m_{ai}(\gamma,\eta))$ (4.4) is defined in terms of continuous field components at face a for an arbitrary aperture angle γ, since its expression contains potential discontinuous components such as derivatives of the field. The proof follows.

According to a local-to-face-a Cartesian coordinate system $X, Y, Z \equiv z$ (see Fig. 1) we have that the continuous components of the field are $T_{YY}, T_{YZ}, T_{XY}, v_X, v_Y, v_Z$, but $\breve{\boldsymbol{\psi}}_{as}(-m_{ai}(\gamma,\eta))$ and thus $\boldsymbol{\psi}_s(v) = \boldsymbol{\psi}_{as}(v)$ are originally defined in terms of $T_{yy}, T_{yz}, T_{xy}, v_x, v_y, v_z$ and their derivatives which in general are discontinuous, see (4.15), (4.4) and (2.16). In fact, the explicit form of $\boldsymbol{\psi}_{as}(v)$ (4.4), using (3.5) and (2.23)-(2.25), is:

$$\boldsymbol{\psi}_{as}(v) = \begin{pmatrix} T_{yy}\cos(\gamma) - T_{xy}\sin(\gamma) \\ \frac{{k_s}^3 T_{yz}\cos(\gamma) + Z_o\sin(\gamma)\left(jD_u v_z {k_s}^2 - 4\alpha_o {k_p}^2 v_x + {k_s}^2(\eta v_z + 3\alpha_o v_x)\right)}{{k_s}^3} \\ \frac{{k_s}^3 T_{xy}\cos(\gamma) + \sin(\gamma)\left(2{k_p}^2(-2jD_u v_x Z_o + k_s T_{yy} - 2Z_o(\alpha_o v_z + \eta v_x)) + {k_s}^2(-k_s T_{yy} + Z_o(4jD_u v_x + 3\alpha_o v_z + 4\eta v_x))\right)}{{k_s}^3} \\ v_x\cos(\gamma) - v_y\sin(\gamma) \\ v_x\sin(\gamma)\left(\frac{2{k_p}^2}{{k_s}^2} - 1\right) + v_y\cos(\gamma) \\ v_z\cos(\gamma) \end{pmatrix} \tag{5.1}$$

with $D_u = \left.\frac{\partial}{\partial u}\right|_{u=0+}$. As a first step to check the properties of (5.1) on face a, we derive expressions for D_u components of the velocity that appears at the 2nd and 3rd components of (5.1). Noting that $D_u = D_x$ and $D_z = -j\alpha_o$, from the 4th and the 8th basic equations reported in (2.15), we have:

$$\begin{aligned} D_u v_x &= \frac{jk_s[2k_p^2(T_{xx} - T_{yy} - T_{zz}) + k_s^2(-2T_{xx} + T_{yy} + T_{zz})]}{8k_p^2 Z_o - 6k_s^2 Z_o}, \\ D_u v_z &= \frac{jk_s T_{xz}}{Z_o} + j\alpha_o v_x. \end{aligned} \tag{5.2}$$

Substituting (5.2) into (5.1), we get an expression of $\boldsymbol{\psi}_{as}(v)$ in terms of $\mathbf{T}$ and $\mathbf{v}$ components without derivatives but still defined in terms of x, y, z. Now, in order to rewrite $\boldsymbol{\psi}_s(v) = \boldsymbol{\psi}_{as}(v) = \boldsymbol{\psi}_s(X, Y = 0)$ only in term of the local continuous components $T_{YY}, T_{YZ}, T_{XY}, v_X, v_Y, v_Z$ (face a, see Fig. 1), we formulate the rotational problem between components along x, y, z with respect to their definition along X, Y, Z. Without loss of generality, assuming $0 < \gamma < \pi$,

$$\mathbb{T} = \mathbb{R}_a^{-1}\,\mathbb{T}_a\,\mathbb{R}_a, \tag{5.3}$$

$$\mathbb{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{xy} & T_{yy} & T_{yz} \\ T_{xz} & T_{yz} & T_{zz} \end{pmatrix}, \; \mathbb{T}_a = \begin{pmatrix} T_{XX} & T_{XY} & T_{XZ} \\ T_{XY} & T_{YY} & T_{YZ} \\ T_{XZ} & T_{YZ} & T_{ZZ} \end{pmatrix}, \; \mathbb{R}_a = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.4}$$

and

$$\mathbf{v} = \mathbb{R}_a^{-1}\,\mathbf{v}_a, \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \quad \mathbf{v}_a = \begin{pmatrix} v_X \\ v_Y \\ v_Z \end{pmatrix}. \tag{5.5}$$

Substituting (5.3) and (5.5) into (5.1) after the application of (5.2), it yields an expression of $\boldsymbol{\psi}_{as}(v)$ in terms of the components $\mathbb{T}_a$ and $\mathbf{v_a}$ in X, Y, Z

$$\boldsymbol{\psi}_{as}(v) = \begin{pmatrix} T_{XY}\sin(\gamma) + T_{YY}\cos(\gamma) \\ \frac{\alpha_o Z_o\left({k_s}^2 - 2{k_p}^2\right)(v_X\sin(2\gamma) + v_Y\cos(2\gamma))}{{k_s}^3} - \frac{\alpha_o v_Y Z_o\left({k_s}^2 - 2{k_p}^2\right)}{{k_s}^3} + \frac{\eta v_Z Z_o\sin(\gamma)}{k_s} + T_{YZ} \\ \boldsymbol{\psi}_{as3}(v) \\ v_X\cos(2\gamma) - v_Y\sin(2\gamma) \\ \frac{v_Y\left({k_p}^2\cos(2\gamma) - {k_p}^2 + {k_s}^2\right) + {k_p}^2 v_X\sin(2\gamma)}{{k_s}^2} \\ v_Z\cos(\gamma) \end{pmatrix}, \tag{5.6}$$

where

$$\begin{aligned}&\boldsymbol{\psi}_{as3}(v)(4{k_p}^2{k_s}^3-3{k_s}^5)={k_s}^3T_{XY}\cos(\gamma)\left(4{k_p}^2-3{k_s}^2\right)+\sin(\gamma)\Big[\alpha_o(-v_Z)Z_o\left(4{k_p}^2-3{k_s}^2\right)^2+\\&+k_s\left(4{k_p}^4(T_{XX}+T_{YY}-T_{ZZ})-2{k_p}^2{k_s}^2(2T_{XX}+4T_{YY}-3T_{ZZ})+{k_s}^4(T_{XX}+4T_{YY}-2T_{ZZ})\right)\Big]+\\&\quad+4\eta Z_o\left(4{k_p}^4-7{k_p}^2{k_s}^2+3{k_s}^4\right)\left(v_Y\sin(\gamma)-v_X\cos(\gamma)\right)\end{aligned}\quad. \tag{5.7}$$

We recall that the procedure aims at finding $\boldsymbol{\psi}_{as}(v)$ in terms of the continuous field $T_{YY}, T_{YZ}, T_{XY}, v_X, v_Y, v_Z$. The result of the proposed substitutions is that the components of $\boldsymbol{\psi}_{as}(v)$ (5.6) are expressed all in terms of the continuous field except the component 3. In fact, from the beginning, the component 3 of (5.1) contains $D_u v_x$ that is represented by the 1st of (5.2) where the discontinuous T_{xx}, T_{zz} are present. The subsequent application of (5.3) and (5.5) do not change the properties $\boldsymbol{\psi}_{as}(v)$ in terms of continuous components and in particular the 3rd component contains the discontinuous components T_{XX}, T_{ZZ} as reported in (5.6) with (5.7). Noting that the basic equations (2.15) are invariant for rotations of the coordinate axes, by applying the 6th of (2.15) in X, Y, Z coordinates we get

$$T_{ZZ}=\frac{k_s\left({k_s}^2-2{k_p}^2\right)(T_{XX}+T_{YY})+2\alpha_o v_Z Z_o\left(4{k_p}^2-3{k_s}^2\right)}{2\left({k_s}^3-{k_p}^2k_s\right)}. \tag{5.8}$$

The substitution of (5.8) into $\boldsymbol{\psi}_{as3}(v)$ (5.7), after mathematical manipulations, yields an expression in terms of continuous field, whose embedding in (5.6) gives a representation of $\boldsymbol{\psi}_{as}(v)$ only in terms of continuous field at face a:

$$\boldsymbol{\psi}_{as}(v)=\begin{pmatrix}T_{XY}\sin(\gamma)+T_{YY}\cos(\gamma)\\ \frac{\alpha_o Z_o\left({k_s}^2-2{k_p}^2\right)(v_X\sin(2\gamma)+v_Y\cos(2\gamma))}{{k_s}^3}-\frac{\alpha_o v_Y Z_o\left({k_s}^2-2{k_p}^2\right)}{{k_s}^3}+\frac{\eta v_Z Z_o\sin(\gamma)}{k_s}+T_{YZ}\\ \frac{\sin(\gamma)\left(4\eta v_Y Z_o\sin(\gamma)(k_p^2-k_s^2)+{k_s}^2(\alpha_o v_Z Z_o-k_s T_{YY})\right)+2\eta v_X Z_o\sin(2\gamma)(k_s-k_p)(k_p+k_s)+{k_s}^3T_{XY}\cos(\gamma)}{{k_s}^3}\\ v_X\cos(2\gamma)-v_Y\sin(2\gamma)\\ \frac{v_Y\left({k_p}^2\cos(2\gamma)-{k_p}^2+{k_s}^2\right)+{k_p}^2v_X\sin(2\gamma)}{{k_s}^2}\\ v_Z\cos(\gamma)\end{pmatrix}. \tag{5.9}$$

From (5.9), we note that $\boldsymbol{\psi}_{as}(v)$ is defined only in term of continuous field component at face a. Now, the application of Laplace transform (4.15) to $\boldsymbol{\psi}_{as}(v)$ yields the explicit expression of the spectral functional equations (4.16) for region 1 in terms of continuous components. We remark that this property is fundamental to easily impose impenetrable boundary conditions and to couple region 1 with other penetrable surrounding regions of arbitrary geometry and in general non-homogeneous to region 1.

The property of the elastic wave motion problem to be formulated in terms of a differential problem (4.2) with sources $\boldsymbol{\psi}_{as}(v)$ (5.9) defined only in term of continuous field on the boundary represents an *equivalence theorem in elasticity* analogous to the well-known equivalence theorem in electromagnetism. In fact, the solution is given by $\tilde{\boldsymbol{\psi}}_t(\eta, v)$ (4.12) through the Green's function formulation only in terms of continuous components on the two faces of the angular region (C_i on face o and $\boldsymbol{\psi}_{as}(v)$ on face a), see (4.12)-(4.14). This property is corresponding to the well-known Schelkunoff's equivalence theorem together the uniqueness theorem in electromagnetics [36] where the equivalent sources are defined in terms of the components of electromagnetic field **E**, **H** tangent (continuous) to (at) the boundaries. A tentative text may be the following.

Equivalence theorem in elasticity: *A field in a lossy region is uniquely specified by the sources within the region plus the continuous components of the fields over the boundary.*

In order to avoid trivial identities for $\alpha_o = 0$ and in order to simplify a little the explicit form of functional equations (4.16), we redefine the reciprocal vectors $\boldsymbol{\nu}_i$ starting from the rows $\mathbb{V}(i,:)$, $i = 1..6$ of (2.33) according to the following scaling (reciprocal vectors as eigenvectors are defined up to a multiplicative constant):

$$\begin{aligned}&\boldsymbol{\nu}_1=\frac{2Z_o\xi_p k_s^2\mathbb{V}(1,:)}{\alpha_o},\ \boldsymbol{\nu}_2=2Z_o\xi_s k_s^2\mathbb{V}(2,:),\ \boldsymbol{\nu}_3=2Z_o\xi_s k_s^2\mathbb{V}(3,:),\\&\boldsymbol{\nu}_4=\frac{2Z_o\xi_p k_s^2\mathbb{V}(4,:)}{\alpha_o},\ \boldsymbol{\nu}_5=2Z_o\xi_s k_s^2\mathbb{V}(5,:),\ \boldsymbol{\nu}_6=2Z_o\xi_s k_s^2\mathbb{V}(6,:).\end{aligned} \tag{5.10}$$

With (5.10), (4.16) take the form (5.11)-(5.13) where the T, v quantities with lowercase subscripts in the LHS of the equations are defined for $u > 0, v = 0_+$ and are Laplace transforms in η, while the T, v quantities with uppercase subscripts are defined for $u = 0_+, v > 0$ and are Laplace transforms in $-m_p, -m_s, -m_s$ respectively in the RHS of (5.11),(5.12),(5.13).

$$\begin{array}{c} k_s\left(-T_{yy}\xi_p + \eta T_{xy} + \alpha_o T_{yz}\right) + Z_o\left[2\xi_p(\eta v_x + \alpha_o v_z) + v_y\left({\alpha_o}^2 + \eta^2 - \xi_s^2\right)\right] = \\ = Z_o[v_Y\left({\alpha_o}^2 + {k_p}^2 - {k_s}^2\right) + v_X\sin(2\gamma)\left(\eta^2 - \xi_p^2\right) + 2\xi_p(\eta v_X\cos(2\gamma) + \\ -\eta v_Y\sin(2\gamma) + \alpha_o v_Z\cos(\gamma)) + v_Y\cos(2\gamma)\left(\eta^2 - \xi_p^2\right) + 2\alpha_o\eta v_Z\sin(\gamma)] + \\ +k_s[-\xi_p(T_{XY}\sin(\gamma) + T_{YY}\cos(\gamma)) + \eta T_{XY}\cos(\gamma) - \eta T_{YY}\sin(\gamma) + \alpha_o T_{YZ}] \end{array}, \tag{5.11}$$

$$\begin{array}{c} k_s\xi_s(\eta T_{xy} + \alpha_o T_{yz}) + k_s T_{yy}\left({\alpha_o}^2 + \eta^2\right) + \\ +Z_o[\xi_s^2(\eta v_x + \alpha_o v_z) + 2v_y\left({\alpha_o}^2 + \eta^2\right)\xi_s - \left({\alpha_o}^2 + \eta^2\right)(\eta v_x + \alpha_o v_z)] = \\ = k_s\xi_s[\eta T_{XY}\cos(\gamma) - \eta T_{YY}\sin(\gamma) + \alpha_o T_{YZ}] + \\ +k_s\left({\alpha_o}^2 + \eta^2\right)[T_{XY}\sin(\gamma) + T_{YY}\cos(\gamma)] + \\ +Z_o\{\xi_s[\xi_s(\eta v_X\cos(2\gamma) - \eta v_Y\sin(2\gamma) + \alpha_o v_Z\cos(\gamma)) + v_X\left({\alpha_o}^2 + 2\eta^2\right)\sin(2\gamma) + \\ +v_Y\left({\alpha_o}^2 + 2\eta^2\right)\cos(2\gamma) + {\alpha_o}^2 v_Y + 2\alpha_o\eta v_Z\sin(\gamma)] + \\ -\left({\alpha_o}^2 + \eta^2\right)[\eta v_X\cos(2\gamma) - \eta v_Y\sin(2\gamma) + \alpha_o v_Z\cos(\gamma)]\} \end{array}, \tag{5.12}$$

$$\begin{array}{l} {k_s}^3 T_{yz} + \xi_s\{Z_o[{k_s}^2 v_z + 2\alpha_o v_y\xi_s - 2\alpha_o(\eta v_x + \alpha_o v_z)] + \alpha_o k_s T_{yy}\} - \alpha_o k_s(\eta T_{xy} + \alpha_o T_{yz}) = \\ = Z_o\{\alpha_o\sin(2\gamma)[v_X\left(-{\alpha_o}^2 - 2\eta^2 + {k_s}^2\right) + 2\eta v_Y\xi_s] - \alpha_o\cos(2\gamma)[v_Y\left({\alpha_o}^2 + 2\eta^2 - {k_s}^2\right) + \\ +2\eta v_X\xi_s] + v_Z\cos(\gamma)\left({k_s}^2 - 2{\alpha_o}^2\right)\xi_s + \eta v_Z\sin(\gamma)\left({k_s}^2 - 2{\alpha_o}^2\right) + \alpha_o v_Y\left({k_s}^2 - {\alpha_o}^2\right)\} + \\ +k_s\{T_{YZ}\left({k_s}^2 - {\alpha_o}^2\right) + \alpha_o\xi_s[T_{XY}\sin(\gamma) + T_{YY}\cos(\gamma)] + \alpha_o\eta[T_{YY}\sin(\gamma) - T_{XY}\cos(\gamma)]\} \end{array}. \tag{5.13}$$

We remark that (5.11)-(5.13) are the functional equations of region 1 for an elastic wave motion problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). These equations, according to our opinion, are deduced and reported for the first time in literature.

In particular, by applying the traction-free boundary conditions ($T_{xy} = T_{yy} = T_{yz} = T_{XY} = T_{YY} = T_{YZ} = 0$), (5.11)-(5.13) becomes GWHEs formulating the 3D elastic wedge problem considered in [17]. This formulation is important because allows to get semi-analytical solutions via Fredholm factorization method as developed by the authors in [4]. According to the authors' opinion, this technique constitutes a very power tool for the accurate approximate solutions of arbitrary WH equations. We remark that the GWHEs are algebraic, while in [17] the solution is obtained by functional equations written in terms of singular integral operators and solved by numerical technique. We assert that the semi-analytic solution using Fredholm factorization method allows physical insights by asymptotics in spectral domain.

(b) Explicit form for region 2

In this subsection, we repeat the procedure reported in subsection 5.(a) for region 2 (see Fig. 1), i.e. $u > 0, v < 0$, but with different aperture angle as reported in Fig. 2(b): the aperture angle of region 2 is γ instead of $\pi - \gamma$ as originally taken in Fig. 1. This difference is of great utility in the analysis of wedge structures with symmetries. For this purpose, we first start on deriving functional equations of region 2 (4.23) with the original aperture angle γ (Fig. 1 and Fig. 2(a)) for an elastic wave motion problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). Second, we apply the change in the aperture angle and the rotation of local reference system. To explicitly represent (4.23) for region 2, we need $\boldsymbol{\nu}_i$ reported in the rows of $\mathbb{V}$ (2.33), the Laplace transform $\tilde{\boldsymbol{\psi}}_t(\eta, 0)$ along $x, u > 0, v = 0$ (face o) and the Laplace transform $\breve{\boldsymbol{\psi}}_{bs}(-m_{bi}(\gamma, \eta))$ along $x, u = 0, v < 0$ (face b). We observe that, while $\tilde{\boldsymbol{\psi}}_t(\eta, 0)$ is continuous at face p by definition (2.16), we need some mathematical manipulations to demonstrate that $\breve{\boldsymbol{\psi}}_{bs}(-m_{bi}(\gamma, \eta))$ (4.18) is defined in terms of continuous field components at face b for an arbitrary aperture angle γ, since its expression contains potential discontinuous components such as derivatives of the field.

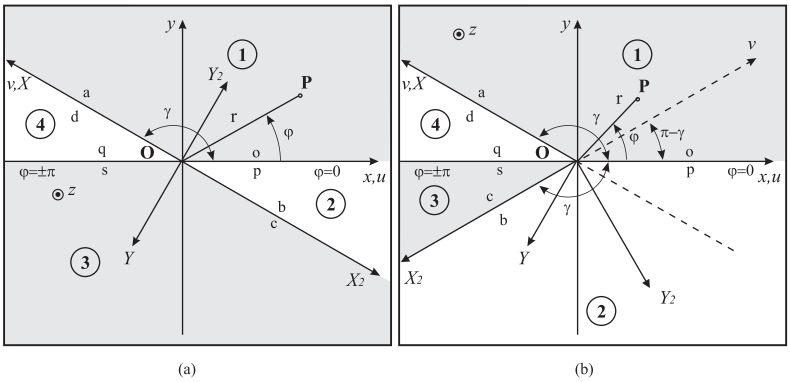

Figure 2. Angular regions and oblique Cartesian coordinates. The left subfigure re-reports Fig. 1 for convenience and it is the reference for the theory developed in the previous sections. The right subfigure shows the new framework of the space divided into four angular regions for wedge structures. We note symmetry between regions 1(3) and 2(4). The figure reports the x, y, z Cartesian coordinates and r, φ, z cylindrical coordinates useful to define the oblique Cartesian coordinate system u, v, z with reference to the angular region 1 $0 < \varphi < \gamma$ with $0 < \gamma < \pi$ and u, v, z with reference to the angular region 2 (only in the right subfigure). The face boundaries are labeled a, b, c, d, o, p, q, s. The figure reports also the local-to-face-a Cartesian coordinate system $X, Y, Z \equiv z$ and the local-to-face-b Cartesian coordinate system $X_2, Y_2, Z_2 \equiv z$ (only in the right subfigure). The $X, Y, Z \equiv z$ and $X_2, Y_2, Z_2 \equiv z$ Cartesian coordinate systems are obtained from x, y, z Cartesian coordinate system by rotation, respectively for a positive γ and $-\gamma$.

According to a local-to-face-b Cartesian coordinate system $X_2, Y_2, Z_2 \equiv z$ (see Fig. 2) we have that the continuous components of the field are $T_{Y2Y2}, T_{Y2Z2}, T_{X2Y2}, v_{X2}, v_{Y2}, v_{Z2}$, but $\breve{\psi}_{bs}(-m_{bi}(\gamma,\eta))$ and thus $\psi_s(v) = \psi_{bs}(v)$ are defined in terms of $T_{yy}, T_{yz}, T_{xy}, v_x, v_y, v_z$ and their derivatives which in general are discontinuous, see (4.22), (4.18) and (2.16). In fact, the explicit form of $\psi_{bs}(v)$ (4.18), using (3.5) and (2.23)-(2.25), yields the same expression of $\psi_{as}(v)$ given in (5.1), even if $\psi_{bs}(v)$ is defined for $v < 0$ and $\psi_{as}(v)$ for $v > 0$. Following the steps done for $\psi_{as}(v)$ in region 1, we derive expressions for D_u components of the velocity appearing in (5.1). Noting that $D_u = D_x$ and $D_z = -j\alpha_o$, from the 4th and the 8th basic equations reported in (2.15), we have (5.2) that substituted into $\psi_{bs}(v)$ yields an expression in terms of $\mathbf{T}$ and $\mathbf{v}$ components without derivatives but still defined in terms of the coordinate system x, y, z.

Now, in order to rewrite $\psi_s(v) = \psi_{bs}(v) = \psi_s(X_2, Y_2 = 0)$ only in term of the local continuous components $T_{Y2Y2}, T_{Y2Z2}, T_{X2Y2}, v_{X2}, v_{Y2}, v_{Z2}$ (face b), we formulate the rotational problem between components along x, y, z with respect to their definition along X_2, Y_2, Z_2. The required rotation in Fig. 2(a) is $-\pi + \gamma$. Now, let us introduce also the change of aperture angle from γ to $\pi - \gamma$ as in the right subfigure of Fig. 2. This change of aperture angle impacts on the definitions of $\mathbb{M}_{ei}$ matrices (due to the replacement of γ with $\pi - \gamma$) and then $\psi_{bs}(v)$ that now becomes different from $\psi_{as}(v)$. In the new region 2 (Fig.2(b)) the rotation relations (5.3)-(5.5) of region 1 are replaced by the relations for region 2 where we have performed the substitution $\gamma \to -\pi + \gamma$ (rotation) and $\gamma \to \pi - \gamma$ (change of aperture angle), thus $\gamma \to -\gamma$. It yields:

$$\mathbb{T} = \mathbb{R}_b^{-1}\, \mathbb{T}_b\, \mathbb{R}_b, \tag{5.14}$$

$$\mathbb{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{xy} & T_{yy} & T_{yz} \\ T_{xz} & T_{yz} & T_{zz} \end{pmatrix}, \ \mathbb{T}_b = \begin{pmatrix} T_{X2X2} & T_{X2Y2} & T_{X2Z2} \\ T_{X2Y2} & T_{Y2Y2} & T_{Y2Z2} \\ T_{X2Z2} & T_{Y2Z2} & T_{Z2Z2} \end{pmatrix}, \ \mathbb{R}_b = \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.15}$$

$$\mathbf{v} = \mathbb{R}_b{}^{-1}\,\mathbf{v}_b, \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}, \quad \mathbf{v}_b = \begin{pmatrix} v_{X2} \\ v_{Y2} \\ v_{Z2} \end{pmatrix}. \tag{5.16}$$

Substituting (5.14) and (5.16) into $\boldsymbol{\psi}_{bs}(v)$ (same expression of $\boldsymbol{\psi}_{as}(v)$ (5.1)) after the application of (5.2) and (5.8) in X_2, Y_2, Z_2 coordinates, it yields an expression of $\boldsymbol{\psi}_{bs}(v)$ in terms of the continuous (at face b) components $T_{Y2Y2}, T_{Y2Z2}, T_{X2Y2}, v_{X2}, v_{Y2}, v_{Z2}$:

$$\boldsymbol{\psi}_{bs}(v) = \begin{pmatrix} T_{X_2Y_2}\sin(\gamma) - T_{Y_2Y_2}\cos(\gamma) \\ \frac{\alpha_o v_{X_2} Z_o \sin(2\gamma)\left(k_s{}^2 - 2k_p{}^2\right) + \alpha_o v_{Y_2} Z_o \cos(2\gamma)\left(2k_p{}^2 - k_s{}^2\right) + \alpha_o v_{Y_2} Z_o\left(k_s{}^2 - 2k_p{}^2\right) + \eta k_s{}^2 v_{Z_2} Z_o \sin(\gamma)}{k_s{}^3} - T_{Y_2Z_2} \\ \frac{\sin(\gamma)\left[4\eta v_{Y_2} Z_o \sin(\gamma)(k_s{}^2 - k_p{}^2) + k_s{}^2(\alpha_o v_{Z_2} Z_o - k_s T_{Y_2Y_2})\right] + 2\eta v_{X_2} Z_o \sin(2\gamma)(k_s{}^2 - k_p{}^2) - k_s{}^3 T_{X_2Y_2}\cos(\gamma)}{k_s{}^3} \\ -v_{X_2}\cos(2\gamma) - v_{Y_2}\sin(2\gamma) \\ \frac{k_p{}^2[v_{X_2}\sin(2\gamma) - v_{Y_2}\cos(2\gamma)] + v_{Y_2}(k_p{}^2 - k_s{}^2)}{k_s{}^2} \\ -v_{Z_2}\cos(\gamma) \end{pmatrix}. \tag{5.17}$$

Now, the application of Laplace transform (4.22) to $\boldsymbol{\psi}_{bs}(v)$ yields the explicit expression of the spectral functional equations (4.16) for region 2 in terms of continuous components.

Again the property of the elastic wave motion problem to be formulated in terms of a differential problem (4.2) with sources $\boldsymbol{\psi}_{bs}(v)$ (5.17) defined only in term of continuous field on the boundary represents an *equivalence theorem in elasticity* for region 2 as discussed in 5(a).

As done for region 1, in order to avoid trivial identities for $\alpha_o = 0$ and in order to simplify a little the explicit form of (4.23), we redefine the reciprocal vectors as reported in (5.10). With (5.10), (4.23) take the form (5.18)-(5.20) where the T, v quantities with lowercase subscripts in the LHS of the equations are defined for $u > 0, v = 0_-$ and are Laplace transforms in η, while the T, v quantities with uppercase subscripts are defined for $u = 0_+, v < 0$ and are Laplace transforms in $-m_{pb}, -m_{sb}, -m_{sb}$ respectively in the RHS of (5.18),(5.19),(5.20). It yields:

$$\begin{array}{c} Z_o\left[2\xi_p(\eta v_x + \alpha_o v_z) - v_y\left(\alpha_o{}^2 + \eta^2 - \xi_s^2\right)\right] - k_s\left(T_{yy}\xi_p + \eta T_{xy} + \alpha_o T_{yz}\right) = \\ = Z_o[-v_{Y2}\left(\alpha_o{}^2 + k_p{}^2 - k_s{}^2\right) + v_{X2}\sin(2\gamma)\left(\eta^2 - \xi_p^2\right) + 2\xi_p(\eta v_{X2}\cos(2\gamma) + \\ \eta v_{Y2}\sin(2\gamma) + \alpha_o v_{Z2}\cos(\gamma)) + v_{Y2}\cos(2\gamma)\left(\xi_p^2 - \eta^2\right) + 2\alpha_o\eta v_{Z2}\sin(\gamma)] + \\ -k_s\left[\xi_p(T_{Y2Y2}\cos(\gamma) - T_{X2Y2}\sin(\gamma)) + \eta T_{X2Y2}\cos(\gamma) + \eta T_{Y2Y2}\sin(\gamma) + \alpha_o T_{Y2Z2}\right] \end{array}, \tag{5.18}$$

$$\begin{array}{c} k_s\xi_s(\eta T_{xy} + \alpha_o T_{yz}) - k_s T_{yy}\left(\alpha_o{}^2 + \eta^2\right) + \\ +Z_o[\xi_s^2(-(\eta v_x + \alpha_o v_z)) + 2v_y\left(\alpha_o{}^2 + \eta^2\right)\xi_s + \left(\alpha_o{}^2 + \eta^2\right)(\eta v_x + \alpha_o v_z)] = \\ = k_s\xi_s[\eta T_{X2Y2}\cos(\gamma) + \eta T_{Y2Y2}\sin(\gamma) + \alpha_o T_{Y2Z2}] + \\ -k_s\left(\alpha_o{}^2 + \eta^2\right)[T_{Y2Y2}\cos(\gamma) - T_{X2Y2}\sin(\gamma)] + \\ +Z_o\{\xi_s[-\xi_s(\eta v_{X2}\cos(2\gamma) + \eta v_{Y2}\sin(2\gamma) + \alpha_o v_{Z2}\cos(\gamma)) - v_{X2}\left(\alpha_o{}^2 + 2\eta^2\right)\sin(2\gamma) \\ +v_{Y2}\left(\alpha_o{}^2 + 2\eta^2\right)\cos(2\gamma) + \alpha_o{}^2 v_{Y2} - 2\sin(\gamma)\alpha_o\eta v_{Z2}] + \\ +\left(\alpha_o{}^2 + \eta^2\right)[\eta v_{X2}\cos(2\gamma) + \eta v_{Y2}\sin(2\gamma) + \alpha_o v_{Z2}\cos(\gamma)]\} \end{array}, \tag{5.19}$$

$$\begin{array}{l} -k_s{}^3 T_{yz} + \xi_s\{Z_o[k_s{}^2 v_z - 2\alpha_o v_y\xi_s - 2\alpha_o(\eta v_x + \alpha_o v_z)] + \alpha_o k_s T_{yy}\} + \alpha_o k_s(\eta T_{xy} + \alpha_o T_{yz}) = \\ = Z_o\{\alpha_o\sin(2\gamma)[v_{X2}\left(-\alpha_o{}^2 - 2\eta^2 + k_s{}^2\right) - 2\alpha_o\eta v_{Y2}\xi_s] + \alpha_o\cos(2\gamma)[v_{Y2}\left(\alpha_o{}^2 + 2\eta^2 - k_s{}^2\right) + \\ -2\eta v_{X2}\xi_s] + v_{Z2}\cos(\gamma)\left(k_s{}^2 - 2\alpha_o{}^2\right)\xi_s + \eta v_{Z2}\sin(\gamma)\left(k_s{}^2 - 2\alpha_o{}^2\right) + \alpha_o v_{Y2}(\alpha_o{}^2 - k_s{}^2)\} + \\ +k_s\{T_{Y2Z2}(\alpha_o{}^2 - k_s{}^2) + \alpha_o\xi_s[T_{Y2Y2}\cos(\gamma) - T_{X2Y2}\sin(\gamma)] + \alpha_o\eta[T_{X2Y2}\cos(\gamma) + T_{Y2Y2}\sin(\gamma)]\} \end{array}. \tag{5.20}$$

We remark that (5.18)-(5.20) are the spectral functional equations of region 2 for an elastic wave motion problem in an isotropic medium at skew (non planar) incidence ($\alpha_o \neq 0$). As cross-validation, we note that (5.18)-(5.20) of region 2 are equivalent to (5.11)-(5.13) of region 1, according to the following replacements dictated by means of symmetry (see Fig. 2):

$$\begin{aligned}\{v_x, v_y, v_z, T_{yy}, T_{xy}, T_{yz}\} &\rightarrow \{v_x, -v_y, v_z, T_{yy}, -T_{xy}, -T_{yz}\},\\ \{v_{X2}, v_{Y2}, v_{Z2}, T_{Y2Y2}, T_{X2Y2}, T_{Y2Z2}\} &\rightarrow \{v_X, -v_Y, v_Z, T_{YY}, -T_{XY}, -T_{YZ}\}.\end{aligned} \tag{5.21}$$

The procedure reported in this Section can be repeated to get the functional equations for regions 3 and 4 following also the explicit mathematical steps described in [1] for em applications.

6. Validation of functional equations for an isotropic angular region with traction-free boundary conditions in the 2D case

The functional equations for the 2D (planar and antiplanar) problems ($\alpha_o = 0$) are a particular case of the ones obtained for the general 3D problem (5.11)-(5.13) and (5.18)-(5.20) respectively for region 1 and region 2 with reference to the right subfigure of Fig. 2.

Taking into consideration region 1, in the following, we demonstrate that the GWHEs obtained from the proposed functional equations while enforcing the traction-free face boundary conditions in the planar angular problem ($\alpha_o = 0$) and the functional equations obtained in [14] by Gautesen's group are identical, although the applied notations are very different from each other and cumbersome to be compared. Moreover, the functional equation for the anti-planar problem are checked with an independent method, too.

We recall that the explicit functional equations for region 1 reported in (5.11)-(5.13) are derived from (4.16). Since functional equations can be written up to multiplicative constant as eigenvectors, to perform the comparison with compact expressions and to avoid the lack o definition of some eigenvectors/reciprocal vectors for $\alpha_o = 0$, we redefine the reciprocal vectors (2.33) as in the following scaling:

$$\boldsymbol{\nu}_1 = \frac{2\xi_p k_s^2 \mathbb{V}(1,:)}{\alpha_o},\ \boldsymbol{\nu}_2 = \frac{2\xi_s k_s^2 \mathbb{V}(2,:)}{\eta};\ \boldsymbol{\nu}_3 = 2\mathbb{V}(3,:),\ \boldsymbol{\nu}_4 = \frac{2\xi_p k_s^2 \mathbb{V}(4,:)}{\alpha_o},\ \boldsymbol{\nu}_5 = \frac{2\xi_s k_s^2 \mathbb{V}(5,:)}{\eta},\ \boldsymbol{\nu}_6 = 2\mathbb{V}(6,:). \tag{6.1}$$

For readability, we report (6.1) in explicit form for $\alpha_o = 0$ in terms of rows of the following matrix:

$$\mathbb{V}_o = \begin{pmatrix} -\frac{k_s\xi_p}{Z_o} & 0 & -\frac{\eta k_s}{Z_o} & 2\eta\xi_p & \xi_s^2-\eta^2 & 0\\ -\frac{\eta k_s}{Z_o} & 0 & \frac{k_s\xi_s}{Z_o} & \eta^2-\xi_s^2 & 2\eta\xi_s & 0\\ 0 & -\frac{k_s}{Z_o\xi_s} & 0 & 0 & 0 & 1\\ -\frac{k_s\xi_p}{Z_o} & 0 & \frac{\eta k_s}{Z_o} & 2\eta\xi_p & \eta^2-\xi_s^2 & 0\\ \frac{\eta k_s}{Z_o} & 0 & \frac{k_s\xi_s}{Z_o} & \xi_s^2-\eta^2 & 2\eta\xi_s & 0\\ 0 & \frac{k_s}{Z_o\xi_s} & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

For $\alpha_o = 0$ we obtain a simplified version of (5.6)

$$\boldsymbol{\psi}_{as}(v) = \begin{pmatrix} T_{XY}\sin(\gamma) + T_{YY}\cos(\gamma)\\ \frac{\eta v_Z \sin(\gamma) Z_o}{k_s} + T_{YZ}\\ \frac{4\eta Z_o \sin(\gamma)({k_p}^2-{k_s}^2)(v_Y\sin(\gamma)-v_X\cos(\gamma))}{{k_s}^3} + T_{XY}\cos(\gamma) - T_{YY}\sin(\gamma)\\ v_X\cos(2\gamma) - v_Y\sin(2\gamma)\\ \frac{v_Y\left({k_p}^2\cos(2\gamma)-{k_p}^2+{k_s}^2\right)+{k_p}^2 v_X\sin(2\gamma)}{{k_s}^2}\\ v_Z\cos(\gamma) \end{pmatrix}. \tag{6.3}$$

With reference to Fig. 1 we now explicit the functional equations (4.16) of an angular region filled by isotropic elastic medium before imposing face boundary conditions in the 2D case.

With $\alpha_o = 0$, the re-scaled reciprocal vectors (6.2), the Laplace transform $\tilde{\psi}_t(\eta, v=0)$ (4.1) of the continuous field (2.17) at face o and the Laplace transform $\breve{\psi}_{as}(\chi)$ (4.15) of the quantity (6.3) expressed in terms of the continuous field at face a, we obtain the following explicit form of the functional equations (4.16):

$$\begin{gathered}\frac{k_s(\eta T_{xy}-T_{yy}\xi_p)}{Z_o}+2\eta v_x\xi_p+v_y\left(\eta^2-\xi_s^2\right)=\sin(2\gamma)[-2\eta\xi_p v_Y-v_X{\xi_p}^2+\eta^2 v_X]+v_Y({k_p}^2-{k_s}^2)+\\+\cos(2\gamma)[-{\xi_p}^2 v_Y+2\eta\xi_p v_X+\eta^2 v_Y]-\frac{k_s\xi_p[T_{XY}\sin(\gamma)+T_{YY}\cos(\gamma)]+\eta k_s[T_{XY}\cos(\gamma)-T_{YY}\sin(\gamma)]}{Z_o}\end{gathered}, \tag{6.4}$$

$$\begin{gathered}\frac{k_s(T_{xy}\xi_s+\eta T_{yy})}{Z_o}-v_x\left(\eta^2-\xi_s^2\right)+2\eta v_y\xi_s=\sin(2\gamma)[2\eta v_X\xi_s-v_Y\xi_s^2+\eta^2 v_Y]+\\+\cos(2\gamma)[v_X\xi_s^2+2\eta v_Y\xi_s-\eta^2 v_X]+\frac{k_s\xi_s[T_{XY}\cos(\gamma)-T_{YY}\sin(\gamma)]+k_s\eta[T_{XY}\sin(\gamma)+T_{YY}\cos(\gamma)]}{Z_o}\end{gathered}, \tag{6.5}$$

$$\frac{k_s T_{yz}}{Z_o\xi_s}+v_z=\frac{k_s T_{YZ}}{Z_o\xi_s}+\frac{\eta v_Z}{\xi_s}\sin(\gamma)+v_Z\cos(\gamma). \tag{6.6}$$

We recall the T, v quantities with lowercase subscripts in the LHS of the equations are defined for $u>0, v=0_+$ and are Laplace transforms in η of $\tilde{\psi}_t(\eta, v=0)$, while the T, v quantities with uppercase subscripts are defined for $u=0_+, v>0$ and are Laplace transforms in $-m_p, -m_s, -m_s$ of $\psi_{as}(v)$ respectively in the RHS of (6.4),(6.5),(6.6).

We note that (6.4) is related to the complex propagation constant $-m_p$ of the principal wave while (6.5),(6.6) are related to $-m_s$, i.e. the one of the secondary waves.

We note also some sort of symmetry between (6.4) and (6.5) except for the additional term $v_Y({k_p}^2-{k_s}^2)$ in (6.4).

Eqs. (6.4),(6.5),(6.6) are functional equations for the general 2D wave motion angular problem in elasticity before imposing boundary conditions, i.e. they represent the planar and anti-planar problems.

To complete the validation with the equations proposed at (4.1) of [14], with reference to region 1 of Fig.1, we impose traction-free face boundary conditions at faces o and a, i.e. the traction $\mathbf{t}=\underline{\mathbf{T}}\cdot\mathbf{n}=\mathbf{0}$ where $\mathbf{n}$ is the unit normal to the face:

$$T_{yy}, T_{yz}, T_{yx}=0 \text{ at face o } (u>0, v=0_+), \quad T_{YY}, T_{YZ}, T_{YX}=0 \text{ at face a } (u=0_+, v>0). \tag{6.7}$$

It yields the following GWHEs:

$$\begin{gathered}2\eta v_x\xi_p+v_y(\eta^2-\xi_s^2)=\sin(2\gamma)[-2\eta\xi_p v_Y+v_X(\eta^2-{\xi_p}^2)]+\\+\cos(2\gamma)[v_Y(\eta^2-{\xi_p}^2)+2\eta\xi_p v_X]+v_Y({k_p}^2-{k_s}^2)\end{gathered}, \tag{6.8}$$

$$-v_x\left(\eta^2-\xi_s^2\right)+2\eta v_y\xi_s=\sin(2\gamma)[2\eta v_X\xi_s-v_Y\xi_s^2+\eta^2 v_Y]+\cos(2\gamma)[v_X\xi_s^2+2\eta v_Y\xi_s-\eta^2 v_X], \tag{6.9}$$

$$v_z=\frac{\eta v_Z}{\xi_s}\sin(\gamma)+v_Z\cos(\gamma). \tag{6.10}$$

where the v quantities with lowercase subscripts in the LHS of (6.8),(6.9),(6.10) are plus functions in η and v quantities with uppercase subscripts in the RHS are minus functions (plus functions) in m_p, m_s, m_s $(-m_p, -m_s, -m_s)$. Both minus and plus functions are Laplace transforms. Standard plus(minus) functions are analytic in the upper(lower) half-plane. We extend the theory to non-standard functions when they have isolated poles due to plane wave sources located in the standard regularity half-plane.

Note that (6.10) is independent from (6.8),(6.9). In fact (6.10) is associated to SH wave in the wave motion problem (antiplanar problem), while (6.8),(6.9) model the coupled problem between P and SV waves (planar problem).

Eq. (6.10) can be checked and validated after imposing the traction-free face boundary conditions with (3.15.5) of [4] where a completely different method specialized on antiplanar problems has been used. Now, let us compare (6.8),(6.9) with (4.1) of [14], reported in original form at (6.11) with (6.12)- (6.13).

$$\begin{aligned} a(\xi)\widehat{u}_1(\xi) - b_1(\xi)\widehat{u}_2(\xi) + \widehat{U}_1(\xi) = f_1(\xi), \\ b_2(\xi)\widehat{u}_1(\xi) + a(\xi)\widehat{u}_2(\xi) + \widehat{U}_2(\xi) = f_2(\xi), \end{aligned} \tag{6.11}$$

$$\begin{aligned} \hat{U}_1(\xi) = (-1)^{\ell}\left[-a(\zeta_1)\,\hat{u}_1(\zeta_1) + \bar{b}_1(\xi)\hat{u}_2(\zeta_1)\right], \quad \ell = 1, 2, (antisym, sym), \\ \hat{U}_2(\xi) = (-1)^{\ell}\left[\bar{b}_2(\xi)\hat{u}_1(\zeta_2) + a(\zeta_2)\,\hat{u}_2(\zeta_2)\right], \quad \ell = 1, 2\,(antisym, sym), \end{aligned} \tag{6.12}$$

$$\begin{aligned} \zeta_{1,2} = \xi\cos\alpha + \gamma_{1,2}(\xi)\sin\alpha, \\ \eta_{1,2} = \xi\sin\alpha - \gamma_{1,2}(\xi)\cos\alpha, \\ \bar{b}_{1,2}(\xi) = 2\zeta_{1,2}\eta_{1,2}. \end{aligned} \tag{6.13}$$

In (6.11) $\widehat{u}_1(\xi), \widehat{u}_2(\xi)$ are one-sided Fourier transforms of unknown displacements on face o (Fig.1) respectively in x, y, ξ is the spectral variable, $a(\xi), b_1(\xi), b_2(\xi)$ are spectral functions and, $\widehat{U}_1(\xi), \widehat{U}_2(\xi)$ are one-sided Fourier transforms of quantities defined in terms of unknown displacements on face a (Fig.1) respectively in $X, -Y$. $f_1(\xi), f_2(\xi)$ model the source of the wave motion problem. In order to compare (6.11) with (6.8),(6.9), we scale all the displacements by $j\omega$ to get the velocities, thus (6.11) hold in homogeneous form ($f_1(\xi), f_2(\xi) = 0$) also interpreting $\widehat{u}_i(\xi)$,$\widehat{U}_i(\xi)$ in terms of velocities. Moreover, we observe that $i = 1, 2$ waves in [14] are respectively associated to SV, P waves, thus we need to compare (6.8),(6.9) respectively with the 2nd and the 1st equation of (6.11). With the help of the definitions given in [14], let us interpret (6.11) in our formalism. Table 1 reports the correspondences for the definition of some quantities in the two works. With Table 1, it is easy to show the equivalence between the LHS of (6.8),(6.9) and the terms in $\widehat{u}_i(\xi)$ in (6.11).

Table 1. Translation of definitions between this work and [14]

[14]	ξ	$\kappa_{1,2}$	α	$\widehat{u}_{1,2}(\xi)$	$\gamma_{1,2}^2 = \kappa_{1,2}^2 - \xi^2$	$a(\xi) = \kappa_1^2 - 2\xi^2$	$b_{1,2}(\xi) = 2\xi\gamma_{1,2}(\xi)$
this paper	η	$k_{s,p}$	γ	$v_{x,y}(\eta)$	$\xi_{s,p}^2 = k_{s,p}^2 - \eta^2$	$\xi_s^2 - \eta^2$	$2\eta\xi_{s,p}$

To complete the comparison we need to check the 1st equation of (6.11) and (6.9) focusing the attention on $\widehat{U}_1(\xi)$ (6.12) and then check the 2nd equation of (6.11) and (6.8) focusing the attention on $\widehat{U}_2(\xi)$ (6.12). Starting from (6.13), $\zeta_{1,2}$ play the roles of $-m_{s,p}$ (4.17) and $\eta_{1,2}$ play the role of $n_{s,p}$. In particular we note that, in our notation,

$$\zeta_{1,2} \to \eta\cos\gamma + \xi_{s,p}\sin\gamma, \quad \eta_{1,2} \to \eta\sin\gamma - \xi_{s,p}\cos\gamma, \tag{6.14}$$

that apart from a sign in the combination of the two terms are respectively $-m_{s,p}$ (4.17) and $n_{s,p}$:

$$m_{s,p} = -\eta\cos\gamma + \xi_{s,p}\sin\gamma, \quad n_{s,p} = \eta\sin\gamma + \xi_{s,p}\cos\gamma. \tag{6.15}$$

Further sign differences appear also in the combination of the quantities between (6.8)-(6.9) and (6.11). We are convinced that these differences are due to different notations in Fourier transforms between engineering (ours, [7] p.XV) and applied mathematics (as in [14]) and, to the different orientation of local coordinate system on face a between our work and [14] where $(X, -Y)$ are selected (see Fig. 1). We note that $\hat{u}_{1,2}(\zeta_1)$ in $\widehat{U}_1(\xi)$ (6.12) for equation (6.11) play the roles of $v_{X,Y}(-m_s)$ for equation (6.9). Let us compare the functional coefficient of $\hat{u}_{1,2}(\zeta_1)$ with the ones of $v_{X,Y}(-m_s)$. With the help of Table 1 and (6.14)-(6.15), for $\hat{u}_1(\zeta_1)$ and $v_X(-m_s)$ we have resp.

$$-a(\zeta_1) = \kappa_1^2 - 2\zeta_1^2 \to k_s^2 - 2m_s^2, \tag{6.16}$$

$$\sin(2\gamma)2\eta\xi_s + \cos(2\gamma)[\xi_s^2 - \eta^2] = k_s^2 - 2m_s^2 \tag{6.17}$$

after some trigonometric manipulation. Again for $\hat{u}_2(\zeta_1)$ and $v_Y(-m_s)$ we have respectively

$$\bar{b}_1(\xi) = 2\zeta_1\eta_1 \to 2m_s n_s, \tag{6.18}$$

$$\sin(2\gamma)[-\xi_s^2 + \eta^2] + \cos(2\gamma)[2\eta\xi_s] = 2m_s n_s. \tag{6.19}$$

Now let us complete the comparison between the 2nd equation of (6.11) and (6.8), focusing the attention on $\widehat{U}_2(\xi)$ (6.12)a nd comparing the functional coefficient of $\hat{u}_{1,2}(\zeta_1)$ in $\widehat{U}_2(\xi)$ with the ones of $v_{X,Y}(-m_p)$. With the help of Table 1 and (6.14)-(6.15), for $\hat{u}_1(\zeta_2)$ and $v_X(-m_p)$ we have respectively

$$\bar{b}_2(\xi) = 2\zeta_2\eta_2 \rightarrow 2m_p n_p, \tag{6.20}$$

$$\sin(2\gamma)[-{\xi_p}^2 + \eta^2] + \cos(2\gamma)[2\eta\xi_p] = 2m_p n_p \tag{6.21}$$

with same calculus done in (6.18)-(6.19). On the contrary, we note that $\hat{u}_2(\zeta_2)$ and $v_Y(-m_p)$ show different properties with respect to (6.16)-(6.17). Their respective functional coefficients are

$$a(\zeta_2) = \kappa_1^2 - 2\zeta_2^2 \rightarrow k_s^2 - 2m_p^2, \tag{6.22}$$

$$\sin(2\gamma)[-2\eta\xi_p] + \cos(2\gamma)[-{\xi_p}^2 + \eta^2] + ({k_p}^2 - {k_s}^2) = k_s^2 - 2m_p^2 \tag{6.23}$$

that are equivalent after some trigonometric manipulation. Note in (6.22)-(6.23) we have the simultaneous presence of SV and P spectral variables and propagation constants and, the presence of additional term $({k_p}^2 - {k_s}^2)$ in the LHS of (6.23) with respect to the LHS of (6.17). This property denotes coupling between SV and P waves.

We conclude by affirming that (6.8),(6.9),(6.10) are the GWHEs for the elastic wave motion angular problem in 2D ($\alpha_o = 0$) with traction-free face boundary conditions that model the planar (6.8),(6.9) and antiplanar (6.10 problems in presence of plane-wave sources or sources located at infinity with the help of the concept of non-standard Laplace transforms (see section 1.4 of [5]).

7. Validation of functional equations through the estimation of characteristic impedances in half-space planar regions

In this Section we further validate the functional equations (5.11)-(5.13) and (5.18)-(5.20) obtained in the general case of 3D angular region problems by computing the characteristic impedances of the half spaces identified as region 1 ($y > 0$) and region 2 ($y < 0$) in Fig. 3 for planar problems.

Fig. 3 shows the half-plane problem (crack) where arbitrary boundary condition can be applied. We recall that GWHEs for practical problems can be derived from (5.11)-(5.13) and (5.18)-(5.20) by applying specific boundary conditions (traction-free, clamped, ...). For example, this method can be used to compare with solutions reported in [34]- [35] for the half-plane problem. In this case, we note that, starting from the general functional equations, by imposing $\gamma = \pi$, we model the half-plane problem via GWHEs that reduce to Classical Wiener-Hopf equations due to the definitions of spectral variables m.

Let us start from region 1, considering (5.11)-(5.13). To model the planar problem, we impose $\gamma = \pi$, $\alpha_o = 0$ and all the continuous z components of the field $\mathbf{T}$ and $\mathbf{v}$ null: $T_{yz} = T_{YZ} = 0$, $v_z = v_Z = 0$. From (5.11)-(5.12) ((5.13) is trivially null in this case) we have

$$\begin{aligned} &Z_o\left((2\eta^2 - {k_s}^2)v_y + 2\eta v_x\xi_p\right) + k_s\left(\eta T_{xy} - T_{yy}\xi_p\right) = Z_o\left((2\eta^2 - {k_s}^2)v_Y + 2\eta v_X\xi_p\right) - k_s\left(\eta T_{XY} - T_{YY}\xi_p\right), \\ &Z_o\left(\left({k_s}^2 - 2\eta^2\right)v_x + 2\eta v_y\xi_s\right) + k_s\left(T_{xy}\xi_s + \eta T_{yy}\right) = Z_o\left(\left({k_s}^2 - 2\eta^2\right)v_X + 2\eta v_Y\xi_s\right) - k_s\left(T_{XY}\xi_s + \eta T_{YY}\right). \end{aligned} \tag{7.1}$$

Now let us focus the attention on the non null continuous field component of $\mathbf{T}$ and $\mathbf{v}$, we have respectively for (2.16) with (4.1) and (5.9) with (4.15):

$$\boldsymbol{\psi}_t = \left(T_{yy}, T_{xy}, v_x, v_y\right)', \; \boldsymbol{\psi}_{as} = \left(-T_{YY}, -T_{XY}, v_X, v_Y\right)'. \tag{7.2}$$

From the definitions of $\boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_{as}$, respectively defined in $x > 0, y = 0$ in x, y coordinates and in $x < 0, y = 0_+$ in X, Y coordinates, we estimate the total fields for $y = 0_+$ as

$$\boldsymbol{\psi}_{0+}^{tot} = \boldsymbol{\psi}_t - \boldsymbol{\psi}_{as} = \left(T_{yy}^{tot}, T_{xy}^{tot}, v_x^{tot}, v_y^{tot}\right)'. \tag{7.3}$$

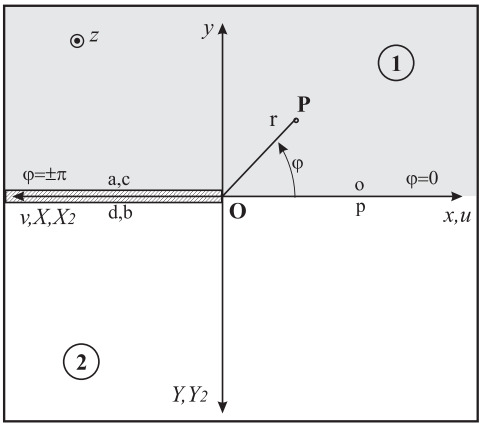

Figure 3. Half-plane planar crack problem with the reference coordinate systems and boundaries adapted from the general configuration reported in Fig. 2 ($X \equiv X_2, Y \equiv Y_2$ local face Cartesian coordinates are reported and are equal in this case due to rotation). The half crack is localized at $x<0, y=0$ and the surrounding space is divided into two rectangular regions: region 1 ($y>0$) and region 2 ($y<0$). In this section we evaluate the characteristic impedances of the half-space regions 1 and 2 that are independent from the boundary conditions on the half-plane and implicitly assume absence of sources localized at finite.

In fact, we note that the local-to-face-a X, Y coordinates have opposite direction with respect to x, y thus the velocity vectors are measured with opposite directions while the tensorial stress components have same directions because of the double inversion.

With the definition of total fields at $y=0_+$ (7.3), from (7.1) we derive expressions of $T_{yy}^{tot}, T_{xy}^{tot}$ in terms of v_x^{tot}, v_y^{tot} that in matrix form yield the matrix characteristic impedance of region 1:

$$\begin{pmatrix} T_{yy}^{tot} \\ T_{xy}^{tot} \end{pmatrix} = \mathbb{Z}_c^+ \begin{pmatrix} v_x^{tot} \\ v_y^{tot} \end{pmatrix}, \quad \mathbb{Z}_c^+ = \begin{pmatrix} \frac{\eta Z_o}{\mathrm{k_s}}\left(2-\frac{\mathrm{ks}^2}{\eta^2+\xi_p\xi_s}\right) & -\frac{\mathrm{k_s} Z_o \xi_s}{\eta^2+\xi_p\xi_s} \\ -\frac{\mathrm{k_s} Z_o \xi_p}{\eta^2+\xi_p\xi_s} & \frac{\eta Z_o}{\mathrm{k_s}}\left(\frac{\mathrm{ks}^2}{\eta^2+\xi_p\xi_s}-2\right) \end{pmatrix}. \tag{7.4}$$

Note that the definition of the characteristic impedance is independent from boundary conditions on the half-plane and implicitly assumes absence of sources localized at finite. The impedance (7.4) is validated with the admittance $\mathbb{Y}_c^+ = (\mathbb{Z}_c^+)^{-1}$ reported in (2.12.5)-(2.12.8) of [4] where, by mistake, a coefficient 2 is missing in (2.12.7) and (2.12.8). We note that while in section 2.12 of [4] the characteristic impedance is evaluated from the homogeneous solution of transverse equations in Fourier domain, in the present work we have used Laplace transforms with boundary conditions that results in a completely different and independent proof.

Now, let us consider region 2 (Fig. 3) and the related functional equations (5.18),(5.19),(5.20) and (5.17) with (4.22). To model the planar problem, we impose $\gamma=\pi$, $\alpha_o=0$ and all the continuous z components of the field $\mathbf{T}$ and $\mathbf{v}$ null: $T_{yz}=T_{YZ}=0$, $v_z=v_Z=0$. From (5.18)-(5.19) ((5.20) is trivially null in this case) we have

$$\begin{aligned} Z_o\left(v_y({k_s}^2-2\eta^2)+2\eta v_x\xi_p\right)-k_s(T_{yy}\xi_p+\eta T_{xy}) &= Z_o\left(v_Y({k_s}^2-2\eta^2)+2\eta v_X\xi_p\right)+k_s(T_{YY}\xi_p+\eta T_{XY}), \\ Z_o\left(v_x(2\eta^2-{k_s}^2)+2\eta v_y\xi_s\right)+k_s(T_{xy}\xi_s-\eta T_{yy}) &= Z_o\left(v_X(2\eta^2-{k_s}^2)+2\eta v_Y\xi_s\right)-k_s(T_{XY}\xi_s-\eta T_{YY}). \end{aligned} \tag{7.5}$$

Now let us focus the attention on the non null continuous field component of $\mathbf{T}$ and $\mathbf{v}$, we have respectively for (2.16) with (4.1) and (5.17) with (4.22):

$$\boldsymbol{\psi}_t = (T_{yy}, T_{xy}, v_x, v_y)', \ \boldsymbol{\psi}_{bs} = (T_{YY}, T_{XY}, -v_X, -v_Y)'. \tag{7.6}$$

From the definitions of $\boldsymbol{\psi}_t$ and $\boldsymbol{\psi}_{bs}$, respectively defined in $x>0, y=0$ in x, y coordinates and in $x<0, y=0_-$ in X, Y coordinates, we estimate the total fields for $y=0_-$ as

$$\boldsymbol{\psi}_{0-}^{tot} = \boldsymbol{\psi}_t + \boldsymbol{\psi}_{bs} = \left(T_{yy}^{tot}, T_{xy}^{tot}, v_x^{tot}, v_y^{tot}\right)'. \tag{7.7}$$

Due to the expressions (7.6), the total field in region 2 (7.7) show a different sign with respect to the expression of region 1 (7.3) to maintain the same physical meaning. With the definition of total fields at $y=0_-$ (7.7), from (7.5) we derive expressions of $T_{yy}^{tot}, T_{xy}^{tot}$ in terms of v_x^{tot}, v_y^{tot} that in matrix form yield the matrix characteristic impedance of region 2:

$$\begin{pmatrix} T_{yy}^{tot} \\ T_{xy}^{tot} \end{pmatrix} = \mathbb{Z}_c^- \begin{pmatrix} -v_x^{tot} \\ -v_y^{tot} \end{pmatrix}, \quad \mathbb{Z}_c^- = \begin{pmatrix} \frac{\eta Z_o}{k_s}\left(\frac{k_s{}^2}{\eta^2+\xi_p\xi_s}-2\right) & -\frac{k_s Z_o \xi_s}{\eta^2+\xi_p\xi_s} \\ -\frac{k_s Z_o \xi_p}{\eta^2+\xi_p\xi_s} & \frac{\eta Z_o}{k_s}\left(2-\frac{k_s{}^2}{\eta^2+\xi_p\xi_s}\right) \end{pmatrix}. \tag{7.8}$$

The impedance (7.8) is validated with the admittance $\mathbb{Y}_c^- = (\mathbb{Z}_c^-)^{-1}$ reported in section 12 at (2.12.5)-(2.12.8) of [4] as discussed for region 1. Note that in (7.8) we have assumed different sign in the velocity with respect to (7.4) of region 1 due to the different direction of propagation in the two regions. Finally, we recall that the method presented in this paper for the calculation of the characteristic impedances is more general and independent from the one reported in [4].

8. Remarks and Conclusions

In this work, we have introduced a general method for the deduction of spectral functional equations and thus GWHEs in angular regions filled by arbitrary linear isotropic homogeneous media in elasticity. The importance to formulate wedge problems with GWHEs in Electromagnetism has been showed in [4]- [5]. We remark that these equations are important also for elastic wedge problems. In particular the functional equations obtained and solved in [14] by Gautesen's group for the planar elastic wedge are GWHEs, although not defined in this way.

The method is based on the original solution of vector differential equations of first order via dyadic Green's function method and on the projection of this solution along the boundaries of the angular region using reciprocal vectors of the pertinent algebraic matrix related to the matrix differential operator. The application of the boundary conditions to the functional equations yields GWHEs for practical problems. We observe that the functional equations are the starting point to develop solutions using WH technique for complex scattering problems.

Using the concept of non-standard Laplace transforms (see section 1.4 of [5]), the validity of the functional equations and of the GWHEs obtained in absence of sources is extended to the total fields in presence of plane-wave sources or in general of sources located at infinity. We observe that the GWHEs in elasticity contains unknowns defined in multiple complex planes $\eta, -m_p, -m_s$ related to P and S waves and this property recall electromagnetic applications (and related solution methods) in media with multiple propagation constants as reported in [27]-[30]. In fact, in this case the reduction of GWHEs to classical WH equations is not possible. Explicit expressions of spectral functional equations in algebraic form are provided in the text in the general case of non planar elastic problems in angular regions with isotropic media and arbitrary boundary conditions and, we remark that, according to our opinion, this is the first time in literature. Validation of the GWHE formulation has been demonstrated by comparison with prestigious literature references reporting special simplified cases in anti-planar and planar problems. The paper demonstrates the flexibility and the advantages of the proposed method, based on first order differential formulation, that is useful for the analysis of complex scattering problem containing angular regions in arbitrarily linear media by changing the matrix operator defined through the fundamental matrices $\mathbb{M}_o, \mathbb{M}_1, \mathbb{M}_2$. The paper shows systematic procedural steps that can be used for arbitrary wave motion problems in different physics.

Data Accessibility. This article has no additional data.

Authors' Contributions. V.D and G.L. co-developed the mathematics, performed the numerical implementation, provided physical interpretations, produced the figures and wrote the article. Both authors approved the final version and agree to be accountable for all aspects of the work.

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Glossary

Table 2. Symbols introduced in the paper

Notation	Description
(x,y,z), (r,φ,z), (u,v,z), (X,Y,Z)	Cartesian, cylindrical, oblique Cartesian, local to face Cartesian coordinates
A, $\mathbf{A}$, $\underline{A}$, $\mathbb{A}$, $\mathcal{A}(\cdot,\cdot)$	scalar, column vector, dyadic, matrix, matrix differential operator
k_p, k_s	propagation constants of P and S waves
$\underline{T}$ ($\mathbf{T}$), $\underline{S}$ ($\mathbf{S}$)	stress tensor (Voigt notation), strain tensor (Voigt notation)
$\mathbf{p}$, $\mathbf{v}$	vector momentum density, vector particle velocity
ρ, λ, μ	material density and Lame's constants
$\underline{\underline{C}}$	Hooke's law as fourth order stiffness tensor
∇_T, ∇_v, $\mathit{\Gamma}_\nabla$	matrix differential operators
$\boldsymbol{\psi}$, θ	vector fields in abstract notation
$\mathbb{W}$	matrix constitutive parameters of media
$\boldsymbol{\psi}_t$	transverse field for a stratification along the y direction
$\mathcal{M}(\frac{\partial}{\partial z},\frac{\partial}{\partial x})$	transversal matrix differential operator for elastic equations
$D_x=\frac{\partial}{\partial x}$	alternative partial derivative notation
α_o	field dependence specified by the factor $e^{-j\alpha_o z}$ due to invariance along z
η	Fourier or Laplace spectral variable according to the position on the text
$\boldsymbol{\Psi}_t(\eta)$	Fourier transform along $x=u$ direction (y, z or v, z dependence is omitted)
$\mathbb{M}(\eta)$	matrix operator in Fourier/Laplace domain in indefinite rectangular region
λ_i, $\mathbf{u}_i$, $\boldsymbol{\nu}_i$	eigenvalues, eigenvector and reciprocal vectors of $\mathbb{M}(\eta)$
ξ_i	different notation of λ_i for propagation's properties, multivalued function
γ	aperture angle of angular regions (Fig. 1)
$\mathbb{M}_e(\gamma,\eta)$	matrix operator in Fourier/Laplace domain in indefinite angular region
λ_{ei}	eigenvalues of $\mathbb{M}_e(\gamma,\eta)$
$\tilde{\boldsymbol{\psi}}_t(\eta,v)$	Laplace transform along $x\equiv u$ of $\boldsymbol{\psi}_t(u,v)$ (omitting z dependence)
$\boldsymbol{\psi}_s(v)$	field components on the face of an angular region in Laplace domain
$\boldsymbol{\psi}_{as}(v)$, $\breve{\boldsymbol{\psi}}_{as}(\chi)$	specialized expression of $\boldsymbol{\psi}_s(v)$ on face a and its Laplace transform
$\underline{G}(v,v')$	dyadic Green's function in Laplace domain for an angular region
m_{ai}	spectral variable for the evaluation of $\breve{\boldsymbol{\psi}}_{as}(\chi)$ along face a in functional eqs