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Real rank geometry of ternary forms

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Abstract We study real ternary forms whose real rank equals the generic complex rank, and we characterize the semialgebraic set of sums of powers representations with that rank. Complete results are obtained for quadrics and cubics. For quintics, we determine the real rank boundary: It is a hypersurface of degree 168. For quartics, sextics and septics, we identify some of the components of the real rank boundary. The real varieties of sums of powers are stratified by discriminants that are derived from hyperdeterminants.

Keywords Real rank · Ternary form · Discriminant

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1 Introduction

Let $\mathbb{R}[x, y, z]_d$ denote the $\binom{d+2}{2}$-dimensional vector space of ternary forms f of degree d. These are homogeneous polynomials of degree d in three unknowns x, y, z, or equivalently,

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symmetric tensors of format $3\times 3\times\cdots\times 3$ with d factors. We are interested in decompositions

$$f(x, y, z) = \sum_{i=1}^{r} \lambda_i \cdot (a_i x + b_i y + c_i z)^d, \tag{1}$$

where $a_i, b_i, c_i, \lambda_i \in \mathbb{R}$ for $i = 1, 2, \ldots, r$. The smallest r for which such a representation exists is the *real rank* of f, denoted $\mathrm{rk}_{\mathbb{R}}(f)$. For d even, the representation (1) has *signature* $(s, r-s)$, for $s \geq r/2$, if s of the λ_i's are positive while the others are negative, or vice versa. The *complex rank* $\mathrm{rk}_{\mathbb{C}}(f)$ is the smallest r such that f has the form (1) where $a_i, b_i, c_i \in \mathbb{C}$. The inequality $\mathrm{rk}_{\mathbb{C}}(f) \leq \mathrm{rk}_{\mathbb{R}}(f)$ always holds and is often strict. For binary forms, this phenomenon is well understood by now, thanks to [7,16]. For ternary forms, explicit regions where the inequality is strict were identified by Blekherman, Bernardi and Ottaviani in [6].

The present paper extends these studies. We focus on ternary forms f that are general in $\mathbb{R}[x, y, z]_d$. The complex rank of such a form is referred to as the *generic rank*. It depends only on d, and we denote it by $R(d)$. The Alexander–Hirschowitz Theorem [2] implies that

$$R(2) = 3,\ \ R(4) = 6,\ \ \text{and}\quad R(d) = \left\lceil \frac{(d+2)(d+1)}{6} \right\rceil \ \text{otherwise.} \tag{2}$$

We are particularly interested in general forms whose minimal decomposition is real. Set

$$\mathcal{R}_d \ = \ \left\{ f \in \mathbb{R}[x, y, z]_d \ : \ \mathrm{rk}_{\mathbb{R}}(f) = R(d) \right\}.$$

This is a full-dimensional semialgebraic subset of $\mathbb{R}[x, y, z]_d$. Its *topological boundary* $\partial\mathcal{R}_d$ is the set-theoretic difference of the closure of $\mathcal{R}_d$ minus the interior of the closure of $\mathcal{R}_d$. Thus, if $f \in \partial\mathcal{R}_d$ then every open neighborhood of f contains a general form of real rank equal to $R(d)$ and also a general form of real rank bigger than $R(d)$. The semialgebraic set $\partial\mathcal{R}_d$ is either empty or pure of codimension 1. The *real rank boundary*, denoted $\partial_{\mathrm{alg}}(\mathcal{R}_d)$, is defined as the Zariski closure of the topological boundary $\partial\mathcal{R}_d$ in the complex projective space $\mathbb{P}(\mathbb{C}[x, y, z]_d) = \mathbb{P}^{\binom{d+2}{2}-1}$. We conjecture that the variety $\partial_{\mathrm{alg}}(\mathcal{R}_d)$ is non-empty and hence has codimension 1, for all $d \geq 4$. This is equivalent to $R(d) + 1$ being a typical rank, in the sense of [6,7,16]. This is proved for $d = 4, 5$ in [6] and for $d = 6, 7, 8$ in this paper.

Our aim is to study these hypersurfaces. The big guiding problem is as follows:

Problem 1.1 Determine the polynomial that defines the real rank boundary $\partial_{\mathrm{alg}}(\mathcal{R}_d)$.

The analogous question for binary forms was answered in [25, Theorem 4.1]. A related and equally difficult issue is to identify all the various open strata in the real rank stratification.

Problem 1.2 Determine the possible real ranks of general ternary forms in $\mathbb{R}[x, y, z]_d$.

This problem is open for $d \geq 4$; the state of the art is the work of Bernardi, Blekherman and Ottaviani in [6]. For binary forms, this question has a complete answer, due to Blekherman [7], building on earlier work of Comon and Ottaviani [16]. See also [25, §4].

For any ternary form f and the generic rank $r = R(d)$, it is natural to ask for the space of all decompositions (1). In the algebraic geometry literature [29,33], this space is denoted $\mathrm{VSP}(f)$ and called the *variety of sums of powers*. By definition, $\mathrm{VSP}(f)$ is the closure of the subscheme of the Hilbert scheme $\mathrm{Hilb}_r(\mathbb{P}^2)$ parametrizing the unordered configurations

$$\left\{(a_1 : b_1 : c_1),\ (a_2 : b_2 : c_2),\ \ldots,\ (a_r : b_r : c_r)\right\} \ \subset \ \mathbb{P}^2 \tag{3}$$

Table 1 Varieties of sums of powers for ternary forms of degree $d = 2, 3, 4, 5, 6, 7, 8$

Ternary forms	$R(d)$	$\mathrm{VSP}(f)$	References
Quadrics	3	del Pezzo threefold V_5	Mukai [27]
Cubics	4	$\mathbb{P}^2$	Dolgachev and Kanev [19]
Quartics	6	Fano threefold V_{22} of genus 12	Mukai [27]
Quintics	7	1 point	Hilbert, Richmond, Palatini, see [33]
Sextics	10	$K3$ surface V_{38} of genus 20	Mukai [28], see also [33]
Septics	12	5 points	Dixon and Stuart [18]
Octics	15	16 points	Ranestad and Schreyer [33]

that can occur in (1). If f is general then the dimension of its variety of sums of powers depends only on d. By counting parameters, the Alexander–Hirschowitz Theorem [2] implies

$$\dim(\mathrm{VSP}(f)) \;=\; \begin{cases} 3 & \text{if } d = 2 \text{ or } 4, \\ 2 & \text{if } d = 0 \pmod 3, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In Table 1, we summarize what is known about the varieties of sums of powers. In two-thirds of all cases, the variety $\mathrm{VSP}(f)$ is finite. It is one point only in the case of quintics, by [26].

We are interested in the semialgebraic subset $\mathrm{SSP}(f)_{\mathbb{R}}$ of those configurations (3) in $\mathrm{VSP}(f)$ whose r points all have real coordinates. This is the *space of real sums of powers*. Note that the space $\mathrm{SSP}(f)_{\mathbb{R}}$ is non-empty if and only if the ternary form f lies in the semialgebraic set $\mathcal{R}_d$. The inclusion of $\overline{\mathrm{SSP}(f)_{\mathbb{R}}}$ in the real variety $\mathrm{VSP}(f)_{\mathbb{R}}$ of real points of $\mathrm{VSP}(f)$ is generally strict. Our aim is to describe these objects as explicitly as possible.

A key player is the *apolar ideal* of the form f. This is the 0-dimensional Gorenstein ideal

$$f^{\perp} = \Big\{ p(x, y, z) \in \mathbb{R}[x, y, z] \;:\; p\big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\big) \text{ annihilates } f(x, y, z) \Big\}. \tag{5}$$

A configuration (3) lies in $\mathrm{VSP}(f)$ if and only if its homogeneous radical ideal is contained in $f^{\perp}$. Hence, points in $\mathrm{SSP}(f)_{\mathbb{R}}$ are 1-dimensional radical ideals in $f^{\perp}$ whose zeros are real.

Another important tool is the middle catalecticant of f, which is defined as follows. For any partition $d = u+v$, consider the bilinear form $C_{u,v}(f) : \mathbb{R}[x, y, z]_u \times \mathbb{R}[x, y, z]_v \to \mathbb{R}$ that maps (p, q) to the real number obtained by applying $(p\cdot q)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ to the polynomial f. We identify $C_{u,v}(f)$ with the matrix that represents the bilinear form with respect to the monomial basis. The *middle catalecticant* $C(f)$ of the ternary form f is precisely that matrix, where we take $u = v = d/2$ when d is even, and $u = (d-1)/2, v = (d+1)/2$ when d is odd. The hypothesis $d \in \{2, 4, 6, 8\}$ ensures that $C(f)$ is square of size equal to $R(d) = \binom{d/2+2}{2}$.

Proposition 1.3 *Let $d \in \{2, 4, 6, 8\}$ and $f \in \mathbb{R}[x, y, z]_d$ be general. The signature of any representation* (1) *coincides with the signature of the middle catalecticant $C(f)$. If $C(f)$*

is positive definite then $\overline{\mathrm{SSP}(f)_{\mathbb{R}}} = \mathrm{VSP}(f)_{\mathbb{R}}$, *and this set is always non-empty provided* $d \leq 4$.

Proof If $f = \sum_{i=1}^{r} \lambda_i \ell_i^d$ as in (1) then $C(f)$ is the sum of the rank one matrices $\lambda_i C(\ell_i^d)$. If $C(f)$ has rank r then its signature is (# positive λ_i, # negative λ_i). The identity $\overline{\mathrm{SSP}(f)_{\mathbb{R}}} = \mathrm{VSP}(f)_{\mathbb{R}}$ will be proved for $d = 2$ in Theorem 2.1, and the same argument works for $d = 4, 6, 8$ as well. The last assertion, for $d \leq 4$, is due to Reznick [35, Theorem 4.6]. □

The structure of the paper is organized by increasing degrees: Section d is devoted to ternary forms of degree d. In Section 2, we determine the threefolds $\mathrm{SSP}(f)_{\mathbb{R}}$ for quadrics, and in Section 3 we determine the surfaces $\mathrm{SSP}(f)_{\mathbb{R}}$ for cubics. Theorem 3.1 summarizes the four cases displayed in Table 2. Section 4 is devoted to quartics f and their real rank boundaries. We present an algebraic characterization of $\mathrm{SSP}(f)_{\mathbb{R}}$ as a subset of Mukai's Fano threefold V_{22}, following [24,27,29,33,36]. In Section 5, we use the uniqueness of the rank 7 decomposition of quintics to determine the irreducible hypersurface $\partial_{\mathrm{alg}}(\mathcal{R}_5)$. We also study the case of septics ($d = 7$), and we discuss VSP_X for arbitrary varieties $X \subset \mathbb{P}^N$. Finally, Section 6 covers all we know about sextics, starting in Theorem 6.1 with a huge component of the boundary $\partial_{\mathrm{alg}}(\mathcal{R}_6)$, and concluding with a case study of the monomial $f = x^2y^2z^2$.

This paper contains numerous open problems and conjectures. We are fairly confident about some of them (like the one stated prior to Problem 1.1). However, others (like Conjectures 4.3 and 5.5) are based primarily on optimism. We hope that all will be useful in inspiring further progress on the real algebraic geometry of tensor decompositions.

2 Quadrics

The real rank geometry of quadratic forms is surprisingly delicate and interesting. Consider a general real quadric f in n variables. We know from linear algebra that $\mathrm{rk}_{\mathbb{R}}(f) = \mathrm{rk}_{\mathbb{C}}(f) = n$. More precisely, if (p, q) is the *signature* of f then, after a linear change of coordinates,

$$f = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad (n = p + q). \tag{6}$$

The stabilizer of f in $\mathrm{GL}(n, \mathbb{R})$ is denoted $\mathrm{SO}(p, q)$. It is called the *indefinite special orthogonal group* when $p, q \geq 1$. We denote by $\mathrm{SO}^+(p, q)$ the connected component of $\mathrm{SO}(p, q)$ containing the identity. Let G denote the stabilizer in $\mathrm{SO}^+(p, q)$ of the set $\big\{\{x_1^2, \ldots, x_p^2\}, \{x_{p+1}^2, \ldots, x_n^2\}\big\}$. In particular, if f is positive definite then we get the group of rotations, $\mathrm{SO}^+(n, 0) = \mathrm{SO}(n)$, and G is the subgroup of rotational symmetries of the n-cube, which has order $2^{n-1}n!$.

Theorem 2.1 *Let f be a rank n quadric of signature (p, q). The space* $\mathrm{SSP}(f)_{\mathbb{R}}$ *can be identified with the quotient* $\mathrm{SO}^+(p, q)/G$. *If the quadric f is definite then* $\mathrm{SSP}(f)_{\mathbb{R}} = \mathrm{VSP}(f)_{\mathbb{R}} = \mathrm{SO}(n)/G$. *In all other cases,* $\overline{\mathrm{SSP}(f)_{\mathbb{R}}}$ *is strictly contained in the real variety* $\mathrm{VSP}(f)_{\mathbb{R}}$.

Proof The analogue of the first assertion over an algebraically closed field appears in [34, Proposition 1.4]. To prove $\mathrm{SSP}(f)_{\mathbb{R}} = \mathrm{SO}^+(p, q)/G$ over $\mathbb{R}$, we argue as follows. Every rank n decomposition of f has the form $\sum_{i=1}^{p} \ell_i^2 - \sum_{j=p+1}^{p+q} \ell_j^2$ and is hence obtained from (6) by an invertible linear transformation $x_j \to \ell_j$ that preserves f. These elements of $\mathrm{GL}(n, \mathbb{R})$ are taken up to sign reversals and permutations of the sets $\{\ell_1, \ldots, \ell_p\}$ and $\{\ell_{p+1}, \ldots, \ell_n\}$.

Suppose that f is not definite, i.e., $p, q \geq 1$. Then we can write $f = 2\ell_1^2 - 2\ell_2^2 + \sum_{j=3}^{n} \pm \ell_j^2$. Over $\mathbb{C}$, with $i = \sqrt{-1}$, this can be rewritten as $f = (\ell_1 + i\ell_2)^2 + (\ell_1 - i\ell_2)^2 +$

Table 2 Four possible types of a real cubic f of form (15) and its quadratic map $F : \mathbb{P}^2 \to \mathbb{P}^2$

	$\lambda < -3$	$-3 < \lambda < 0$	$0 < \lambda < 6$	$6 < \lambda$
f	Hyperbolic	Not hyperbolic	Not hyperbolic	Not hyperbolic
$H(f)$	Not hyperbolic	Hyperbolic	Hyperbolic	Hyperbolic
$C(f)$	Hyperbolic	Hyperbolic	Not hyperbolic	Hyperbolic
$\#F^{-1}(\bullet)_{\mathbb{R}}$	4, 2, 0	4, 2, 4	4, 2	4, 2, 0
$\mathrm{SSP}(f)_{\mathbb{R}}$	Disk	Disk $\sqcup$ Möbius strip	Disk	Disk
Oriented matroid	$(+,+,+,+)$	$(+,+,+,+) \sqcup (+,+,-,-)$	$(+,+,+,-)$	$(+,+,+,+)$

$\sum_{j=3}^{n} \pm\ell_j^2$. This decomposition represents a point in $\mathrm{VSP}(f)_{\mathbb{R}} \backslash \overline{\mathrm{SSP}(f)_{\mathbb{R}}}$. There is an open set of such points.

Let f be definite and consider any point in $\mathrm{VSP}(f)_{\mathbb{R}}$. It corresponds to a decomposition

$$f = \sum_{j=1}^{k} \big((a_{2j-1}+ib_{2j-1})(\ell_{2j-1}+i\ell_{2j})^2 + (a_{2j}+ib_{2j})(\ell_{2j-1}-i\ell_{2j})^2\big) \; + \sum_{j=2k+1}^{n} c_j \ell_j^2,$$

where $\ell_1, \ldots, \ell_n$ are independent real linear forms and the a's and b's are in $\mathbb{R}$. By rescaling ℓ_{2j} and ℓ_{2j-1}, we obtain $a_{2j-1} + ib_{2j-1} = 1$. Adding the right-hand side to its complex conjugate, we get $c_j \in \mathbb{R}$ and $a_{2j} + ib_{2j} = 1$. The catalecticant $C(f)$ is the matrix that represents f. A change of basis shows that $C(f)$ has $\geq k$ negative eigenvalues, hence $k = 0$. □

The geometry of the inclusion $\mathrm{SSP}(f)_{\mathbb{R}}$ into $\mathrm{VSP}(f)_{\mathbb{R}}$ is already quite subtle in the case of binary forms, i.e., $n = 2$. We call $f = a_0x^2 + a_1xy + a_2y^2$ *hyperbolic* if its signature is $(1, 1)$. Otherwise f is definite. These two cases depend on the sign of the discriminant $a_0a_2 - 4a_1^2$.

Corollary 2.2 *Let f be a binary quadric of rank 2. If f is definite then* $\mathrm{SSP}(f)_{\mathbb{R}} = \mathrm{VSP}(f)_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{R}}$. *If f is hyperbolic then* $\mathrm{SSP}(f)_{\mathbb{R}}$ *is an interval in the circle* $\mathrm{VSP}(f)_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{R}}$.

Proof The apolar ideal $f^{\perp}$ is generated by two quadrics q_1, q_2 in $\mathbb{R}[x, y]_2$. Their pencil $\mathbb{P}(f_2^{\perp})$ is $\mathrm{VSP}(f) \simeq \mathbb{P}^1$. A real point $(u : v) \in \mathbb{P}^1_{\mathbb{R}} = \mathrm{VSP}(f)_{\mathbb{R}}$ may or may not be in $\mathrm{SSP}(f)_{\mathbb{R}}$. The fibers of the map $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$ given by (q_1, q_2) consist of two points, corresponding to the decompositions $f = \ell_1^2 \pm \ell_2^2$. The fiber over $(u : v)$ consists of the roots of the quadric $uq_2 - vq_1$. If f is definite, then both roots are always real. Otherwise the discriminant with respect to (x, y), which is a quadric in (u, v), divides $\mathbb{P}^1_{\mathbb{R}}$ into $\mathrm{SSP}(f)_{\mathbb{R}}$ and its complement. □

Example 2.3 Fix the hyperbolic quadric $f = x^2 - y^2$. We take $q_1 = xy$ and $q_2 = x^2 + y^2$. The quadric $uq_2 - vq_1 = u(x^2+y^2) - vxy$ has two real roots if and only if $(2u-v)(2u+v) < 0$. Hence $\mathrm{SSP}(f)_{\mathbb{R}}$ is the interval in $\mathbb{P}^1_{\mathbb{R}}$ defined by $-1/2 < u/v < 1/2$. In the topological description in Theorem 2.1, the group G is trivial, and $\mathrm{SSP}(f)_{\mathbb{R}}$ is identified with the group

$$\mathrm{SO}^+(1,1) \quad = \quad \left\{ \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \; : \; \alpha \in \mathbb{R} \right\}.$$

The homeomorphism between $\mathrm{SO}^+(1,1)$ and the interval between $-1/2$ and $1/2$ is given by

$$\alpha \;\mapsto\; \frac{u}{v} \;=\; \frac{\cosh(\alpha)\cdot\sinh(\alpha)}{\cosh(\alpha)^2+\sinh(\alpha)^2}.$$

The resulting factorization $u(x^2+y^2) - vxy = \big(\sinh(\alpha)x - \cosh(\alpha)y\big)\big(\cosh(\alpha)x - \sinh(\alpha)y\big)$ yields the decomposition $f = \big(\cosh(\alpha)x + \sinh(\alpha)y\big)^2 - \big(\sinh(\alpha)x + \cosh(\alpha)y\big)^2$.

It is instructive to examine the topology of the family of curves $\mathrm{SSP}(f)_{\mathbb{R}}$ as f runs over the projective plane $\mathbb{P}^2_{\mathbb{R}} = \mathbb{P}(\mathbb{R}[x,y]_2)$. This plane is divided by an oval into two regions:

(i) the interior region $\{a_0a_2 - 4a_1^2 < 0\}$ is a disk, and it parametrizes the definite quadrics;
(ii) the exterior region $\{a_0a_2 - 4a_1^2 > 0\}$ is a Möbius strip, consisting of hyperbolic quadrics.

Over the disk, the circles $\mathrm{VSP}(f)_{\mathbb{R}}$ provide a trivial $\mathbb{P}^1_{\mathbb{R}}$–fibration. Over the Möbius strip, there is a twist. Namely, if we travel around the disk, along an $\mathbb{S}^1$ in the Möbius strip, then the two endpoints of $\mathrm{SSP}(f)_{\mathbb{R}}$ get switched. Hence, here we get the twisted circle bundle.

The topic of this paper is ternary forms, so we now fix $n = 3$. A real ternary form of rank 3 is either definite or hyperbolic. In the definite case, the normal form is $f = x^2 + y^2 + z^2$, and $\mathrm{SSP}(f)_{\mathbb{R}} = \mathrm{VSP}(f)_{\mathbb{R}} = \mathrm{SO}(3)/G$, where G has order 24. In the hyperbolic case, the normal form is $f = x^2 + y^2 - z^2$, and $\overline{\mathrm{SSP}(f)_{\mathbb{R}}} \subsetneq \mathrm{VSP}(f)_{\mathbb{R}} = \overline{\mathrm{SO}^+(2,1)/G}$, where G has order 4. These spaces are three-dimensional, and they sit inside the complex Fano threefold V_5, as seen in Table 1. We follow [29,34] in developing our algebraic approach to $\mathrm{SSP}(f)_{\mathbb{R}}$. This sets the stage for our study of ternary forms of degree $d \geq 4$ in the later sections.

Fix $S = \mathbb{R}[x,y,z]$ and $f \in S_2$ a quadric of rank 3. The apolar ideal $f^\perp \subset S$ is artinian, Gorenstein, and it has five quadratic generators. Its minimal free resolution has the form

$$0 \longrightarrow S(-5) \longrightarrow S(-3)^5 \xrightarrow{A} S(-2)^5 \longrightarrow S \longrightarrow 0. \tag{7}$$

By the Buchsbaum-Eisenbud structure theorem, we can choose bases so that the matrix A is skew-symmetric. The entries are linear, so $A = xA_1 + yA_2 + zA_3$ where A_1, A_2, A_3 are real skew-symmetric 5×5-matrices. More invariantly, the matrices A_1, A_2, A_3 lie in $\bigwedge^2 f_3^\perp \simeq \mathbb{R}^{10}$. The five quadratic generators of the apolar ideal $f^\perp$ are the 4×4-sub-Pfaffians of A.

The three points $(a_i : b_i : c_i)$ in a decomposition (1) are defined by three of the five quadrics. Hence, $\mathrm{VSP}(f)$ is identified with a subvariety of the Grassmannian $\mathrm{Gr}(3,5)$, defined by the condition that the three quadrics are the minors of a 2×3 matrix with linear entries. Equivalently, the chosen three quadrics need to have two linear syzygies. After taking a set of five minimal generators of $f^\perp$ containing three such quadrics, the matrix A has the form

$$A = \begin{pmatrix} \star & T \\ -T^t & 0 \end{pmatrix}. \tag{8}$$

Here, 0 is the zero 2×2 matrix and T is a 3×2 matrix of linear forms. The 2×2-minors of T—which are also Pfaffians of A—are the three quadrics defining the points $(a_i : b_i : c_i)$.

Proposition 2.4 *The threefold* $\mathrm{VSP}(f)$ *is the intersection of the Grassmannian* $\mathrm{Gr}(3,5)$*, in its Plücker embedding in* $\mathbb{P}(\bigwedge^3 f_3^\perp) \simeq \mathbb{P}^9$*, with the* 6*-dimensional linear subspace*

$$\mathbb{P}^6_A = \big\{ U \in \mathbb{P}^9 \ : \ U \wedge A_1 = U \wedge A_2 = U \wedge A_3 = 0 \big\}. \tag{9}$$

Proof This fact was first observed by Mukai [27]. See also [33, §1.5]. If $U = u_1 \wedge u_2 \wedge u_3$ lies in this intersection then the matrix A has the form (8) for any basis that contains u_1, u_2, u_3. □

Note that any general codimension 3 linear section of $\mathrm{Gr}(3, 5)$ arises in this manner. In other words, we can start with three skew-symmetric 5×5-matrices A_1, A_2, A_3 and obtain $\mathrm{VSP}(f) = \mathrm{Gr}(3, 5) \cap \mathbb{P}^6_A$ for a unique quadratic form f. In algebraic geometry, this Fano threefold is denoted V_5. It has degree 5 in $\mathbb{P}^9$ and is known as the *quintic del Pezzo threefold*.

Our space $\mathrm{SSP}(f)_{\mathbb{R}}$ is a semialgebraic subset of the real Fano threefold $\mathrm{VSP}(f)_{\mathbb{R}} \subset \mathbb{P}^9_{\mathbb{R}}$. If f is hyperbolic, then the inclusion is strict. We now extend Example 2.3 to this situation.

Example 2.5 We shall compute the algebraic representation of $\mathrm{SSP}(f)_{\mathbb{R}}$ for $f = x^2+y^2-z^2$. The apolar ideal $f^{\perp}$ is generated by the 4×4 Pfaffians of the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & x & -y & z & 0 \\ -x & 0 & -z & y & -y \\ y & z & 0 & 0 & -x \\ -z & -y & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 \end{pmatrix} = x(e_{12}-e_{35})-y(e_{13}-e_{24}+e_{25})+z(e_{14}-e_{23}). \quad (10)$$

Here $e_{ij} = e_i \wedge e_j$. This is in the form (8). We fix affine coordinates on $\mathrm{Gr}(3, 5)$ as follows:

$$U \;=\; \text{rowspan of} \begin{pmatrix} 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & c & d \\ 0 & 0 & 1 & e & g \end{pmatrix}. \quad (11)$$

If we write p_{ij} for the signed 3×3-minors obtained by deleting columns i and j from this 3×5-matrix, then we see that $\mathrm{VSP}(f) = \mathbb{P}^6_A \cap \mathrm{Gr}(3, 5)$ is defined by the affine equations

$$\begin{aligned} p_{12} - p_{35} &= ad - bc + e = 0, \\ p_{13} - p_{24} + p_{25} &= be - ag + d + c = 0, \\ p_{14} - p_{23} &= b + de - cg = 0. \end{aligned} \quad (12)$$

We now transform (10) into the coordinate system given by U and its orthogonal complement:

$$\begin{pmatrix} \star & T \\ -T^t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & c & d \\ 0 & 0 & 1 & e & g \\ a & c & e & -1 & 0 \\ b & d & g & 0 & -1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & c & d \\ 0 & 0 & 1 & e & g \\ a & c & e & -1 & 0 \\ b & d & g & 0 & -1 \end{pmatrix}. \quad (13)$$

The lower right 2×2-block is zero precisely when (12) holds. The upper right block equals

$$T \;=\; \begin{pmatrix} (be+c)x + (-ac+bc-e)y - (a^2+1)z & (bg+d)x + (-ad+bd-g)y - abz \\ (de-a)x + (-c^2+cd-1)y - (ac+e)z & (dg-b)x + (-cd+d^2+1)y - (bc+g)z \\ egx + (-ce+cg+a)y - (ae-c)z & (g^2+1)x + (-de+dg+b)y - (be-d)z \end{pmatrix}.$$

Writing $T = xT_1 + yT_2 + zT_3$, we regard T as a $2 \times 3 \times 3$ tensor with slices T_1, T_2, T_3 whose entries are quadratic polynomials in a, b, c, d, e, g. The *hyperdeterminant* of that tensor equals

$$\begin{aligned} \mathrm{Det}(T) &= \mathrm{discr}_w\big(\mathrm{Jac}_{x,y,z}(\, T \cdot \tbinom{1}{w}\,)\big) \qquad \text{(by Schläfli's formula [31, §5])} \\ &= 27a^8c^2d^6g^4+54a^8c^2d^4g^6+27a^8c^2d^2g^8+\cdots-4d^2+2e^2-6eg-8g^2-1. \end{aligned} \quad (14)$$

In general, the expected degree of the hyperdeterminant of this form is 24. In this case, after some cancellations occur, this is a polynomial in 6 variables of degree 20 with 13956 terms. Now, consider any real point (a, b, c, d, e, g) that satisfies (12). The 2×2-minors of T define three points ℓ_1, ℓ_2, ℓ_3 in the complex projective plane $\mathbb{P}^2$. These three points are all real if and only if $\mathrm{Det}(T) < 0$.

Our derivation establishes the following result for the hyperbolic quadric $f = x^2+y^2-z^2$. The solutions of (12) correspond to the decompositions $f = \ell_1^2+\ell_2^2-\ell_3^2$, as described above.

Corollary 2.6 *In affine coordinates on the Grassmannian* $\mathrm{Gr}(3, 5)$*, the real threefold* $\mathrm{VSP}(f)_{\mathbb{R}}$ *is defined by the quadrics* (12)*. The affine part of* $\mathrm{SSP}(f)_{\mathbb{R}} \simeq \mathrm{SO}^+(2, 1)/G$ *is the semialgebraic subset of points* $(a, \ldots, e, g)$ *at which the hyperdeterminant* $\mathrm{Det}(T)$ *is negative.*

We close this section with an interpretation of hyperdeterminants (of next-to-boundary format) as *Hurwitz forms* [38]. This will be used in later sections to generalize Corollary 2.6.

Proposition 2.7 *The hyperdeterminant of format* $m \times n \times (m+n-2)$ *equals the Hurwitz form (in dual Stiefel coordinates) of the variety of* $m \times (m+n-2)$*-matrices of rank* $\leq m-1$*. The maximal minors of such a matrix whose entries are linear forms in n variables define* $\binom{m+n-2}{n-1}$ *points in* $\mathbb{P}^{n-1}$*, and the above hyperdeterminant vanishes when two points coincide.*

Proof Let X be the variety of $m \times (m+n-2)$-matrices of rank $\leq m-1$. By [22, Theorem 3.10, Section 14.C], the Chow form of X equals the hyperdeterminant of boundary format $m \times n \times (m+n-1)$. The derivation can be extended to next-to-boundary format, and it shows that the $m \times n \times (m+n-2)$ hyperdeterminant is the Hurwitz form of X. The case $m = n = 3$ is worked out in [38, Example 4.3]. □

In this paper, we are concerned with the case $n = 3$. In Corollary 2.6 we took $m = 2$.

Corollary 2.8 *The hyperdeterminant of format* $3 \times m \times (m+1)$ *is an irreducible homogeneous polynomial of degree* $12\binom{m+1}{3}$*. It serves as the discriminant for ideals of* $\binom{m+1}{2}$ *points in* $\mathbb{P}^2$*.*

Proof The formula $12\binom{m+1}{3}$ is derived from the generating function in [22, Theorem 14.2.4], specialized to 3-dimensional tensors in [31, §4]. For the geometry see [31, Theorem 5.1]. □

3 Cubics

The case $d = 3$ was studied by Banchi [4]. He gave a detailed analysis of the real ranks of ternary cubics $f \in \mathbb{R}[x, y, z]_3$ with focus on the various special cases. In this section, we consider a general real cubic f. We shall prove the following result on its real decompositions.

Theorem 3.1 *The semialgebraic set* $\mathrm{SSP}(f)_{\mathbb{R}}$ *is either a disk in the real projective plane or a disjoint union of a disk and a Möbius strip. The two cases are characterized in Table 2. The algebraic boundary of* $\mathrm{SSP}(f)_{\mathbb{R}}$ *is an irreducible sextic curve that has nine cusps.*

Our point of departure is the following fact which is well known, e.g., from [4, §5] or [6].

Proposition 3.2 *The real rank of a general ternary cubic is* $R(3) = 4$*, so it agrees with the complex rank. Hence, the closure of* $\mathcal{R}_3$ *is all of* $\mathbb{R}[x, y, z]_3$*, and its boundary* $\partial\mathcal{R}_3$ *is empty.*

Proof Every smooth cubic curve in the complex projective plane $\mathbb{P}^2$ can be transformed, by an invertible linear transformation $\tau \in \mathrm{PGL}(3, \mathbb{C})$, into the *Hesse normal form* (cf. [3]):

$$f \;=\; x^3 + y^3 + z^3 + \lambda xyz. \tag{15}$$

Suppose that the given cubic is defined over $\mathbb{R}$. It is known classically that the matrix τ can be chosen to have real entries. In particular, the parameter λ will be real. Also, $\lambda \neq -3$; otherwise, the curve would be singular. Banchi [4] observed that $24(\lambda+3)^2 f$ is equal to

$$\begin{aligned}&\big[(6+\lambda)x - \lambda y - \lambda z\big]^3 + \big[(6+\lambda)y - \lambda x - \lambda z\big]^3 + \big[(6+\lambda)z - \lambda x - \lambda y\big]^3\\ &\quad + \lambda(\lambda^2 + 6\lambda + 36)\big[x+y+z\big]^3.\end{aligned}$$

By applying τ^{-1}, one obtains the decomposition for the original cubic. The entries of the transformation matrix $\tau \in \mathrm{PGL}(3, \mathbb{R})$ can be written in radicals in the coefficients of f. The corresponding Galois group is solvable and has order 432. It is the automorphism group of the Hesse pencil; see e.g., [3, Remark 4.2] or [15, Section 2]. □

Remark 3.3 The Hesse normal form (15) is well suited for this real structure. For any fixed isomorphism class of a real elliptic curve over $\mathbb{C}$, there are two isomorphism classes over $\mathbb{R}$, by [37, Proposition 2.2]. We see this by considering the j-invariant of the Hesse curve:

$$j(f) = -\frac{\lambda^3(\lambda-6)^3\left(\lambda^2+6\lambda+36\right)^3}{(\lambda+3)^3\left(\lambda^2-3\lambda+9\right)^3}. \tag{16}$$

For a fixed real value of $j(f)$, this equation has two real solutions λ_1 and λ_2. These two elliptic curves are isomorphic over $\mathbb{C}$ but not over $\mathbb{R}$. They are distinguished by the sign of the degree 6 invariant T of ternary cubics, which takes the following value for the Hesse curve:

$$T(f) = 1 - \frac{4320\lambda^3 + 8\lambda^6}{6^6}. \tag{17}$$

If $T(f) = 0$ then the two curves differ by the sign of the Aronhold invariant. This proves that any real smooth cubic is isomorphic over $\mathbb{R}$ to exactly one element of the Hesse pencil.

An illustrative example is the Fermat curve $x^3 + y^3 + z^3$. It is unique over $\mathbb{C}$, but it has two distinct real models, corresponding to $\lambda = 0$ or 6. The case $\lambda = 6$ is isomorphic over $\mathbb{R}$ to $g = x^3 + (y+iz)^3 + (y-iz)^3$. This real cubic satisfies $\mathrm{rk}_{\mathbb{C}}(g) = 3$ but $\mathrm{rk}_{\mathbb{R}}(g) = 4$. Here, the real surface $\mathrm{VSP}(g)_{\mathbb{R}}$ is non-empty, but its semialgebraic subset $\mathrm{SSP}_{\mathbb{R}}(g)$ is empty.

We now construct the isomorphism $\mathrm{VSP}(f) \simeq \mathbb{P}^2$ for ternary cubics f as shown in Table 1. The apolar ideal $f^\perp$ is a complete intersection generated by three quadrics q_0, q_1, q_2. We denote this *net* of quadrics by $f_2^\perp$. Conversely, any such complete intersection determines a unique cubic f. The linear system $f_2^\perp$ defines a branched $4:1$ covering of projective planes:

$$F : \mathbb{P}^2 \to \mathbb{P}^2,\ (x:y:z) \mapsto \big(q_0(x,y,z) : q_1(x,y,z) : q_2(x,y,z)\big).$$

We regard F as a map from $\mathbb{P}^2$ to the Grassmannian $\mathrm{Gr}(2, f_2^\perp)$ of 2-dimensional subspaces of $f_2^\perp \simeq \mathbb{C}^3$. It takes a point ℓ to the pencil of quadrics in $f_2^\perp$ that vanish at ℓ. The fiber of F is the base locus of that pencil. Let $B \subset \mathbb{P}^2$ be the branch locus of F. This is a curve of degree six. The fiber of F over any point in $\mathbb{P}^2 \backslash B$ consists of four points $\ell_1, \ell_2, \ell_3, \ell_4$, and these determine decompositions $f = \ell_1^3 + \ell_2^3 + \ell_3^3 + \ell_4^3$. In this manner, the rank 4 decompositions of f are in bijection with the points of $\mathbb{P}^2 \backslash B$. We conclude that $\mathrm{VSP}(f) = \mathrm{Gr}(2, f_2^\perp) \simeq \mathbb{P}^2$.

Second proof of Proposition 3.2 We follow a geometric argument, due to De Paolis in 1886, as presented in [4, §5] and [6, §3]. Let $H(f)$ be the Hessian of f, i.e., the 3×3 determinant of second partial derivatives of f. We choose a real line ℓ_1 that intersects the cubic $H(f)$ in three distinct real points. The line ℓ_1 is identified with its defining linear form and hence with a point in the dual $\mathbb{P}^2$. That $\mathbb{P}^2$ is the domain of F. We may assume that $F(\ell_1)$ is not in the branch locus B. There exists a decomposition $f = \ell_1^3 + \ell_2^3 + \ell_3^3 + \ell_4^3$, where $\ell_2, \ell_3, \ell_4 \in \mathbb{C}[x, y, z]_1$. We claim that the ℓ_i have real coefficients. Let $\partial_p(f)$ be the polar conic of f with respect to $p = \ell_1 \cap \ell_2$. This conic is a $\mathbb{C}$-linear combination of ℓ_3^2 and ℓ_4^2. It is singular at the point $\ell_3 \cap \ell_4$. In particular, p belongs to ℓ_1 and to the cubic $H(f)$. Hence, p is a real point, the conic $\partial_p(f)$ is real, and its singular point $\ell_3 \cap \ell_4$ is real. The latter point is distinct from $p = \ell_1 \cap \ell_2$ because f is smooth. After relabeling, all pairwise intersection points of the lines $\ell_1, \ell_2, \ell_3, \ell_4$ are distinct and real. Hence the lines themselves are real.

The key step in the second proof is the choice of the line ℓ_1. In practice, this is done by sampling linear forms ℓ_1 from $\mathbb{R}[x, y, z]_1$ until $H(f) \cap \ell_1$ consists of three real points p. For each of these, we compute the singular point of the conic $\partial_p(f)$ and connect it to p by a line. This gives the lines $\ell_2, \ell_3, \ell_4 \in \mathbb{R}[x, y, z]_1$. The advantage of this method is that the coordinates of the ℓ_i live in a cubic extension and are easy to express in terms of radicals.

In order to choose the initial line ℓ_1 more systematically, we must understand the structure of $\mathrm{SSP}(f)_{\mathbb{R}}$. This is our next topic. By definition, $\mathrm{SSP}(f)_{\mathbb{R}}$ is the locus of real points $p \in \mathbb{P}^2 = \mathrm{Gr}(2, f_2^{\perp})$ for which the fiber $F^{-1}(p)$ is fully real. Such points p have the form $p = F(\ell)$ where ℓ is a line that meets the Hessian cubic $H(f)$ in three distinct real points.

Example 3.4 Let f be the Hesse cubic (15). The net $f_2^{\perp}$ is spanned by the three quadrics

$$q_0 = \lambda x^2 - 6yz, \quad q_1 = \lambda y^2 - 6xz, \quad \text{and } q_2 = \lambda z^2 - 6xy.$$

These quadrics define the map $F : \mathbb{P}^2 \to \mathbb{P}^2$. We use coordinates $(x : y : z)$ on the domain $\mathbb{P}^2$ and coordinates $(a : b : c)$ on the image $\mathbb{P}^2$. The branch locus B of F is the sextic curve

$$\begin{aligned} &177147\lambda^4(a^6 + b^6 + c^6) - (1458\lambda^8 - 157464\lambda^5 + 4251528\lambda^2)(a^4bc + ab^4c + abc^4) \\ &+ (36\lambda^{10} - 5832\lambda^7 - 39366\lambda^4 - 5668704\lambda)(a^3b^3 + a^3c^3 + b^3c^3) \\ &- (\lambda^{12} - 216\lambda^9 - 61236\lambda^6 + 3621672\lambda^3 + 8503056)a^2b^2c^2. \end{aligned}$$

We regard the Hessian $H(f)$ as a curve in the image $\mathbb{P}^2$. This cubic curve equals

$$H(f) = a^3 + b^3 + c^3 - \frac{\lambda^3 + 108}{3\lambda^2} abc. \tag{18}$$

The ramification locus of the map F is the Jacobian of the net of quadrics:

$$C(f) = \det \begin{pmatrix} \frac{\partial q_0}{\partial x} & \frac{\partial q_0}{\partial y} & \frac{\partial q_0}{\partial z} \\ \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} & \frac{\partial q_1}{\partial z} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} & \frac{\partial q_2}{\partial z} \end{pmatrix} = x^3 + y^3 + z^3 + \frac{54 - \lambda^3}{9\lambda} xyz. \tag{19}$$

This cubic is known classically as the *Cayleyan* of f; see [3, Prop. 3.3] and [21, eqn. (3.27)].

We note that the dual of the cubic $C(f)$ is the sextic B. The preimage of $B = C(f)^{\vee}$ under F is a non-reduced curve of degree 12. It has multiplicity 2 on the Cayleyan $C(f)$. The other component is the sextic curve dual to the Hessian $H(f)$. That sextic equals

$$\begin{aligned} H(f)^{\vee} = &-2187\lambda^8(x^6+y^6+z^6)+(162\lambda^{10}+34992\lambda^7+1889568\lambda^4)(x^4yz+xy^4z+xyz^4) \\ &+ (-12\lambda^{11} + 486\lambda^8 - 41990\lambda^5 - 15116544\lambda^2)(x^3y^3 + x^3z^3 + y^3z^3) + \\ &+ (\lambda^{12} - 2484\lambda^9 - 244944\lambda^6 + 5038848\lambda^3 + 136048896)x^2y^2z^2. \end{aligned}$$

So, we constructed four curves: the cubic $C(f)$ and the sextic $H(f)^\vee$ in the domain $\mathbb{P}^2 = \{(x:y:z)\}$, and the cubic $H(f)$ and the sextic $B = C(f)^\vee$ in the image $\mathbb{P}^2 = \{(a:b:c)\}$. ◇

A smooth cubic f in the real projective plane is either a connected curve, namely a pseudoline, or it has two connected components, namely a pseudoline and an oval. In the latter case, f is *hyperbolic*. The cubic in the Hesse pencil (15) is singular for $\lambda = -3$, it is hyperbolic if $\lambda < -3$, and it is not hyperbolic if $\lambda > -3$. This trichotomy is the key for understanding $\mathrm{SSP}(f)_\mathbb{R}$. However, we must consider this trichotomy also for the Hessian cubic $H(f)$ in (18) and for the Cayleyan cubic $C(f)$ in (19). The issue is whether their Hesse parameters $-\frac{\lambda^3+108}{3\lambda^2}$ and $\frac{54-\lambda^3}{9\lambda}$ are bigger or smaller than the special value -3. The values at which the behavior changes are $\lambda = -3, 0, 6$. Table 2 summarizes the four possibilities.

Three possible hyperbolicity behaviors are exhibited by the three cubics f, $H(f)$, $C(f)$. One of these behaviors leads to two different types, seen in the second and fourth column in Table 2. These two types are distinguished by the fibers of the map $F : \mathbb{P}^2 \to \mathbb{P}^2$. These fibers are classified by the connected components in the complement of the Cayleyan $C(f)$. There are three such components if $C(f)$ is hyperbolic and two otherwise. The fifth row in Table 2 shows the number of real points over these components. For $6 < \lambda$, there are no real points over one component; here, the general fibers have 4, 2 or 0 real points. However, for $-3 < \lambda < 0$, all fibers contain real points; here, the general fibers have 4, 2 or 4 real points.

Proof of Theorem 3.1 After a coordinate change by a matrix $\tau \in \mathrm{PGL}(3, \mathbb{R})$, we can assume that the cubic f is in the Hesse pencil (15). Hence so are the associated cubics $H(f)$ and $C(f)$. If we change the parameter λ so that all three cubics remain smooth, then the real topology of the map F is unchanged. This gives four different types for $\mathrm{SSP}(f)_\mathbb{R}$, the locus of fully real fibers. The sextic B divides the real projective plane into two or three connected components, depending on whether its dual cubic $C(f) = B^\vee$ is hyperbolic or not.

Figures 1, 2, 3 and 4 illustrate the behavior of the map F in the four cases given by the columns in Table 2. Each figure shows the plane $\mathbb{P}^2$ with coordinates $(x : y : z)$ on the left and the plane $\mathbb{P}^2$ with coordinates $(a : b : c)$ on the right. The map F takes the left plane onto the right plane. The two planes are dual to each other. In particular, points on the left correspond to lines on the right. Each of the eight drawings shows a cubic curve and a sextic curve. The two curves on the left are dual to the two curves on the right.

In each right diagram, the thick red curve is the branch locus B and the thin blue curve is the Hessian $H(f)$. In each left diagram, the turquoise curve is the Cayleyan $C(f) = B^\vee$, and the thick blue curve is the sextic $H(f)^\vee$ dual to the Hessian. Each of the eight cubics has either two or one connected components, depending on whether the curve is hyperbolic or not. The complement of the cubic in $\mathbb{P}^2_\mathbb{R}$ has three or two connected components. The diagrams verify the hyperbolicity behavior stated in the third and fourth row of Table 2. Note that each sextic curve has the same number of components in $\mathbb{P}^2_\mathbb{R}$ as its dual cubic.

Consider the three cases where $C(f)$ is hyperbolic. These are in Figs. 1, 2 and 3. Here, $\mathbb{P}^2_\mathbb{R}\backslash B$ has three connected components. The fibers of F could have 0, 2 or 4 real points on these three regions. The innermost region has four real points in its fibers. It is bounded by the triangular connected component of the (red) branch curve B, which is dual to the pseudoline of $C(f)$. This innermost region is connected and contractible: it is a disk in $\mathbb{P}^2_\mathbb{R}$.

If $\lambda \notin [-3, 0]$ then this disk is exactly our set $\mathrm{SSP}(f)_\mathbb{R}$. This happens in Figs.1 and 3. However, the case $\lambda \in (-3, 0)$ is different. This case is depicted in Fig. 2. Here, we see that $\mathrm{SSP}(f)_\mathbb{R}$ consists of two regions. First, there is the disk as before, and second, we have the outermost region. This region is bounded by the oval that is shown as two unbounded branches on the right in Fig. 2. That region is homeomorphic to a Möbius strip in $\mathbb{P}^2_\mathbb{R}$. The key observation is that the fibers of F over that Möbius strip consist of four real points.

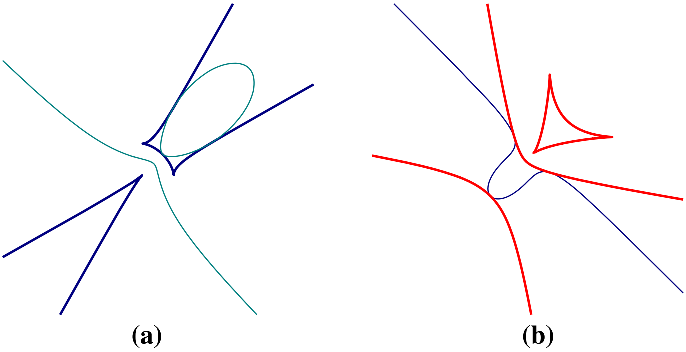

Fig. 1 Ramification and branching for $\lambda < -3$. The domain $\mathbb{P}^2 = \{(x : y : z)\}$. is shown in (**a**). The domain $\mathbb{P}^2 = \{(a : b : c)\}$ is shown in (**b**). The triangular region in (**b**) is $\mathrm{SSP}(f)_{\mathbb{R}}$

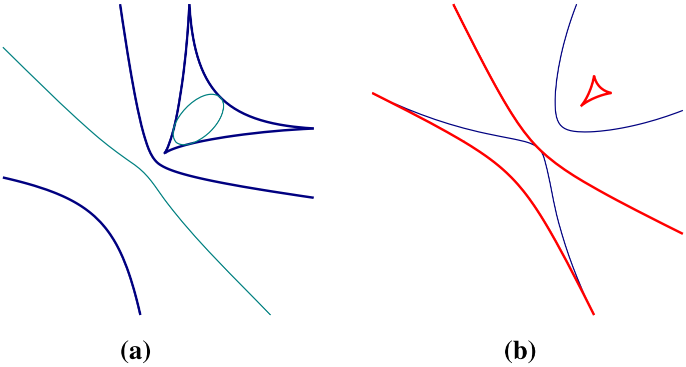

Fig. 2 Ramification and branching for $-3 < \lambda < 0$. The locus $\mathrm{SSP}(f)_{\mathbb{R}}$ is bounded by the (*red*) sextic curve on the right. It consists of the triangular disk and the Möbius strip (color figure online)

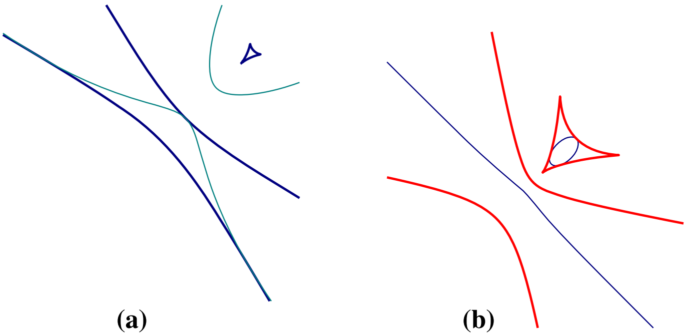

Fig. 3 Ramification and branching for $\lambda > 6$. The *triangular region* is $\mathrm{SSP}(f)_{\mathbb{R}}$

Figure 2 reveals something interesting for the decompositions $f = \sum_{i=1}^{4} \ell_i^3$. These come in two different types, for $\lambda \in (-3, 0)$, one for each of the two connected components of $\mathrm{SSP}(f)_{\mathbb{R}}$. Over the disk, all four lines ℓ_i intersect the Hessian $H(f)$ only in its pseudoline. Over the Möbius strip, the ℓ_i intersect the oval of $H(f)$ in two points and the third intersection point is on the pseudoline. Compare this with Fig. 3: the Hessian $H(f)$ is also hyperbolic,

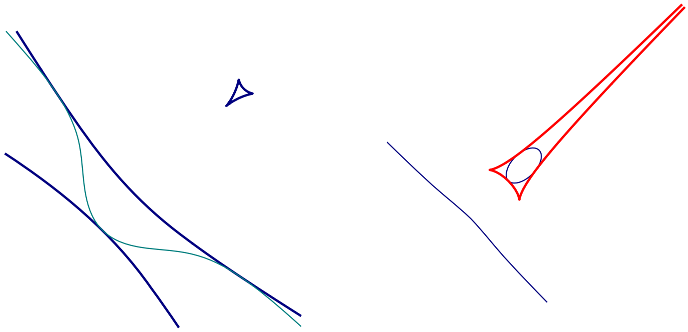
(a) (b)

Fig. 4 Ramification and branching for $0 < \lambda < 6$. The *triangular region* is $\mathrm{SSP}(f)_{\mathbb{R}}$

but all decompositions are of the same type: three lines ℓ_i intersect $H(f)$ in two points of its oval and one point of its pseudoline, while the fourth line intersects $H(f)$ only in its pseudoline.

It remains to consider the case when $C(f)$ is not hyperbolic. This is shown in Fig. 4. The branch curve $B = C(f)^\vee$ divides $\mathbb{P}^2_{\mathbb{R}}$ into two regions, one disk and one Möbius strip. The former corresponds to fibers with four real points, and the latter corresponds to fibers with two real points. We conclude that $\mathrm{SSP}(f)_{\mathbb{R}}$ is a disk also in this last case. We might note, as a corollary, that all fibers of $F : \mathbb{P}^2 \to \mathbb{P}^2$ contain real points, provided $0 < \lambda < 6$.

For all four columns of Table 2, the algebraic boundary of the set $\mathrm{SSP}(f)_{\mathbb{R}}$ is the branch curve B. This is a sextic with nine cusps because it is dual to the smooth cubic $C(f)$.

One may ask for the topological structure of the $4:1$ covering over $\mathrm{SSP}(f)_{\mathbb{R}}$. Over the disk, our map F is $4:1$. It maps four disjoint disks. Each linear form in the corresponding decompositions $f = \sum_{i=1}^{4} \ell_i^3$ comes from one of the four regions seen in the left pictures:

(i) in Fig. 1, inside the region bounded by $H(f)^\vee$ and cut into four by $C(f)$;
(ii) in Fig. 2, inside the spiky triangle bounded by $H(f)^\vee$ and cut into four by $C(f)$;
(iii) in Figs. 3 and 4, one inside the triangle bounded by $H(f)^\vee$ and the others in the region bounded by the other component of $H(f)^\vee$ cut into three regions by $C(f)$.

The situation is even more interesting over the Möbius strip. We can continuously change the set $\{\ell_1, \ell_2, \ell_3, \ell_4\}$, reaching in the end the same as at the beginning, but cyclicly permuted.

Remark 3.5 Given a ternary cubic f with rational coefficients, how to decide whether $\mathrm{SSP}(f)_{\mathbb{R}}$ has one or two connected components? The classification in Table 2 can be used for this task as follows. We first compute the j-invariant of f and then we substitute the rational number $j(f)$ into (16). This gives a polynomial of degree 4 in the unknown λ. That polynomial has two distinct real roots $\lambda_1 < \lambda_2$, provided $j(f) \notin \{0, 1728\}$. They satisfy $\lambda_1 < \lambda_2 < -3$, or $0 < \lambda_1 < \lambda_2 < 6$, or ($-3 < \lambda_1 < 0$ and $6 < \lambda_2$). Consider the involution that swaps λ_1 with λ_2. This fixes the case in Fig. 1, and the case in Fig. 4, but it swaps the cases in Figs. 2 and 3. Thus this involution preserves the hyperbolicity behavior. We get two connected components, namely both the disk and Möbius strip, only in the last case. The correct λ is identified by comparing the sign of the degree six invariant $T(f)$, as in (17).

Example 3.6 The following cubic is featured in the statistics context of [39, Example 1.1]:

$$f = \det \begin{pmatrix} x+y+z & x & y \\ x & x+y+z & z \\ y & z & x+y+z \end{pmatrix} = \frac{4}{3}(x+y+z)^3$$
$$-\frac{2}{3}(x+y)^3 - \frac{2}{3}(x+z)^3 - \frac{2}{3}(y+z)^3.$$

Its j-invariant equals $j(f) = 16384/5$. The corresponding real Hesse curves have parameters $\lambda_1 = -13.506\ldots$ and $\lambda_2 = -5.57\ldots$, so we are in the case of Fig. 1. Indeed, the curve $V(f)$ is hyperbolic, as seen in [39, Figure 1]. Hence $\mathrm{SSP}(f)_{\mathbb{R}}$ is a disk, shaped like a spiky triangle. The real decomposition above is right in its center. Moreover, we can check that $T(f) < 0$. Hence λ_1 provides the unique curve in the Hesse pencil that is isomorphic to f over $\mathbb{R}$. ◇

Remark 3.7 The value 1728 for the j-invariant plays a special role. A real cubic f is hyperbolic if $j(f) > 1728$, and it is not hyperbolic if $j(f) < 1728$. Applying this criterion to a given cubic along with its Hessian and Cayleyan is useful for the classification in Table 2.

What happens for $j(f) = 1728$? Here, the two real forms of the complex curve $V(f)$ differ: one is hyperbolic and the other one is not. For example, $f_1 = x^3 - xz^2 - y^2z$ is hyperbolic and $f_2 = x^3 + xz^2 - y^2z$ is not hyperbolic. These two cubics are isomorphic over $\mathbb{C}$, with $j(f_1) = j(f_2) = 1728$, and they are also isomorphic to their Hessians and Cayleyans.

We find noteworthy that the topology of $\mathrm{SSP}(f)_{\mathbb{R}}$ can distinguish between the two real forms of an elliptic curve. This happens when $j(f) < 1728 < \min\{j(H(f)), j(C(f))\}$. Here the two real forms of the curve correspond to the second and fourth column in Table 2.

We close this section by explaining the last row of Table 2. It concerns the *oriented matroid* [5] of the configuration $\{(a_i, b_i, c_i) : i = 1, \ldots, r\}$ in the decompositions (1). For $d = 3$ the underlying matroid is always *uniform*. This is the content of the following lemma.

Lemma 3.8 *Consider a ternary cubic $f = \sum_{i=1}^4 \ell_i^3$ whose apolar ideal $f^\perp$ is generated by three quadrics. Then any three of the linear forms $\ell_1, \ell_2, \ell_3, \ell_4$ are linearly independent.*

Proof Suppose ℓ_1, ℓ_2, ℓ_3 are linearly dependent. They are annihilated by a linear operator p as in (5). Let q_1 and q_2 be independent linear operators that annihilate ℓ_4. Then pq_1 and pq_2 are independent quadratic operators annihilating f. Adding a third quadric would not lead to a complete intersection. This is a contradiction, since $f^\perp$ is a complete intersection. □

In the situation of Lemma 3.8, there is unique vector $v = (v_1, v_2, v_3, v_4) \in (\mathbb{R}\backslash\{0\})^4$ satisfying $v_1 = 1$ and $\sum_{i=1}^4 v_i \ell_i = 0$. The oriented matroid of $(\ell_1, \ell_2, \ell_3, \ell_4)$ is the sign vector $\big(+, \mathrm{sign}(v_2), \mathrm{sign}(v_3), \mathrm{sign}(v_4)\big) \in \{-, +\}^4$. Up to relabeling there are only three possibilities:

$(+, +, +, +)$: the four vectors ℓ_i contain the origin in their convex hull;
$(+, +, +, -)$: the triangular cone spanned by ℓ_1, ℓ_2, ℓ_3 in $\mathbb{R}^3$ contains ℓ_4;
$(+, +, -, -)$: the cone spanned by $\ell_1, \ell_2, \ell_3, \ell_4$ is the cone over a quadrilateral.

For a general cubic f, every point in $\mathrm{SSP}(f)_{\mathbb{R}}$ is mapped to one of the three sign vectors above. By continuity, this map is constant on each connected component of $\mathrm{SSP}(f)_{\mathbb{R}}$. The last row in Table 2 shows the resulting map from the five connected components to the three oriented matroids. Two of the fibers have cardinality one. For instance, the fiber over $(+, +, -, -)$ is the Möbius strip in $\mathrm{SSP}(f)_{\mathbb{R}}$. This is the first of the following two cases.

Corollary 3.9 *For a general ternary cubic f, we have the following equivalences:*

(i) *The space $\mathrm{SSP}(f)_{\mathbb{R}}$ is disconnected if and only if f is isomorphic over $\mathbb{R}$ to a cubic of the form $x^3 + y^3 + z^3 + (ax + by - cz)^3$ where a, b, c are positive real numbers.*

(ii) *The Hessian $H(f)$ is hyperbolic and the Cayleyan $C(f)$ is not hyperbolic if and only if f is isomorphic to $x^3 + y^3 + z^3 + (ax + by + cz)^3$ where a, b, c are positive real numbers.*

Proof The sign patterns $(+, +, -, -)$ and $(+, +, +, -)$ occur in the second and third column in Table 2, respectively. The corollary is a reformulation of that fact. The sign pattern $(+, +, +, +)$ in columns 1,2 and 4 corresponds to cubics $x^3 + y^3 + z^3 - (ax + by + cz)^3$. □

Remark 3.10 The fiber over the oriented matroid $(+, +, +, +)$ consists of cubics of the form $x^3 + y^3 + z^3 - (ax + by + cz)^3$, where $a, b, c > 0$. It may seem surprising that this space has three components in Table 2. One can pass from one component to another, for instance, by passing through singular cubics, like $24xyz = (x + y + z)^3 + (x - y - z)^3 + (-x + y - z)^3 + (-x - y + z)^3$.

4 Quartics

In this section, we fix $d = 4$ and we consider a general ternary quartic $f \in \mathbb{R}[x, y, z]_4$. We have $r = R(4) = 6$ and $\dim(\mathrm{VSP}(f)) = 3$, so the quartic f admits a threefold of decompositions

$$\begin{aligned} f(x, y, z) = \lambda_1(a_1x + b_1y + c_1z)^4 + \lambda_2(a_2x + b_2y + c_2z)^4 \\ + \cdots + \lambda_6(a_6x + b_6y + c_6z)^4. \end{aligned} \tag{20}$$

By the *signature* of f, we mean the signature of $C(f)$. This makes sense by Proposition 1.3.

According to the Hilbert–Burch Theorem, the radical ideal I_T of the point configuration $\{(a_i : b_i : c_i)\}_{i=1,\dots,6}$ is generated by the 3×3-minors of a 4×3-matrix $T = T_1x+T_2y+T_3z$, where $T_1, T_2, T_3 \in \mathbb{R}^{4\times3}$. We interpret T also as a $3 \times 3 \times 4$-tensor with entries in $\mathbb{R}$, or as a 3×3-matrix of linear forms in 4 variables. The determinant of the latter matrix defines the cubic surface in $\mathbb{P}^3$ that is the blow-up of the projective plane $\mathbb{P}^2$ at the six points. The apolar ideal $f^\perp$ is generated by seven cubics, and $I_T \subset f^\perp$ is generated by four of these.

Mukai [29] showed that $\mathrm{VSP}(f)$ is a Fano threefold of type V_{22}, and a more detailed study of this threefold was undertaken by Schreyer in [36]. The topology of the real points in that Fano threefold was studied by Kollár and Schreyer in [24]. Inside that real locus lives the semialgebraic set we are interested in. Namely, $\mathrm{SSP}(f)_{\mathbb{R}}$ represents the set of radical ideals I_T, arising from tensors $T \in \mathbb{R}^{3\times3\times4}$, such that $I_T \subset f^\perp$ and the variety $V(I_T)$ consists of six real points in $\mathbb{P}^2$. This is equivalent to saying that the cubic surface of T has 27 real lines.

Disregarding the condition $I_T \subset f^\perp$ for the moment, this reality requirement defines a full-dimensional, connected semialgebraic region in the tensor space $\mathbb{R}^{3\times3\times4}$. The algebraic boundary of that region is defined by the hyperdeterminant $\mathrm{Det}(T)$, which is an irreducible homogeneous polynomial of degree 48 in the 36 entries of T. Geometrically, this hyperdeterminant is the Hurwitz form in [38, Example 4.3]. This is Proposition 2.7 for $m = n = 3$.

We are interested in those ternary forms f whose apolar ideal $f^\perp$ contains I_T for some T in the region described above. Namely, we wish to understand the semialgebraic set

$$\mathcal{R}_4 = \left\{ f \in \mathbb{R}[x, y, z]_4 \ :\ \mathrm{SSP}(f)_{\mathbb{R}} \neq \emptyset \right\}.$$

The following is a step toward understanding the algebraic boundary $\partial_{\mathrm{alg}}(\mathcal{R}_4)$ of this set.

Theorem 4.1 *The algebraic boundary* $\partial_{\mathrm{alg}}(\mathcal{R}_4)$ *is a reducible hypersurface in the* $\mathbb{P}^{14}$ *of quartics. One of its irreducible components has degree* 51*; that component divides the quartics of signature* (5, 1)*. Another irreducible component divides the region of hyperbolic quartics.*

Proof By [6, Example 4.6], $\partial_{\mathrm{alg}}(\mathcal{R}_4)$ is non-empty, so it must be a hypersurface. We next identify the component of degree 51. The *anti-polar* of a quartic f, featured in [6, §5.1] and in [20], is defined by the following rank 1 update of the middle catalecticant:

$$\Omega(f)(a,b,c) \;:=\; \det\big(C(f+\ell^4)\big) \,-\, \det\big(C(f)\big) \quad \text{for} \quad \ell = ax+by+cz. \tag{21}$$

Writing "Adj" for the adjoint matrix, the Matrix Determinant Lemma implies

$$\Omega(f)(a,b,c) \;:=\; (a^2,ab,ac,b^2,bc,c^2)\cdot \mathrm{Adj}(C(f))\cdot (a^2,ab,ac,b^2,bc,c^2)^T. \tag{22}$$

The coefficients of the anti-polar quartic $\Omega(f)$ are homogeneous polynomials of degree 5 in the coefficients of f. The discriminant of $\Omega(f)$ is a polynomial of degree $135 = 27\times 5$ in the coefficients of f. A computation reveals that this factors as $\det(C(f))^{14}$ times an irreducible polynomial $\mathrm{Bdisc}(f)$ of degree 51. We call $\mathrm{Bdisc}(f)$ the *Blekherman discriminant* of f.

We claim that $\mathrm{Bdisc}(f)$ is an irreducible component of $\partial_{\mathrm{alg}}(\mathcal{R}_4)$. Let f be a general quartic of signature (5, 1). Then $\det(C(f))$ is negative, and the quartic $\Omega(f)$ is non-singular. We claim that $\mathrm{rk}_{\mathbb{R}}(f) = 6$ if and only if the curve $\Omega(f)$ has a real point. The only-if direction is proved in a more general context in Lemma 6.4. For the if-direction, we note that the anti-polar quartic curve divides $\mathbb{P}^2_{\mathbb{R}}$ into regions where $\Omega(f)$ has opposite signs. Hence, we can find ℓ such that $\det\big(C(f+\ell^4)\big) = 0$, and therefore $\mathrm{rk}_{\mathbb{R}}(f) = 6$. Examples in [6, §5.1] show that $\mathrm{rk}_{\mathbb{R}}(f)$ can be either 6 or 7. We conclude that, among quartics of signature (5, 1), the boundary of $\mathcal{R}_4$ is given by the Blekherman discriminant $\mathrm{Bdisc}(f)$ of degree 51.

To prove that $\partial_{\mathrm{alg}}(\mathcal{R}_4)$ is reducible, we consider the following pencil of quartics:

$$\begin{aligned} f_t = {} & (6x-4y+17z)^4 + (4x-16y-5z)^4 + (20x+2y-19z)^4 - (15y-17z)^4 \\ & - (13x+14y+9z)^4 - (16x-6y-18z)^4 + t\cdot(-2x^4+2x^3z-x^2y^2+2x^2yz \\ & + x^2z^2 + xy^3 - xy^2z - 2xyz^2 + xz^3 + y^4 + y^3z + 2y^2z^2 - 2yz^3 - 2z^4). \end{aligned}$$

At $t = 0$, we obtain a quartic f_0 of signature (3, 3) that has real rank 6. One checks that f_0 is smooth and hyperbolic. We substitute f_t into the invariant of degree 51 derived above, and we note that the resulting univariate polynomial in t has no positive real roots. So, the ray $\{f_t\}$ given by $t \geq 0$ does not intersect the boundary component we already identified.

For positive parameters t, the discriminant of f_t is nonzero, until t reaches $\tau_1 = 6243.83\ldots$. This means that f_t is smooth hyperbolic for real parameters t between 0 and τ_1. On the other hand, the rank of the middle catalecticant $C(f_t)$ drops from 6 to 5 when t equals $\tau_0 = 3103.22\ldots$. Hence, for $\tau_0 < t < \tau_1$, the quartics f_t are hyperbolic and of signature (4, 2). By [6, Corollary 4.8], these quartics have real rank at least 7. This means that the half-open interval given by $(0, \tau_0]$ crosses the boundary of $\mathcal{R}_4$ in a new irreducible component. □

Remark 4.2 One of the starting points of this project was the question whether $\mathrm{rk}_{\mathbb{R}}(f) \geq 7$ holds for all hyperbolic quartics f. This was shown to be false in [6, Remark 4.9]. The above quartic f_0 is an alternative counterexample, with an explicit rank 6 decomposition over $\mathbb{Q}$.

We believe that, in the proof above, the crossing takes place at τ_0, and that this newly discovered component is simply the determinant of the catalecticant $\det(C(f))$. But we have not been able to certify this. Similar examples suggest that also the discriminant $\mathrm{disc}(f)$ itself appears in the real rank boundary. Based on this, we propose the following conjecture.

Conjecture 4.3 *The real rank boundary $\partial_{\mathrm{alg}}(\mathcal{R}_4)$ for ternary quartics is a reducible hypersurface of degree $84 = 6 + 27 + 51$ in $\mathbb{P}^{14}$. It has three irreducible components, namely the determinant of the catalecticant, the discriminant and the Blekherman discriminant. Algebraically,*

$$\partial_{\mathrm{alg}}(\mathcal{R}_4) = \det(C(f)) \cdot \mathrm{disc}(f) \cdot \mathrm{Bdisc}(f).$$

The construction of the Blekherman discriminant extends to the case when f is a sextic or octic; see Lemma 6.4. For quartics f, we can use it to prove $\mathrm{rk}_{\mathbb{R}}(f) > 6$ also when the signature is (4, 2) or (3, 3). We illustrate this for the quartic given by four distinct lines.

Example 4.4 We claim that $f = xyz(x + y + z)$ has $\mathrm{rk}_{\mathbb{R}}(f) = 7$. For the upper bound, note

$$12f = x^4 + y^4 + z^4 - (x + y)^4 - (x + z)^4 - (y + z)^4 + (x + y + z)^4.$$

The catalecticant $C(f)$ has signature (3, 3). The anti-polar quartic $\Omega(f) = a^2b^2 + a^2c^2 + b^2c^2 - a^2bc - ab^2c - abc^2$ is nonnegative. By [6, Section 5.1], we have $\mathrm{rk}_{\mathbb{R}}(f) = 7$. $\diamond$

If f is a general ternary quartic of real rank 6, then $\mathrm{SSP}(f)_{\mathbb{R}}$ is an open semialgebraic set inside the threefold $\mathrm{VSP}(f)_{\mathbb{R}}$. Our next goal is to derive an algebraic description of this set. We begin by reviewing some of the relevant algebraic geometry found in [17,27,29,33,36].

The Fano threefold $\mathrm{VSP}(f)$ in its anti-canonical embedding is a subvariety of the Grassmannian $\mathrm{Gr}(4, 7)$ in its Plücker embedding in $\mathbb{P}^{34}$. It parametrizes 4-dimensional subspaces of $f_3^{\perp} \simeq \mathbb{R}^7$ that can serve to span I_T. In other words, the Fano threefold $\mathrm{VSP}(f)$ represents quadruples of cubics in $f_3^{\perp}$ that arise from a $3 \times 3 \times 4$-tensor T as described above. Explicitly, $\mathrm{VSP}(f)$ is the intersection of $\mathrm{Gr}(4, 7)$ with a linear subspace $\mathbb{P}^{13}_A$ in $\mathbb{P}^{34}$. This is analogous to Proposition 2.4, but more complicated. The resolution of the apolar ideal $f^{\perp}$ has the form

$$0 \longrightarrow S(-7) \longrightarrow S(-4)^7 \xrightarrow{A} S(-3)^7 \longrightarrow S \longrightarrow 0.$$

By the Buchsbaum-Eisenbud structure theorem, we can write $A = xA_1 + yA_2 + zA_3$ where A_1, A_2, A_3 are real skew-symmetric 7×7-matrices. In other words, the matrices A_1, A_2, A_3 lie in $\bigwedge^2 f_3^{\perp}$. The seven cubic generators of the ideal $f^{\perp}$ are the 6×6-sub-Pfaffians of A.

The ambient space $\mathbb{P}^{34}$ for the Grassmannian $\mathrm{Gr}(4, 7)$ is the projectivation of the 35-dimensional vector space $\bigwedge^4 f_3^{\perp}$. The matrices A_1, A_2, A_3 determine the following subspace:

$$\mathbb{P}^{13}_A := \left\{ U \in \bigwedge^4 f_3^{\perp} \;:\; U \wedge A_1 = U \wedge A_2 = U \wedge A_3 = 0 \;\text{ in }\; \bigwedge^6 f_3^{\perp} \right\}. \tag{23}$$

Each constraint $U \wedge A_i = 0$ gives seven linear equations, for a total of 21 linear equations.

Lemma 4.5 *The Fano threefold* $\mathrm{VSP}(f)$ *of degree* 22 *is the intersection of the Grassmannian* $\mathrm{Gr}(4, 7)$ *with the linear space* $\mathbb{P}^{13}_A$*. Its defining ideal is generated by* 45 *quadrics, namely the* 140 *quadratic Plücker relations defining* $\mathrm{Gr}(4, 7)$ *modulo the* 21 *linear relations in* (23).

Proof This description of the Fano threefold V_{22} was given by Ranestad and Schreyer in [33] and by Dinew, Kapustka and Kapustka in [17, Section 2.3]. We verified the numbers 22 and 45 by a direct computation. $\square$

It is important to note that we can turn this construction around and start with any three general skew-symmetric 7×7-matrices A_1, A_2, A_3. Then the 6×6-sub-Pfaffians of $xA_1 + yA_2 + zA_3$ generate a Gorenstein ideal whose socle generator is a ternary quartic f.

This correspondence shows how to go from rank 6 decompositions of f to points U in $\mathrm{VSP}(f) = \mathrm{Gr}(4,7) \cap \mathbb{P}^{13}_A \subset \mathbb{P}^{34}$. Given the configuration $\mathbb{X} = \{(a_i : b_i : c_i)\}$ in (20), the point U is the space of cubics that vanish on $\mathbb{X}$. Conversely, given any point U in $\mathrm{VSP}(f)$, we can choose a basis of $f_3^\perp$ such that our 7×7-matrices A_i have the form analogous to (8):

$$A_i = \begin{pmatrix} \star & T_i \\ -T_i^t & 0 \end{pmatrix} \quad \text{for } i = 1, 2, 3.$$

Here T_i is a 4×3-matrix, and 0 is the zero 3×3-matrix. The four 3×3-minors of the 4×3-matrix $T = xT_1 + yT_2 + zT_3$ are among the seven 6×6-sub-Pfaffians of $A = xA_1 + yA_2 + zA_3$. These four cubics define the six points in the decomposition (20).

We are now ready to extend the real geometry in Corollary 2.6 from quadrics to quartics. Let $V = (v_{ij})$ be a 4×3-matrix of unknowns. These serve as affine coordinates on $\mathrm{Gr}(4, 7)$. Each point is the row span of the 4×7-matrix $U = \begin{pmatrix} \mathrm{Id}_4 & V \end{pmatrix}$. This is analogous to (11).

Proceeding as in (13), we consider the skew-symmetric 7×7-matrix

$$\begin{pmatrix} \star & T \\ -T^t & 0 \end{pmatrix} \;=\; \begin{pmatrix} \mathrm{Id}_4 & V \\ V^t & -\mathrm{Id}_3 \end{pmatrix} \cdot A \cdot \begin{pmatrix} \mathrm{Id}_4 & V \\ V^t & -\mathrm{Id}_3 \end{pmatrix}. \tag{24}$$

Its entries are linear forms in x, y, z whose coefficients are quadratic polynomials in the 12 affine coordinates v_{ij}. Vanishing of the lower right 3×3-matrix defines $\mathrm{VSP}(f)$. The matrix T is identified with a $4 \times 3 \times 3$-tensor whose entries are quadratic polynomials in the v_{ij}.

Theorem 4.6 *Let f be a general ternary quartic of real rank* 6. *Using the affine coordinates v_{ij} on* $\mathrm{Gr}(4, 7)$, *the threefold* $\mathrm{VSP}(f)_\mathbb{R}$ *is defined by nine quadratic equations in* $\mathbb{R}^{12}$. *If f has signature* $(6, 0)$ *then* $\mathrm{SSP}(f)_\mathbb{R}$ *equals* $\mathrm{VSP}(f)_\mathbb{R}$. *If* $\overline{\mathrm{SSP}(f)_\mathbb{R}}$ *is a proper subset of* $\mathrm{VSP}(f)_\mathbb{R}$ *then its algebraic boundary has degree* 84. *It is the hyperdeterminant of the* $4\times3\times3$-*tensor* T.

Proof The description of $\mathrm{VSP}(f)$ in affine coordinates follows from Lemma 4.5. The equations in (23) mean that the linear map given by A vanishes on the kernel of U. This translates into the condition that the lower right 3×3-matrix in (24) is zero. Each of the 3 coefficients of the 3 upper diagonal matrix entries must vanish, for a total of 9 quadratic equations.

If f has signature $(6, 0)$ then we know from Proposition 1.3 that $\mathrm{SSP}(f)_\mathbb{R} = \mathrm{VSP}(f)_\mathbb{R}$. In general, a point (v_{ij}) of $\mathrm{VSP}(f)_\mathbb{R}$ lies in $\mathrm{SSP}(f)_\mathbb{R}$ if and only if all six zeros of the ideal I_T are real points in $\mathbb{P}^2$. The boundary of $\mathrm{SSP}(f)_\mathbb{R}$ is given by those (v_{ij}) for which two of these zeros come together in $\mathbb{P}^2$ and form a complex conjugate pair. The Zariski closure of that boundary is the hypersurface defined by the hyperdeterminant $\mathrm{Det}(T)$, by Proposition 2.7.

The hyperdeterminant of format $4 \times 3 \times 3$ has degree 48 in the tensor entries. For our tensor T, the entries are inhomogeneous polynomials of degree 2, so the degree of $\mathrm{Det}(T)$ is bounded above by $96 = 2 \times 48$. A direct computation reveals that the actual degree is 84. The degree drop from 96 to 84 is analogous to the drop from 24 to 20 witnessed in (14). □

At present, we do not know whether the hyperdeterminantal boundary always exists:

Conjecture 4.7 *If the quartic f has real rank* 6 *and its signature is* $(3, 3)$, $(4, 2)$ *or* $(5, 1)$, *then the semialgebraic set* $\overline{\mathrm{SSP}(f)_\mathbb{R}}$ *is strictly contained in the variety* $\mathrm{VSP}(f)_\mathbb{R}$.

Our next tool for studying $\mathrm{VSP}(f)$ is another quartic curve, derived from f, and endowed with a distinguished even theta characteristic θ. Recall that there is a unique (up to scaling) invariant of ternary cubics in degree 4. This is the *Aronhold invariant*, which vanishes on cubics g with $\mathrm{rk}_\mathbb{C}(g) \leq 3$. For the given quartic f, the *Aronhold quartic* $S(f)$ is defined by

$$S(f)(p) \;:=\; \text{the Aronhold invariant evaluated at } \partial_p(f). \tag{25}$$

Following [19], we call f the *Scorza quartic* of $S(f)$. Points on $S(f)$ correspond to lines in the threefold $\mathrm{VSP}(f)$. To see this, consider any decomposition $f = \sum_{i=1}^{6} \ell_i^4$, representing a point in $\mathrm{VSP}(f)$. This point lies on a $\mathbb{P}^1$ in $\mathrm{VSP}(f)$ if and only if three of the lines $\ell_1, \ell_2, \ldots, \ell_6$ meet. Indeed, if $a \in \ell_1 \cap \ell_2 \cap \ell_3$ in $\mathbb{P}^2$ then $\partial_p(f)$ is a sum of three cubes, i.e., $p \in S(f)$. We may regard ℓ_1, ℓ_2, ℓ_3 as linear forms in two variables, so that $\ell_1^4 + \ell_2^4 + \ell_3^4$ is a binary quartic. This binary quartic has a $\mathbb{P}^1$ of rank 4 decomposition, each giving a decomposition of f, with ℓ_4, ℓ_5, ℓ_6 fixed. The resulting line in $\mathrm{VSP}(f) \subset \mathbb{P}^{13}_A$ is the set of 4-planes U containing the $\mathbb{P}^3$ of cubics $Q \cdot a$, where Q is a quadric vanishing on ℓ_4, ℓ_5, ℓ_6. This gives all lines on $\mathrm{VSP}(f)$.

One approach we pursued is the relationship of the real rank of f with its topology in $\mathbb{P}^2_{\mathbb{R}}$. A classical result of Klein and Zeuthen, reviewed in [32, Theorem 1.7], states that there are six types of smooth plane quartics in $\mathbb{P}^2_{\mathbb{R}}$, and these types form connected subsets of $\mathbb{P}^{14}_{\mathbb{R}}$:

$$\text{4 ovals, 3 ovals, 2 non-nested ovals, hyperbolic, 1 oval, empty curve.} \tag{26}$$

We consider the pairs of types given by a general quartic f and its Aronhold quartic $S(f)$.

Proposition 4.8 *Among the* 36 *pairs of topological types* (26) *of smooth quartic curves in the real projective plane* $\mathbb{P}^2_{\mathbb{R}}$*, at least* 30 *pairs are realized by a quartic* f *and its Aronhold quartic* $S(f)$*. Every pair not involving the hyperbolic type is realizable as* $\big(f, S(f)\big)$.

Proof This was established by exhaustive search. We generated random quartic curves using various sampling schemes, and this led to 30 types. The six missing types are listed in Conjecture 4.9. For a concrete example, here is an instance where f and $S(f)$ are empty:

$$\begin{aligned} f = {} & (3x^2 + 5zx - 5yx - 5z^2 - 3yz)^2 + (7x^2 + 7zx - 7yx + z^2 - yz - 5y^2)^2 \\ & + (5x^2 + 7zx + 7yx - 8z^2 - 2yz + 2y^2)^2 \end{aligned}$$

At the other end of the spectrum, let us consider

$$\begin{aligned} f \;= {} & 1439x^4 + 1443y^4 - 2250(x^2 + y^2)z^2 + 3500x^2y^2 + 817z^4 - x^3z - x^3y \\ & - 5x^2z^2 - 7x^2yz - 4xz^3 - 6xyz^2 + xy^2z + 6xy^3 - 3yz^3 + 5y^2z^2 + 7y^3z \end{aligned}$$

For this quartic, both f and $S(f)$ have 28 real bitangents, so they consist of four ovals. □

Conjecture 4.9 *If a smooth quartic* f *on* $\mathbb{P}^2_{\mathbb{R}}$ *is hyperbolic then its Aronhold quartic* $S(f)$ *is either empty or has two ovals. If* f *consists of three or four ovals then* $S(f)$ *is not hyperbolic.*

We now describe the construction in [29] of eight distinguished *Mukai decompositions*

$$f = \ell_1^4 + \ell_2^4 + \ell_3^4 + \ell_{12}^4 + \ell_{13}^4 + \ell_{23}^4. \tag{27}$$

The configuration $\ell_{12}, \ell_{23}, \ell_{13}$ is a *biscribed triangle* of the Aronhold quartic $S(f)$. Being a biscribed triangle means that ℓ_{ij} is tangent to $S(f)$ at a point q_{ij}, the lines ℓ_{ij} and ℓ_{ik} meet at a point q_i on the curve $S(f)$, and the line ℓ_i is spanned by q_{ij} and q_{ik}.

Let $D = q_1 + q_2 + q_3 + q_{12} + q_{13} + q_{23}$ be a divisor on the Aronhold quartic $S(f)$. The biscribed triangle $\ell_{12}\ell_{13}\ell_{23}$ is a *contact cubic* [32, §2], and $2D$ is its intersection divisor with $S(f)$. The associated theta characteristic is given by $\theta \sim q_{12} + q_{13} + q_1 - q_{23}$. Each of the points $q_{12}, q_{13}, q_{23} \in S(f)$ represents a line on the Fano threefold $\mathrm{VSP}(f) \subset \mathbb{P}^{13}_A$. The pairs (q_{12}, q_{13}), (q_{12}, q_{23}), (q_{13}, q_{23}) lie in the *Scorza correspondence*, as defined in [19,36]. Indeed, the corresponding second derivatives of f are ℓ_{12}^2, ℓ_{13}^2 and ℓ_{23}^2, so the lines q_{12}, q_{13}, q_{23} on $\mathrm{VSP}(f)$ intersect pairwise. In fact, there is a point of $\mathrm{VSP}(f)$ on all three lines, namely (27).

Example 4.10 We illustrate the concepts above, starting with the skew-symmetric matrix

$$A = A_1x + A_2y + A_3z$$
$$= \begin{pmatrix} 0 & -x+y+3z & z & y+z & x & -x & 0 \\ x-y-3z & 0 & x-y+3z & -x-3y+z & -x+z & 2x-2y & y-z \\ -z & -x+y-3z & 0 & x+y+z & 0 & y & -y \\ -y-z & x+3y-z & -x-y-z & 0 & z & -3z & z \\ -x & x-z & 0 & -z & 0 & 0 & 0 \\ x & -2x+2y & -y & 3z & 0 & 0 & 0 \\ 0 & -y+z & y & -z & 0 & 0 & 0 \end{pmatrix}.$$

Its 6×6 Pfaffians generate the apolar ideal $f^{\perp}$. Orthogonal to this is the rank 6 quartic

$$f \;=\; x^4+y^4+z^4+(x+y)^4+(y+z)^4+(z+x)^4.$$

The upper right 4×3-block of A has rank 2 precisely on these six points $\ell_1, \ell_2, \ell_3, \ell_{12}, \ell_{13}, \ell_{23}$. Here, $q_1 = (-1{:}1{:}1), q_2 = (1{:}-1{:}1), q_3 = (1{:}1{:}-1), q_{12} = (0{:}0{:}1), q_{13} = (0{:}1{:}0), q_{23} = (1{:}0{:}0)$. The theta characteristic θ on the Aronhold quartic $S(f)$ is defined by the contact cubic $(x+y)(x+z)(y+z)$. This is the lower right 3×3-minor in the determinantal representation

$$S(f) \;=\; \det\begin{pmatrix} 5x+5y+5z & x-y & x-z & y-z \\ x-y & x+y+z & x & -y \\ x-z & x & x+y+z & z \\ y-z & -y & z & x+y+z \end{pmatrix}.$$

This matrix is constructed from the contact cubic by the method in [32, Proposition 2.2].

We write $\mathrm{VSP}(f)^{\mathrm{Muk}}$ for the subvariety of $\mathrm{VSP}(f)$ given by Mukai decompositions (27). Mukai [29] showed that $\mathrm{VSP}(f)^{\mathrm{Muk}}$ is a finite set with eight elements. We are interested in

$$\mathrm{VSP}(f)^{\mathrm{Muk}}_{\mathbb{R}} \;:=\; \mathrm{VSP}(f)^{\mathrm{Muk}} \cap \mathrm{VSP}(f)_{\mathbb{R}} \quad \text{and} \quad \mathrm{SSP}(f)^{\mathrm{Muk}}_{\mathbb{R}} \;:=\; \mathrm{VSP}(f)^{\mathrm{Muk}} \cap \mathrm{SSP}(f)_{\mathbb{R}}.$$

One idea we had for certifying $\mathrm{rk}_{\mathbb{R}}(f) = 6$ is to compute the eight points in $\mathrm{VSP}(f)^{\mathrm{Muk}}$. If (27) is fully real for one of them then we are done. Unfortunately, this algorithm may fail. The semialgebraic set of quartics with real Mukai decompositions is strictly contained in $\mathcal{R}_4$:

Proposition 4.11 *There exist quartics f of real rank 6 such that $\mathrm{SSP}(f)^{\mathrm{Muk}}_{\mathbb{R}}$ is empty.*

Proof Consider the eight Mukai decompositions (27) of the following quartic of real rank 6:

$$\begin{aligned} f \;=\; & (21x+y+9z)^4 + (14x-13y+14z)^4 + (13x+5y-7z)^4 \\ & + (2x-5y-13z)^4 \;-\; (12x+15y-9z)^4 \;-\; (12x+21z)^4. \end{aligned}$$

The lines ℓ_i and ℓ_j in (27) intersect in the point $q_{ij} \in S(f)$. If both lines are real then so is q_{ij}. But, a computation shows that $S(f)$ has no real points. This implies $\mathrm{SSP}(f)^{\mathrm{Muk}}_{\mathbb{R}} = \emptyset$. □

Example 4.12 We close this section by discussing the *Blum–Guinand quartics* in [11]. Set

$$f_{a,b,m} \;:=\; (x^2+y^2-a^2z^2-b^2z^2)(m^4x^2+y^2-m^2(a^2+b^2)z^2) + a^2b^2(m^2-1)^2z^4.$$

The parameters a, b, m satisfy $0 < b < a$ and $\sqrt{b/a} < m < \sqrt{a/b}$. Blum–Guinand quartics have 28 real bitangents, so they consist of four ovals. Figure 5 shows this for $a = 20$, $b = 8$

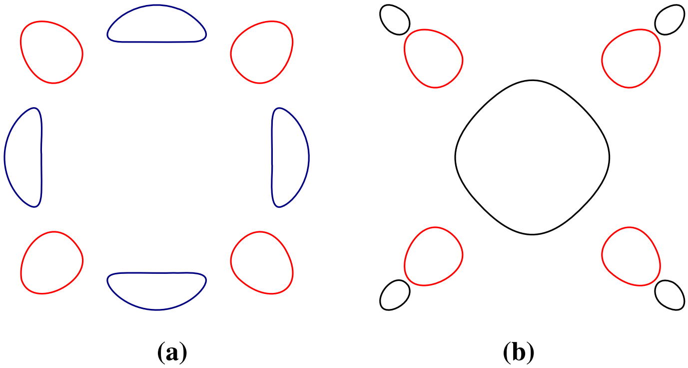

Fig. 5 Left picture shows a Blum–Guinand quartic in blue and its Aronhold quartic in red. The Aronhold quartic does not meet the sextic covariant, shown in *black on the right* (color figure online)

and $m = \sqrt{\frac{20}{8}} - 0.1$. The diagram on the left has $f_{a,b,m}$ in blue and $S(f_{a,b,m})$ in red. The sextic covariant, defined by (25) but with S replaced by the sextic invariant T, is shown on the right in black. The cubic $\partial_p(f_{a,b,m})$ has real rank 4 for all $p \in S(f_{a,b,m})$; cf. Remark 3.3. For the chosen parameters, for each direction there exists a line that meets the Blum quartic at 4 real points. We can conclude that $\mathrm{rk}_{\mathbb{R}}(f_{a,b,m}) \geq 7$. In general, the signature is (4,2) if $\sqrt{2} - 1 < m < \sqrt{2} + 1$ and (3,3) otherwise. The picture seen in Figure 5 can change. For instance, the Aronhold quartic $S(f_{a,b,m})$ has no real points when $a = 70$, $b = 8$, $m = 6/5$. $\diamond$

5 Quintics and septics

A general ternary quintic $f \in \mathbb{R}[x, y, z]_5$ has complex rank $R(5) = 7$. The decomposition

$$f = \ell_1^5 + \ell_2^5 + \ell_3^5 + \ell_4^5 + \ell_5^5 + \ell_6^5 + \ell_7^5 \tag{28}$$

is unique by a classical result of Hilbert, Richmond and Palatini. Oeding and Ottaviani [30] explained how to compute the seven linear forms ℓ_i by realizing them as eigenvectors of a certain $3\times3\times3$-tensor. Inspired by [33, §1.5], we propose the following alternative algorithm:

Algorithm 5.1 Input*: A general ternary quintic f*. Output*: The decomposition* (28)*.*

1. *Compute the apolar ideal $f^{\perp}$. It is generated by one quartic and four cubics g_1, g_2, g_3, g_4.*
2. *Compute the syzygies of $f^{\perp}$. Find the unique linear syzygy (l_1, l_2, l_3, l_4) on the cubics.*
3. *Compute a vector $(c_1, c_2, c_3, c_4) \in \mathbb{R}^4\backslash\{0\}$ that satisfies $c_1l_1 + c_2l_2 + c_3l_3 + c_4l_4 = 0$.*
4. *Let J be the ideal generated by the cubics $c_2g_1 - c_1g_2$, $c_3g_2 - c_2g_3$ and $c_4g_3 - c_3g_4$. Compute the variety $V(J)$ in $\mathbb{P}^2$. It consists precisely of the points dual to $\ell_1, \ell_2, \ldots, \ell_7$.*

To prove the correctness of this algorithm, we recall what is known about the ideal J of seven points in $\mathbb{P}^2$. The ideal J is Cohen–Macaulay of codimension 2, so it is generated by the maximal minors of its Hilbert–Burch matrix T. According to [1, Theorem 5.1], this matrix has the following form if and only if no six of the seven points lie on a conic:

$$T = \begin{pmatrix} q_1 & l_1 \\ q_2 & l_2 \\ q_3 & l_3 \end{pmatrix}. \tag{29}$$

Here l_1, l_2, l_3 are independent linear forms and q_1, q_2, q_3 are quadratic forms in x, y, z.

Proposition 5.2 *Algorithm 5.1 computes the unique decomposition of a general quintic f. In the resulting representation (28), no six of the seven lines ℓ_i are tangent to a conic.*

Proof Let f be a general quintic. The apolar ideal $f^\perp$ in S is generated by four cubics g_1, g_2, g_3, g_4 and one quartic h. This ideal is Gorenstein of codimension 3. The Buchsbaum-Eisenbud structure theorem implies that the minimal free resolution of $f^\perp$ has the form

$$0 \longrightarrow S(-8) \longrightarrow S(-4) \oplus S(-5)^4 \xrightarrow{A} S(-4) \oplus S(-3)^4 \xrightarrow{B} S \longrightarrow 0.$$

The matrix A is skew-symmetric of size 5×5, i.e.,

$$A = \begin{pmatrix} 0 & q_{12} & q_{13} & q_{14} & l_1 \\ -q_{12} & 0 & q_{23} & q_{24} & l_2 \\ -q_{13} & -q_{23} & 0 & q_{34} & l_3 \\ -q_{14} & -q_{24} & -q_{34} & 0 & l_4 \\ -l_1 & -l_2 & -l_3 & -l_4 & 0 \end{pmatrix}. \tag{30}$$

Here the l_i's are linear forms and the q_{ij}'s are quadrics. As described above and in Section 2, we should find a 5×5 matrix U such that the lower right 2×2 submatrix of $U \cdot A \cdot U^t$ is the zero matrix. Since f is general, we may assume that the l_i span $\mathbb{R}[x, y, z]_1$. After relabeling if necessary, we can write $c_1 l_1 + c_2 l_2 + c_3 l_3 + l_4 = 0$ for some scalars c_1, c_2, c_3. Setting $c_4 = 1$, these are the scalars in Step 3 of Algorithm 5.1. Let

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We perform row and column operations by the following right and left multiplication:

$$A' = U \cdot A \cdot U^t = \begin{pmatrix} 0 & q_{12} & q_{13} & q'_{14} & l_1 \\ -q_{12} & 0 & q_{23} & q'_{24} & l_2 \\ -q_{13} & -q_{23} & 0 & q'_{34} & l_3 \\ -q'_{14} & -q'_{24} & -q'_{34} & 0 & 0 \\ -l_1 & -l_2 & -l_3 & 0 & 0 \end{pmatrix}.$$

The inverse column operation on the row vector B of minimal generators of $f^\perp$ gives

$$B' = \left(g_1\, g_2\, g_3\, g_4\, h\right) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -c_1 & -c_2 & -c_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \left(g_1 - c_1 g_4 \;\; g_2 - c_2 g_4 \;\; g_3 - c_3 g_4 \;\; g_4 \;\; h\right).$$

Let $J = \langle g_i - c_i g_4 \ : \ i = 1, 2, 3\rangle$ denote the ideal generated by the first three cubics. This is the ideal in Step 4 of the algorithm. We claim that $V(J)$ consists of seven points in $\mathbb{P}^2$.

By construction, we have $B' \cdot A' = 0$, and the columns of A' span the syzygies on B'. The entries of B' are the 4×4 sub-Pfaffians of A'. The first three entries are the sub-Pfaffians that involve the last two rows and columns. These three 4×4 Pfaffians are the 2×2-minors of

$$T = \begin{pmatrix} q'_{14} & l_1 \\ q'_{24} & l_2 \\ q'_{34} & l_3 \end{pmatrix}.$$

This is a Hilbert–Burch matrix for the ideal J. Hence J is an ideal of seven points in $\mathbb{P}^2$. Moreover, since l_1, l_2 and l_3 are linearly independent, no six points of them lie on a conic. Dually, this means that no six of the seven lines ℓ_i used in (28) are tangent to a conic. □

It is easy to decide whether the real rank of a given ternary quintic f is 7 or not. Namely, one computes the unique complex decomposition (28) and checks whether it is real. The real rank boundary corresponds to transition points where two of the linear forms in (28) come together and form a complex conjugate pair. The following is our main result on quintics.

Theorem 5.3 *The algebraic boundary $\partial_{\mathrm{alg}}(\mathcal{R}_5)$ of the set $\mathcal{R}_5 = \{f : rk_{\mathbb{R}}(f) = 7\}$ is an irreducible hypersurface of degree* 168 *in the $\mathbb{P}^{20}$ of quintics. It has the parametric representation*

$$g = \ell_1^5 + \ell_2^5 + \ell_3^5 + \ell_4^5 + \ell_5^5 + \ell_6^4\ell_7, \quad \textit{where } \ell_1, \ldots, \ell_7 \in \mathbb{R}[x, y, z]_1. \tag{31}$$

Proof The parametrization (31) defines a unirational variety Y in $\mathbb{P}^{20}$. The Jacobian of this parametrization is found to have corank 1. This means that Y has codimension 1 in $\mathbb{P}^{20}$. Hence Y is an irreducible hypersurface, defined by a unique (up to sign) irreducible homogeneous polynomial Φ in 21 unknowns, namely the coefficients of a ternary quintic.

Let g be a real quintic (31) that is a general point in Y. For $\epsilon \to 0$, the real quintics $(\ell_6 + \epsilon\ell_7)^5 - \ell_6^5$ and $(i\ell_6 + \epsilon\ell_7)^5 + (-i\ell_6 + \epsilon\ell_7)^5$ converge to the special quintic $\ell_6^4\ell_7$ in $\mathbb{P}^{20}_{\mathbb{R}}$. Hence any small neighborhood of g in $\mathbb{P}^{20}_{\mathbb{R}}$ contains quintics of real rank 7 and quintics of real rank $\geq$ 8. This implies that Y lies in the algebraic boundary $\partial_{\mathrm{alg}}(\mathcal{R}_5)$. Since Y is irreducible and has codimension 1, it follows that $\partial_{\mathrm{alg}}(\mathcal{R}_5)$ exists and has Y as an irreducible component.

We carried out an explicit computation to determine that the (possibly reducible) hypersurface $\partial_{\mathrm{alg}}(\mathcal{R}_5)$ has degree 168. This was done as follows. Fix the field $K = \mathbb{Q}(t)$, where t is a new variable. We picked random quintics f_1 and f_2 in $\mathbb{Q}[x, y, z]_5$, and we ran Algorithm 5.1 for $f = f_1 + tf_2 \in K[x, y, z]_5$. Step 4 returned a homogeneous ideal J in $K[x, y, z]$ that defines 7 points in $\mathbb{P}^2$ over the algebraic closure of K. By eliminating each of the three variables, we obtain binary forms of degree 7 in $K[x, y]$, $K[x, z]$ and $K[y, z]$. Their coefficients are polynomials of degree 35 in t. The discriminant of each binary form is a polynomial in $\mathbb{Q}[t]$ of degree $420 = 12 \times 35$. The greatest common divisor of these three discriminants is a polynomial $\Psi(t)$ of degree 168. We checked that $\Psi(t)$ is irreducible in $\mathbb{Q}[t]$.

By definition, Φ is an irreducible homogeneous polynomial with integer coefficients in the 21 coefficients of a general quintic f. Its specialization $\Phi(f_1 + tf_2)$ is a non-constant polynomial in $\mathbb{Q}[t]$, of degree $\deg(X)$ in t. That polynomial divides $\Psi(t)$. Since the latter is irreducible, we conclude that $\Phi(f_1 + tf_2) = \gamma \cdot \Psi(t)$, where γ is a nonzero rational number. Hence Φ has degree 168. We conclude that $\deg(Y) = 168$, and therefore $Y = \partial_{\mathrm{alg}}(\mathcal{R}_5)$. □

Theorem 5.3 was stated for a very special situation, namely ternary quintics. We shall now describe a geometric generalization. Let X be any irreducible projective variety in the complex projective space $\mathbb{P}^N$ that is defined over $\mathbb{R}$ and whose real points are Zariski dense. We set $d = \dim(X)$. The *generic rank* is the smallest integer r such that the rth secant variety $\sigma_r(X)$ equals $\mathbb{P}^N$. Given $f \in \mathbb{P}^N$, we define $\mathrm{VSP}_X(f)$ to be the closure in the Hilbert scheme $\mathrm{Hilb}_r(X)$ of the set of configurations of r distinct points in X whose span contains f. Now, VSP stands for **v**ariety of **s**ums of **p**oints. This object agrees with that studied by

Gallet, Ranestad and Villamizar in [23]. It differs from more inclusive definitions seen in other articles. In particular, if $N = \binom{d+2}{2} - 1$ and $X = \nu_d(\mathbb{P}^2)$ is the dth Veronese surface then $\mathrm{VSP}_X(f) = \mathrm{VSP}(f)$. In this case, we recover the familiar **v**ariety of **s**ums of powers.

The objects of real algebraic geometry studied in this paper generalize in a straightforward manner. We write $\mathrm{VSP}_X(f)_{\mathbb{R}}$ for the variety of real points in $\mathrm{VSP}_X(f)$, and we define $\mathrm{SSP}_X(f)_{\mathbb{R}}$ to be the semialgebraic subset of all f that lie in an $(r-1)$-plane spanned by r real points in X. Following Blekherman and Sinn [9], we are interested in generic points in $\mathbb{P}^N_{\mathbb{R}}$ whose real rank equals the generic complex rank. These comprise the semialgebraic set $\mathcal{R}_X = \{f \in \mathbb{P}^N_{\mathbb{R}} : \mathrm{SSP}_X(f)_{\mathbb{R}} \neq \emptyset\}$. The topological boundary $\partial\mathcal{R}_X$ is the closure of $\mathcal{R}_X$ minus the interior of that closure. If X has more than one typical real rank, then $\partial\mathcal{R}_X$ is non-empty and its Zariski closure $\partial_{\mathrm{alg}}(\mathcal{R}_X)$ is a hypersurface in $\mathbb{P}^N$. This hypersurface is the *real rank boundary* we are interested in.

Example 5.4 Let $N = 20$ and $X = \nu_5(\mathbb{P}^2)$ the fifth Veronese surface in $\mathbb{P}^{20}$. Then $r = 7$ and $\partial_{\mathrm{alg}}(\mathcal{R}_X)$ equals the irreducible hypersurface of degree 168 described in Theorem 5.3.

This example generalizes as follows. Let $X \subset \mathbb{P}^N$ as above, and let $\tau(X)$ denote its *tangential variety*. By definition, $\tau(X)$ is the closure of the union of all lines that are tangent to X. We also consider the $(r-2)$nd secant variety $\sigma_{r-2}(X)$. The expected dimensions are

$$\dim\big(\tau(X)\big) = 2d \quad \text{and} \quad \dim\big(\sigma_{r-2}(X)\big) = (r-2)d + r - 3.$$

We are interested in the join of the two varieties, denoted $\sigma_{r-2}(X)\star\tau(X)$. This is an irreducible projective variety of expected dimension $rd + r - 2$ in $\mathbb{P}^N$. It comes with a distinguished parametrization, generalizing that in (31) for the Veronese surface of Example 5.4. The following generalization of Theorem 5.3 explains the geometry of the real rank boundary:

Conjecture 5.5 *Suppose $rd + r = N$ and $\mathrm{VSP}_X(f)$ is finite for general f. Then $\sigma_{r-2}(X) \star \tau(X)$ is an irreducible component of $\partial_{\mathrm{alg}}(\mathcal{R}_X)$. Equality holds when $\mathrm{VSP}_X(f)$ is a point.*

One difficulty in proving this conjecture is that we do not know how to control interactions among the distinct decompositions $f = f_1 + \cdots + f_r$ of a general point $f \in \mathbb{P}^N$ into r points $f_1, \ldots, f_r$ on the variety X. Moreover, we do not even know that $\partial_{\mathrm{alg}}(\mathcal{R}_X)$ is non-empty.

To illustrate Conjecture 5.5, we prove it in the case when X is the 7th Veronese surface. The parameters are $d = 2$, $N = 35$, and $r = 12$. We return to the previous notation, so f is a general ternary form in $\mathbb{R}[x, y, z]_7$. Here we can show that 13 is indeed a typical real rank.

Proposition 5.6 *The real rank boundary $\partial_{\mathrm{alg}}(\mathcal{R}_7)$ is a non-empty hypersurface in $\mathbb{P}^{35}$ with one of the components equal to the join of the tenth secant variety and the tangential variety.*

Proof For a general septic f, the minimal free resolution of the apolar ideal $f^{\perp}$ equals

$$0 \longrightarrow S(-10) \longrightarrow S(-6)^5 \xrightarrow{A} S(-4)^5 \longrightarrow S \longrightarrow 0. \tag{32}$$

This is as in (7), but now the entries of the skew-symmetric 5×5-matrix A are quadratic. To find the five rank 12 decompositions of f, we proceed as in Example 2.5: we solve the matrix equation (13). The matrix entry in position $(4, 5)$ is a homogeneous quadric in x, y, z whose six coefficients are inhomogeneous quadrics in the unknowns a, b, c, d, e, g. These coefficients must vanish. This system of six equations in six unknowns defines $\mathrm{VSP}(f)$ in $\mathbb{C}^6$. It has precisely five solutions. For each of these solutions, we consider the upper right 3×2-matrix T. Its entries are quadrics in x, y, z. The 2×2-minors of T define the desired 12 points in $\mathbb{P}^2$.

We apply the above algorithm to the point in $\sigma_{10}(X) \star \tau(X)$ given by the septic

$$\begin{aligned} f := & (11x+13y-12z)(-18x+13y-16z)^6+(2x-12y+z)^7+(3x-13y-7z)^7 \\ & +(-6x+5y+15z)^7+(16x+5y+14z)^7+(18x-19y-9z)^7+(-4x+10y-18z)^7 \\ & +(11x+9y+10z)^7+(-19x+15y-z)^7+(-4x-20y-16z)^7+(2x+20y-18z)^7. \end{aligned}$$

This septic f has complex rank 12, but its real rank is larger. The output of our decomposition algorithm shows that four of the five decompositions are not fully real. This remains true in a small neighborhood of f. Near the point $(11x+13y-12z)(-18x+13y-16z)^6$ in the tangential variety $\tau(X)$, some septics have real rank 2 and some others have real rank 3. Hence, the above decomposition of f can change from purely real to a decomposition that contains complex linear forms. The same holds for all nearby points in the join variety. We conclude that a general point of the join in a small neighborhood of f belongs to $\partial(\mathcal{R}_7)$. □

Our algorithm for septics f computes the five elements in $\mathrm{VSP}(f)$ along with the 12 linear forms in each of the five decompositions $f = \sum_{i=1}^{12} \ell_i^7$. It outputs 60 points in $\mathbb{P}^2$. These come in 5 unlabeled groups of 12 unlabeled points in $\mathbb{P}^2$. Here are two concrete instances.

Example 5.7 First, consider the septic $f = \sum_{i=1}^{12} \ell_i^7$ of real rank 12 that is defined by

$$\begin{array}{lll} \ell_1 = -7x+14y+3z, & \ell_2 = -13x-12y+20z, & \ell_3 = -7x-5y-18z, \\ \ell_4 = 12x-16y+17z, & \ell_5 = -8x+16y+7z, & \ell_6 = 15x-8y+2z, \\ \ell_7 = 13x-7y-11z, & \ell_8 = -x-3y-3z, & \ell_9 = 18x-4y-7z, \\ \ell_{10} = 19x-9y+7z, & \ell_{11} = 15y-17z, & \ell_{12} = -19x-2y+11z. \end{array}$$

Here, all 60 points in $\mathbb{P}^2$ are real. This means that the variety of sums of powers is fully real, and the twelve ℓ_i in each of the five decompositions are real: $\mathrm{VSP}(f) = \mathrm{VSP}(f)_{\mathbb{R}} = \mathrm{SSP}(f)_{\mathbb{R}}$.

Second, consider the real septic $f = \sum_{i=1}^{6}(\ell_i^7 + \overline{\ell}_i^7)$ defined by the complex linear forms

$$\begin{array}{ll} \ell_1 = 8x+11y-15z+i(13x+15y+17z), & \ell_2 = -16x+6y+11z+i(-4x+19y+16z), \\ \ell_3 = 5x+13y-3z+i(-2x+4y+5z), & \ell_4 = 8x+8y-7z+i(-13x-12y-8z), \\ \ell_5 = -5x-20y-15z+i(-8x+18y+7z), & \ell_6 = -14x-18y-7z+i(-9x-2y+19z). \end{array}$$

This satisfies $\mathrm{VSP}(f)_{\mathbb{R}} \neq \mathrm{SSP}(f)_{\mathbb{R}} = \emptyset$. None of the 60 points in $\mathbb{P}^2$ are real. Our two examples are extremal. One can easily find other septics f with $\mathrm{VSP}(f)_{\mathbb{R}} \neq \mathrm{SSP}(f)_{\mathbb{R}}$.

6 Sextics

We now consider ternary forms of degree six. The generic complex rank for sextics is $R(6) = 10$. Our first result states that both 10 and 11 are typical real ranks, in the sense of [6,7,16].

Theorem 6.1 *The algebraic boundary $\partial_{\mathrm{alg}}(\mathcal{R}_6)$ is a hypersurface in the $\mathbb{P}^{27}$ of ternary sextics. One of its irreducible components is the dual to the Severi variety of rational sextics.*

Proof We use notation and results from [8]. Let $P_{3,6}$ denote the convex cone of nonnegative rational sextics and $\Sigma_{3,6}$ the subcone of sextics that are sums of squares of cubics over $\mathbb{R}$. The dual cone $\Sigma_{3,6}^{\vee}$ consists of sextics f whose middle catalecticant $C(f)$ is positive semidefinite. Its subcone $P_{3,6}^{\vee}$ is spanned by sixth powers of linear forms. It is known as the *Veronese orbitope*. The difference $\Sigma_{3,6}^{\vee} \backslash P_{3,6}^{\vee}$ is a full-dimensional semialgebraic subset of $\mathbb{R}[x, y, z]_6$.

We claim that general sextics f in that set have real rank ≥ 11. Let f be a general sextic in $\Sigma_{3,6}^{\vee}\backslash P_{3,6}^{\vee}$. Suppose that $\mathrm{rk}_{\mathbb{R}}(f) = 10$. The middle catalecticant $C(f)$ is positive definite. Proposition 1.3 tells us that the signature of any representation (1) is $(10, 0)$. This means that f lies in the Veronese orbitope $P_{3,6}^{\vee}$. This is a contradiction to the hypothesis, and we conclude $\mathrm{rk}_{\mathbb{R}}(f) \geq 11$. Using [6, Theorem 1.1], this means that 11 is a typical rank.

Consider the algebraic boundary of the Veronese orbitope $P_{3,6}^{\vee}$. One of its two components is the determinant of the catalecticant $C(f)$, which is the algebraic boundary of the spectrahedron $\Sigma_{3,6}^{\vee}$. The other component is the dual of the Zariski closure of the set of extreme rays of $P_{3,6}\backslash\Sigma_{3,6}$. That Zariski closure was shown in [8, Theorem 2] to be equal to the Severi variety of rational sextics, which has codimension 10 and degree 26312976 in $\mathbb{P}^{27}$. Every generic boundary point of $P_{3,6}^{\vee}$ that is not in the spectrahedron $\Sigma_{3,6}^{\vee}$ represents a linear functional whose maximum over $P_{3,6}$ occurs at a point in the Severi variety.

A result of Choi, Lam and Reznick (cf. [8, Proposition 7]) states that every general supporting hyperplane of $P_{3,6}^{\vee}$ touches the Veronese surface in precisely 10 rays. Every form in the cone spanned by these rays has real rank ≤ 10. Consider the subset of $P_{3,6}^{\vee}$ obtained by replacing each of the 10 rays by a small neighborhood. This defines a full-dimensional subset of forms $f \in P_{3,6}^{\vee}$ that satisfy $\mathrm{rk}_{\mathbb{R}}(f) = 10$. By construction, this subset must intersect the boundary of $P_{3,6}^{\vee}$ in a relatively open set. Its Zariski closure is the hypersuface dual to the Severi variety. We conclude that this dual is an irreducible component of $\partial_{\mathrm{alg}}(\mathcal{R}_6)$. □

Remark 6.2 The same proof applies also for octics ($d = 8$), ensuring that the algebraic boundary $\partial_{\mathrm{alg}}(\mathcal{R}_8)$ exists. Indeed, $R(8) = 15$ coincides with the size of the middle catalecticant f, and we can conclude that every octic in $\Sigma_{3,8}^{\vee}\backslash P_{3,8}^{\vee}$ has real rank bigger than 15. However, for even integers $d \geq 10$, this argument no longer works, because the generic complex rank exceeds the size of the middle catalecticant. In symbols, $R(d) > \binom{d/2+2}{2}$. New ideas are needed to establish the existence of the hypersurface $\partial_{\mathrm{alg}}(\mathcal{R}_d)$ for $d \geq 9$.

We record the following upper bounds on the real ranks of general ternary forms.

Proposition 6.3 *Let $m(d)$ be the maximal typical rank of a ternary form of degree d. Then*

$$m(d) \;\leq\; \min\Big(\binom{d+1}{2} - 2,\; 2R(d)\Big).$$

In particular, typical real ranks for ternary sextics are between 10 *and* 19.

Proof The same argument as in [6, Proposition 6.2] shows $m(d) \leq m(d-1) + d$. The binomial bound follows by induction. The bound $2R(d)$ comes from [10, Theorem 3]. □

The anti-polar construction in (21) extends to f of degree $d = 6$ and $d = 8$. We define

$$\Omega(f)(a, b, c) \;=\; \det\big(C(f + \ell^d)\big) \;-\; \det\big(C(f)\big) \quad \text{for} \quad \ell = ax + by + cz.$$

Lemma 6.4 *Let f be a general ternary form of degree $d \in \{4, 6, 8\}$ that is not in the cone $P_d^{\vee}$ spanned by dth powers. If $\Omega(f)$ has no real zeros than the real rank of f exceeds $R(d)$.*

Proof Suppose that f is of minimal generic rank $R(d)$. Since f is not a sum of dth powers of linear forms over $\mathbb{R}$, by Proposition 1.3, there exists a real linear form $\ell = ax + by + cz$ such that the catalecticant matrix of $f + \ell^d$ is degenerate; hence, $\Omega(f)(\ell) = -\det\big(C(f)\big)$. On the other hand, there exists ℓ' such that the catalecticant matrix of $-f + \ell'^d$ also drops rank, so $\Omega(f)(\ell') = -\Omega(-f)(\ell') = \det\big(C(f)\big)$. Hence the real curve defined by $\Omega(f)$ is non-empty. □

Let $f \in \mathbb{R}[x, y, z]_6$ be general, with $\mathrm{rk}_{\mathbb{C}}(f) = R(6) = 10$. In what follows we derive the algebraic representation of the K3 surface $\mathrm{VSP}(f)$ and its semialgebraic subset $\mathrm{SSP}(f)_{\mathbb{R}}$. The apolar ideal $f^{\perp}$ is generated by nine quartics. The minimal free resolution of $f^{\perp}$ equals

$$S \to S(-5)^9 \xrightarrow{A} S(-4)^9 \to S \to 0.$$

Here A is a skew-symmetric 9×9-matrix with linear entries. Its 8×8 sub-Pfaffians generate $f^{\perp}$. We write $A = A_1 x + A_2 y + A_3 z$ where $A_i \in \bigwedge^2 f_4^{\perp} \simeq \bigwedge^2 \mathbb{R}^9$. The variety $\mathrm{VSP}(f)$ is a K3 surface of genus 20 and degree 38. See [28,33] for details and proofs. The following representation, found in [33, Theorem 1.7 (iii)], is analogous to Proposition 2.4 and Lemma 4.5.

Proposition 6.5 *The surface* $\mathrm{VSP}(f)$ *is the intersection of the Grassmannian* $\mathrm{Gr}(5, 9)$*, in its Plücker embedding in* $\mathbb{P}(\bigwedge^5 f_4^{\perp}) \simeq \mathbb{P}^{125}$*, with the* 20*-dimensional linear subspace*

$$\mathbb{P}^{20}_A = \left\{ U \in \mathbb{P}^{125} \,:\, U \wedge A_1 = U \wedge A_2 = U \wedge A_3 = 0 \right\}.$$

Inside this space, $\mathrm{VSP}(f)$ *is cut out by* 153 *quadrics, obtained from the Plücker relations.*

For any sextic f, we can compute the surface $\mathrm{VSP}(f)$ explicitly, by the method explained for quadrics in Example 2.5. Namely, as in (11), we introduce local coordinates on $\mathrm{Gr}(5, 9)$. The equations defining $\mathbb{P}^{20}_A$ translate into quadrics in the 20 local coordinates. In analogy to (13), we transform the 9×9-matrix A into $\begin{pmatrix} \star & T \\ -T^t & 0 \end{pmatrix}$. Here T is a 5×4-matrix of linear forms whose 4×4 minors define the ten points in $\mathbb{P}^2$ in the representation $f = \sum_{i=1}^{10} \ell_i^6$. We can study $\mathrm{SSP}(f)_{\mathbb{R}}$ and its boundary inside the real K3 surface $\mathrm{VSP}(f)_{\mathbb{R}}$ by means of the hyperdeterminant for $m = 4$ in Corollary 2.8. The following example demonstrates this.

Example 6.6 Consider the sextic ternary $f = \sum_{i+j+k=3} (ix + jy + kz)^6$, where $i, j, k \in \mathbb{Z}_{\geq 0}$. It is given by a real rank 10 representation. Consider the 9×9 matrix of linear forms in some minimal free resolution of the apolar ideal $f^{\perp}$. We transform this matrix into

$$A = \begin{bmatrix} 0 & l_1 & l_2 & l_3 & l_4 & l_5 & l_6 & l_7 & l_8 \\ -l_1 & 0 & l_9 & l_{10} & l_{11} & l_{12} & l_{13} & l_{14} & l_{15} \\ -l_2 & -l_9 & 0 & l_{16} & l_{17} & l_{18} & l_{19} & l_{20} & l_{21} \\ -l_3 & -l_{10} & -l_{16} & 0 & l_{22} & l_{23} & l_{24} & l_{25} & l_{26} \\ -l_4 & -l_{11} & -l_{17} & -l_{22} & 0 & l_{27} & l_{28} & l_{29} & l_{30} \\ -l_5 & -l_{12} & -l_{18} & -l_{23} & -l_{27} & 0 & 0 & 0 & 0 \\ -l_6 & -l_{13} & -l_{19} & -l_{24} & -l_{28} & 0 & 0 & 0 & 0 \\ -l_7 & -l_{14} & -l_{20} & -l_{25} & -l_{29} & 0 & 0 & 0 & 0 \\ -l_8 & -l_{15} & -l_{21} & -l_{26} & -l_{30} & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $l_1 = \frac{6885}{631}x - \frac{4050}{631}y - \frac{175770}{631}z$, $l_2 = -\frac{4050}{631}x + \frac{3240}{631}y + \frac{37665}{631}z$ $l_3 = -\frac{810}{631}z$, $l_4 = \frac{324}{631}x + \frac{810}{631}y + \frac{2025}{631}z$, $l_5 = -\frac{5}{631}x$, $l_6 = \frac{21}{631}x$, $l_7 = -\frac{4}{631}x$, $l_8 = 0$, $l_9 = -\frac{67230}{631}x - \frac{67230}{631}y - \frac{2791260}{631}z$, $l_{10} = \frac{3240}{631}x - \frac{4050}{631}y + \frac{37665}{631}z$, $l_{11} = -\frac{55728}{631}x - \frac{70308}{631}y - \frac{13446}{631}z$, $l_{12} = \frac{25}{631}x - \frac{25}{631}z$, $l_{13} = -\frac{1482}{631}x + \frac{818}{631}y + \frac{251}{631}z$, $l_{14} = \frac{668}{631}x - \frac{852}{631}y + z$, $l_{15} = \frac{10}{631}y - \frac{20}{631}z$, $l_{16} = -\frac{4050}{631}x + \frac{6885}{631}y - \frac{175770}{631}z$, $l_{17} = \frac{70308}{631}x + \frac{55728}{631}y + \frac{13446}{631}z$, $l_{18} = -\frac{10}{631}x + \frac{20}{631}z$, $l_{19} = \frac{852}{631}x - \frac{668}{631}y - z$, $l_{20} = -\frac{818}{631}x + \frac{1482}{631}y - \frac{251}{631}z$, $l_{21} = -\frac{25}{631}y + \frac{25}{631}z$, $l_{22} = -\frac{810}{631}x - \frac{324}{631}y - \frac{2025}{631}z$, $l_{23} = 0$, $l_{24} = \frac{4}{631}y$, $l_{25} = -\frac{21}{631}y$, $l_{26} = \frac{5}{631}y$, $l_{27} = \frac{5}{631}z$, $l_{28} = -\frac{430}{631}z$, $l_{29} = \frac{430}{631}z$, $l_{30} = -\frac{5}{631}z$.

The upper right 5×4-submatrix of A drops rank precisely on the ten points $(i : j : k) \in \mathbb{P}^2$ where $i + j + k = 3$ in nonnegative integers.

We introduce local coordinates on $\mathrm{Gr}(5, 9)$ as follows. Let U be the row span of $\begin{pmatrix} \mathrm{Id}_5 & V \end{pmatrix}$, where $V = (v_{ij})$ is a 5×4 matrix of unknowns. We transform A into the coordinate system given by U and its orthogonal complement:

$$\begin{pmatrix} \star & T \\ -T^t & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_5 & V \\ V^t & -\mathrm{Id}_4 \end{pmatrix} \cdot A \cdot \begin{pmatrix} \mathrm{Id}_5 & V \\ V^t & -\mathrm{Id}_4 \end{pmatrix}.$$

We proceed as in Example 2.5. The lower right 4×4 block is zero whenever the corresponding 18 quadrics in the 20 local coordinates vanish. The submatrix T is a $3 \times 4 \times 5$ tensor. Its hyperdeterminant $\mathrm{Det}(T)$ is the discriminant for our problem. By Corollary 2.8, this is a polynomial of degree 120 in the entries of T. The specialization of $\mathrm{Det}(T)$ to the 20 local coordinates has degree ≤ 240. That polynomial represents the algebraic boundary of $\mathrm{SSP(f)}_{\mathbb{R}}$ inside $\mathrm{VSP}(f)_{\mathbb{R}}$, similarly to Corollary 2.6.

In the paper, we focused on general ternary forms. Special cases are also very interesting:

Example 6.7 Consider the monomial $f = x^2y^2z^2$. By [14], this has $\mathrm{rk}_{\mathbb{R}}(f) \leq 13$ because

$$\begin{aligned} 360f = {} & 4(x^6 + y^6 + z^6) + (x + y + z)^6 + (x + y - z)^6 + (x - y + z)^6 + (x - y - z)^6 \\ & - 2\big[(x + y)^6 + (x - y)^6 + (x + z)^6 + (x - z)^6 + (y + z)^6 + (y - z)^6\big]. \end{aligned}$$

The apolar ideal is $f^{\perp} = \langle x^3, y^3, z^3 \rangle$. The radical ideal generated by general cubics in $f^{\perp}$,

$$C_1 = \alpha x^3 + \beta y^3 + \gamma z^3 \quad \text{and} \quad C_2 = \alpha' x^3 + \beta' y^3 + \gamma' z^3, \tag{33}$$

proves that $\mathrm{rk}_{\mathbb{C}}(f) = 9$. We can replace C_1 and C_2 by two linear combinations that are binomials, say $C_1' = x^3 + \delta z^3$ and $C_2' = y^3 + \epsilon z^3$. For any choice of δ and ϵ, at most three of the nine points of $V(C_1', C_2') \subset \mathbb{P}^2$ are real. This shows $\mathrm{rk}_{\mathbb{R}}(f) \geq 10$.

We next prove $\mathrm{rk}_{\mathbb{R}}(f) \geq 11$. Assume that $p_1, p_2, \ldots, p_{10} \in \mathbb{P}^2_{\mathbb{R}}$ give a rank 10 decomposition of f. Consider a pencil of cubics $C_1 + t \cdot C_2$ passing through $p_1, \ldots, p_8$. We may assume $p_{10} \notin V(C_1, C_2)$ and $C_2(p_9) = 0$. We claim that C_2 vanishes also at p_{10}. Indeed, C_2 acts by differentiation and gives $C_2(f) = \lambda_{10} C_2(p_{10}) \ell_{10}^3$ where $\lambda_{10} \in \mathbb{R}^*$ and $C_2(p_{10})$ is the evaluation at p_{10}. However, $C_2(f) = C_2(x^2y^2z^2)$ contains none of the pure powers x^3, y^3, z^3 and so C_2 passes through p_{10} as well. We now know that C_1 vanishes at neither p_9 nor p_{10}. Differentiation gives

$$C_1(f) = \alpha \ell_9^3 + \beta \ell_{10}^3, \qquad \text{where } \alpha = \lambda_9 C_1(p_9), \beta = \lambda_{10} C_1(p_{10}) \text{ and } \lambda_9, \lambda_{10} \in \mathbb{R}^*.$$

Let $\ell_9 = a_9 x + b_9 y + c_9 z$ and $\ell_{10} = a_{10} x + b_{10} y + c_{10} z$. The coefficients of x^3, y^3, z^3 in the cubic $C_1(f)$ vanish. Hence

$$\alpha a_9^3 + \beta a_{10}^3 = \alpha b_9^3 + \beta b_{10}^3 = \alpha c_9^3 + \beta c_{10}^3 = 0.$$

Over $\mathbb{R}$, these equations imply $a_9 = -(\beta/\alpha)^{1/3} a_{10}$, $b_9 = -(\beta/\alpha)^{1/3} b_{10}$, $c_9 = -(\beta/\alpha)^{1/3} c_{10}$. So, p_9 and p_{10} are the same point in $\mathbb{P}^2$. This is a contradiction and we conclude $\mathrm{rk}_{\mathbb{R}}(f) \geq 11$.

At present we do not know whether the real rank of f is 11, 12 or 13. The argument above can be extended to establish the following result: *if* $\mathrm{rk}_{\mathbb{R}}(f) \leq 12$ *then there exists a decomposition* (1) *whose points* $(a_i : b_i : c_i)$ *all lie on the Fermat cubic* $V(x^3 + y^3 + z^3)$.

Let us explore the possibility $\mathrm{rk}_{\mathbb{R}}(f) = 12$. Then it is likely that the $(a_i : b_i : c_i)$ are the complete intersection of a cubic and a quartic. They can be assumed to have the form:

$$x^3 + y^3 + 1 = ax^4 + bx^3y + cx^3 + dx + ey + 1 = 0. \tag{34}$$

Hence determining the real rank of f leads directly to the following easy-to-state question: *Can we find real constants* a, b, c, d, e *such that all* 12 *solutions to the equations* (34) *are real?* If the answer is "yes" then we can conclude $\mathrm{rk}_{\mathbb{R}}(f) \leq 12$. Otherwise, we cannot reach a conclusion. A systematic approach to this real root classification problem is via the discriminant of the system (34). This discriminant is a polynomial of degree 24 in a, b, c, d, e. We would need to explore the connected components of the complement of this hypersurface in $\mathbb{R}^5$. For further reading on the rank geometry of monomials, we refer to [12,13]. $\diamond$

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