A rational polynomial chaos formulation for the uncertainty quantification of linear circuits in the frequency domain

A Rational Polynomial Chaos Formulation for the Uncertainty Quantification of Linear Circuits in the Frequency Domain

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Summary. This paper discusses the general form of the transfer functions of linear lumped circuits. It is shown that an arbitrary transfer function defined on such circuits has a functional dependence on individual circuit parameters that is rational, with multi-linear numerator and denominator. This result is then used to introduce a suitable model for the uncertainty quantification of this class of circuits, namely a rational polynomial chaos expansion in place of the conventional single expansion.

1 Introduction

In recent years, the polynomial chaos expansion (PCE) method [1] received a wide attention by the macromodeling and model-order reduction community thanks to its superior computational efficiency over blind and brute-force Monte Carlo (MC) in the uncertainty quantification of stochastic circuits [2]. While in many application scenarios the method was demonstrated to provide very high accuracy with a very limited expansion order, the modeling of resonant and/or distributed circuits may require large orders and the accuracy of the calculated PCE coefficients may be impaired by the large variability of the outputs.

In this paper, we show that the general form of any transfer function defined for a linear lumped circuit is rational w.r.t. both frequency and element values, and specifically that both the numerator and denominator are multi-linear with the latter. Recently, this feature motivated the introduction of a rational PCE model [3], as opposed to the conventional single expansion that is used in most of the applications, including the ones in electrical engineering. The proposed model turns out to be exact for lumped circuits and, by extension, more accurate also for distributed ones. While the aforementioned theoretical result is somewhat well-known in electrical engineering [4], it is not accessible explicitly and unambiguously in the desired form. A rigorous and formal derivation is therefore provided in this paper.

2 General Form of the Transfer Functions of Linear Lumped Circuits

The main objective of this section is to prove that a transfer function defined for a linear lumped circuit is a rational function in which both the numerator and denominator are polynomial functions of the complex frequency s and multi-linear functions[1] of the element values, generically denoted with vector $\xi \in \mathbb{R}^d$.

The derivations are exemplified based on the impedance matrix of a generic multi-port circuit. Starting from a modified nodal analysis (MNA) formulation of the circuit equations [5], the impedance matrix is expressed as

$$Z(s;\xi) = \mathscr{B}^{\mathsf{T}}\mathscr{Y}^{-1}(s,\xi)\mathscr{B} = \frac{\mathbf{N}(s;\xi)}{\mathsf{D}(s;\xi)}, \tag{1}$$

where matrix $\mathscr{Y}$ includes the static and dynamic MNA matrices, whereas matrix $\mathscr{B}$ contains the incidence matrix of the port inputs. In (1), the scalar denominator D coincides with the determinant of $\mathscr{Y}$, whereas each element of the numerator $\mathbf{N}$ is a linear combination of the determinants of the submatrices (minors) obtained from $\mathscr{Y}$ by deleting one row and one column.

After some manipulations, it is possible to show that each entry of the impedance matrix can be expressed as

$$Z_{ij}(s,\xi) = \frac{\mathsf{N}_{ij}(s,\xi)}{\mathsf{D}(s,\xi)} = \frac{\sum_k \mathsf{n}_k(s)\prod_{m=1}^{d}\xi_m^{\alpha_{km}}}{\sum_k \mathsf{d}_k(s)\prod_{m=1}^{d}\xi_m^{\alpha_{km}}}, \tag{2}$$

where $\alpha_{km} = \{0,1\}\ \forall k,m$, whereas n_k and d_k are polynomials in s.

3 Polynomial Chaos Expansion

In the uncertainty quantification of electrical circuits, a single PCE is normally used to model stochastic outputs of interest (see, e.g., [2]):

$$Z(s,\xi) \approx \sum_{\ell=1}^{L} Z_\ell(s)\varphi_\ell(\xi), \tag{3}$$

where φ_ℓ are multivariate polynomials in the parameters ξ. However, given the functional form (2) of

[1] A multi-linear function is a multivariate polynomial in which each term in the monomials has degree at most one.

the transfer functions, it is argued that a rational PCE model, i.e.,

$$Z(s,\xi) \approx \frac{\sum_{\ell=1}^{L} N_\ell(s)\varphi_\ell(\xi)}{1+\sum_{\ell=2}^{L} D_\ell(s)\varphi_\ell(\xi)} \tag{4}$$

provides a better model than (3).[2] This was effectively demonstrated based on a number of application examples in [3]. In particular, the model is exact for lumped circuits, provided that the polynomial basis $\{\varphi_\ell\}_{\ell=1}^{L}$ includes all the multi-linear terms appearing in (2). By extension, the model turns out to be more accurate, albeit not exact, also for distributed circuits.

4 Numerical Results

As an example, the proposed model is applied to the stochastic characterization of scattering parameters S_{11}, S_{21}, S_{31}, and S_{41} in the distributed circuit of Fig. 1, which includes three sections of coupled microstrip lines. In the first scenario, the variability is assumed on lumped components, i.e., the three capacitors and the inductor. Hence, the proposed rational model is in this case virtually exact. Indeed, the maximum errors on the standard deviation collected in the first part of Table 1 shows that the rational model exhibits an accuracy that is about four orders of magnitude better than the conventional PCE model.

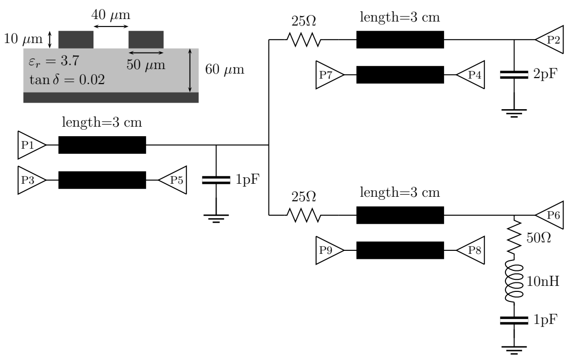

Fig. 1. Example test case: distributed circuit with coupled microstrip lines.

In the second scenario, the uncertainty is assumed to be on two geometrical parameters of the microstrip lines, namely the trace gap and the trace length. The results of the stochastic analysis are illustrated in Fig. 2 and are obtained with a conventional PCE of order six and with a rational PCE of order three. Data collected in Table 1 show that the proposed rational model is one to two orders of magnitude more accurate than the conventional PCE.

[2] The first coefficient in the denominator of (4) is set to one to remove indeterminacy.

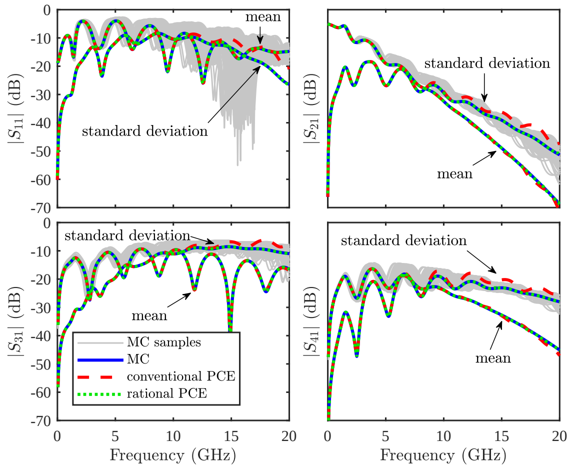

Fig. 2. Mean and standard deviation of the four scattering parameters under study for the scenario with uncertainty in the microstrip line geometry.

Table 1. Maximum errors on the standard deviation over frequency obtained with the conventional and proposed models.

	First scenario	
	conventional model	rational model
S_{11}	9.803×10^{-4}	1.601×10^{-8}
S_{21}	6.236×10^{-4}	1.606×10^{-8}
S_{31}	1.966×10^{-4}	1.459×10^{-8}
S_{41}	3.000×10^{-4}	6.139×10^{-9}
	Second scenario	
	conventional model	rational model
S_{11}	1.154×10^{-1}	4.699×10^{-3}
S_{21}	5.538×10^{-3}	2.310×10^{-4}
S_{31}	1.556×10^{-1}	1.600×10^{-3}
S_{41}	3.687×10^{-2}	4.461×10^{-3}

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