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Using the κ-exponential to deform Gumbel and Gompertz probability distributions

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Here we propose the use of the κ-exponential to deform Gumbel and the Gompertz distributions. The κ-exponential is a function of κ-statistics, a statistics which has been developed by G. Kaniadakis, Politecnico di Torino, in the framework of special relativity, and used for statistical analyses involving power law tailed distributions. The exponentiated κ-Gumbel function is also considered, to have a further deformation of the probability distribution.

Torino, April 12, 2022

Keywords: Gumbel distribution, Gompertz distribution, exponentiated Gumbel distribution. K-exponential, \kappa-exponential, k-exponential

In a previous discussion [Sparavigna, 2022], we have compared the Weibull distribution [Weibull, 1939, 1951] and the κ-Weibull distribution, that has been recently used to analyse time series of Covid-19 infections [Kaniadakis et al., 2020],[Sparavigna, 2021].

Here we consider the Gumbel and the Gompertz distributions and propose a deformation of them in the form of κ-Gumbel and κ-Gompertz distributions.

The Gumbel distribution is used to model the distribution of the maximum (or the minimum) of a number of samples of various distributions. It is also known as the log-Weibull distribution and the double exponential distribution. The Gumbel distribution is named after Emil Julius Gumbel (1891–1966), based on his original papers describing the distribution [Gumbel, 1935, 1941].

The cumulative distribution is given by:

$$F(x|\mu,\beta)=\exp(-\exp(-(x-\mu)/\beta)) \qquad (1)$$

The probability density function is obtained by: $f(x|\mu,\beta)=dF(x|\mu,\beta)/dx$.

Let us consider the analogue of the exponential in the κ-statistics [Kaniadakis, 2001, 2002, 2013]. The κ-exponential is defined in the following manner:

$$\exp_\kappa(x) = \left(\sqrt{1+\kappa^2 x^2} + \kappa x\right)^{1/\kappa} \qquad (2)$$

This function has been obtained in the framework of the κ-calculus [Kaniadakis, 2013], a calculus which has its roots in special relativity and is used for statistical analyses involving power law tailed statistical distributions. The κ-deformed exponential function is a continuous one parameter deformation of the Euler exponential function. Then, if we use the κ-exponential, the shape of the Gumbel distribution can be widely changed. When $\kappa \to 0$, we find the Gumbel distribution.

Then, let us define the cumulative function of the κ-Gumbel:

$$F_\kappa(x|\mu,\beta) = \exp_\kappa(-\exp_\kappa(-(x-\mu)/\beta)) \qquad (3)$$

Here in the two following figures, the cumulative and probability density functions.

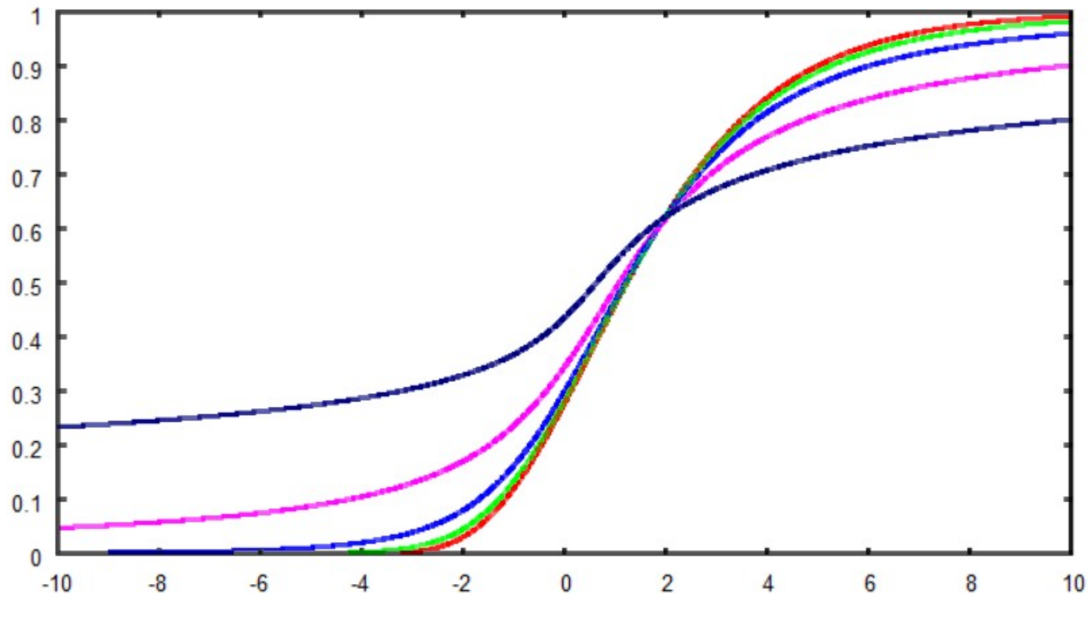

Fig.1 – The cumulative function of the Gumbel distribution (in red), and of the κ-Gumbel for different values of parameter κ ($\kappa=0.25$ green, $\kappa=0.5$ blue, $\kappa=1.0$ violet, $\kappa=2.0$ dark blue). Parameters are $\mu=0.5$, $\beta=2.0$.

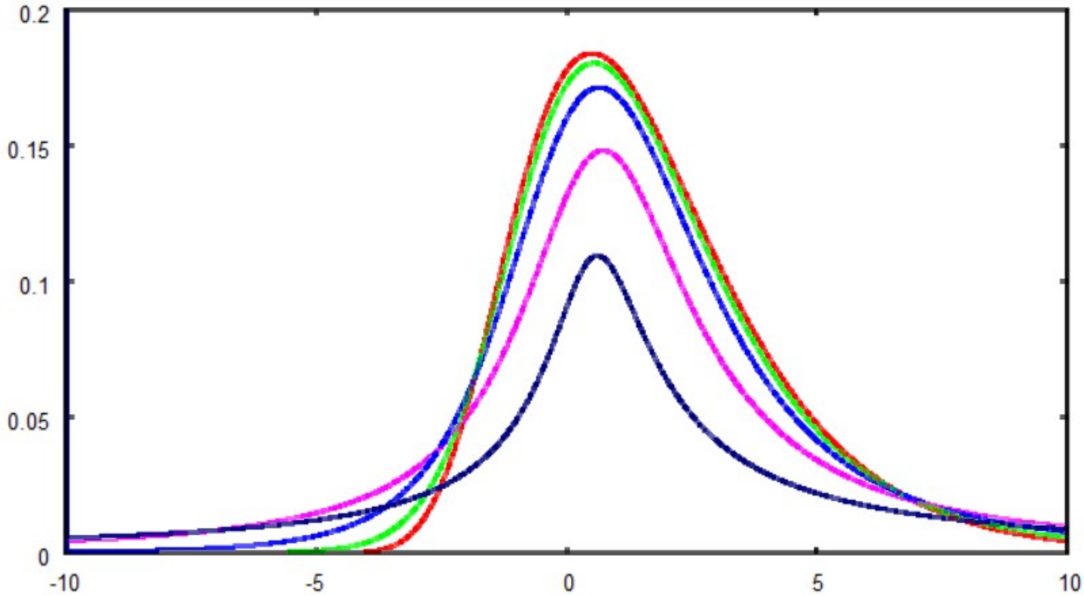

Fig.2 – The probability density function of the Gumbel distribution (in red), and of the κ-Gumbel for different values of parameter κ ($\kappa=0.25$ green, $\kappa=0.5$ blue, $\kappa=1.0$ violet, $\kappa=2.0$ dark blue). Parameters are $\mu=0.5, \beta=2.0$.

The Gumbel distribution is related to the Gompertz distribution: when its density is first reflected about the origin and then restricted to the positive half line, a Gompertz function is obtained.

The Gompertz distribution is a continuous probability distribution, named after Benjamin Gompertz (1779 – 1865). The cumulative function is:

$$F(x|b,\beta)=1-\exp(-\beta(\exp(bx)-1)) \qquad (4)$$

with two parameters b,β .

The κ-deformed cumulative function is:

$$F_\kappa(x|b,\beta)=1-\exp_\kappa(-\beta(\exp_\kappa(bx)-1)) \qquad (5)$$

In the following two figures, the cumulative and probability density functions.

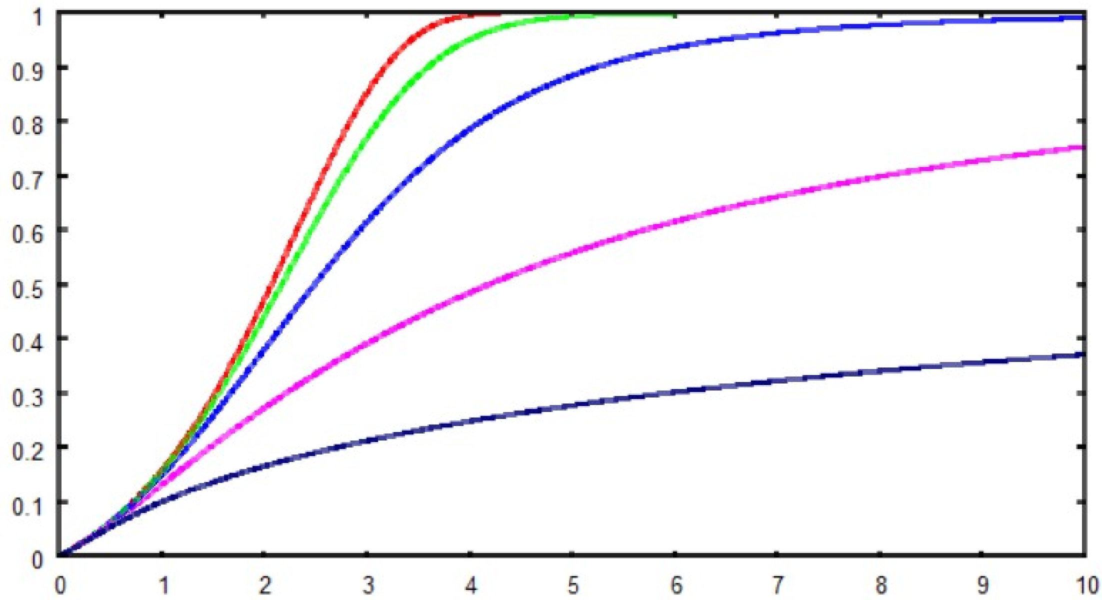

Fig.3 – The cumulative function of the Gompertz distribution (in red), and of the κ-Gompertz for different values of parameter κ ($\kappa=0.25$ green, $\kappa=0.5$ blue, $\kappa=1.0$ violet, $\kappa=2.0$ dark blue). Parameters are $b=1.0$, $\beta=0.1$.

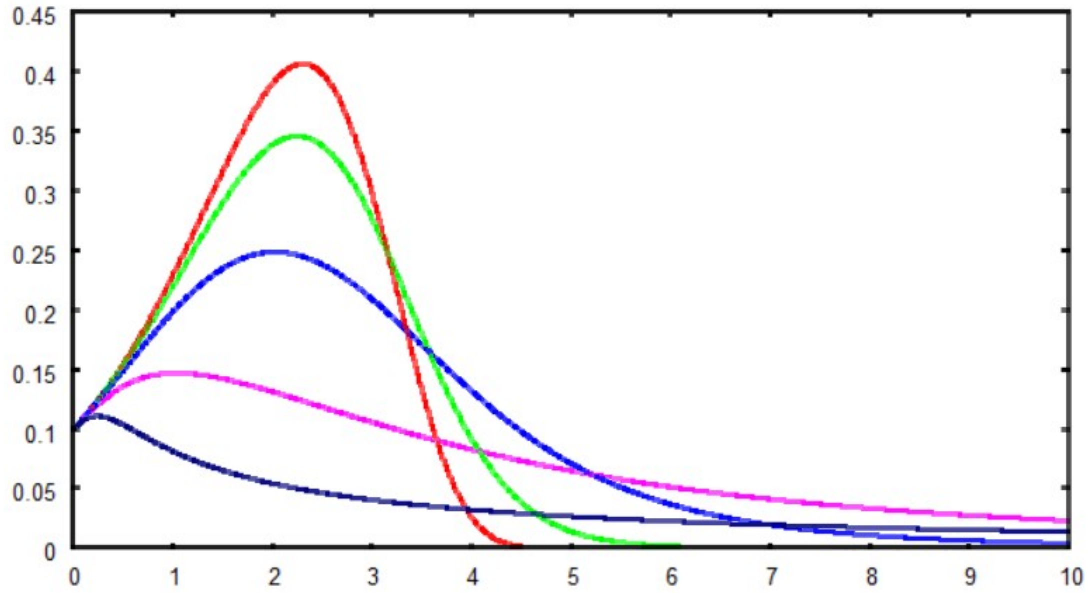

Fig.4 – The probability density function of the Gompertz distribution (in red), and of the κ-Gompertz for different values of parameter κ ($\kappa=0.25$ green, $\kappa=0.5$ blue, $\kappa=1.0$ violet, $\kappa=2.0$ dark blue). Parameters are $b=1.0$, $\beta=0.1$.

Exponentiated Gumbel distribution

In [Nadarajah, 2006], it is proposed an exponentiated Gumbel distribution, in the framework of climate applications. In the introduction of the article, the authors tells that the Gumbel distribution is perhaps the most widely applied distribution for climate modeling. Application include the global warning problems, fllod frequency analysis, offshore modeling, rainfall modeling, and wind speed modeling.

In [Nadarajah, 2006], the cumulative $F(x|\mu,\beta)=\exp(-\exp(-(x-\mu)/\beta))$ becomes:

$$F(x|\mu,\beta)=1-[1-\exp(-\exp(-(x-\mu)/\beta))]^{\alpha} \quad (6)$$

We can compare this exponentiated Gumbel distribution with an exponentiated κ-Gumbel distribution:

$$F_\kappa(x|\mu,\beta)=1-\left[1-\exp_\kappa(-\exp_\kappa(-(x-\mu)/\beta))\right]^\alpha \quad (7)$$

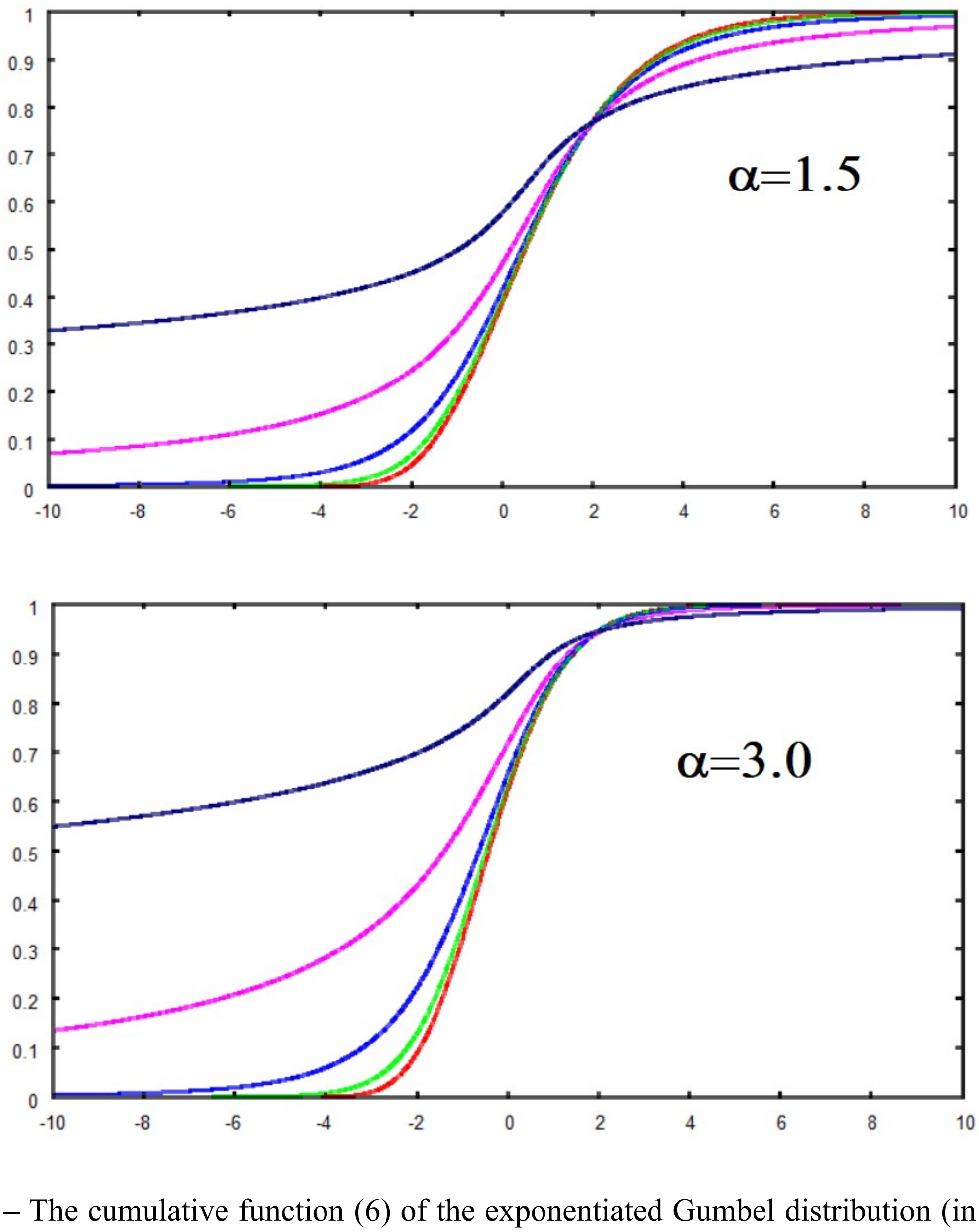

Fig. 5 – The cumulative function (6) of the exponentiated Gumbel distribution (in red), and of the exponentiated (7) κ-Gumbel for different values of parameter κ ($\kappa=0.25$ green, $\kappa=0.5$ blue, $\kappa=1.0$ violet, $\kappa=2.0$ dark blue). Parameters are $\mu=0.5$, $\beta=2.0$. The values of α is given in the panels.

We can see, comparing the panels of Figures 1 and 5, that the exponentiation changes the intersection of the Gumbel cumulative function with the κ-Gumbel.

It is also interesting to plot probability density function of the exponentiated Gumbel (6) and κ-Gumbel (7) distributions. We can see that the exponentiation is deforming, in a relevant manner, the κ-Gumbel.

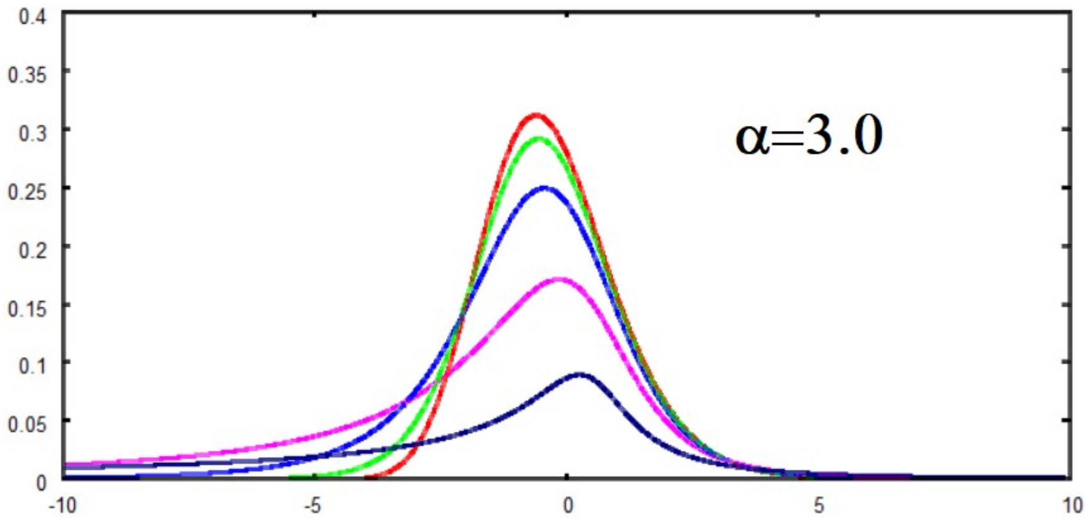

Fig. 5 – The probability density function corresponding to the cumulative function (6) of the exponentiated Gumbel distribution (in red), and of the exponentiated (7) κ-Gumbel for different values of parameter κ ($\kappa=0.25$ green, $\kappa=0.5$ blue, $\kappa=1.0$ violet, $\kappa=2.0$ dark blue). Parameters are $\mu=0.5$, $\beta=2.0$. The value of α is given in the panel.

The κκ-Gumbel with different κ parameters

We can also use two different values of the κ parameter:

$$F_{\kappa_2,\kappa_1}(x|\mu,\beta)=\exp_{\kappa_2}(-\exp_{\kappa_1}(-(x-\mu)/\beta)) \quad (8)$$

In the following figure, the cumulative function (upper panel) and the pdf (lower panel) of the Gumbel distribution (in red) are given with parameters $\mu=0.0$, $\beta=1.0$. The same parameter are used for the κκ-Gumbel. The green curve has parameters $\kappa_1=0.05$, $\kappa_2=2.0$ and the blue curve has . $\kappa_1=2.0$, $\kappa_2=0.05$.

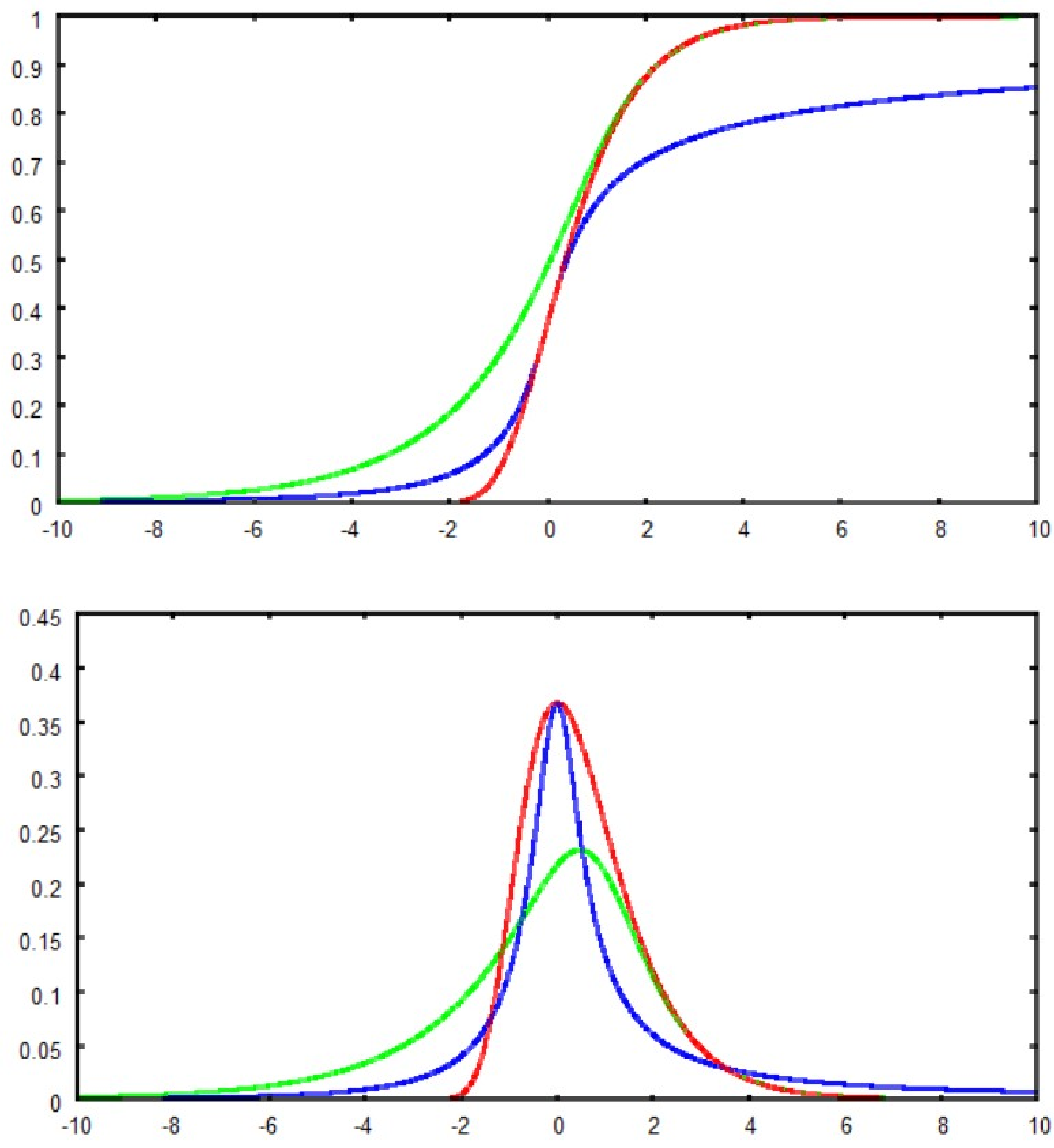

Fig. 6 – Gumbel (red) and κκ-Gumbel

(green, $\kappa_1=0.05$, $\kappa_2=2.0$; blue, $\kappa_1=2.0$, $\kappa_2=0.05$).

Discussion

In article [Hristopulos et al. 2015], the weakest-link scaling is used in the reliability analysis. The discussion considers Weibull and κ-Weibull functions and the Gamma function. In the conclusion of the article, we can find mentioned the Gumbel distribution as an "extreme value distribution that fits within the weak-scaling formalism". "The Gumbel is characterized by a different mathematical expression and size scaling than the Weibull distribution. There is both theoretical and experimental evidence (e.g., in the fracture of brittle ceramics) that in certain cases, the Gumbel is a more suitable asymptotic form than the Weibull". The authors add that it is not immediate to apply the same arguments that led to the κ-Weibull distribution to the Gumbel distribution and that this is an open subject of research.

References

1. Gumbel, E. J. (1935), Les valeurs extrêmes des distributions statistiques. Annales de l'Institut Henri Poincaré, 5 (2): 115–158
2. Gumbel E. J. (1941). The return period of flood flows. The Annals of Mathematical Statistics, 12, 163–190.
3. Hristopulos, D. T., Petrakis, M. P., & Kaniadakis, G. (2015). Weakest-link scaling and extreme events in finite-sized systems. Entropy, 17(3), 1103-1122.
4. Kaniadakis, G. (2001). Non-linear kinetics underlying generalized statistics. Physica A: Statistical mechanics and its applications, 296(3-4), 405-425.
5. Kaniadakis, G. (2002). Statistical mechanics in the context of special relativity. Physical review E, 66(5), 056125.
6. Kaniadakis G. (2013). Theoretical Foundations and Mathematical Formalism of the Power-Law Tailed Statistical Distributions. Entropy, Volume 15, Issue 10. Pages 3983-4010. https://doi.org/10.3390/e15103983
7. Kaniadakis, G., Baldi, M. M., Deisboeck, T. S., Grisolia, G., Hristopulos, D. T., Scarfone, A. M., Sparavigna, A. C., Wada, T. and Lucia, U., 2020. The κ-statistics approach to epidemiology. Scientific Reports, 10(1), pp.1-14. https://www.nature.com/articles/s41598-020-76673-3
8. Nadarajah, S. (2006). The exponentiated Gumbel distribution with climate application. Environmetrics: The official journal of the International Environmetrics Society, 17(1), 13-23.
9. Sparavigna, Amelia Carolina. (2021). A decomposition of waves in time series

of data related to Covid-19, applied to study the role of Alpha variant in the spread of infection. Zenodo. https://doi.org/10.5281/zenodo.5745008

10. Sparavigna, Amelia Carolina. (2022). Weibull, κ-Weibull and other probability distributions. Zenodo. https://doi.org/10.5281/zenodo.6425210
11. Weibull, W. (1939). A statistical theory of strength of materials. The Royal Swedish Institute of Engineering Research (Ingenors Vetenskaps Akadiens Handlingar), Proc. No. 151, Stockholm.
12. Weibull, Waloddi (1951). A Statistical Distribution Function of Wide Applicability. Journal of Applied Mechanics, American Society of Mechanical Engineers, 1951. Volume 18, Pages 293-297. https://hal.archives-ouvertes.fr/hal-03112318