

Lp spectral multipliers on the free group $N_{3,2}$

Original

Lp spectral multipliers on the free group $N_{3,2}$ / Martini, A.; Muller, D.. - In: STUDIA MATHEMATICA. - ISSN 0039-3223. - STAMPA. - 217:1(2013), pp. 41-55. [10.4064/sm217-1-3]

Availability:

This version is available at: 11583/2949496 since: 2022-01-21T17:37:28Z

Publisher:

POLISH ACAD SCIENCES INST MATHEMATICS-IMPAN

Published

DOI:10.4064/sm217-1-3

Terms of use:

openAccess

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

GENERICO preprint/submitted version accettata

This article has been accepted for publication in STUDIA MATHEMATICA, published by POLISH ACAD SCIENCES INST MATHEMATICS-IMPAN.

(Article begins on next page)

L^p SPECTRAL MULTIPLIERS ON THE FREE GROUP $N_{3,2}$

ALESSIO MARTINI AND DETLEF MÜLLER

ABSTRACT. Let L be a homogeneous sublaplacian on the 6-dimensional free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators. We prove a theorem of Mihlin-Hörmander type for the functional calculus of L , where the order of differentiability $s > 6/2$ is required on the multiplier.

1. INTRODUCTION

The free 2-step nilpotent Lie group $N_{3,2}$ on 3 generators is the simply connected, connected nilpotent Lie group defined by the relations

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2,$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ is a basis of its Lie algebra (that is, the Lie algebra of the left-invariant vector fields on $N_{3,2}$). In exponential coordinates, $N_{3,2}$ can be identified with $\mathbb{R}_x^3 \times \mathbb{R}_y^3$, where the group law is given by

$$(x, y) \cdot (x', y') = (x + x', y + y' + x \wedge x' / 2)$$

and $x \wedge x'$ denotes the usual vector product of $x, x' \in \mathbb{R}^3$. The family $(\delta_t)_{t>0}$ of automorphic dilations of $N_{3,2}$, defined by

$$\delta_t(x, y) = (tx, t^2y),$$

turns $N_{3,2}$ into a stratified group of homogeneous dimension $Q = 9$.

Let L be a homogeneous sublaplacian on $N_{3,2}$; without loss of generality, we may assume that $L = -(X_1^2 + X_2^2 + X_3^2)$. L is a self-adjoint operator on $L^2(N_{3,2})$, hence a functional calculus for L is defined via spectral integration and, for all Borel functions $F : \mathbb{R} \rightarrow \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(N_{3,2})$ whenever the “spectral multiplier” F is a bounded function. Here we are interested in giving a sufficient condition for the L^p -boundedness (for $p \neq 2$) of the operator $F(L)$, in terms of smoothness properties of the multiplier F .

Let $W_2^s(\mathbb{R})$ denote the L^2 Sobolev space of (fractional) order s . Then our main result reads as follows.

2010 *Mathematics Subject Classification.* Primary 43A22; Secondary 42B15.

Key words and phrases. nilpotent Lie groups, spectral multipliers, sublaplacians, Mihlin-Hörmander multipliers, singular integral operators.

Theorem 1.1. *Suppose that a function $F : \mathbb{R} \rightarrow \mathbb{C}$ satisfies*

$$\sup_{t>0} \|\eta F(t\cdot)\|_{W_2^s} < \infty$$

for some $s > 6/2$ and some nonzero $\eta \in C_c^\infty(]0, \infty[)$. Then the operator $F(L)$ is of weak type $(1, 1)$ and bounded on $L^p(N_{3,2})$ for all $p \in]1, \infty[$.

Observe that the general multiplier theorem for homogeneous sublaplacians on stratified Lie groups by Christ [3] and Mauceri and Meda [16] requires the stronger regularity condition $s > Q/2 = 9/2$. To the best of our knowledge, in the case of $N_{3,2}$ none of the results and techniques known so far allowed one to go below the condition $s > Q/2$. Our result pushes the regularity assumption down to $s > d/2 = 6/2$, where $d = 6$ is the topological dimension of $N_{3,2}$. We conjecture that this condition is sharp.

The problem of L^p -boundedness for spectral multipliers on nilpotent Lie groups has a long history, and the theorem by Christ and Mauceri and Meda is itself an improvement of a series of previous results (see, e.g., [4, 8, 5]). Nevertheless it is still an open question, whether the homogeneous dimension in the smoothness condition may always be replaced by the topological dimension.

It has been known for a long time [10, 17] that such an improvement of the multiplier theorem holds true in the case of the Heisenberg and related groups (more precisely, for direct products of Métivier and abelian groups; see also [11, 14]). This class of groups, however, does not include $N_{3,2}$, nor any free 2-step nilpotent group $N_{n,2}$ on n generators (see [20, §3] for a definition), except for the smallest one, $N_{2,2}$, which is the 3-dimensional Heisenberg group. The free groups $N_{n,2}$ have in a sense the maximal structural complexity among 2-step groups, since every 2-step nilpotent Lie group is a quotient of a free one. Our result should then hopefully shed some new light and contribute to the understanding of the problem for general 2-step nilpotent Lie groups.

2. STRATEGY OF THE PROOF

The sublaplacian L is a left-invariant operator on $N_{3,2}$, hence any operator of the form $F(L)$ is left-invariant too. Let $\mathcal{K}_{F(L)}$ then denote the convolution kernel of $F(L)$. As shown, e.g., in [14, Theorem 4.6], the previous Theorem 1.1 is a consequence of the following L^1 -estimate.

Proposition 2.1. *For all $s > 6/2$, for all compact sets $K \subseteq]0, \infty[$, and for all functions $F : \mathbb{R} \rightarrow \mathbb{C}$ such that $\text{supp } F \subseteq K$,*

$$(2.1) \quad \|\mathcal{K}_{F(L)}\|_1 \leq C_{K,s} \|F\|_{W_2^s}.$$

Let $|\cdot|_\delta$ be any δ_t -homogeneous norm on $N_{3,2}$; take, e.g., $|(x, y)|_\delta = |x| + |y|^{1/2}$. The crucial estimate in the proof of [16] of the general theorem for stratified groups, that is,

$$(2.2) \quad \|(1 + |\cdot|_\delta)^\alpha \mathcal{K}_{F(L)}\|_2 \leq C_{K,\alpha,\beta} \|F\|_{W_2^\beta}$$

for all $\alpha \geq 0$ and $\beta > \alpha$, implies (2.1) when $s > 9/2$, by Hölder's inequality. In order to push the condition down to $s > 6/2$, here we prove an enhanced version of (2.2), that is,

$$(2.3) \quad \|(1 + |\cdot|_\delta)^\alpha w^r \mathcal{K}_{F(L)}\|_2 \leq C_{K,\alpha,\beta,r} \|F\|_{W_2^\beta},$$

for some “extra weight” function w on $N_{3,2}$, and suitable constraints on the exponents α, β, r .

A similar approach is adopted in the mentioned works on the Heisenberg and related groups. However, in [17] the extra weight w is the full weight $1 + |\cdot|_\delta$, while [10] employs the weight $w(x, y) = 1 + |x|$. Here instead the weight $w(x, y) = 1 + |y|$ is used, and (2.3) is proved under the conditions $\alpha \geq 0$, $0 \leq r < 3/2$, $\beta > \alpha + r$ (see Proposition 4.6 below).

The proof of (2.3) when $\alpha = 0$ is based on a careful analysis exploiting identities for Laguerre polynomials, somehow in the spirit of [4, 17, 19], but with additional complexity due, inter alia, to the simultaneous use of generalized Laguerre polynomials of different types. The estimate for arbitrary α is then recovered by interpolation with (2.2). An analogous strategy is followed in [15], where identities for Hermite polynomials are used in order to prove a sharp spectral multiplier theorem for Grushin operators.

3. A JOINT FUNCTIONAL CALCULUS

It is convenient for us to embed the functional calculus for the sublaplacian L in a larger functional calculus for a system of commuting left-invariant differential operators on $N_{3,2}$. Specifically, the operators

$$(3.1) \quad L, -iY_1, -iY_2, -iY_3$$

are essentially self-adjoint and commute strongly, hence they admit a joint functional calculus (see, e.g., [13]).

If \mathbf{Y} denotes the “vector of operators” $(-iY_1, -iY_2, -iY_3)$, then we can express the convolution kernel $\mathcal{K}_{G(L,\mathbf{Y})}$ of the operator $G(L, \mathbf{Y})$ in terms of Laguerre functions (cf. [7]). Namely, for all $n, k \in \mathbb{N}$, let

$$L_n^{(k)}(u) = \frac{u^{-k} e^u}{n!} \left(\frac{d}{du} \right)^n (u^{k+n} e^{-u})$$

be the n -th Laguerre polynomial of type k , and define

$$\mathcal{L}_n^{(k)}(t) = (-1)^n e^{-t} L_n^{(k)}(2t).$$

Further, for all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\xi \in \mathbb{R}^3$, define ξ_{\parallel}^{η} and ξ_{\perp}^{η} by

$$\xi_{\parallel}^{\eta} = \langle \xi, \eta/|\eta| \rangle, \quad \xi_{\perp}^{\eta} = \xi - \xi_{\parallel}^{\eta} \eta/|\eta|.$$

Proposition 3.1. *Let $G : \mathbb{R}^4 \rightarrow \mathbb{C}$ be in the Schwartz class, and set*

$$(3.2) \quad m(n, \mu, \eta) = G((2n+1)|\eta| + \mu^2, \eta),$$

for all $n \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^3$ with $\eta \neq 0$. Then

$$\begin{aligned} \mathcal{K}_{G(L, \mathbf{Y})}(x, y) &= \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{N}} m(n, \xi_{\parallel}^{\eta}, \eta) \mathcal{L}_n^{(0)}(|\xi_{\perp}^{\eta}|^2/|\eta|) e^{i\langle \xi, x \rangle} e^{i\langle \eta, y \rangle} d\xi d\eta. \end{aligned}$$

Proof. For all $\eta \in \mathbb{R}^3 \setminus \{0\}$, choose a unit vector $E_{\eta} \in \eta^{\perp}$, and set $\bar{E}_{\eta} = (\eta/|\eta|) \wedge E_{\eta}$; moreover, for all $x \in \mathbb{R}^3$, denote by x_1^{η} , x_2^{η} , x_{\parallel}^{η} the components of x with respect to the positive orthonormal basis E^{η} , \bar{E}^{η} , $\eta/|\eta|$ of \mathbb{R}^3 .

For all $\eta \in \mathbb{R}^3 \setminus \{0\}$ and all $\mu \in \mathbb{R}$, an irreducible unitary representation $\pi_{\eta, \mu}$ of $N_{3,2}$ on $L^2(\mathbb{R})$ is defined by

$$\pi_{\eta, \mu}(x, y)\phi(u) = e^{i\langle \eta, y \rangle} e^{i|\eta|(u+x_1^{\eta}/2)x_2^{\eta}} e^{i\mu x_{\parallel}^{\eta}} \phi(x_1^{\eta} + u)$$

for all $(x, y) \in N_{3,2}$, $u \in \mathbb{R}$, $\phi \in L^2(\mathbb{R})$. Following, e.g., [1, §2], one can see that these representations are sufficient to write the Plancherel formula for the group Fourier transform of $N_{3,2}$, and the corresponding Fourier inversion formula:

$$(3.3) \quad f(x, y) = (2\pi)^{-5} \int_{\mathbb{R}^3 \setminus \{0\}} \int_{\mathbb{R}} \text{tr}(\pi_{\eta, \mu}(x, y) \pi_{\eta, \mu}(f)) |\eta| d\mu d\eta$$

for all $f : N_{3,2} \rightarrow \mathbb{C}$ in the Schwartz class and all $(x, y) \in N_{3,2}$, where $\pi_{\eta, \mu}(f) = \int_{N_{3,2}} f(z) \pi_{\eta, \mu}(z^{-1}) dz$.

Fix $\eta \in \mathbb{R}^3 \setminus \{0\}$ and $\mu \in \mathbb{R}$. The operators (3.1) are represented in $\pi_{\eta, \mu}$ as

$$(3.4) \quad d\pi_{\eta, \mu}(L) = -\partial_u^2 + |\eta|^2 u^2 + \mu^2, \quad d\pi_{\eta, \mu}(-iY_j) = \eta_j.$$

If h_n is the n -th Hermite function, that is,

$$h_n(t) = (-1)^n (n! 2^n \sqrt{\pi})^{-1/2} e^{t^2/2} \left(\frac{d}{dt} \right)^n e^{-t^2},$$

and $\tilde{h}_{\eta, n}$ is defined by

$$\tilde{h}_{\eta, n}(u) = |\eta|^{1/4} h_n(|\eta|^{1/2} u),$$

then $\{\tilde{h}_{\eta,n}\}_{n \in \mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R})$, made of joint eigenfunctions of the operators (3.4); in fact,

$$(3.5) \quad \begin{aligned} d\pi_{\eta,\mu}(L)\tilde{h}_{\eta,n} &= (|\eta|(2n+1) + \mu^2)\tilde{h}_{\eta,n}, \\ d\pi_{\eta,\mu}(-iY_j)\tilde{h}_{\eta,n} &= \eta_j\tilde{h}_{\eta,n}. \end{aligned}$$

Moreover the corresponding diagonal matrix coefficients $\varphi_{\eta,\mu,n}$ of $\pi_{\eta,\mu}$ are given by

$$\begin{aligned} \varphi_{\eta,\mu,n}(x, y) &= \langle \pi_{\eta,\mu}(x, y)\tilde{h}_{\eta,n}, \tilde{h}_{\eta,n} \rangle \\ &= e^{i\langle \eta, y \rangle} e^{i\mu x_\parallel^\eta} |\eta|^{1/2} \int_{\mathbb{R}} e^{i|\eta|u x_2^\eta} h_n(|\eta|^{1/2}(u + x_1^\eta/2)) h_n(|\eta|^{1/2}(u - x_1^\eta/2)) du. \end{aligned}$$

The last integral is essentially the Fourier-Wigner transform of the pair (h_n, h_n) , whose Fourier transform has a particularly simple expression (cf. [9, formula (1.90)]); the parity of the Hermite functions then yields

$$\begin{aligned} \varphi_{\eta,\mu,n}(x, y) &= e^{i\langle \eta, y \rangle} e^{i\mu x_\parallel^\eta} \frac{(-1)^n}{\pi|\eta|} \int_{\mathbb{R}^2} e^{iv_2 x_2^\eta} e^{iv_1 x_1^\eta} \\ &\quad \times \int_{\mathbb{R}} e^{-it(2v_1/|\eta|^{1/2})} h_n(t + v_2/|\eta|^{1/2}) h_n(t - v_2/|\eta|^{1/2}) dt dv, \end{aligned}$$

that is,

$$(3.6) \quad \varphi_{\eta,\mu,n}(x, y) = \frac{1}{\pi|\eta|} e^{i\langle \eta, y \rangle} e^{i\mu x_\parallel^\eta} \int_{\mathbb{R}^2} e^{iv_1 x_1^\eta} e^{iv_2 x_2^\eta} \mathcal{L}_n^{(0)}(|v|^2/|\eta|) dv$$

(see [21, Theorem 1.3.4] or [9, Theorem 1.104]).

Note that $\mathcal{K}_{G(L, \mathbf{Y})} \in \mathcal{S}(N_{3,2})$ since $G \in \mathcal{S}(\mathbb{R}^4)$ (see [2, Theorem 5.2] or [12, §4.2]). Moreover

$$\pi_{\eta,\mu}(\mathcal{K}_{G(L, \mathbf{Y})})\tilde{h}_{\eta,n} = G(|\eta|(2n+1) + \mu^2, \eta)\tilde{h}_{\eta,n}$$

by (3.5) and [18, Proposition 1.1], hence

$$\langle \pi_{\eta,\mu}(x, y)\pi_{\eta,\mu}(\mathcal{K}_{G(L, \mathbf{Y})})\tilde{h}_{\eta,n}, \tilde{h}_{\eta,n} \rangle = m(n, \mu, \eta) \varphi_{\eta,\mu,n}(x, y).$$

Therefore, by (3.3) and (3.6),

$$\begin{aligned} \mathcal{K}_{G(L, \mathbf{Y})}(x, y) &= (2\pi)^{-5} \int_{\mathbb{R}^3 \setminus \{0\}} \int_{\mathbb{R}} \sum_{n \in \mathbb{N}} m(n, \mu, \eta) \varphi_{\eta,\mu,n}(x, y) |\eta| d\mu d\eta \\ &= \frac{2}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{n \in \mathbb{N}} m(n, \xi_3, \eta) e^{i\langle \eta, y \rangle} e^{i\langle \xi, (x_1^\eta, x_2^\eta, x_\parallel^\eta) \rangle} \mathcal{L}_n^{(0)}((\xi_1^2 + \xi_2^2)/|\eta|) d\xi d\eta. \end{aligned}$$

The conclusion follows by a change of variable in the inner integral. \square

4. WEIGHTED ESTIMATES

For convenience, set $\mathcal{L}_n^{(k)} = 0$ for all $n < 0$. The following identities are easily obtained from the properties of Laguerre polynomials (see, e.g., [6, §10.12]).

Lemma 4.1. *For all $k, n, n' \in \mathbb{N}$ and $t \in \mathbb{R}$,*

$$(4.1) \quad \mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) + \mathcal{L}_n^{(k+1)}(t),$$

$$(4.2) \quad \frac{d}{dt} \mathcal{L}_n^{(k)}(t) = \mathcal{L}_{n-1}^{(k+1)}(t) - \mathcal{L}_n^{(k+1)}(t),$$

$$(4.3) \quad \int_0^\infty \mathcal{L}_n^{(k)}(t) \mathcal{L}_{n'}^{(k)}(t) t^k dt = \begin{cases} \frac{(n+k)!}{2^{k+1}n!} & \text{if } n = n', \\ 0 & \text{otherwise.} \end{cases}$$

We introduce some operators on functions $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$:

$$\begin{aligned} \tau f(n, \mu, \eta) &= f(n+1, \mu, \eta), \\ \delta f(n, \mu, \eta) &= f(n+1, \mu, \eta) - f(n, \mu, \eta), \\ \partial_\mu f(n, \mu, \eta) &= \frac{\partial}{\partial \mu} f(n, \mu, \eta), \\ \partial_\eta^\alpha f(n, \mu, \eta) &= \left(\frac{\partial}{\partial \eta} \right)^\alpha f(n, \mu, \eta), \end{aligned}$$

for all $\alpha \in \mathbb{N}^3$. For all multiindices $\alpha \in \mathbb{N}^3$, we denote by $|\alpha|$ its length $\alpha_1 + \alpha_2 + \alpha_3$. We set moreover $\langle t \rangle = 2|t| + 1$ for all $t \in \mathbb{R}$.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$, $\tau^l f$ is null for all sufficiently large $l \in \mathbb{N}$; hence the operator $1 + \tau$, when restricted to the set of compactly supported functions, is invertible, with inverse given by

$$(1 + \tau)^{-1} f = \sum_{l \in \mathbb{N}} (-1)^l \tau^l f,$$

and therefore the operator $(1 + \tau)^q$ is well-defined for all $q \in \mathbb{Z}$.

Proposition 4.2. *Let $G : \mathbb{R}^4 \rightarrow \mathbb{C}$ be smooth and compactly supported in $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and let $m(n, \mu, \eta)$ be defined by (3.2). For all $\alpha \in \mathbb{N}^3$,*

$$(4.4) \quad \begin{aligned} & \int_{N_{3,2}} |y^\alpha \mathcal{K}_{G(L, \mathbf{Y})}(x, y)|^2 dx dy \\ & \leq C_\alpha \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1 + \tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta)|^2 \\ & \quad \times \mu^{2b_\iota} |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \langle n \rangle^{|\beta^\iota|} d\mu d\eta, \end{aligned}$$

where I_α is a finite set and, for all $\iota \in I_\alpha$,

- $\gamma^\iota \in \mathbb{N}^3$, $l_\iota, k_\iota \in \mathbb{N}$, $\gamma^\iota \leq \alpha$, $\min\{1, |\alpha|\} \leq |\gamma^\iota| + l_\iota + k_\iota \leq |\alpha|$,

- $b_l \in \mathbb{N}$, $\beta^\nu \in \mathbb{N}^3$, $b_l + |\beta^\nu| = l_\nu + 2k_\nu$, $|\gamma^\nu| + l_\nu + b_l \leq |\alpha|$.

Proof. Proposition 3.1 and integration by parts allow us to write

$$(4.5) \quad y^\alpha \mathcal{K}_{G(L, \mathbf{Y})}(x, y) = \frac{2i^{|\alpha|}}{(2\pi)^6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\left(\frac{\partial}{\partial \eta} \right)^\alpha \sum_{n \in \mathbb{N}} m(n, \xi_\parallel^\eta, \eta) \mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2/|\eta|) \right] e^{i\langle \xi, x \rangle} e^{i\langle \eta, y \rangle} d\xi d\eta.$$

From the definition of ξ_\parallel^η and ξ_\perp^η , the following identities are not difficult to obtain:

$$(4.6) \quad \begin{aligned} \frac{\partial}{\partial \eta_j} \xi_\parallel^\eta &= (\xi_\perp^\eta)_j \frac{1}{|\eta|}, & \frac{\partial}{\partial \eta_j} (\xi_\perp^\eta)_k &= -\xi_\parallel^\eta \frac{\partial}{\partial \eta_j} \frac{\eta_k}{|\eta|} - (\xi_\perp^\eta)_j \frac{\eta_k}{|\eta|^2}, \\ \frac{\partial}{\partial \eta_j} \frac{|\xi_\perp^\eta|^2}{|\eta|} &= -\xi_\parallel^\eta (\xi_\perp^\eta)_j \frac{2}{|\eta|^2} - |\xi_\perp^\eta|^2 \frac{\eta_j}{|\eta|^3}. \end{aligned}$$

The multiindex notation will also be used as follows:

$$(\xi_\perp^\eta)^\beta = (\xi_\perp^\eta)_1^{\beta_1} (\xi_\perp^\eta)_2^{\beta_2} (\xi_\perp^\eta)_3^{\beta_3}$$

for all $\xi, \eta \in \mathbb{R}$, with $\eta \neq 0$, and all $\beta \in \mathbb{N}^3$; consequently

$$|\xi_\perp^\eta|^2 = (\xi_\perp^\eta)^{(2,0,0)} + (\xi_\perp^\eta)^{(0,2,0)} + (\xi_\perp^\eta)^{(0,0,2)}.$$

Via these identities, one can prove inductively that, for all $\alpha \in \mathbb{N}^3$,

$$(4.7) \quad \begin{aligned} &\left(\frac{\partial}{\partial \eta} \right)^\alpha \sum_{n \in \mathbb{N}} m(n, \xi_\parallel^\eta, \eta) \mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2/|\eta|) \\ &= \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}} \partial_\eta^{\gamma^\iota} \partial_\mu^{\delta^\iota} \delta^{k_\iota} m(n, \xi_\parallel^\eta, \eta) (\xi_\parallel^\eta)^{b_\iota} (\xi_\perp^\eta)^{\beta^\iota} \Theta_\iota(\eta) \mathcal{L}_n^{(k_\iota)}(|\xi_\perp^\eta|^2/|\eta|), \end{aligned}$$

where I_α , γ^ι , l_ι , k_ι , b_ι , β^ι are as in the statement above, while $\Theta_\iota : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}$ is smooth and homogeneous of degree $|\gamma^\iota| - |\alpha| - k_\iota$. For the inductive step, one employs Leibniz' rule, and when a derivative hits a Laguerre function, the identity (4.2) together with summation by parts is used.

Note that, for all compactly supported $f : \mathbb{N} \times \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$,

$$\sum_{n \in \mathbb{N}} f(n, \mu, \eta) \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau) f(n, \mu, \eta) \mathcal{L}_n^{(k+1)}(t),$$

by (4.1). Since $1 + \tau$ is invertible, simple manipulations and iteration yield the more general identity

$$\sum_{n \in \mathbb{N}} f(n, \mu, \eta) \mathcal{L}_n^{(k)}(t) = \sum_{n \in \mathbb{N}} (1 + \tau)^{k' - k} f(n, \mu, \eta) \mathcal{L}_n^{(k')}(t),$$

for all $k, k' \in \mathbb{N}$. This formula allows us to adjust in (4.7) the type of the Laguerre functions to the exponent of ξ_\perp , and to obtain that

$$\begin{aligned} & \left(\frac{\partial}{\partial \eta} \right)^\alpha \sum_{n \in \mathbb{N}} m(n, \xi_\parallel^\eta, \eta) \mathcal{L}_n^{(0)}(|\xi_\perp^\eta|^2 / |\eta|) \\ &= \sum_{\iota \in I_\alpha} \sum_{n \in \mathbb{N}} \partial_\eta^{\gamma_\iota} \partial_\mu^{\iota_\mu} \delta^{k_\iota} (1 + \tau)^{|\beta_\iota| - k_\iota} m(n, \xi_\parallel^\eta, \eta) \\ & \quad \times (\xi_\parallel^\eta)^{b_\iota} (\xi_\perp^\eta)^{\beta_\iota} \Theta_\iota(\eta) \mathcal{L}_n^{(|\beta_\iota|)}(|\xi_\perp^\eta|^2 / |\eta|), \end{aligned}$$

By plugging this identity into (4.5) and exploiting Plancherel's formula for the Fourier transform, the finiteness of I_α and the triangular inequality, we get that

$$\begin{aligned} & \int_{N_{3,2}} |y^\alpha \mathcal{K}_{G(L, \mathbf{Y})}(x, y)|^2 dx dy \\ & \leq C_\alpha \sum_{\iota \in I_\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left| \sum_{n \in \mathbb{N}} \partial_\eta^{\gamma_\iota} \partial_\mu^{\iota_\mu} \delta^{k_\iota} (1 + \tau)^{|\beta_\iota| - k_\iota} m(n, \mu, \eta) \mathcal{L}_n^{(|\beta_\iota|)}(|\zeta|^2 / |\eta|) \right|^2 \\ & \quad \times \mu^{2b_\iota} |\zeta|^{2|\beta_\iota|} |\eta|^{2|\gamma_\iota| - 2|\alpha| - 2k_\iota} d\zeta d\mu d\eta \end{aligned}$$

A passage to polar coordinates in the ζ -integral and a rescaling then give that

$$\begin{aligned} & \int_{N_{3,2}} |y^\alpha \mathcal{K}_{G(L, \mathbf{Y})}(x, y)|^2 dx dy \\ & \leq C_\alpha \sum_{\iota \in I_\alpha} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \int_0^\infty \left| \sum_{n \in \mathbb{N}} \partial_\eta^{\gamma_\iota} \partial_\mu^{\iota_\mu} \delta^{k_\iota} (1 + \tau)^{|\beta_\iota| - k_\iota} m(n, \mu, \eta) \mathcal{L}_n^{(|\beta_\iota|)}(s) \right|^2 s^{|\beta_\iota|} ds \\ & \quad \times \mu^{2b_\iota} |\eta|^{2|\gamma_\iota| - 2|\alpha| - 2k_\iota + |\beta_\iota| + 1} d\mu d\eta, \end{aligned}$$

and the conclusion follows by applying the orthogonality relations (4.3) for the Laguerre functions to the inner integral. \square

Note that $\tau f(\cdot, \mu, \eta)$, $\delta f(\cdot, \mu, \eta)$ depend only on $f(\cdot, \mu, \eta)$; in other words, τ and δ can be considered as operators on functions $\mathbb{N} \rightarrow \mathbb{C}$. The next lemma will be useful in converting finite differences into continuous derivatives.

Lemma 4.3. *Let $f : \mathbb{N} \rightarrow \mathbb{C}$ have a smooth extension $\tilde{f} : [0, \infty[\rightarrow \mathbb{C}$, and let $k \in \mathbb{N}$. Then*

$$\delta^k f(n) = \int_{J_k} \tilde{f}^{(k)}(n + s) d\nu_k(s)$$

for all $n \in \mathbb{N}$, where $J_k = [0, k]$ and ν_k is a Borel probability measure on J_k .

In particular

$$|\delta^k f(n)|^2 \leq \int_{J_k} |\tilde{f}^{(k)}(n + s)|^2 d\nu_k(s)$$

for all $n \in \mathbb{N}$.

Proof. Iterated application of the fundamental theorem of integral calculus gives

$$\delta^k f(n) = \int_{[0,1]^k} \tilde{f}^{(k)}(n + s_1 + \cdots + s_k) ds.$$

The conclusion follows by taking as ν_k the push-forward of the uniform distribution on $[0,1]^k$ via the map $(s_1, \dots, s_k) \mapsto s_1 + \cdots + s_k$, and by Hölder's inequality. \square

We give now a simplified version of the right-hand side of (4.4), in the case where we restrict to the functional calculus for the sublaplacian L alone. In order to avoid divergent series, however, it is convenient at first to truncate the multiplier along the spectrum of \mathbf{Y} .

Lemma 4.4. *Let $\chi \in C_c^\infty(\mathbb{R})$ be supported in $[1/2, 2]$, $K \subseteq]0, \infty[$ be compact and $M \in]0, \infty[$. If $F : \mathbb{R} \rightarrow \mathbb{C}$ is smooth and supported in K , and $F_M : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ is given by*

$$F_M(\lambda, \eta) = F(\lambda) \chi(|\eta|/M),$$

then, for all $r \in [0, \infty[$,

$$\int_{N_{3,2}} ||y|^r \mathcal{K}_{F_M(L, \mathbf{Y})}(x, y)|^2 dx dy \leq C_{K, \chi, r} M^{3-2r} \|F\|_{W_2^r}^2.$$

Proof. We may restrict to the case $r \in \mathbb{N}$, the other cases being recovered a posteriori by interpolation. Hence we need to prove that

$$(4.8) \quad \int_{N_{3,2}} |y^\alpha \mathcal{K}_{F_M(L, \mathbf{Y})}(x, y)|^2 dx dy \leq C_{K, \chi, \alpha} M^{3-2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2$$

for all $\alpha \in \mathbb{N}^3$. On the other hand, if

$$m(n, \mu, \eta) = F(|\eta|\langle n \rangle + \mu^2) \chi(|\eta|/M),$$

then the left-hand side of (4.8) can be majorized by (4.4), and we are reduced to proving that

$$(4.9) \quad \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{k_\iota} \delta^{k_\iota} (1 + \tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta)|^2 \mu^{2b_\iota} |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \\ \times \langle n \rangle^{|\beta^\iota|} d\mu d\eta \leq C_{K, \chi, \alpha} M^{3-2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2$$

for all $\iota \in I_\alpha$.

Consider first the case $|\beta^\iota| \geq k_\iota$. A smooth extension $\tilde{m} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$ of m is defined by

$$\tilde{m}(t, \mu, \eta) = F(|\eta|(2t + 1) + \mu^2) \chi(|\eta|/M).$$

Then, by Lemma 4.3,

$$\begin{aligned} & \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1 + \tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta) \\ &= \sum_{j=0}^{|\beta^\iota| - k_\iota} \binom{|\beta^\iota| - k_\iota}{j} \int_{J_\iota} \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(n + j + s, \mu, \eta) d\nu_\iota(s), \end{aligned}$$

where $J_\iota = [0, k_\iota]$ and ν_ι is a suitable probability measure on J_ι ; consequently (4.9) will be proved if we show that

$$(4.10) \quad \sum_{n \in \mathbb{N}} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(n + s, \mu, \eta)|^2 \mu^{2b_\iota} |\eta|^{2|\gamma^\iota| - 2|\alpha| - 2k_\iota + |\beta^\iota| + 1} \\ \times \langle n \rangle^{|\beta^\iota|} d\mu d\eta \leq C_{K, \chi, \alpha} M^{3 - 2|\alpha|} \|F\|_{W_2^{|\alpha|}}^2$$

for all $s \in [0, |\beta^\iota|]$. On the other hand, it is easily proved inductively that

$$\begin{aligned} & \partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(t, \mu, \eta) \\ &= \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota| - l_\iota + r} \sum_{q=0}^{|\gamma^\iota| - v} \Psi_{l, v, q}(\eta) \langle t \rangle^v \mu^{2r - l_\iota} M^{-q} F^{(k_\iota + v + r)}(|\eta| \langle t \rangle + \mu^2) \chi^{(q)}(|\eta|/M) \end{aligned}$$

for all $t \geq 0$, where $\Psi_{l, v, q} : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}$ is smooth and homogeneous of degree $k_\iota + v + q - |\gamma^\iota|$; hence

$$(4.11) \quad |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{k_\iota} \tilde{m}(t, \mu, \eta)|^2 \leq C_{\chi, \alpha} \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota| - l_\iota + r} M^{2k_\iota + 2v - 2|\gamma^\iota|} \langle t \rangle^{2v} \mu^{4r - 2l_\iota} \\ \times |F^{(k_\iota + v + r)}(|\eta| \langle t \rangle + \mu^2)|^2 \tilde{\chi}(|\eta|/M),$$

where $\tilde{\chi}$ is the characteristic function of $[1/2, 2]$, and we are using the fact that $|\eta| \sim M$ in the region where $\tilde{\chi}(|\eta|/M) \neq 0$. Consequently the left-hand

side of (4.10) is majorized by

$$\begin{aligned}
C_{\chi,\alpha} & \sum_{r=\lceil l_i/2 \rceil}^{l_i} \sum_{v=0}^{|\gamma^t|} M^{2v-2|\alpha|+|\beta^t|+1} \sum_{n \in \mathbb{N}} \langle n \rangle^{|\beta^t|} \langle n+s \rangle^{2v} \\
& \times \int_{\mathbb{R}^3} \int_{\mathbb{R}} |F^{(k_i+v+r)}(|\eta| \langle n+s \rangle + \mu^2)|^2 \mu^{2b_i+4r-2l_i} \tilde{\chi}(|\eta|/M) d\mu d\eta \\
& \leq C_{\chi,\alpha} \sum_{r=\lceil l_i/2 \rceil}^{l_i} \sum_{v=0}^{|\gamma^t|} M^{2v-2|\alpha|+|\beta^t|+3} \sum_{n \in \mathbb{N}} \langle n+s \rangle^{|\beta^t|+2v} \\
& \times \int_0^\infty \int_0^\infty |F^{(k_i+v+r)}(\rho \langle n+s \rangle + \mu^2)|^2 \mu^{2b_i+4r-2l_i} \tilde{\chi}(\rho/M) d\mu d\rho \\
& \leq C_{\chi,\alpha} \sum_{r=\lceil l_i/2 \rceil}^{l_i} \sum_{v=0}^{|\gamma^t|} M^{2v-2|\alpha|+|\beta^t|+3} \int_0^\infty \int_0^\infty |F^{(k_i+v+r)}(\rho + \mu^2)|^2 \\
& \times \mu^{2b_i+4r-2l_i} \sum_{n \in \mathbb{N}} \langle n+s \rangle^{|\beta^t|+2v-1} \tilde{\chi}(\rho/(\langle n+s \rangle M)) d\mu d\rho,
\end{aligned}$$

by passing to polar coordinates and rescaling. The last sum in n is easily controlled by $(\rho/M)^{|\beta^t|+2v}$, hence the left-hand side of (4.10) is majorized by

$$\begin{aligned}
C_{\chi,\alpha} M^{3-2|\alpha|} & \sum_{r=\lceil l_i/2 \rceil}^{l_i} \sum_{v=0}^{|\gamma^t|} \int_0^\infty \int_0^\infty |F^{(k_i+v+r)}(\rho + \mu^2)|^2 \mu^{2b_i+4r-2l_i} \rho^{|\beta^t|+2v} d\mu d\rho \\
& \leq C_{K,\chi,\alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_i/2 \rceil}^{l_i} \sum_{v=0}^{|\gamma^t|} \sup_{u \in [0, \max K]} \int_0^\infty |F^{(k_i+v+r)}(\rho + u)|^2 d\rho,
\end{aligned}$$

because $2b_i + 4r - 2l_i \geq 0$ and $|\beta^t| + 2v \geq 0$ if r and v are in the range of summation, and $\text{supp } F \subseteq K$. Since moreover $k_i + v + r \leq k_i + |\gamma^t| + l_i \leq |\alpha|$, the last integral is dominated by $\|F\|_{W_2^{|\alpha|}}^2$ uniformly in r, v, u , and (4.10) follows.

Consider now the case $|\beta^t| < k_i$. Via the identity

$$(1 + \tau)^{-1} = (1 - \tau)(1 - \tau^2)^{-1} = -\delta(1 - \tau^2)^{-1} = -\delta \sum_{j=0}^{\infty} \tau^{2j},$$

together with Lemma 4.3, we obtain that

$$\begin{aligned}
(4.12) \quad & \partial_\eta^{\gamma^t} \partial_\mu^{l_i} \delta^{k_i} (1 + \tau)^{|\beta^t| - k_i} m(n, \mu, \eta) \\
& = (-1)^{k_i - |\beta^t|} \sum_{j=0}^{\infty} \binom{j+k_i-|\beta^t|-1}{k_i-|\beta^t|-1} \int_{J_i} \partial_\eta^{\gamma^t} \partial_\mu^{l_i} \partial_t^{2k_i-|\beta^t|} \tilde{m}(n+2j+s, \mu, \eta) d\nu_i(s),
\end{aligned}$$

where $J_i = [0, 2k_i - |\beta^t|]$ and ν_i is a suitable probability measure on J_i . Note that, because of the assumptions on the supports of F and χ , the sum on j

in the right-hand side of (4.12) is a finite sum, that is, the j -th summand is nonzero only if $\langle n + 2j \rangle \leq 2M^{-1} \max K$; consequently, by applying the Cauchy-Schwarz inequality to the sum in j , and by (4.11),

$$\begin{aligned} & |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \delta^{k_\iota} (1 + \tau)^{|\beta^\iota| - k_\iota} m(n, \mu, \eta)|^2 \\ & \leq C_{K, \alpha} M^{1+2|\beta^\iota| - 2k_\iota} \sum_{j=0}^{\infty} \int_{J_\iota} |\partial_\eta^{\gamma^\iota} \partial_\mu^{l_\iota} \partial_t^{2k_\iota - |\beta^\iota|} \tilde{m}(n + 2j + s, \mu, \eta)|^2 d\nu_\iota(s) \\ & \leq C_{K, \chi, \alpha} \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{1+2k_\iota+2v-2|\gamma^\iota|} \sum_{j=0}^{\infty} \int_{J_\iota} \langle n + 2j + s \rangle^{2v} \mu^{4r-2l_\iota} \\ & \quad \times |F^{(2k_\iota - |\beta^\iota| + v+r)}(|\eta| \langle n + 2j + s \rangle + \mu^2)|^2 \tilde{\chi}(|\eta|/M) d\nu_\iota(s). \end{aligned}$$

Remember that $|\eta| \sim M$ in the region where $\tilde{\chi}(|\eta|/M) \neq 0$. Hence the left-hand side of (4.9) is majorized by

$$\begin{aligned} & C_{K, \chi, \alpha} \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \int_{J_\iota} \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{N}} \langle n + 2j + s \rangle^{2v} \langle n \rangle^{|\beta^\iota|} \int_{\mathbb{R}^3} \int_{\mathbb{R}} M^{2+2v-2|\alpha|+|\beta^\iota|} \\ & \quad \times \mu^{2b_\iota+4r-2l_\iota} |F^{(2k_\iota - |\beta^\iota| + v+r)}(|\eta| \langle n + 2j + s \rangle + \mu^2)|^2 \tilde{\chi}(|\eta|/M) d\mu d\eta d\nu_\iota(s) \\ & \leq C_{K, \chi, \alpha} \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \int_{J_\iota} \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{N}} \langle n + 2j + s \rangle^{2v+|\beta^\iota|} \int_0^\infty \int_0^\infty M^{4+2v-2|\alpha|+|\beta^\iota|} \\ & \quad \times \mu^{2b_\iota+4r-2l_\iota} |F^{(2k_\iota - |\beta^\iota| + v+r)}(\rho \langle n + 2j + s \rangle + \mu^2)|^2 \tilde{\chi}(\rho/M) d\mu d\rho d\nu_\iota(s) \\ & \leq C_{K, \chi, \alpha} \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} M^{4+2v-2|\alpha|+|\beta^\iota|} \int_0^\infty \int_0^\infty |F^{(2k_\iota - |\beta^\iota| + v+r)}(\rho + \mu^2)|^2 \\ & \quad \times \mu^{2b_\iota+4r-2l_\iota} \int_{J_\iota} \sum_{(n, j) \in \mathbb{N}^2} \frac{\tilde{\chi}(\rho / (\langle n + 2j + s \rangle M))}{\langle n + 2j + s \rangle^{1-2v-|\beta^\iota|}} d\nu_\iota(s) d\mu d\rho, \end{aligned}$$

by passing to polar coordinates and rescaling. The sum in (n, j) is dominated by $(\rho/M)^{2v+|\beta^\iota|+1}$, uniformly in $s \in J_\iota$, and moreover $\text{supp } F \subseteq K$. Therefore the left-hand side of (4.9) is majorized by

$$C_{K, \chi, \alpha} M^{3-2|\alpha|} \sum_{r=\lceil l_\iota/2 \rceil}^{l_\iota} \sum_{v=0}^{|\gamma^\iota|} \sup_{u \in [0, \max K]} \int_0^\infty |F^{(2k_\iota - |\beta^\iota| + v+r)}(\rho + u)|^2 d\rho.$$

On the other hand, $b_\iota + |\beta^\iota| = l_\iota + 2k_\iota$, hence $2k_\iota - |\beta^\iota| + v + r \leq 2k_\iota - |\beta^\iota| + |\gamma^\iota| + l_\iota = b_\iota + |\gamma^\iota| \leq |\alpha|$ if r and v are in the range of summation, therefore the last integral is dominated by $\|F\|_{W_2^{|\alpha|}}^2$ uniformly in r, v, u , and (4.9) follows. \square

Proposition 4.5. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be smooth and such that $\text{supp } F \subseteq K$ for some compact set $K \subseteq]0, \infty[$. For all $r \in [0, 3/2[$,*

$$\int_{N_{3,2}} |(1 + |y|)^r \mathcal{K}_{F(L)}(x, y)|^2 dx dy \leq C_{K,r} \|F\|_{W_2^r}^2.$$

Proof. Take $\chi \in C_c^\infty(]0, \infty[)$ such that $\text{supp } \chi \subseteq [1/2, 2]$ and $\sum_{k \in \mathbb{Z}} \chi(2^{-k}t) = 1$ for all $t \in]0, \infty[$. Note that, if (λ, η) belongs to the joint spectrum of L, \mathbf{Y} , then $|\eta| \leq \lambda$. Therefore, if $k_K \in \mathbb{Z}$ is sufficiently large so that $2^{k_K-1} > \max K$, and if F_M is defined for all $M \in]0, \infty[$ as in Lemma 4.4, then

$$F(L) = \sum_{k \in \mathbb{Z}, k \leq k_K} F_{2^k}(L, \mathbf{Y})$$

(with convergence in the strong sense). Hence an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}(L, \mathbf{Y})}$ given by Lemma 4.4. If $r < 3/2$, then the series $\sum_{k \leq k_K} (2^k)^{3/2-r}$ converges, thus

$$\int_{N_{3,2}} ||y|^r \mathcal{K}_{F(L)}(x, y)|^2 dx dy \leq C_{K,r} \|F\|_{W_2^r}^2.$$

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. \square

Recall that $|\cdot|_\delta$ denotes a δ_t -homogeneous norm on $N_{3,2}$, thus $|(x, y)|_\delta \sim |x| + |y|^{1/2}$. Interpolation then allows us to improve the standard weighted estimate for a homogeneous sublaplacian on a stratified group.

Proposition 4.6. *Let $F : \mathbb{R} \rightarrow \mathbb{C}$ be smooth and such that $\text{supp } F \subseteq K$ for some compact set $K \subseteq]0, \infty[$. For all $r \in [0, 3/2[$, $\alpha \geq 0$ and $\beta > \alpha + r$,*

$$(4.13) \quad \int_{N_{3,2}} |(1 + |(x, y)|_\delta)^\alpha (1 + |y|)^r \mathcal{K}_{F(L)}(x, y)|^2 dx dy \leq C_{K,\alpha,\beta,r} \|F\|_{W_2^\beta}^2.$$

Proof. Note that $1 + |y| \leq C(1 + |(x, y)|_\delta)^2$. Hence, in the case $\alpha \geq 0$, $\beta > \alpha + 2r$, the inequality (4.13) follows by the standard estimate [16, Lemma 1.2]. On the other hand, if $\alpha = 0$ and $\beta \geq r$, then (4.13) is given by Proposition 4.5. The full range of α and β is then obtained by interpolation (cf. the proof of [16, Lemma 1.2]). \square

We can finally prove the fundamental L^1 -estimate, and consequently Theorem 1.1.

Proof of Proposition 2.1. Take $r \in]9/2 - s, 3/2[$. Then $s - r > 3/2 + 3 - 2r$, hence we can find $\alpha_1 > 3/2$ and $\alpha_2 > 3 - 2r$ such that $s - r > \alpha_1 + \alpha_2$.

Therefore, by Proposition 4.6 and Hölder's inequality,

$$\|\mathcal{K}_{F(L)}\|_1^2 \leq C_{k,s} \|F\|_{W_2^s}^2 \int_{N_{3,2}} (1 + |(x, y)|_\delta)^{-2\alpha_1 - 2\alpha_2} (1 + |y|)^{-2r} dx dy.$$

The integral on the right-hand side is finite, because $2\alpha_1 > 3$, $\alpha_2 + 2r > 3$, and

$$(1 + |(x, y)|_\delta)^{-2\alpha_1 - 2\alpha_2} (1 + |y|)^{-2r} \leq C_s (1 + |x|)^{-2\alpha_1} (1 + |y|)^{-\alpha_2 - 2r},$$

and we are done. \square

Acknowledgements. The first-named author gratefully acknowledges the support of the Alexander von Humboldt Foundation.

REFERENCES

- [1] F. Astengo, M. Cowling, B. Di Blasio, and M. Sundari, *Hardy's uncertainty principle on certain Lie groups*, J. London Math. Soc. (2) 62 (2000), 461–472.
- [2] F. Astengo, B. Di Blasio, and F. Ricci, *Gelfand pairs on the Heisenberg group and Schwartz functions*, J. Funct. Anal. 256 (2009), 1565–1587.
- [3] M. Christ, *L^p bounds for spectral multipliers on nilpotent groups*, Trans. Amer. Math. Soc. 328 (1991), 73–81.
- [4] L. De Michele and G. Mauceri, *L^p multipliers on the Heisenberg group*, Michigan Math. J. 26 (1979), 361–371.
- [5] L. De Michele and G. Mauceri, *H^p multipliers on stratified groups*, Ann. Mat. Pura Appl. (4) 148 (1987), 353–366.
- [6] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions. Vol. II*, Robert E. Krieger Publishing Co. Inc., Melbourne, Fla., 1981.
- [7] V. Fischer and F. Ricci, *Gelfand transforms of $SO(3)$ -invariant Schwartz functions on the free group $N_{3,2}$* , Ann. Inst. Fourier (Grenoble) 59 (2009), 2143–2168.
- [8] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J., 1982.
- [9] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, Princeton, NJ, 1989.
- [10] W. Hebisch, *Multiplier theorem on generalized Heisenberg groups*, Colloq. Math. 65 (1993), 231–239.
- [11] W. Hebisch and J. Zienkiewicz, *Multiplier theorem on generalized Heisenberg groups. II*, Colloq. Math. 69 (1995), 29–36.

- [12] A. Martini, *Algebras of differential operators on Lie groups and spectral multipliers*, Tesi di perfezionamento (PhD thesis), Scuola Normale Superiore, Pisa, 2010, arXiv:1007.1119.
- [13] A. Martini, *Spectral theory for commutative algebras of differential operators on Lie groups*, J. Funct. Anal. 260 (2011), 2767–2814.
- [14] A. Martini, *Analysis of joint spectral multipliers on Lie groups of polynomial growth*, Ann. Inst. Fourier (Grenoble) 62 (2012), 1215–1263.
- [15] A. Martini and D. Müller, *A sharp multiplier theorem for Grushin operators in arbitrary dimensions* (2012), to appear in Rev. Mat. Iberoam., arXiv:1210.3564.
- [16] G. Mauceri and S. Meda, *Vector-valued multipliers on stratified groups*, Rev. Mat. Iberoam. 6 (1990), 141–154.
- [17] D. Müller and E. M. Stein, *On spectral multipliers for Heisenberg and related groups*, J. Math. Pures Appl. (9) 73 (1994), 413–440.
- [18] D. Müller, *A restriction theorem for the Heisenberg group*, Ann. of Math. (2) 131 (1990), 567–587.
- [19] D. Müller, F. Ricci, and E. M. Stein, *Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups. II*, Math. Z. 221 (1996), 267–291.
- [20] L. P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. 137 (1976), 247–320.
- [21] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, vol. 42, Princeton University Press, Princeton, NJ, 1993.

MATHEMATISCHES SEMINAR, C.-A.-UNIVERSITÄT ZU KIEL, LUDEWIG-MEYN-STR.
4, D-24118 KIEL, GERMANY
E-mail address: `martini@math.uni-kiel.de`

MATHEMATISCHES SEMINAR, C.-A.-UNIVERSITÄT ZU KIEL, LUDEWIG-MEYN-STR.
4, D-24118 KIEL, GERMANY
E-mail address: `mueller@math.uni-kiel.de`