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18 October 2022

Information Rate Optimization for Joint Relay and Link in Non-Regenerative MIMO Channels*

Giorgio Taricco

***Abstract*—The optimization of the Relay Transform Matrix (RTM) in a two-hop relay network with an average relay power constraint and perfect channel state information at the relay is addressed in this paper. The study considers the most general case in terms of number of transmit and receive antennas at the source, relay, and destination, with arbitrary correlation of the received noise at all terminals. The optimization problem is reduced to a manageable convex form, which is solved by linear algebra transformation and the KKT equations. A parametric solution is given, which yields the power constraint and the capacity achieved with uncorrelated transmitted data. The solution is shown to be amenable to a *water-filling-like* algorithmic implementation, which extends earlier literature results addressing the case without the direct link. Simulation results are reported concerning a Rayleigh relay network where, in particular, the role of the direct link SNR is precisely assessed.**

***Index Terms*—Relay networks. Information rate, Relay Transform Matrix. Convex Optimization. Water-filling.**

I. Introduction

The use of relays in wireless communications has been considered for several decades as an important tool to extend the coverage of radio networks. Relaying helps to combat the effects of fading and shadowing which prevent the signal to reach the destination in harsh environments. From an information theoretical point of view, the basic three-terminal channel model (source-relay-destination) was introduced by [1], [2] who studied the achievable rate under several operating conditions. More recent results, mostly based on single-antenna systems, have been proposed in this framework [3], [4]. Relaying based on Multiple-Input Multiple-Output (MIMO) wireless terminals has been studied in [5], [6]. Specifically, joint transmission and reception at the relay has been studied in [5] but, as pointed out in [6], this approach exposes the system to unwanted side effects determined by the fact that, typically, the transmitted signal power at the relay overshadows the power of the received signal. As a result, a more practical approach consists of keeping the reception and transmission processes at the relay orthogonal with respect to each other. As an example, time-orthogonality can be implemented by operating the system in a two-hop mode.

As far as it concerns the nature of the signal transmitted by the relay, alternative approaches have been proposed, which can be classified as *regenerative* or *non-regenerative*. The former are labeled as decode-and-forward (DF) and the latter as amplify-and-forward scheme (AF) schemes. For single-antenna systems, it has been observed that AF schemes are advantageous in terms of achievable diversity order with respect to DF schemes while the situation is not clearly understood as far as capacity is concerned. Nevertheless, non-regenerative AF schemes present a number of benefits that make them worth of being considered rather than DF schemes [6]. More recently, it has been pointed out that the AF schemes enables to retain the soft information of the transmitted signal and guarantees a limited signal delay at the same time [7]. An important role in the relaying performance is played by the Relay Transform Matrix (RTM), which transforms the received signal into the transmitted signal by a matrix multiplication.

According to the classification presented in [6], there are three basic operating modes for a MIMO relay system: $i)$ Direct Link without Relay; $ii)$ Relay without Direct Link; and $iii)$ Relay with Direct Link. The optimal RTM (in terms of capacity optimization) was obtained for the second operating mode in [6]. The optimal RTM for the more comprehensive third operating mode was claimed to be unknown in [6, p.1400] and was derived in [7] in order to maximize the overall Signal-to-Noise Ratio (SNR).

In this work we present an algorithm to derive the RTM optimizing the capacity of the two-hop relay channel network. The case considered here is more general than [6], [7] for several reasons. First of all, we allow the number of transmit and receive antennas at the relay to be different. Moreover, we consider the case of correlated noise at the receivers (not addressed in [6]). Finally, the solution presented here applies to the joint link and relay transmission case (labeled as "Case (C) Relay With Direct Link" in [6]), recognized as an *open problem* by the authors of [6]. The solution is parametric and, for a given relay power constraint, can be obtained by a water-filling-like algorithm, bearing some similarity with the one presented in [6, Sec.IV] (which is nevertheless not applicable to this case). We compare the results obtained by numerical simulation with those from [6, (B) Relay Without Direct Link] by forcing the direct link channel matrix to the all-zero matrix.

II. System Model

We consider a MIMO relay network consisting of three nodes: the source (S) equipped with t transmit antennas; the destination (D), equipped with r receive antenna; and the relay (R), equipped with u transmit and s receive antennas. The channel matrices corresponding to the three different links of interest are labeled as $\boldsymbol{H}_0$ (S→D), $\boldsymbol{H}_1$ (S→R), and $\boldsymbol{H}_2$ (R→D). The system operates in *two-hop relaying* mode: the source transmits during the first hop and the relay during the second hop. The average power transmitted by the source and the relay are upper bounded by P_1 and P_2, respectively. We

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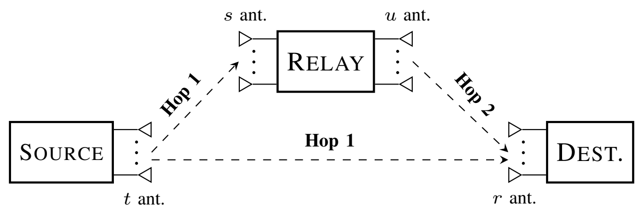

Fig. 1. System block diagram. Transmission occurs in two time/frequency slots (hops) so that the received signal at the destination arrives alternately from the source and from the relay.

assume that the relay applies a $u \times s$ *Relay Transformation Matrix* (RTM) $\boldsymbol{X}$ to the received signal before forwarding it to the destination in the second hop. The resulting channel equations are given as follows:

$$\begin{array}{rlll} \boldsymbol{y}_0 = & \boldsymbol{H}_0\boldsymbol{x} & +\boldsymbol{z}_0 & \text{(Hop 1, S}\rightarrow\text{D)} \\ \boldsymbol{y}_1 = & \boldsymbol{H}_1\boldsymbol{x} & +\boldsymbol{z}_1 & \text{(Hop 1, S}\rightarrow\text{R)} \\ \boldsymbol{y}_2 = & \boldsymbol{H}_2\boldsymbol{X}\boldsymbol{y}_1 & +\boldsymbol{z}_2 & \text{(Hop 2, R}\rightarrow\text{D)} \\ = & \boldsymbol{H}_2\boldsymbol{X}\boldsymbol{H}_1\boldsymbol{x} & +\boldsymbol{H}_2\boldsymbol{X}\boldsymbol{z}_1 + \boldsymbol{z}_2 & \end{array} \quad (1)$$

The received noise vectors are distributed as[1]

$$\boldsymbol{z}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{R}_i), \qquad i = 0, 1, 2,$$

with positive definite correlation matrices and $\boldsymbol{R}_0 = \boldsymbol{R}_2$. As far as the overall information transmission from source to destination is concerned, the equivalent joint vector channel equation is given by

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{H}_0 \\ \boldsymbol{H}_2\boldsymbol{X}\boldsymbol{H}_1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} \boldsymbol{z}_0 \\ \boldsymbol{H}_2\boldsymbol{X}\boldsymbol{z}_1 + \boldsymbol{z}_2 \end{pmatrix} \quad (2)$$

The noise correlation can be removed by pre-multiplying the Gaussian vectors by their own covariance matrices, so that we get the equivalent uncorrelated channel equation:

$$\tilde{\boldsymbol{y}} = \begin{pmatrix} \tilde{\boldsymbol{H}}_0 \\ \tilde{\boldsymbol{H}}_2\tilde{\boldsymbol{X}}\tilde{\boldsymbol{H}}_1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} \tilde{\boldsymbol{z}}_0 \\ \tilde{\boldsymbol{H}}_2\tilde{\boldsymbol{X}}\tilde{\boldsymbol{z}}_1 + \tilde{\boldsymbol{z}}_2 \end{pmatrix} \quad (3)$$

where we defined

$$\begin{array}{ll} \tilde{\boldsymbol{H}}_i \triangleq \boldsymbol{R}_i^{-1/2}\boldsymbol{H}_i & i = 0, 1, 2 \\ \tilde{\boldsymbol{X}} \triangleq \boldsymbol{X}\boldsymbol{R}_1^{1/2} & \\ \tilde{\boldsymbol{z}}_i \triangleq \boldsymbol{R}_i^{-1/2}\boldsymbol{z}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_r) & i = 0, 2 \\ \tilde{\boldsymbol{z}}_1 \triangleq \boldsymbol{R}_1^{-1/2}\boldsymbol{z}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_s) & \end{array}$$

with $\boldsymbol{R}_0 \equiv \boldsymbol{R}_2$. After decorrelating the second hop noise components, the channel equation can also be written as follows:

$$\tilde{\boldsymbol{y}} = \begin{pmatrix} \tilde{\boldsymbol{H}}_0 \\ (\tilde{\boldsymbol{H}}_2\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}^{\mathsf{H}}\tilde{\boldsymbol{H}}_2^{\mathsf{H}} + \boldsymbol{I}_r)^{-1/2}\tilde{\boldsymbol{H}}_2\tilde{\boldsymbol{X}}\tilde{\boldsymbol{H}}_1 \end{pmatrix} \boldsymbol{x} + \tilde{\boldsymbol{z}} \quad (4)$$

where $\tilde{\boldsymbol{z}} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_{2r})$. This channel equation leads directly to the equation representing the capacity of the relay network reported in the following eq. (5).

[1]The notation $\boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is associated to the circularly-symmetric complex Gaussian distribution of the random vector $\boldsymbol{z}$ and the corresponding pdf is defined by $f_{\boldsymbol{z}}(\boldsymbol{z}) = \det(\pi\boldsymbol{\Sigma})^{-1}\exp[-(\boldsymbol{z}-\boldsymbol{\mu})^{\mathsf{H}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{z}-\boldsymbol{\mu})]$.

III. RTM OPTIMIZATION

In this section we address the calculation of the optimum RTM based on the assumption that the relay *knows* all the channel matrices involved in eq. (1). Specifically, we look for the RTM which maximizes the two-hop relay channel capacity.

A. Optimum RTM

In the absence of Channel State Information at the Transmitter (CSIT), the capacity is achieved when $\boldsymbol{x} \sim \mathcal{CN}(\boldsymbol{0}, \frac{P_1}{t}\boldsymbol{I}_t)$ and is given by

$$C = \log_2 \det \Big\{ \boldsymbol{I}_t + \frac{P_1}{t}\Big[\tilde{\boldsymbol{H}}_0^{\mathsf{H}}\tilde{\boldsymbol{H}}_0 + \tilde{\boldsymbol{H}}_1^{\mathsf{H}}\tilde{\boldsymbol{X}}^{\mathsf{H}}\tilde{\boldsymbol{H}}_2^{\mathsf{H}} (\boldsymbol{I}_r + \tilde{\boldsymbol{H}}_2\tilde{\boldsymbol{X}}\tilde{\boldsymbol{X}}^{\mathsf{H}}\tilde{\boldsymbol{H}}_2^{\mathsf{H}})^{-1}\tilde{\boldsymbol{H}}_2\tilde{\boldsymbol{X}}\tilde{\boldsymbol{H}}_1 \Big] \Big\} \quad (5)$$

The average power constraint at the relay can be expressed in terms of $\tilde{\boldsymbol{X}}$ and the channel matrices as follows:

$$\begin{aligned} &\mathrm{tr}\left\{ \boldsymbol{X}\boldsymbol{R}_1\boldsymbol{X}^{\mathsf{H}} + \frac{P_1}{t}\boldsymbol{X}\boldsymbol{H}_1\boldsymbol{H}_1^{\mathsf{H}}\boldsymbol{X}^{\mathsf{H}} \right\} \\ &\quad = \mathrm{tr}\left\{ \tilde{\boldsymbol{X}}\left(\boldsymbol{I}_s + \frac{P_1}{t}\tilde{\boldsymbol{H}}_1\tilde{\boldsymbol{H}}_1^{\mathsf{H}} \right)\tilde{\boldsymbol{X}}^{\mathsf{H}} \right\} \le P_2. \quad (6) \end{aligned}$$

The optimum RTM (maximizing the capacity (5) under the constraint (6)) is given by the following Theorem.

Theorem 1 *Given the two-hop MIMO relay network described by eqs.* (1) *with average source and relay power constraints* P_1 *and* P_2*, the optimum (capacity-maximizing) RTM* $\boldsymbol{X}$ *is given by*

$$\boldsymbol{X} = \widetilde{\boldsymbol{U}}_B\widetilde{\boldsymbol{\Lambda}}_B^{-1/2}\widetilde{\boldsymbol{\Lambda}}^{1/2}\widetilde{\boldsymbol{U}}_A^{\mathsf{H}}\boldsymbol{R}_1^{-1/2}, \quad (7)$$

where we define the matrices

$$\boldsymbol{A} \triangleq \tilde{\boldsymbol{H}}_1\left(\frac{t}{P_1}\boldsymbol{I}_t + \tilde{\boldsymbol{H}}_0^{\mathsf{H}}\tilde{\boldsymbol{H}}_0 + \tilde{\boldsymbol{H}}_1^{\mathsf{H}}\tilde{\boldsymbol{H}}_1 \right)^{-1}\tilde{\boldsymbol{H}}_1^{\mathsf{H}}$$

$$\boldsymbol{B} \triangleq \tilde{\boldsymbol{H}}_2^{\mathsf{H}}\tilde{\boldsymbol{H}}_2, \qquad \boldsymbol{C} \triangleq \boldsymbol{I}_s + \frac{P_1}{t}\tilde{\boldsymbol{H}}_1\tilde{\boldsymbol{H}}_1^{\mathsf{H}},$$

and calculate the following "thin" unitary diagonalizations (UD's) [8, Th. 7.3.2]:[2]

$$\boldsymbol{A} = \widetilde{\boldsymbol{U}}_A\widetilde{\boldsymbol{\Lambda}}_A\widetilde{\boldsymbol{U}}_A^{\mathsf{H}}, \qquad \boldsymbol{B} = \widetilde{\boldsymbol{U}}_B\widetilde{\boldsymbol{\Lambda}}_B\widetilde{\boldsymbol{U}}_B^{\mathsf{H}}.$$

Here, $\widetilde{\boldsymbol{\Lambda}}_A$ *is the diagonal matrix of positive singular values of* $\boldsymbol{A}$ *and* $\widetilde{\boldsymbol{\Lambda}}_B$ *is the diagonal matrix of positive eigenvalues of* $\boldsymbol{B}$*. We also denote* $\rho_B \triangleq \mathrm{rank}(\boldsymbol{B}) \le \min(u, r)$*. The matrix* $\widetilde{\boldsymbol{\Lambda}}$ *is obtained by solving the convex optimization problem*

$$\begin{cases} \min\limits_{\boldsymbol{x} \ge \boldsymbol{0}} & -\sum\limits_{i=1}^{\rho} \ln\left\{ 1 - \frac{\alpha_i}{1 + x_i} \right\} \\ \text{s.t.} & \sum\limits_{i=1}^{\rho} \beta_i x_i \le P_2, \quad x_i \ge 0, i = 1, \ldots, \rho \end{cases} \quad (8)$$

[2]Throughout this paper, we assume that the diagonal elements of UD's are always sorted in nonincreasing.

where $\rho \triangleq \min(s, \rho_B)$ *and*

$$\alpha_i \triangleq |(\widetilde{\boldsymbol{\Lambda}}_A)_{i,i}|^2, \qquad \beta_i \triangleq \frac{(\boldsymbol{U}_A^{\mathsf{H}}\boldsymbol{C}\boldsymbol{U}_A)_{i,i}}{(\widetilde{\boldsymbol{\Lambda}}_B)_{i,i}}, i = 1, \ldots, \rho$$

$$\boldsymbol{x} \triangleq (x_1, \ldots, x_\rho)^{\mathsf{T}}, \qquad \widetilde{\boldsymbol{\Lambda}} \triangleq \mathrm{diag}(x_1, \ldots, x_\rho, \underbrace{0, \ldots, 0}_{\rho_B - \rho}).$$

Proof: The proof of this theorem is not included for space limitations. ■

B. Parametric and Water-Filling solution

It is interesting to note that the optimization problem considered in Theorem 1 is amenable to a closed-form parametric solution based on a positive independent variable ξ. This solution is illustrated by the following equations. First, we define the auxiliary functions

$$\varphi_i(\xi) \triangleq \left\{ \frac{\alpha_i}{2} - 1 + \sqrt{\frac{\alpha_i^2}{4} + \frac{\alpha_i \xi}{\beta_i}} \right\}_+ \tag{9}$$

where $\{\cdot\}_+ \triangleq \max(0, \cdot)$. These functions yields the components of the vector $\boldsymbol{x}$ defined in Theorem 1, i.e., $x_i = \varphi_i(\xi)$, which are required to build the optimum RTM. Based on this definition, we obtain two parametric equations (with real parameter $\xi > 0$) for the average relay power and the corresponding capacity:

$$P_2 = \sum_{i=1}^{\rho} \beta_i \varphi_i(\xi) \tag{10}$$

$$C = \log_2 \det \left\{ \boldsymbol{I}_t + \frac{P_1}{t}(\tilde{\boldsymbol{H}}_0^{\mathsf{H}}\tilde{\boldsymbol{H}}_0 + \tilde{\boldsymbol{H}}_1^{\mathsf{H}}\tilde{\boldsymbol{H}}_1) \right\} + \sum_{i=1}^{\rho} \log_2 \left\{ 1 - \frac{\alpha_i}{1 + \varphi_i(\xi)} \right\} \tag{11}$$

The two expressions are obtained by solving the KKT equations corresponding to optimization problem (8). Their derivation is given in App. A.

In order to solve eq. (10), one could divide the positive real line by the thresholds $\xi_i \triangleq (1 - \alpha_i)\beta_i$ and use the fact that $x_i = 0$ whenever $\xi \leq \xi_i$. This approach is closely related to the water-filling algorithm for orthogonal additive Gaussian channels [11].

IV. Numerical Results

In this section we collect some numerical results to illustrate the theory.

A. Validation of the results

Here, for the purpose of validation, we compare our results with [6, Figs. 3 and 4]. We assume that $t = r = u = s = M = 4$ and plot the ergodic capacity by corresponding to iid normalized Rayleigh channel matrices (all the entries of $\boldsymbol{H}_1, \boldsymbol{H}_2$ are iid and $\mathcal{CN}(0, 1)$ distributed while $\boldsymbol{H}_0 \equiv \boldsymbol{0}$).

In accordance with [6], the received noise vectors are uncorrelated with covariance matrices $\boldsymbol{R}_i = \sigma_i^2 \boldsymbol{I}_M$ for $i = 1, 2$, and we define the SNR's as

$$\rho_1 \triangleq \frac{P_1}{M\sigma_1^2}, \quad \rho_2 \triangleq \frac{P_2}{M\sigma_2^2}.$$

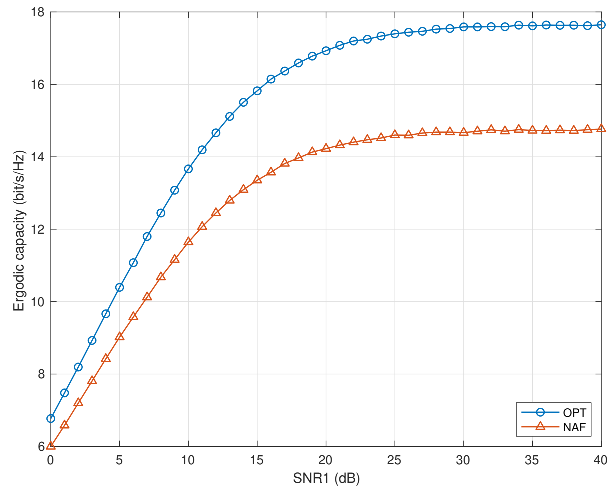

Fig. 2. Plot of the ergodic capacity vs. ρ_1 (denoted SNR1) with $\rho_2 = 10$ dB, iid Rayleigh fading and two types of Relay Transfer Matrices. *i*) OPT: optimum RTM. *ii*) NAF: Naive Amplify and Forward.

Accordingly, the capacity (5) simplifies to

$$C = \log_2 \det \left\{ \boldsymbol{I}_M + \frac{P_1}{P_2}\rho_2 \boldsymbol{H}_1^{\mathsf{H}}\boldsymbol{X}^{\mathsf{H}}\boldsymbol{H}_2^{\mathsf{H}} \left(\boldsymbol{I}_M + \frac{P_1}{P_2}\frac{\rho_2}{\rho_1}\boldsymbol{H}_2\boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}\boldsymbol{H}_2^{\mathsf{H}} \right)^{-1} \boldsymbol{H}_2\boldsymbol{X}\boldsymbol{H}_1 \right\} \tag{12}$$

and the relay power constraint to

$$\mathrm{tr}\left\{ \boldsymbol{X}\left(\boldsymbol{I}_s + \rho_1 \boldsymbol{H}_1\boldsymbol{H}_1^{\mathsf{H}} \right)\boldsymbol{X}^{\mathsf{H}} \right\} = M\frac{P_2}{P_1}\rho_1. \tag{13}$$

Equations (12) and (13) show that the relay-power constrained capacity is independent of P_1, P_2 for given values of ρ_1, ρ_2.

Figs. 2 and 3 show the ergodic capacity vs. the SNR ρ_1 (resp. ρ_2) with $\rho_2 = 10$ dB (resp. $\rho_1 = 10$ dB) and iid Rayleigh fading channel matrices. Two types of RTM's are considered: *i*) Optimum RTM (OPT) and *ii*) Naive Amplify and Forward (NAF), corresponding to $\boldsymbol{X} = \alpha \boldsymbol{I}_M$ for properly calculated α. The results are in perfect agreement with the corresponding curves reported in [6, Figs. 3 and 4].

B. Example with direct link

Here we extend the scenario considered in Section IV-A to the case obtained by adding the direct link to the previous scenario. Again, we assume that $t = r = u = s = M = 4$ and plot the ergodic capacity by corresponding to iid normalized Rayleigh channel matrices (but in this case all the entries of $\boldsymbol{H}_0, \boldsymbol{H}_1, \boldsymbol{H}_2$ are iid $\mathcal{CN}(0, 1)$ distributed). The received noise vectors are uncorrelated with covariance matrices $\boldsymbol{R}_i = \sigma_i^2 \boldsymbol{I}_M$ for $i = 0, 1, 2$, and we define the SNR's as

$$\rho_0 \triangleq \frac{P_1}{M\sigma_0^2}, \quad \rho_1 \triangleq \frac{P_1}{M\sigma_1^2}, \quad \rho_2 \triangleq \frac{P_2}{M\sigma_2^2}.$$

In this case, the capacity (5) can be written as

$$C = \log_2 \det \left\{ \boldsymbol{I}_M + \rho_0 \boldsymbol{H}_0^{\mathsf{H}}\boldsymbol{H}_0 + \frac{P_1}{P_2}\rho_2 \boldsymbol{H}_1^{\mathsf{H}}\boldsymbol{X}^{\mathsf{H}}\boldsymbol{H}_2^{\mathsf{H}} \left(\boldsymbol{I}_M + \frac{P_1}{P_2}\frac{\rho_2}{\rho_1}\boldsymbol{H}_2\boldsymbol{X}\boldsymbol{X}^{\mathsf{H}}\boldsymbol{H}_2^{\mathsf{H}} \right)^{-1} \boldsymbol{H}_2\boldsymbol{X}\boldsymbol{H}_1 \right\} \tag{14}$$

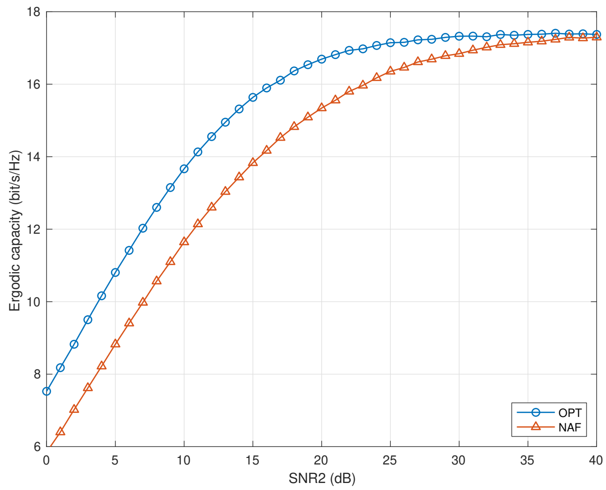

Fig. 3. Plot of the ergodic capacity vs. ρ_2 (denoted SNR2) with $\rho_1 = 10$ dB, iid Rayleigh fading and two types of Relay Transfer Matrices. $i)$ OPT: optimum RTM. $ii)$ NAF: Naive Amplify and Forward.

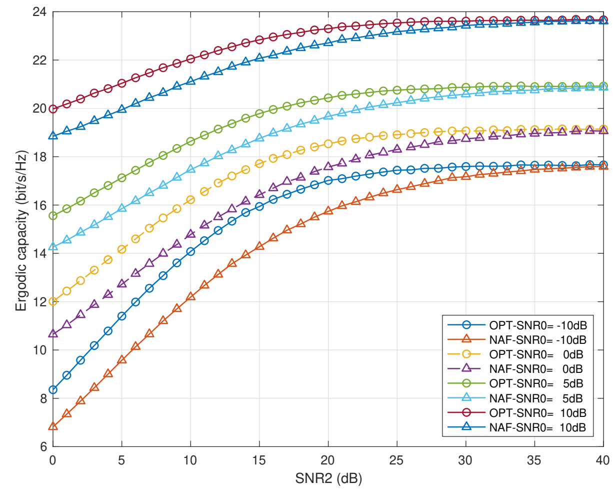

Fig. 4. Plot of the ergodic capacity vs. ρ_2 (denoted SNR2) with $\rho_0 = -10, -5, 0, 10$ db, $\rho_1 = 10$ dB, iid Rayleigh fading and two types of Relay Transfer Matrices. $i)$ OPT: optimum RTM. $ii)$ NAF: Naive Amplify and Forward.

and the relay power remains the same as in eq. (13). Again, we notice that the relay-power constrained capacity is independent of P_1, P_2 for given ρ_0, ρ_1, ρ_2.

Fig. 4 illustrates the ergodic capacity behavior vs. the SNR ρ_2 with $\rho_1 = 10$ dB and different values of direct link SNR ρ_0: $-10, -5, 0, 10$ dB. Again, the OPT and NAF RTM's are considered. It is interesting to note that, when ρ_0 takes on its lowest values, -10 dB, the results are very close to those reported in Fig. 2 (and, obviously, [6, Fig. 3]). In fact, this corresponds to having a weak signal on the direct link while most of the power arrives through the relay.

However, the situation changes radically when we increase ρ_0, as illustrated by the curves. Therefore, the results presented allow to assess more clearly the trade-offs implied by the availability of even a weak power component coming through the direct link, instead of disregarding it completely. This is one of the key values of this contribution.

Fig. 5 illustrates the ergodic capacity vs. ρ_0 with $\rho_1 = 10$ dB and different values of ρ_2 from 0 to 30 dB. The curves show a monotonic increase of the capacity vs. the direct link SNR ρ_0. As expected, the curves saturate when the relay-to-destination SNR ρ_2 grows sufficiently high.

These numerical results are useful to assess whether the RTM optimization is worth being pursued in a two-hop relay network or rather the relay should just amplify and forward the received signal vector. Under the stated assumptions, there is a sizable advantage, as long as the link SNR, ρ_0, is within 10 dB and the relay-to-destination SNR, ρ_2, is within 20 dB (while $\rho_1 = 10$ dB).

V. Conclusions

In this work we considered the optimization of the Relay Transformation Matrix (RTM) in a two-hop relay network with an average relay power constraint. The problem was already addressed in the literature, in a more limited form. A solution for the case of a pure relay network (without the direct link) was given in [6]. The complete relay with direct link network

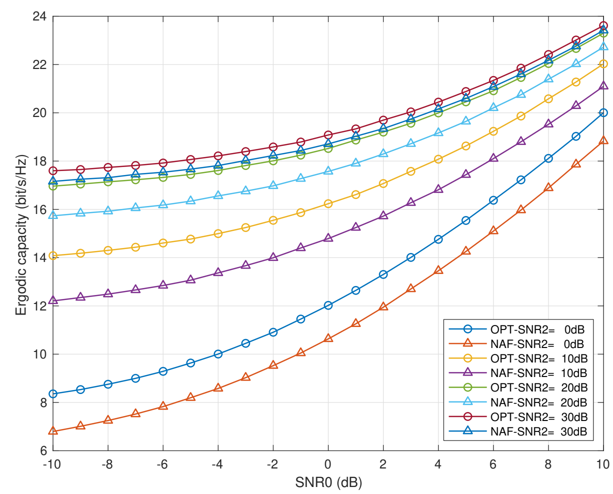

Fig. 5. Plot of the ergodic capacity vs. ρ_0 (denoted SNR0) with $\rho_2 = 0, 10, 20, 30$ db, $\rho_1 = 10$ dB, iid Rayleigh fading and two types of Relay Transfer Matrices. $i)$ OPT: optimum RTM. $ii)$ NAF: Naive Amplify and Forward.

was assessed in [7] but the optimization criterion was that of maximizing the OSTBC capacity, which corresponds to maximizing the equivalent receiver SNR.

The optimum RTM has been derived as the solution of a convex optimization problem combined with a number of linear algebra transformations based the unitary diagonalization of certain matrices. The optimization problem has been solved by the resorting to the corresponding KKT equations. A complete parametric solution has been derived which provides the average relay transmitted power constraint and the relay network capacity as two functions of a single scalar positive parameter. For a given average relay transmitted power, this solution leads to a *water-filling*-like implementation.

Simulation results have been presented to compare the capacity achieved by the optimum RTM and by the straightforward "naive" implementation consisting in letting the relay forward the received signal without alteration. It turns out that there is a considerable advantage entailed by RTM optimization in a range of SNR values of practical interest.

Appendix A
Parametric solution of optimization problem (8)

Referring to the definition of the optimization problem reported in eq. (8) of Theorem (1), we obtain the relevant KKT equations by defining the Lagrangian function of the problem as follows:

$$\mathcal{L}(\boldsymbol{x},\lambda_0,\lambda_1,\ldots,\lambda_\rho) = -\sum_{i=1}^{\rho} \ln\left\{1-\frac{\alpha_i}{1+x_i}\right\} + \lambda_0(\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{x}-P_2) - \sum_{i=1}^{\rho}\lambda_i x_i.$$

where $0<\alpha_i<1, \beta_i>0, i=1,\ldots,\rho$. The KKT equations are obtained according to [10, Sec. 5.5.3]. First, we take the partial derivatives of the Lagrangian function with respect to the variables x_i, for $i=1,\ldots,\rho$:

$$\frac{\partial\mathcal{L}}{\partial x_i} = \frac{1}{1+x_i} - \frac{1}{1-\alpha_i+x_i} + \lambda_0\beta_i - \lambda_i$$

Then, we have the following KKT equations:

$$\begin{aligned}
\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{x}-P_2 &\le 0 \\
\lambda_0(\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{x}-P_2) &= 0 \\
\lambda_0 &\ge 0 \\
-x_i &\le 0 \qquad & i=1,\ldots,\rho \\
\lambda_i x_i &= 0 \qquad & i=1,\ldots,\rho \\
\lambda_i &\ge 0 \qquad & i=1,\ldots,\rho \\
\frac{\partial\mathcal{L}}{\partial x_i} &= 0 \qquad & i=1,\ldots,\rho
\end{aligned}$$

We can see that the objective function

$$f(\boldsymbol{x}) \triangleq -\sum_{i=1}^{\rho}\ln\left\{1-\frac{\alpha_i}{1+x_i}\right\}$$

is convex for $\boldsymbol{x}\ge\mathbf{0}$ because

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\alpha_i(2-\alpha_i+2x_i)}{(1+x_i)^2(1-\alpha_i+x_i)^2} \ge 0.$$

The mixed derivatives $\partial^2 f/(\partial x_i \partial x_j) = 0$ for all $i \ne j$. Therefore, we have a convex optimization problem. We can see that Slater's condition is satisfied, so that the KKT equations are sufficient for optimality.

The constraint $\boldsymbol{\beta}^{\mathsf{T}}\boldsymbol{x}-P_2\le 0$ is achieved with equality since $f(\boldsymbol{x})$ is decreasing with every x_i. Therefore, we have $\lambda_0 \ge 0$.

Finally, we obtain from the gradient equations:

$$\frac{1}{1-\alpha_i+x_i} - \frac{1}{1+x_i} = \lambda_0\beta_i - \lambda_i, \quad i=1,\ldots,\rho.$$

For a given $\lambda_0\ge 0$, recalling that $\lambda_i\ge 0, x_i\ge 0, \lambda_i x_i = 0$, there are two possible cases

- $\lambda_i = 0$, which implies that the equation is equivalent to
$$x_i^2 + (2-\alpha_i)x_i + 1 - \alpha_i - \frac{\alpha_i}{\lambda_0\beta_i} = 0$$
Since $0<\alpha_i<1$, a solution $x_i>0$ exists only if
$$1-\alpha_i-\frac{\alpha_i}{\lambda_0\beta_i} < 0 \Longrightarrow \lambda_0 < \frac{\alpha_i}{(1-\alpha_i)\beta_i}$$
and is given by
$$x_i = \frac{\alpha_i}{2} - 1 + \sqrt{\frac{\alpha_i^2}{4}+\frac{\alpha_i}{\lambda_0\beta_i}}.$$
- $\lambda_i > 0$, which implies that $x_i = 0$ to satisfy the KKT condition $\lambda_i x_i = 0$. Hence,
$$1-\alpha_i-\frac{\alpha_i}{\lambda_0\beta_i-\lambda_i} = 0$$
and
$$\lambda_0 = \frac{\alpha_i}{(1-\alpha_i)\beta_i} + \frac{\lambda_i}{\beta_i} > \frac{\alpha_i}{(1-\alpha_i)\beta_i},$$

Summarizing, we can write the solution as

$$x_i = \left\{\frac{\alpha_i}{2} - 1 + \sqrt{\frac{\alpha_i^2}{4}+\frac{\alpha_i}{\lambda_0\beta_i}}\right\}_+$$

where $\{\cdot\}_+ \triangleq \max(0,\cdot)$. The unknown $\lambda_0\ge 0$ can be found by solving the nonlinear equation

$$P_2 = \sum_{i=1}^{\rho}\beta_i\left\{\frac{\alpha_i}{2} - 1 + \sqrt{\frac{\alpha_i^2}{4}+\frac{1}{\lambda_0\beta_i}}\right\}_+$$

A unique solution always exists because the rhs is a monotonically decreasing function of λ_0, which is identically equal to 0 when $\lambda_0 \ge \max_{1\le i\le\rho}\frac{1}{(1-\alpha_i)\beta_i}$. Setting $\xi \triangleq 1/\lambda_0$ yields the parametric solution reported in eqs. (9) to (11).

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