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Académie des sciences

Comptes Rendus
Mathématique

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Volume 359, issue 10 (2021), p. 1239-1250

<https://doi.org/10.5802/crmath.255>

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www.centre-mersenne.org

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https://doi.org/10.5802/crmath.255

Harmonic analysis / *Analyse harmonique*

Uniform pointwise estimates for ultraspherical polynomials

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Abstract. We prove pointwise bounds for two-parameter families of Jacobi polynomials. Our bounds imply estimates for a class of functions arising from the spectral analysis of distinguished Laplacians and sub-Laplacians on the unit sphere in arbitrary dimension, and are instrumental in the proof, discussed in a companion paper, of sharp multiplier theorems for those operators.

Mathematical subject classification (2010). 33C45, 33C55, 42C05, 58J50.

Funding. The first and the second author were partially supported by GNAMPA (Project 2018 "Operatori e disuguaglianze integrali in spazi con simmetrie" and Project 2020 "Alla frontiera tra l'analisi complessa in più variabili e l'analisi armonica"). Part of this research was carried out while the third author was visiting the Dicea, Università di Padova, Italy, as a recipient of a "Visiting Scientist 2019" grant; he gratefully thanks the Università di Padova for the support and hospitality. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Manuscript received 10th May 2021, accepted 8th August 2021.

1. Introduction

The primary purpose of this work is to prove pointwise estimates for a family of functions that are fundamentally related to the spectral analysis of spherical Laplacians and sub-Laplacians and expressed in terms of ultraspherical polynomials. More specifically, for a fixed $d \in \mathbb{N}$, $d \geq 2$, we consider the functions

$$X^d_{\ell,m}(x) = c_{\ell m}(1-x^2)^{m/2-(d-2)/4} P^{(m,m)}_{\ell-m-1/2}(x). \tag{1}$$

Here $\ell \in \mathbb{N}_d := \mathbb{N} + (d-1)/2$, $m \in \mathbb{N}_{d-1}$, $m \leq \ell$, $x \in [-1,1]$, the symbol $P^{(\alpha,\beta)}_j$ denotes the Jacobi polynomial of degree $j \in \mathbb{N}$ and indices $\alpha, \beta > -1$, and $c_{\ell m}$ is the normalization constant given by

$$c_{\ell m} = \frac{\left[\ell\,\Gamma(\ell-m+1/2)\,\Gamma(\ell+m+1/2)\right]^{1/2}}{2^m\,\Gamma(\ell+1/2)} \tag{2}$$

and chosen so that

$$\int_{-1}^{1} |X^d_{\ell,m}(x)|^2\,(1-x^2)^{(d-2)/2}\,\mathrm{d}x = 1, \tag{3}$$

see [32, (4.3.3)].

The functions $X^d_{\ell,m}$ are instrumental in the recursive construction of orthonormal bases of $L^2(\mathbb{S}^d)$, $\mathbb{S}^d$ denoting the unit sphere in $\mathbb{R}^{1+d}$, made of spherical harmonics. Namely, for all $k \geq 1$ and $m \in \mathbb{N}_k$, let $\mathscr{H}^m(\mathbb{S}^k)$ denote the space of spherical harmonics (that is, restrictions to the spherical surface of harmonic polynomials) of degree $m-(k-1)/2$ on the unit sphere in $\mathbb{R}^{1+k}$. Moreover, for all functions f on $\mathbb{S}^{d-1}$, let us define the function $X^d_{\ell,m} \otimes f$ on $\mathbb{S}^d$ by

$$(X^d_{\ell,m} \otimes f)((\cos\psi)\omega, \sin\psi) = X^d_{\ell,m}(\sin\psi) f(\omega),$$

for all $\omega \in \mathbb{S}^{d-1}$ and $\psi \in [-\pi/2, \pi/2]$ (this definition makes sense almost everywhere on $\mathbb{S}^d$; actually, when $m > (d-2)/2$, it makes sense everywhere, because $X^d_{\ell,m}(\pm 1) = 0$ in that case). Then, for all $\ell \in \mathbb{N}_d$ and $m \in \mathbb{N}_{d-1}$ such that $m \leq \ell$, the map $f \mapsto X^d_{\ell,m} \otimes f$ is an isometric embedding of $\mathscr{H}^m(\mathbb{S}^{d-1})$ into $\mathscr{H}^\ell(\mathbb{S}^d)$ (with respect to the Hilbert space structures induced by $L^2(\mathbb{S}^{d-1})$ and $L^2(\mathbb{S}^d)$ respectively), and indeed we have the orthogonal direct sum decomposition

$$\mathscr{H}^\ell(\mathbb{S}^d) = \bigoplus_{m \leq \ell} X^d_{\ell,m} \otimes \mathscr{H}^m(\mathbb{S}^{d-1}). \tag{4}$$

This construction is classical and can be found in several places in the literature, modulo some minor notational differences (see, e.g., [35, Chapter IX] or [10, Chapter XI]).

In order to obtain pointwise estimates for $X^d_{\ell,m}(x)$, it is natural to seek bounds for the (d-independent) functions

$$Y_{\ell,m}(x) = c_{\ell,m}(1-x^2)^{m/2}\, P^{(m,m)}_{\ell-m-1/2}(x),$$

with $(\ell, m) \in (\mathbb{N}/2)^2$ and $\ell - m - 1/2 \in \mathbb{N}$. Upper bounds for Jacobi polynomials $P^{(\alpha,\beta)}_j$, that are uniform with respect to α, β and j in suitable ranges, have recently attracted a considerable interest. For a brief account of these bounds, with particular emphasis on Bernstein-type inequalities, we refer to [24]; for some earlier results on ultraspherical polynomials and the strictly related associated Legendre functions, see [20, 21]. For recent contributions, focusing on the uniformity with respect to the indices, we refer to works of Haagerup and Schlichtkrull [13], Koornwinder, Kostenko and Teschl [16], and Krasikov [17, 18]. In the particular case $d = 2$, some relevant upper bounds for the classical spherical harmonics may be found in [6, 12, 28].

Most of the aforementioned results give uniform weighted estimates for suitably normalised families of Jacobi polynomials $P^{(\alpha,\beta)}_j$, where the weight depends on the type (α, β) and is independent of the degree j. In contrast, the estimates that we obtain here take into consideration, for each individual function $Y_{\ell,m}$, the position of the "transition points" $\pm a_{\ell,m}$ (see (7) below) that separate the regions of oscillation and decay of $Y_{\ell,m}$ on $[-1, 1]$. Estimates of this nature, that describe with a certain precision the behaviour of the function near the transition points, turn out to be essential ingredients in the proof of a sharp spectral multiplier theorem for Grushin operators on the unit sphere $\mathbb{S}^d$, whose spectral decomposition can be expressed in terms of spherical harmonics. In the case $d = 2$, this problem was studied in [7], where pointwise estimates of this type were proved for the functions $X^2_{\ell,m}$. The present paper confirms the validity of similar estimates for the functions $X^d_{\ell,m}$ with arbitrary $d \geq 2$; details on their application to the proof of a multiplier theorem are given in the companion paper [8], where the estimates of the present paper are crucially used to prove "weighted spectral cluster bounds" for spherical Grushin operators (see Remark 2 below). We also refer to [15, Section 8] for the discussion of estimates of the kind proved in the present paper for a different family of Jacobi polynomials (namely, $P^{(\alpha,\beta)}_j$, with $\alpha \neq \beta$ and only one fixed between α and β).

As in the case $d = 2$, our approach detects a discrepancy in the behaviour of $X^d_{\ell,m}$, depending on whether m is smaller or larger than $\epsilon\ell$ for some fixed $\epsilon \in (0,1)$. This corresponds to the fact that, if $m \le \epsilon\ell$, the functions in (1) are asymptotically related to Bessel functions, while for $m \ge \epsilon\ell$ their asymptotical behaviour is described by Hermite polynomials. Indeed a crucial tool in the proof of our pointwise bounds for $X^d_{\ell,m}$ is provided by the precise asymptotic approximations of ultraspherical polynomials in terms of Bessel functions and Hermite polynomials previously obtained by Boyd and Dunster and by Olver [5, 26]. We point out that estimates for Hermite and Bessel functions of a similar character to those considered here are available in the literature (see, e.g., [2, 4]), but they apply to one-parameter families; in contrast, here we obtain uniform estimates for two-parameter families of ultraspherical polynomials. Similarly, but in a different context, [9] presents a robust approach that applies to orthonormal expansions associated to second-order ODE on the real line, yielding estimates that are uniform with respect to an additional scale parameter.

We would like to point out that part of the results obtained here can be deduced from estimates in [17, 18]. In those works, the author employs a different approach, yielding very precise bounds with explicit constants for Jacobi polynomials $P_j^{(\alpha,\alpha)}$, where α need not be integer or half-integer. On the other hand, due to the various constraints on α and j in [17, 18], those estimates do not appear to cover the whole range of indices that we consider here. In addition, the results in [17,18] mainly focus on the oscillatory region, and do not appear to provide comparably precise information on the behaviour beyond the transition points.

Some of the proofs presented here are similar to those given in [7, Section 3], but several variations and new ideas are required when $d > 2$. As a matter of fact, even in the case $d = 2$, here we obtain a substantially stronger decay beyond the transition points in the Hermite regime compared to the one proved in [7]. When comparing results, one should take into account a slight change of notation, since ℓ in [7] corresponds to $\ell - 1/2$ here.

Let us introduce, for all $d \in \mathbb{N}$, $d \ge 2$, the index set

$$I_d = \{(\ell, m) : \ell \in \mathbb{N}_d,\ m \in \mathbb{N}_{d-1},\ \ell \ge m\}. \tag{5}$$

Moreover, for all $\ell, m \in \mathbb{N}/2$ with $\ell \ne 0$ and $0 \le m \le \ell$, we define the points $a_{\ell,m}, b_{\ell,m} \in [0,1]$ by

$$b_{\ell,m} = \frac{m}{\ell} \tag{6}$$

and

$$a^2_{\ell,m} = 1 - b^2_{\ell,m} = \frac{(\ell - m)(\ell + m)}{\ell^2}. \tag{7}$$

One should think of $\pm a_{\ell,m}$ as the values of $x \in [-1,1]$ corresponding to the transition points for $X^d_{\ell,m}(x)$, while $b_{\ell,m}$ corresponds to the transition points after the change of variables $y = \sqrt{1 - x^2}$.

In the statement below, and throughout the paper, for two given nonnegative quantities A and B, we use the notation "$A \lesssim B$" to indicate that $A \le CB$ for some positive constant C. We also write $A \simeq B$ as shorthand for $A \lesssim B$ and $B \lesssim A$. Variants such as $\lesssim_k$ and $\simeq_k$ are used to indicate that the implicit constants may depend on the parameter k.

Theorem 1. *Let $d \in \mathbb{N}$, $d \ge 2$. For all $\epsilon \in (0,1)$, there exists $c \in (0,1)$ such that, for all $(\ell, m) \in I_d$, if $m \ge \epsilon\ell$, then*

$$|X^d_{\ell,m}(x)| \lesssim_{d,\epsilon} \begin{cases} (\ell^{-1} + |x^2 - a^2_{\ell,m}|)^{-1/4} & \text{for all } x \in [-1,1], \\ |x|^{-1/2}(1 - x^2)^{(c\ell - (d-2)/4)_+} & \text{for } |x| \ge 2\, a_{\ell,m}, \end{cases} \tag{8}$$

while, if $m \le \epsilon\ell$, then

$$|X^d_{\ell,m}(x)| \lesssim_{d,\epsilon} \begin{cases} y^{-(d-2)/2}\big(\ell^{-2}(1+m)^{4/3} + |y^2 - b^2_{\ell,m}|\big)^{-1/4} & \text{for all } x \in [-1,1], \\ \ell^{(d-1)/2}\, 2^{-m} & \text{if } y \le b_{\ell,m}/(2e), \end{cases} \tag{9}$$

where $y = \sqrt{1 - x^2}$.

The above estimates will be derived from a series of bounds for the d-independent functions $Y_{\ell,m}$ stated in Propositions 4, 6, 8, and 9. It is important to remark that the dependence on d of the above estimates is not only due to the factor $(1-x^2)^{-d/4}$ in (1), but also to the range of indices I_d.

Remark 2. As observed above, Theorem 1 is an essential tool in [8] in order to prove "weighted spectral cluster bounds" of the form

$$\sum_{\substack{\ell_d \geq \ldots \geq \ell_k \\ \epsilon \ell_d \leq \ell_k \\ \ell_d^2 - \ell_k^2 \in [i^2,(i+1)^2]}} \ell_k^{k-1-\gamma} \prod_{j=k+1}^{d} \left| X^j_{\ell_j,\ell_{j-1}}(x_j)\right|^2 \lesssim_{d,k,\alpha,\epsilon} i^{d-1-\gamma} \min\{i, |\vec{x}|^{-1}\}^{k-\gamma},$$

$$\sum_{\substack{\ell_d \geq \ldots \geq \ell_k \\ \ell_k \leq \epsilon \ell_d \\ \ell_d^2 - \ell_k^2 \in [i^2,(i+1)^2]}} \ell_k^{k-1} \prod_{j=k+1}^{d} \left| X^j_{\ell_j,\ell_{j-1}}(x_j)\right|^2 \lesssim_{d,k,\epsilon} i^{d-1}$$

for all $1 \leq k < d$, $\gamma \in [0,k)$, $\epsilon \in (0,1)$, $\vec{x} = (x_{k+1},\ldots,x_d) \in [0,1]^{d-k}$ and $i \in \mathbb{N} \setminus \{0\}$. We refer to [8] for a discussion of the significance of these bounds in relation to the spectral theory of spherical Grushin operators of the form $\mathscr{L}_{d,k} = \Delta_d - \Delta_k$, where Δ_d is the usual Laplace–Beltrami operator on $\mathbb{S}^d$, while Δ_k is the "partial Laplacian" corresponding to the subsphere $\mathbb{S}^k \times \{0\}$. Here we only mention that, in the unweighted case $\gamma = 0$, the above bounds imply the spectral projection estimate

$$\left\| \chi_{[i,i+1]}(\sqrt{\mathscr{L}_{d,k}}) \right\|_{L^1 \to L^2} \lesssim_{d,k} i^{d+k-1}, \tag{10}$$

which is a sub-elliptic analogue (see also [11]) of the Agmon–Avakumovič–Hörmander spectral cluster bounds for the Laplace–Beltrami operator,

$$\left\| \chi_{[i,i+1]}(\sqrt{\Delta_d}) \right\|_{L^1 \to L^2} \lesssim_{d,k} i^{d-1}$$

[14, 29, 30]; the weighted case $\gamma > 0$ of the above bounds is nevertheless crucial in the proof of the sharp multiplier theorem for $\mathscr{L}_{d,k}$ in [8]. A more general discussion of the problem of proving sharp multiplier theorems for sub-elliptic operators can be found, e.g., in [22] and references therein.

2. Notation and preliminaries

By the symbol $P_j^{(\alpha,\beta)}$ we shall denote the Jacobi polynomial of degree $j \in \mathbb{N}$ and indices $\alpha, \beta > -1$, defined by means of Rodrigues' formula:

$$P_j^{(\alpha,\beta)}(x) = \frac{(-1)^j}{2^j\, j!} (1-x)^{-\alpha}(1+x)^{-\beta} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \left((1-x)^{\alpha+j}(1+x)^{\beta+j}\right)$$

for $x \in (-1,1)$. We recall, in particular, the symmetry relation

$$P_j^{(\alpha,\beta)}(x) = (-1)^j P_j^{(\beta,\alpha)}(x),$$

for $j \in \mathbb{N}$, $\alpha, \beta > -1$ and $x \in \mathbb{R}$.

In the case $\alpha = \beta$, Jacobi polynomials reduce to ultraspherical polynomials [32, (4.7.1)]. In particular, by using the relation between Jacobi polynomials and associated Legendre functions (Ferrers functions), namely,

$$P_k^{(\alpha,\alpha)}(x) = \frac{2^\alpha \Gamma(\alpha+k)}{k!} (1-x^2)^{-\alpha/2} \mathrm{P}^{-\alpha}_{\alpha+k}(x)$$

for $x \in (-1,1)$, $k \in \mathbb{N}$, $\alpha \geq 0$ (see [1, formulas 14.3.1, 14.3.3, 15.8.1 and 18.5.7]), we can write the functions $X^d_{\ell,m}$ as follows:

$$X^d_{\ell,m}(x) = \sqrt{\frac{\ell\,\Gamma(\ell+m+1/2)}{\Gamma(\ell-m+1/2)}}(1-x^2)^{-(d-2)/4}\mathrm{P}^{-m}_{\ell-1/2}(x). \tag{11}$$

Let now $I = \{(\ell,m) \in (\mathbb{N}/2)^2 : \ell - m - 1/2 \in \mathbb{N}\}$. For $(\ell,m) \in I$, define

$$\begin{aligned} Y_{\ell,m}(x) &= c_{\ell m}(1-x^2)^{m/2} P^{(m,m)}_{\ell-m-1/2}(x) \\ &= \sqrt{\frac{\ell\,\Gamma(\ell+m+1/2)}{\Gamma(\ell-m+1/2)}}\mathrm{P}^{-m}_{\ell-1/2}(x). \end{aligned} \tag{12}$$

Note that, if $d \geq 2$ and $m \in \mathbb{N}_{d-1}$, then

$$X^d_{\ell,m}(x) = (1-x^2)^{-(d-2)/4} Y_{\ell,m}(x). \tag{13}$$

3. Results from representation theory

We recall some well known facts concerning the spectral theory of the Laplace–Beltrami operator Δ_d on the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{1+d}$. For a detailed account of the theory we refer to [31, Chapter 4] or [3, Chapter 5].

The operator Δ_d is essentially self-adjoint on $L^2(\mathbb{S}^d)$, with discrete spectrum. The symbol $\mathscr{H}^\ell(\mathbb{S}^d)$ will denote the eigenspace of Δ_d corresponding to the eigenvalue

$$\lambda^d_\ell := (\ell + (d-1)/2)(\ell - (d-1)/2), \tag{14}$$

where $\ell \in \mathbb{N}_d$. It is well-known that $\mathscr{H}^\ell(\mathbb{S}^d)$ consists of all spherical harmonics of degree $\ell' = \ell - (d-1)/2 \in \mathbb{N}$, that is, of all restrictions to $\mathbb{S}^d$ of homogeneous harmonic polynomials on $\mathbb{R}^{1+d}$ of degree ℓ'.

The following facts on the spaces $\mathscr{H}^\ell(\mathbb{S}^d)$ are standard.

(i) Since Δ_d is self-adjoint, its eigenspaces are mutually orthogonal in $L^2(\mathbb{S}^d)$, i.e.,

$$\mathscr{H}^{\ell_1}(\mathbb{S}^d) \perp \mathscr{H}^{\ell_2}(\mathbb{S}^d)$$

for $\ell_1, \ell_2 \in \mathbb{N}_d$, $\ell_1 \neq \ell_2$.

(ii) Each $\mathscr{H}^\ell(\mathbb{S}^d)$ is a finite-dimensional space of dimension

$$\dim(\mathscr{H}^\ell(\mathbb{S}^d)) = \binom{\ell'+d}{\ell'} - \binom{\ell'+d-2}{\ell'-2} = \frac{2\ell'+d-1}{d-1}\binom{\ell'+d-2}{d-2} \tag{15}$$

for $\ell = \ell' + (d-1)/2 \in \mathbb{N}_d$ (the last identity in (15) only makes sense when $d > 1$). In particular

$$\dim(\mathscr{H}^\ell(\mathbb{S}^d)) \simeq_d \ell^{d-1} \tag{16}$$

Here and subsequently, we adhere to the convention that $0^0 = 1$, so that this estimate is also valid when $d = 1$.

(iii) The spaces $\mathscr{H}^\ell(\mathbb{S}^d)$ are $\mathrm{O}(d+1)$-invariant for every $\ell \in \mathbb{N}_d$.

(iv) The representation of $\mathrm{O}(d+1)$ on the space $\mathscr{H}^\ell(\mathbb{S}^d)$ is irreducible.

Next, we introduce a system of "cylindrical coordinates" on $\mathbb{S}^d$, $d \geq 2$. For all $\omega \in \mathbb{S}^{d-1}$ and $x \in [-1,1]$, one defines the point $\lfloor x,\omega \rceil \in \mathbb{S}^d$ as

$$\lfloor x,\omega \rceil = (\sqrt{1-x^2}\,\omega, x). \tag{17}$$

Then (17) yields a "system of coordinates" on $\mathbb{S}^d$, modulo null sets, since, apart from $x = \pm 1$, the map $(\omega,x) \mapsto \lfloor \omega,x \rceil$ is a diffeomorphism onto its image, which is the sphere with the two poles removed.

In these coordinates, the spherical measure σ_d on $\mathbb{S}^d$ is given by

$$d\sigma_d(\lfloor\omega,x\rceil) = (1-x^2)^{(d-2)/2}\, dx\, d\sigma_{d-1}(\omega),$$

where σ_{d-1} is the spherical measure on $\mathbb{S}^{d-1}$. We recall that

$$\sigma_d(\mathbb{S}^d) = \frac{(d+1)\pi^{(d+1)/2}}{\Gamma((d+3)/2)}. \tag{18}$$

The following formula, proved in [31, Chapter 4, Corollary 2.9], will be repeatedly used throughout the paper: if E^d_ℓ is any orthonormal basis of $\mathscr{H}^\ell(\mathbb{S}^d)$, then

$$\sum_{Z\in E^d_\ell} |Z(z)|^2 = \left(\sigma_d(\mathbb{S}^d)\right)^{-1} \dim(\mathscr{H}^\ell(\mathbb{S}^d)) \tag{19}$$

for all $z \in \mathbb{S}^d$.

The above-mentioned properties as a whole imply a universal bound for $Y_{\ell,m}(x)$, which will be useful, in particular, in the Bessel regime.

Proposition 3. *For all $(\ell,m) \in I$ and all $x \in [-1,1]$,*

$$Y_{\ell,m}(x)^2 \lesssim (1-x^2)^m \frac{\ell}{\sqrt{m+1}} \binom{\ell-1/2+m}{2m}. \tag{20}$$

Moreover

$$Y_{\ell,m}(x)^2 \lesssim \begin{cases} \ell^{1/2} & \text{if } m \in \mathbb{N}, \\ (1-x^2)^{1/2}\ell/m^{1/2} & \text{if } m \in \mathbb{N}+1/2. \end{cases} \tag{21}$$

Proof. Let $\ell \in \mathbb{N}_d$, $d \geq 2$. By the decomposition (4), if $K^d_\ell : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ is the integral kernel of the orthogonal projection of $L^2(\mathbb{S}^d)$ onto $\mathscr{H}^\ell(\mathbb{S}^d)$, then

$$K^d_\ell(\lfloor x,\omega\rceil, \lfloor x',\omega'\rceil) = \sum_{\substack{m\leq\ell\\ m\in\mathbb{N}_{d-1}}} X^d_{\ell,m}(x) X^d_{\ell,m}(x') K^{d-1}_m(\omega,\omega').$$

Hence, in light of (19),

$$\frac{\dim(\mathscr{H}^\ell(\mathbb{S}^d))}{\sigma_d(\mathbb{S}^d)} = \sum_{\substack{m\leq\ell\\ m\in\mathbb{N}_{d-1}}} X^d_{\ell,m}(x)^2 \frac{\dim(\mathscr{H}^m(\mathbb{S}^{d-1}))}{\sigma_{d-1}(\mathbb{S}^{d-1})} \tag{22}$$

and in particular

$$Y_{\ell,m}(x)^2 = (1-x^2)^{(d-2)/2} X^d_{\ell,m}(x)^2 \leq (1-x^2)^{(d-2)/2} \frac{\dim(\mathscr{H}^\ell(\mathbb{S}^d))}{\dim(\mathscr{H}^m(\mathbb{S}^{d-1}))} \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{\sigma_d(\mathbb{S}^d)} \tag{23}$$

for all $(\ell,m) \in I_d$. Now, for a given $(\ell,m) \in I$, the estimates (20) and (21) follow from (23) by choosing $d \geq 2$ to be, respectively, the largest and the smallest possible so that $(\ell,m) \in I_d$, and using (18) and (15). □

4. The Bessel regime

In this section we prove some pointwise estimates for $Y_{\ell,m}$ and $X^d_{\ell,m}$ in the range $m \leq \epsilon\ell$, for some $\epsilon \in (0,1)$.

First, from the bound (20) we readily derive an estimate that is particularly effective in the region where $y = \sqrt{1-x^2} \ll b_{\ell,m}$.

Proposition 4. *Let $\epsilon \in (0,1)$. For all $(\ell,m) \in I$ such that $m \leq \epsilon\ell$, and for all $x \in [-1,1]$,*

$$\left|Y_{\ell,m}(x)\right| \lesssim_\epsilon b^{-(m+1/2)}_{\ell,m} (ye)^m, \tag{24}$$

where $y = \sqrt{1-x^2}$.

Proof. For $m = 0$ the estimate is trivial, so we may assume $m > 0$. The universal bound (20) implies that for all $x \in [0,1]$ and all $(\ell, m) \in I$, with $0 < m \le \epsilon\ell$,

$$\begin{aligned} Y_{\ell,m}(x)^2 &\lesssim y^{2m} \frac{\ell}{\sqrt{m}} \binom{\ell - 1/2 + m}{2m} \\ &\lesssim_\epsilon y^{2m} \frac{\ell}{\sqrt{m}} \frac{1}{\sqrt{2\pi(2m)}} \left(\frac{(\ell - 1/2 + m)e}{2m}\right)^{2m} \\ &\lesssim y^{2m} \frac{\ell}{m} \left(\frac{\ell\, e}{m}\right)^{2m}, \end{aligned}$$

as a consequence of Stirling's approximation. This proves (24). □

A more precise estimate in the region where $y \gtrsim b_{\ell,m}$ can be derived from a uniform asymptotic approximation for the associated Legendre functions $\mathrm{P}^{-m}_{\ell-1/2}$ in terms of Bessel functions, previously proved in [5]. This was shown in [7, Proposition 3.5] in the case where m is integer. The case where m is half-integer can be treated similarly, however the proof requires a number of modifications, mainly due to the fact that the proof in [7] exploits certain estimates for spherical harmonics on $\mathbb{S}^2$ from [6], which do not directly apply to the case where m is not an integer. The proof presented below, instead, applies irrespective of whether m is integer, and exploits the following bound from [19] for the Bessel function of the first kind J_ν of order $\nu \in (-1,\infty)$.

Lemma 5. *There exists $b \in (0,1)$ such that, for all $\nu \in (0,\infty)$ and $z \in \mathbb{R}$,*

$$|J_\nu(z)| \le b\nu^{-1/3}.$$

By combining this bound with the results of [5] we can prove the following estimate.

Proposition 6. *Let $\epsilon \in (0,1)$. The following bounds hold for all $(\ell, m) \in I$ such that $m \le \epsilon\ell$, and for all $x \in [-1,1]$:*

$$\left|Y_{\ell,m}(x)\right| \lesssim_\epsilon \left(\frac{(1+m)^{4/3}}{\ell^2} + |y^2 - b^2_{\ell,m}|\right)^{-1/4}, \tag{25}$$

where $y = \sqrt{1-x^2}$.

Proof. Without loss of generality we may assume $x \ge 0$. Following the proof of [7, Proposition 3.5], by using the results of [5] we can write

$$\begin{aligned} |y^2 - b^2_{\ell,m}|^{1/4}\, Y_{\ell,m}(x) = \tilde{\varkappa}_{\ell,m} \left|\ell^2 \zeta_{\ell,m}(x) - m^2\right|^{1/4} \\ \times \left[J_m(\ell\,\zeta_{\ell,m}(x)^{1/2}) + E_m^{-1} M_m(\ell\,\zeta_{\ell,m}(x)^{1/2})\,\mathcal{O}(\ell^{-1})\right], \end{aligned} \tag{26}$$

uniformly in $x \in [0,1]$ and $(\ell, m) \in I$ with $m \le \epsilon\ell$. Here $y = \sqrt{1-x^2}$ and $\tilde{\varkappa}_{\ell,m} \simeq 1$ uniformly in $(\ell, m) \in I$; moreover, $E_m^{-1} M_m$ is the pointwise quotient of the auxiliary functions M_m and E_m introduced in [5, Section 3] and $\zeta_{\ell,m} : [0,1] \to [0, \zeta_{\ell,m}(0)]$ is the decreasing bijection satisfying $\zeta_{\ell,m}(a_{\ell,m}) = b^2_{\ell,m}$ and implicitly defined by

$$\int_{b^2_{\ell,m}}^{\zeta_{\ell,m}(x)} \frac{(\xi - b^2_{\ell,m})^{1/2}}{2\xi}\, \mathrm{d}\xi = \int_x^{a_{\ell,m}} \frac{(a^2_{\ell,m} - s^2)^{1/2}}{1-s^2}\, \mathrm{d}s \qquad (0 \le x \le a_{\ell,m}), \tag{27}$$

$$\int_{\zeta_{\ell,m}(x)}^{b^2_{\ell,m}} \frac{(b^2_{\ell,m} - \xi)^{1/2}}{2\xi}\, \mathrm{d}\xi = \int_{a_{\ell,m}}^{x} \frac{(s^2 - a^2_{\ell,m})^{1/2}}{1-s^2}\, \mathrm{d}s \qquad (a_{\ell,m} \le x \le 1). \tag{28}$$

Notice that ℓ in [7] corresponds to $\ell - 1/2$ here.

The same argument as in [7] (see formula (3.20) there) shows that the right-hand side of (26) is uniformly bounded, thus yielding that

$$|Y_{\ell,m}(x)| \lesssim_\epsilon |y^2 - b^2_{\ell,m}|^{-1/4}, \tag{29}$$

uniformly in $x \in [0,1]$ and $(\ell, m) \in I$ with $m \le \epsilon\ell$. Hence the proof of (25) will be complete if we show that

$$|Y_{\ell,m}(x)| \lesssim_\epsilon \ell^{1/2}(1+m)^{-1/3} \tag{30}$$

for all $(\ell, m) \in I$ with $m \le \epsilon\ell$ and $x \in [0,1]$. Actually, we need only consider the case where $b_{\ell,m}/2 \le y \le b_{\ell,m}(1+\delta m^{-2/3})$ for some $\delta > 0$, for otherwise (30) easily follows from (29). In this case, $y \simeq m/\ell$, and therefore $|Y_{\ell,m}(x)| \lesssim \ell^{1/2}$ by (21); hence, in proving (30), we need only consider $m \ge m_0$ for some $m_0 > 0$.

Now, as discussed in [5, Section 3], the identity

$$E_m^{-1} M_m(z) = \sqrt{2} J_m(z)$$

holds for all $z \in [0, X_m]$, where X_m is a positive real number defined in [5, (3.4)] and satisfying

$$X_m \ge m \tag{31}$$

for all $m \ge 0$ by [23, Corollary 1 applied with $\theta = 3\pi/4$], as well as

$$X_m = m + 2cm^{1/3} + \mathcal{O}(m^{-1/3})$$

as $m \to \infty$, for some $c \in (0,1)$ [25, Chapter 12, Example 1.1, p. 438]. In particular

$$X_m \ge m(1 + cm^{-2/3}) \tag{32}$$

for all $m \ge m_0$, for a suitable $m_0 > 0$. Moreover, (26) implies that

$$|y^2 - b_{\ell,m}^2|^{1/4} |Y_{\ell,m}(x)| \lesssim_\epsilon |\ell^2 \zeta_{\ell,m}(x) - m^2|^{1/4} |J_m(\ell\zeta_{\ell,m}(x)^{1/2})| \tag{33}$$

uniformly for all $(\ell, m) \in I$ with $m \le \epsilon\ell$ and $x \in [0,1]$ satisfying $\ell\zeta_{\ell,m}(x)^{1/2} \le X_m$.

We now recall from [7, (3.24)] the inequality

$$\zeta_{\ell,m}(x)^{1/2} \le y \tag{34}$$

for all $x \in [a_{\ell,m}, 1]$. Further, we claim that

$$\frac{\zeta_{\ell,m}(x) - b_{\ell,m}^2}{y^2 - b_{\ell,m}^2} \simeq_\epsilon 1 \tag{35}$$

for all $x \in [0,1]$ with $b_{\ell,m}/2 \le y \le \epsilon^{-1/2} b_{\ell,m}$.

Assuming the claim, from (35) we deduce that, for all $(\ell, m) \in I$ and $x \in [0,1]$, if $m \le \epsilon\ell$ and $b_{\ell,m}/2 \le y \le b_{\ell,m}(1+\delta m^{-2/3})$ for some $\delta \in (0,1)$, then

$$\zeta_{\ell,m}(x) \le b_{\ell,m}^2(1 + c_\epsilon \delta m^{-2/3}),$$

whence, by (32),

$$\ell\zeta_{\ell,m}(x)^{1/2} \le m(1 + c_\epsilon \delta m^{-2/3}) \le X_m$$

provided δ is chosen sufficiently small and $m \ge m_0$ for some sufficiently large m_0. Therefore, from (35) and (33) and Lemma 5 we deduce that

$$|Y_{\ell,m}(x)| \lesssim_\epsilon \ell^{1/2} m^{-1/3}$$

for all $(\ell, m) \in I$ and $x \in [0,1]$ satisfying $m_0 \le m \le \epsilon\ell$ and $b_{\ell,m}/2 \le y \le b_{\ell,m}(1+\delta m^{-2/3})$. This completes the proof of (30).

We are left with the proof of the claim (35). Assume first that $b_{\ell,m} \le y \le \epsilon^{-1/2} b_{\ell,m}$. Then, by (34), $b_{\ell,m} \le \zeta_{\ell,m}^{1/2}(x) \le \epsilon^{-1/2} b_{\ell,m}$ as well, and moreover $\sqrt{1-\epsilon^{1/2}} \le x \le a_{\ell,m} \le 1$ (here we use that $b_{\ell,m} \le \epsilon$). Consequently, from (27) we deduce that

$$\int_{b_{\ell,m}^2}^{\zeta_{\ell,m}(x)} (\xi - b_{\ell,m}^2)^{1/2}\,\mathrm{d}\xi \simeq_\epsilon \int_x^{a_{\ell,m}} (a_{\ell,m}^2 - s^2)^{1/2}\,\mathrm{d}s \simeq_\epsilon \int_{x^2}^{a_{\ell,m}^2} (a_{\ell,m}^2 - t)^{1/2}\,\mathrm{d}t, \tag{36}$$

that is,

$$(\zeta_{\ell,m}(x) - b_{\ell,m}^2)^{3/2} \simeq_\epsilon (a_{\ell,m}^2 - x^2)^{3/2} = (y^2 - b_{\ell,m}^2)^{3/2}, \tag{37}$$

which gives (35) in this case. In the case where $b_{\ell,m}/2 \le y \le b_{\ell,m}$, instead, by (28) we first deduce that

$$\frac{b_{\ell,m}}{2\sqrt{2}} \log_+ \left(\frac{b_{\ell,m}^2}{2\zeta_{\ell,m}(x)} \right) \le \int_{\min\{\zeta_{\ell,m}(x), b_{\ell,m}^2/2\}}^{b_{\ell,m}^2/2} \frac{(b_{\ell,m}^2 - \xi)^{1/2}}{2\xi} \, \mathrm{d}\xi \le \frac{4}{b_{\ell,m}^2} \int_{a_{\ell,m}}^{x} (s^2 - a_{\ell,m}^2)^{1/2} \, \mathrm{d}s \simeq_\epsilon b_{\ell,m}^{-2} (x^2 - a_{\ell,m}^2)^{3/2} \lesssim b_{\ell,m}$$

(here we used that $1 \ge x \ge a_{\ell,m} \ge \sqrt{1-\epsilon^2}$), whence

$$c_\epsilon b_{\ell,m} \le \zeta_{\ell,m}(x)^{1/2} \le b_{\ell,m}$$

for some $c_\epsilon \in (0,1)$. Now the analogues of (36) and (37) can be derived by using (28) in place of (27), giving (35) in this case as well. □

Propositions 4 and 6 immediately yield the second part of Theorem 1.

Corollary 7. *Let $d \in \mathbb{N}$, $d \ge 2$, and $\epsilon \in (0,1)$. For all $(\ell, m) \in I_d$, if $m \le \epsilon\ell$, then*

$$|X_{\ell,m}^d(x)| \lesssim_{\epsilon,d} \begin{cases} y^{-(d-2)/2} \left(\frac{(1+m)^{4/3}}{\ell^2} + |y^2 - b_{\ell,m}^2| \right)^{-1/4} & \text{for all } x \in [-1,1], \\ 2^{-m}\, \ell^{(d-1)/2} & \text{if } y \le b_{\ell,m}/2e, \end{cases} \tag{38}$$

where $y = \sqrt{1-x^2}$.

Proof. The first inequality is an immediate consequence of (13) and (25). Moreover, if $m \in \mathbb{N}_{d-1}$ and $y \le b_{\ell,m}/2e$, then

$$\left| X_{\ell,m}^d(x) \right| \lesssim_\epsilon \left(b_{\ell,m}/2e \right)^{m-(d-2)/2} b_{\ell,m}^{-m-1/2} e^m \lesssim_d 2^{-m} \ell^{(d-1)/2},$$

proving the second bound in (38). □

5. The Hermite regime

In this section we prove pointwise estimates for both $Y_{\ell,m}$ and $X_{\ell,m}^d$ as $m \ge \epsilon\ell$ for some $\epsilon \in (0,1)$. In this range, we can apply a uniform asymptotic approximation of $\mathrm{P}_{\ell-1/2}^{-m}$ for large ℓ in terms of Hermite functions previously proved by Olver [26, 27]. Indeed, the same argument used in the proof of [7, Proposition 3.3], which is based on Olver's approximation, as well as standard estimates for Hermite functions [2, 33] and the uniform estimate for Jacobi polynomials of Haagerup and Schlichtkrull [13], can be applied to prove the following estimate.

Proposition 8. *Let $\epsilon \in (0,1)$. Then for all $(\ell, m) \in I$ with $m \ge \epsilon\ell$ and for all $x \in [-1,1]$*

$$\left| Y_{\ell,m}(x) \right| \lesssim_\epsilon \left(\ell^{-1} + |x^2 - a_{\ell,m}^2| \right)^{-1/4}. \tag{39}$$

By combining this estimate with ODE techniques we can obtain a stronger decay estimate in the region where $|x| \gg a_{\ell,m}$.

Proposition 9. *For all $K \in (1,\infty)$ there exists $c \in (0,1)$ such that, for all $\epsilon \in (0,1)$ and $m_0 \in \mathbb{N}/2$, if $(\ell, m) \in I$ is such that $m \ge \max\{\epsilon\ell, m_0\}$, then*

$$\left| Y_{\ell,m}(x) \right| \lesssim_{\epsilon,m_0,K} |x|^{-1/2} (1-x^2)^{\max\{c\epsilon\ell, m_0\}/2} \tag{40}$$

whenever $x \in (-1,1)$ and $|x| \ge K a_{\ell,m}$.

Proof. Note that, if $m \le 1$, then $\ell \lesssim_\epsilon 1$ and the desired estimate trivially follows from (20). So in what follows we may assume $m > 1$. For a similar reason, we may also assume that $\ell \ge \ell(m_0, K)$ for some large $\ell(m_0, K)$ to be specified later. Further, due to parity, we need only prove the estimate for $x \ge 0$.

Recall (see, e.g., [26, (2.1)]) that the function $L(x) = (1-x^2)^{1/2} Y_{\ell,m}(x)$ satisfies the ODE

$$L''(x) = Q(x)L(x) \tag{41}$$

on the interval $(-1,1)$, where

$$Q(x) = Q_{\ell,m}(x) = \frac{\ell^2(x^2 - a_{\ell,m}^2) - (3+x^2)/4}{(1-x^2)^2} = (\ell^2 - 1/4)\frac{x^2 - \bar{x}_{\ell,m}^2}{(1-x^2)^2}, \tag{42}$$

with $a_{\ell,m}$ defined as in (7), and

$$\bar{x}_{\ell,m} = \sqrt{\frac{\ell^2 - m^2 + 3/4}{\ell^2 - 1/4}} \in [a_{\ell,m}, 8a_{\ell,m}] \tag{43}$$

for all $(\ell, m) \in I$. Note that $\bar{x}_{\ell,m} < 1$ (since $m > 1$), and $Q(x) > 0$ whenever $|x| > \bar{x}_{\ell,m}$. In addition, since $m > 1$, from (12) we deduce that

$$\lim_{x\to 1} L(x) = \lim_{x\to 1} L'(x) = 0. \tag{44}$$

We now claim that $L(x)L'(x) < 0$ for all $x \ge \bar{x}_{\ell,m}$. Indeed, $L(x)$ and $L'(x)$ cannot vanish simultaneously, because L is a nontrivial solution of a second order linear ODE. Moreover, by (44), $L(x)L'(x)$ cannot be positive for any $x > \bar{x}_{\ell,m}$ (otherwise by (41) the function L would be positive and increasing, or negative and decreasing, on the interval $(x,1)$, and would not tend to zero). Finally one cannot have $L(x)L'(x) = 0$ for any $x \ge \bar{x}_{\ell,m}$ (because for any larger x one would find the situation that we have just ruled out).

Note also that Q is strictly increasing for $x \ge 0$. We can then apply the argument in [34, Section 8.2] and conclude that, for $x > x_* > \bar{x}_{\ell,m}$,

$$|L(x)| \le |L(x_*)| \exp\left(-\int_{x_*}^{x} Q(u)^{1/2}\,\mathrm{d}u\right). \tag{45}$$

From (42) we deduce that, if $x^2 \ge (1-\eta^2)^{-1}\bar{x}_{\ell,m}^2$ for some $\eta \in (0,1)$, then

$$Q(x)^{1/2} \ge \eta\sqrt{\ell^2 - 1/4}\,\frac{x}{1-x^2},$$

and consequently, for $x > x_* \ge (1-\eta^2)^{-1/2}\bar{x}_{\ell,m}$,

$$\int_{x_*}^{x} Q(u)^{1/2}\,\mathrm{d}u \ge \frac{\eta}{2}\sqrt{\ell^2 - 1/4}\int_{x_*^2}^{x^2}\frac{\mathrm{d}u}{1-u} = \frac{\eta}{2}\sqrt{\ell^2 - 1/4}\log\frac{1-x_*^2}{1-x^2}.$$

Hence (45) yields

$$|Y_{\ell,m}(x)| \le |Y_{\ell,m}(x_*)|\left(\frac{1-x^2}{1-x_*^2}\right)^{(\eta\sqrt{\ell^2-1/4}-1)/2}.$$

Note that, if we take $x^2 \ge (1-\delta)^{-1}x_*^2$ for some $\delta \in (0,1)$, then $1 - x_*^2 \ge 1 - (1-\delta)x^2 \ge (1-x^2)^{1-\delta}$, by Bernoulli's inequality, whence

$$|Y_{\ell,m}(x)| \le |Y_{\ell,m}(x_*)|\,(1-x^2)^{\delta(\eta\sqrt{\ell^2-1/4}-1)/2}. \tag{46}$$

Finally, let us remark that $\ell^2 - m^2 \ge (\ell+m)/2$ for all $(\ell,m) \in I$. Consequently, by (43), $\bar{x}_{\ell,m}/a_{\ell,m} \to 1$ as $\ell \to \infty$ uniformly in m, so there exists $\ell_{K,\eta} \in \mathbb{N}/2$ such that

$$\bar{x}_{\ell,m}/a_{\ell,m} \in [1, K^{1/3}], \qquad \eta\sqrt{\ell^2 - 1/4} - 1 \ge \eta\ell/2, \tag{47}$$

for all $(\ell,m) \in I$ with $\ell \ge \ell_{K,\eta}$. Moreover

$$a_{\ell,m}^2 \ge 1/(2\ell) \tag{48}$$

for all $(\ell, m) \in I$, and therefore, for any $\alpha > 0$,

$$|x|/a_{\ell,m} \lesssim \ell^{1/2}|x| \lesssim_\alpha \exp(\alpha\ell x^2) \leq (1-x^2)^{-\alpha\ell}. \tag{49}$$

Now, since $m \geq \epsilon\ell$, if we take $x_* = (1-\eta^2)^{-1/2}\bar{x}_{\ell,m}$, then $x_* \geq (1-\eta^2)^{-1/2}a_{\ell,m}$ and

$$|Y_{\ell,m}(x_*)| \lesssim_{\epsilon,\eta} a_{\ell,m}^{-1/2} \tag{50}$$

by (39). Hence, by (46), (50) and (49), if $x^2 \geq (1-\delta)^{-1}(1-\eta^2)^{-1}\bar{x}^2_{\ell,m}$, then

$$\begin{aligned}|Y_{\ell,m}(x)| &\lesssim_{\epsilon,\eta} a_{\ell,m}^{-1/2}(1-x^2)^{\delta(\eta\sqrt{\ell^2-1/4}-1)/2}\\ &\lesssim_\alpha |x|^{-1/2}(1-x^2)^{\delta(\eta\sqrt{\ell^2-1/4}-1-\alpha\ell)/2}.\end{aligned}$$

As a consequence, by (47), if we take δ and η so that $1-\delta = 1-\eta^2 = K^{-2/3}$, $\alpha = \eta/4$ and $c = \delta\eta/4$, then

$$|Y_{\ell,m}(x)| \lesssim_{\epsilon,K} |x|^{-1/2}(1-x^2)^{c\ell/2},$$

whenever $x \geq Ka_{\ell,m}$, $m \geq \epsilon\ell$ and $\ell \geq \ell_{K,\eta}$. This proves the desired estimate (40) for all $\ell \geq \ell(m_0,K) = \max\{\ell_{K,\eta}, m_0/c\}$. □

The first part of Theorem 1 is a consequence of the following result.

Corollary 10. *Let $d \in \mathbb{N}$, $d \geq 2$. For all $K \in (1,\infty)$, there exists $c \in (0,1)$ such that, for all $\epsilon \in (0,1)$, for all $(\ell,m) \in I_d$, if $m \geq \epsilon\ell$ then*

$$|X^d_{\ell,m}(x)| \lesssim_{\epsilon,K,d} \begin{cases}(\ell^{-1}+|x^2-a^2_{\ell,m}|)^{-1/4} & \text{for all } x \in [-1,1],\\ |x|^{-1/2}(1-x^2)^{(c\epsilon\ell-(d-2)/2)_+/2} & \text{if } |x| \geq K\,a_{\ell,m}.\end{cases} \tag{51}$$

Proof. In light of (13), the second estimate in (51) immediately follows from Proposition 9 applied with $m_0 = (d-2)/2$. Let now $\bar{\epsilon} = (1-\epsilon^2)^{1/2}$ and note that $a_{\ell,m} \leq \bar{\epsilon}$ whenever $m \geq \epsilon\ell$. By Proposition 9 applied with $\bar{\epsilon}^{-1/2}$ in place of K, we also deduce that

$$|X_{\ell,m}(x)| \lesssim_{\epsilon,d} |x|^{-1/2} \lesssim_\epsilon a_{\ell,m}^{-1/2}$$

whenever $|x| \geq \bar{\epsilon}^{-1/2}a_{\ell,m}$, and in particular whenever $|x| \geq \bar{\epsilon}^{1/2}$. In view of (48), this proves the first estimate in (51) whenever $|x| \geq \bar{\epsilon}^{1/2}$. Since $\bar{\epsilon} \in (0,1)$, the same estimate for $|x| \leq \bar{\epsilon}^{1/2}$ immediately follows from Proposition 8 and (13). □

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