On a basic mean value theorem with explicit exponents

ON A BASIC MEAN VALUE THEOREM WITH EXPLICIT EXPONENTS

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ABSTRACT. In this paper we follow a paper from A. Sedunova [5] regarding R. C. Vaughan's basic mean value Theorem [6] to improve and complete a more general demonstration for a suitable class of arithmetic functions as started by A. C. Cojocaru and M. R. Murty [2]. As an application we derive a basic mean value Theorem for the von Mangoldt generalized functions.

1. INTRODUCTION

In 1980 R. C. Vaughan [6] proved the basic mean value Theorem

Theorem 1.1.

$$\sum_{q\leq Q}\frac{q}{\phi(q)}\sum_{\chi \bmod q}^{*}\max_{y\leq x}\Big|\sum_{n\leq y}\Lambda(n)\chi(n)\Big| \ll (x+x^{\frac{5}{6}}Q+x^{\frac{1}{2}}Q^2)\log^4 x,$$

where Λ is the von Mangoldt function and the sum is restricted to primitive characters.

This result was a major tool for R. C. Vaughan to prove with elementary methods the Bombieri-Vinogradov Theorem. Recently A. Sedunova [5] improved the exponent of the logarithm using a weighted version of Vaughan's identity and an estimate due to M. B. Barban and P. P. Vehov [1] related to Selberg's sieve. A. C. Cojocaru and M. R. Murty in [2] proved a more general Theorem than the basic mean value Theorem. We will follow their proof improving the results adapting Sedunova's method. Using the main Theorem 2.1 we will be able to prove a basic mean value Theorem for the generalized von Mangoldt function $\Lambda_k = \mu\star\log^k$, precisely

Theorem 1.2. *For each $k\in\mathbb{N}$, $\epsilon>0$ it holds*

$$\sum_{q\leq Q}\frac{q}{\phi(q)}\sum_{\chi \bmod q}^{*}\max_{y\leq x}\Big|\sum_{n\leq y}\Lambda_k(n)\chi(n)\Big| \ll_k \left(x+x^{\frac{13}{14}+\epsilon}Q+x^{\frac{1}{2}}Q^2\right)\log^{k+1} x.$$

Notation. Given $A\subset\mathbb{R}$, with $\mathbb{1}_A$ we denote the characteristic function of A, when we write $\mathbb{1}$ we suppose $A=\{1\}$. Given an arithmetic function $f:\mathbb{N}\to\mathbb{C}$ and two real numbers $U<V$, we write $f_{\leq U}$ for $f\cdot\mathbb{1}_{[1,U]}$, $f_{>V}$ for $f\cdot(1-\mathbb{1}_{[1,V]})$ and with $f_{(U,V]}$ for $f\cdot\mathbb{1}_{(U,V]}$. We use the standard Vinogradov notation $\ll$ and when the implicit constant does depend on something we specify it. The quantities Q, M_1, M_2, N_1, N_2 are always some functions that depend on x, when we use the $\ll$ notation we assume $x\to+\infty$.

2. MAIN RESULT

Let us indicate the class of arithmetic functions

$$\mathscr{D}=\Big\{D:\mathbb{N}\to\mathbb{C}:\sum_{n\leq x}|D(n)|^2\ll x\log^\alpha x \ \textit{ for some } \alpha\geq 0\Big\} \tag{2.1}$$

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and for $D \in \mathscr{D}$ let

$$\alpha_D = \inf\Big\{\alpha \geq 0 : \sum_{n\leq x} |D(n)|^2 \ll x \log^\alpha x\Big\}. \tag{2.2}$$

We will need also information about the average of $|D(n)|/n^k$ for $k \in [0,1]$. Let us indicate

$$\beta_D(k) = \inf\Big\{\beta \geq 0 : \sum_{n\leq x} \frac{|D(n)|}{n^k} \ll_k x^{1-k} \log^\beta x\Big\}. \tag{2.3}$$

It is straightforward that if $D \in \mathscr{D}$ then $\beta_D(k) < +\infty$ for all $k \in [0,1]$, we will give a precise bound in Lemma 3.1.
From now on we consider two arithmetic functions $f, g : \mathbb{N} \to \mathbb{C}$ with $f(1) \neq 0$. We define μ_f, Λ_{fg} as

$$\mathbb{1} = \mu_f \star f, \tag{2.4}$$

$$\Lambda_{fg} = \mu_f \star g. \tag{2.5}$$

In particular μ_f is the convolution inverse of f: it exists and is unique since $f(1) \neq 0$. We can understand better these definitions with the help of the associated formal Dirichlet series: if

$$G(s) = \sum_{n\geq 1} \frac{g(n)}{n^s}, \quad F(s) = \sum_{n\geq 1} \frac{f(n)}{n^s};$$

then

$$\frac{G(s)}{F(s)} = \sum_{n\geq 1} \frac{\Lambda_{fg}(n)}{n^s}, \quad \frac{1}{F(s)} = \sum_{n\geq 1} \frac{\mu_f(n)}{n^s}.$$

The benchmark case is clearly when

$$f = 1, \quad g = \log, \quad \mu_f = \mu, \quad \Lambda_{fg} = \Lambda.$$

We are interested in estimates for

$$\sum_{q\leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\leq x} \Big|\sum_{n\leq y} \Lambda_{fg}(n)\chi(n)\Big|.$$

We have two trivial bounds. Using the triangle inequality we obtain, for each $\epsilon > 0$,

$$\begin{aligned}\sum_{q\leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\leq x} \Big|\sum_{n\leq y} \Lambda_{fg}(n)\chi(n)\Big| &\leq \sum_{q\leq Q} q \sum_{n\leq x} |\Lambda_{fg}(n)| \\ &\ll xQ^2 \log^{\beta_{\Lambda_{fg}}(0)+\epsilon} x. \end{aligned} \tag{2.6}$$

Using the Cauchy-Schwarz inequality we obtain, for each $\epsilon > 0$,

$$\begin{aligned}\sum_{q\leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\leq x} \Big|\sum_{n\leq y} \Lambda_{fg}(n)\chi(n)\Big| &\leq \sum_{q\leq Q} q \Big(\sum_{n\leq x} |\Lambda_{fg}(n)|^2\Big)^{\frac{1}{2}} \Big(\sum_{n\leq x} 1\Big)^{\frac{1}{2}} \\ &\ll xQ^2 \log^{\frac{\alpha_{\Lambda_{fg}}}{2}+\epsilon} x. \end{aligned} \tag{2.7}$$

We can improve these inequalities assuming further hypotheses for f, g, μ_f and Λ_{fg}.

Theorem 2.1. *We suppose that g, f, μ_f and Λ_{fg}, as defined before, satisfy the following hypotheses:*

(H1) $g : \mathbb{N} \to \mathbb{R}^+$ *is an increasing function;*
(H2) f, μ_f, $\Lambda_{fg} \in \mathscr{D}$*;*
(H3) *there exist $\theta_f, \gamma_f \in [0,1]$ such that, for any non-principal primitive Dirichlet character χ mod q*

$$\sum_{n\leq x} f(n)\chi(n) \ll x^{\theta_f} q^{\frac{1}{2}} \log q + x^{\gamma_f};$$

(H4) *for each* $1 \le V_1 < V_2$ *there exists a bounded function* $\eta(b) = \eta(b; V_1, V_2)$ *such that* $\eta(b) = 1$ *for* $b \le V_1$, $\eta(b) = 0$ *for* $b > V_2$ *and*

$$\sum_{n=1}^{V} \Big| \big((\mu_f \cdot \eta) \star f \big)(n) \Big|^2 \ll \frac{V}{\log(\frac{V_2}{V_1})}.$$

Then for each $\epsilon > 0$, $U_0 \le U_1, V_1 < V_2$ *it holds*

$$\sum_{q \le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y \le x} \Big| \sum_{n \le y} \Lambda_{fg}(n) \chi(n) \Big| \ll H(x, Q, U_0, U_1, V_1, V_2)$$

and we have

$$\begin{aligned} H(x,Q,U_0,U_1,V_1,V_2) \ll{}& U_1 Q^2 \log^{\beta_{\Lambda_{fg}}(0)+\epsilon} U_1 \\ &+ x^{\theta_f} (U_0 V_2)^{1-\theta_f} Q^{\frac{5}{2}} \log^{\beta_{\mu_f}(\theta_f)+\beta_{\Lambda_{fg}}(\theta_f)+1+\epsilon} (U_0 V_2 Q) \\ &+ x^{\gamma_f} (U_0 V_2)^{1-\gamma_f} Q^2 \log^{\beta_{\mu_f}(\gamma_f)+\beta_{\Lambda_{fg}}(\gamma_f)+\epsilon} (U_0 V_2) \\ &+ x \log^{\beta_f(0)+\beta_{\mu_f}(1)+\beta_{\Lambda_{fg}}(1)+\epsilon} (x U_0 V_2) \\ &+ \Big(\big(x^{\frac{1}{2}} Q^2 + x \big) \log U_1 + x^{\frac{1}{2}} Q \Big(U_1^{\frac{1}{2}} + \frac{x^{\frac{1}{2}}}{U_0^{\frac{1}{2}}} \Big) \Big) \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+1+\epsilon} x}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})} \\ &+ g(x) V_2 Q^{\frac{5}{2}} \log^{\beta_{\mu_f}(0)+1+\epsilon} (V_2 Q) + g(x) x \log^{\beta_{\mu_f}(1)+\epsilon} V_2 \\ &+ \Big(\big(x^{\frac{1}{2}} Q^2 + x \big) \log x + x Q \Big(\frac{1}{V_1^{\frac{1}{2}}} + \frac{1}{U_1^{\frac{1}{2}}} \Big) \Big) \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+1+\epsilon} x}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})}. \end{aligned}$$

In particular if all the α *and* β *reach the minima in definitions 2.2 and 2.3, then the claim holds with* $\epsilon = 0$.

Corollary 2.2. *Assuming the same hypotheses as in Theorem 2.1*

$$\sum_{q \le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y \le x} \Big| \sum_{n \le y} \Lambda_{fg}(n) \chi(n) \Big| \ll ML$$

where M *is the main term and* L *is the logarithmic term, precisely*

$$\begin{aligned} M = \max\Big\{ & U_1 Q^2,\ x^{\theta_f} (U_0 V_2)^{1-\theta_f} Q^{\frac{5}{2}},\ x^{\gamma_f} (U_0 V_2)^{1-\gamma_f} Q^2,\ x,\ x^{\frac{1}{2}} Q^2, \\ & \frac{xQ}{U_0^{\frac{1}{2}}},\ x^{\frac{1}{2}} U_1^{\frac{1}{2}} Q,\ V_2 Q^{\frac{5}{2}},\ \frac{xQ}{V_1^{\frac{1}{2}}} \Big\} \end{aligned}$$

and

$$\begin{aligned} L = \max\Big\{ & \log^{\beta_{\Lambda_{fg}}(0)+\epsilon} U_1,\ \log^{\beta_{\mu_f}(\theta_f)+\beta_{\Lambda_{fg}}(\theta_f)+1+\epsilon} (U_0 V_2 Q), \\ & \log^{\beta_{\mu_f}(\gamma_f)+\beta_{\Lambda_{fg}}(\gamma_f)+\epsilon} (U_0 V_2), \log^{\beta_f(0)+\beta_{\mu_f}(1)+\beta_{\Lambda_{fg}}(1)+\epsilon} (x U_0 V_2), \\ & \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+2+\epsilon} (x U_1)}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})},\ g(x) \log^{\beta_{\mu_f}(0)+1+\epsilon} (V_2 Q),\ g(x) \log^{\beta_{\mu_f}(1)+\epsilon} V_2, \\ & \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+2+\epsilon} x}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})} \Big\}. \end{aligned}$$

3. Preparation for the proof

First we prove a Lemma that guarantees us that if $D \in \mathscr{D}$ then $\beta_D(k)$ is bounded for all $k \in [0,1]$.

Lemma 3.1. *If $D \in \mathscr{D}$ then*

$$\beta_D(k) \le \frac{\alpha_D}{2} + \mathbb{1}(k).$$

Proof. This follows easily using partial summation and the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{n\le x}\frac{|D(n)|}{n^k} &= \frac{1}{x^k}\sum_{n\le x}|D(n)| + k\int_1^x\Big(\sum_{n\le t}|D(n)|\Big)\frac{dt}{t^{k+1}} \\ &\ll_k \frac{1}{x^k}\Big(\sum_{n\le x}1\Big)^{\frac12}\Big(\sum_{n\le x}|D(n)|^2\Big)^{\frac12} + \int_1^x\Big(\sum_{n\le t}1\Big)^{\frac12}\Big(\sum_{n\le t}|D(n)|^2\Big)^{\frac12}\frac{dt}{t^{k+1}} \\ &\ll_k x^{1-k}\log^{\frac{\alpha_D}{2}+\epsilon}x + \log^{\frac{\alpha_D}{2}+\epsilon}x\int_1^x\frac{dt}{t^k},\end{aligned}$$

for each $\epsilon > 0$. So we have the claim distinguishing $k = 1$ from the other cases. □

This is typically far from the best exponent, for example $\Lambda \in \mathscr{D}$ with $\alpha_\Lambda = 1$, Lemma 3.1 provides us the bound $\beta_\Lambda(0) \le 1/2$ but the prime number Theorem claims that $\beta_\Lambda(0) = 0$. Another example rises from Mertens' formula

$$\sum_{n\le x}\frac{\Lambda(n)}{n} = \log x + O(1)$$

and so $\beta_\Lambda(1) = 1$ but with the Lemma 3.1 we can only obtain $\beta_\Lambda(1) \le 3/2$. However with our kind of generalization it can't be done better than Lemma 3.1, for example the function identically 1 is in $\mathscr{D}$ with $\alpha_1 = 0$, $\beta_1(0) = 0$ and $\beta_1(1) = 1$.
As in the classic proof of the basic mean value Theorem we need a modified multiplicative large sieve inequality.

Theorem 3.2. *Let f_1, f_2 be two arithmetic function, then*

$$\begin{aligned}&\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le M_1M_2}\Big|\sum_{n\le y}(f_{1\le M_1}\star f_{2\le M_2})(n)\chi(n)\Big| \\ &\ll (Q^2+M_1)^{\frac12}(Q^2+M_2)^{\frac12}\Big(\sum_{n\le M_1}|f_1(n)|^2\Big)^{\frac12}\Big(\sum_{n\le M_2}|f_2(n)|^2\Big)^{\frac12}\log(M_1M_2).\end{aligned}$$

For the proof see Lemma 2 of [6]. If we have to estimate sums like

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}(f_{1(N_1,M_1]}\star f_2)(n)\chi(n)\Big|,$$

with $f_1, f_2 \in \mathscr{D}$ and $M_1/N_1 \ll x$, using directly Theorem 3.2 is not in general convenient. Indeed writing

$$\max_{y\le x}\Big|\sum_{n\le y}(f_{1(N_1,M_1]}\star f_2)(n)\chi(n)\Big| = \max_{y\le x}\Big|\sum_{n\le y}(f_{1(N_1,M_1]}\star f_{2\le\frac{x}{N_1}})(n)\chi(n)\Big|$$

we obtain a bound like

$$\begin{aligned}&\ll (Q+M_1^{\frac12})\Big(Q+\frac{x^{\frac12}}{N_1^{\frac12}}\Big)M_1^{\frac12}\frac{x^{\frac12}}{N_1^{\frac12}}\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x \\ &= \Big(x^{\frac12}Q^2\Big(\frac{M_1}{N_1}\Big)^{\frac12} + x\frac{M_1}{N_1} + x^{\frac12}Q\Big(\frac{M_1}{N_1}\Big)^{\frac12}\Big(M_1^{\frac12}+\frac{x^{\frac12}}{N_1^{\frac12}}\Big)\Big)\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x. \qquad (3.1)\end{aligned}$$

Combining a dicotomic method with Theorem 3.2 we can find a better bound when $\log M_1 \ll M_1/N_1$.

Lemma 3.3. Given $f_1, f_2 \in \mathscr{D}$, M_1, N_1 such that $M_1/N_1 \ll x$ and $\epsilon > 0$,

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}(f_{1(N_1,M_1]}\star f_2)(n)\chi(n)\Big|$$
$$\ll \bigg(\big(x^{\frac12}Q^2+x\big)\log M_1 + x^{\frac12}Q\Big(M_1^{\frac12}+\frac{x^{\frac12}}{N_1^{\frac12}}\Big)\bigg)\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x. \tag{3.2}$$

Proof. The estimate (3.1) is good when $M_1 \asymp N_1$ The idea is to split the interval $(N_1, M_1]$ in subintervals of the type $[T, 2T]$ and then apply Theorem 3.2 at each of this subintervals. For $T \le x$

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}(f_{1(T,2T]}\star f_2)(n)\chi(n)\Big|$$
$$= \sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}(f_{1(T,2T]}\star f_{2\le \frac{x}{T}})(n)\chi(n)\Big|$$
$$\ll (Q+T^{\frac12})\Big(Q+\frac{x^{\frac12}}{T^{\frac12}}\Big)T^{\frac12}\frac{x^{\frac12}}{T^{\frac12}}\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x \tag{3.3}$$
$$= \bigg(x^{\frac12}Q^2+x+x^{\frac12}Q\Big(T^{\frac12}+\frac{x^{\frac12}}{T^{\frac12}}\Big)\bigg)\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x. \tag{3.4}$$

We choose $T = N_1 2^k$ by varying $k \in \mathscr{S} \subset \mathbb{N}$ such that

$$(N_1, M_1] \subset \bigcup_{k\in\mathscr{S}}[N_1 2^k, N_1 2^{k+1}]$$

and $|\mathscr{S}|$ is minimum. In general the inclusion will be proper, to avoid problems and to be able to use the triangle inequality we extend to zero f_1 in the external points to $(N_1, M_1]$, i.e. we define $\tilde{f_1} = f_{1(N_1,M_1]}$. Now using the triangle inequality

$$\Big|\sum_{n\le y}(f_{1(N_1,M_1]}\star f_2)(n)\chi(n)\Big| = \Big|\sum_{n\le y}(\tilde{f_1}\star f_2)(n)\chi(n)\Big|$$
$$\le \sum_{\substack{T=N_1 2^k\\ k\in\mathscr{S}}}\Big|\sum_{n\le y}(\tilde{f}_{1(T,2T]}\star f_2)(n)\chi(n)\Big|.$$

Since $T \in [N_1, 2M_1]$, with (3.4) we can conclude

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}(f_{1(N_1,M_1]}\star f_2)(n)\chi(n)\Big|$$
$$\ll \bigg(\big(x^{\frac12}Q^2+x\big)|\mathscr{S}| + x^{\frac12}Q\Big(M_1^{\frac12}+\frac{x^{\frac12}}{N_1^{\frac12}}\Big)\bigg)\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x$$

and since $|\mathscr{S}| \ll \log M_1$, we obtain the claim. □

Remark 3.4. We remark that the previous Lemma is useful also when we have to estimate

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}(f_{1>N_1}\star f_{2>N_2})(n)\chi(n)\Big|,$$

indeed we can take $M_1 = x/N_2$ and obtain the bound

$$\ll \bigg(\big(x^{\frac12}Q^2+x\big)\log x + xQ\Big(\frac{1}{N_1^{\frac12}}+\frac{1}{N_2^{\frac12}}\Big)\bigg)\log^{\frac{\alpha_{f_1}+\alpha_{f_2}}{2}+1+\epsilon}x \tag{3.5}$$

3.1 Weighted Vaughan's identity. We want to use a decomposition formula for Λ_{fg} using a weight $\eta : \mathbb{N} \to \mathbb{C}$ such that $\eta(b) = 1$ for $b \le V_1$ as A. Sedunova did in [5]. We know the classic Vaughan's identity

$$\begin{aligned}\Lambda_{fg} &= \Lambda_{fg_{\le U_1}} - \Lambda_{fg_{\le U_1}} \star \mu_{f_{\le V_1}} \star f + \mu_{f_{\le V_1}} \star g + \Lambda_{fg_{>U_1}} \star \mu_{f_{>V_1}} \star f \\ &= \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4\end{aligned}$$

that follows from (2.4) and (2.5), indeed

$$\begin{aligned}\Lambda_{fg} &= \Lambda_{fg_{\le U_1}} + \Lambda_{fg} - \Lambda_{fg_{\le U_1}} = \Lambda_{fg_{\le U_1}} + \mu_f \star g - \Lambda_{fg_{\le U_1}} \star \mu_f \star f \\ &= \Lambda_{fg_{\le U_1}} + \mu_{f_{\le V_1}} \star g + \mu_{f_{>V_1}} \star g - \Lambda_{fg_{\le U_1}} \star \mu_{f_{\le V_1}} \star f - \Lambda_{fg_{\le U_1}} \star \mu_{f_{>V_1}} \star f \\ &= \Lambda_{fg_{\le U_1}} - \Lambda_{fg_{\le U_1}} \star \mu_{f_{\le V_1}} \star f + \mu_{f_{\le V_1}} \star g + \mu_{f_{>V_1}} \star (g - \Lambda_{fg_{\le U_1}} \star f)\end{aligned}$$

and we use that from (2.4) and (2.5) follows also

$$g = \Lambda_{fg} \star f. \tag{3.6}$$

We claim that, more in general

Lemma 3.5. *For every $\eta : \mathbb{N} \to \mathbb{C}$ such that $\eta(b) = 1$ for every $b \le V_1$*

$$\begin{aligned}\Lambda_{fg} &= \Lambda_{fg_{\le U_1}} - \Lambda_{fg_{\le U_1}} \star (\mu_f \cdot \eta) \star f + (\mu_f \cdot \eta) \star g + \Lambda_{fg_{>U_1}} \star \big(\mu_f \cdot (1 - \eta)\big) \star f \\ &= \Lambda_1' + \Lambda_2' + \Lambda_3' + \Lambda_4'.\end{aligned}$$

Proof. We observe, using essentially that $\eta(b) = 1$ for every $b \le V_1$,

$$\begin{aligned}\Lambda_1' &= \Lambda_1, \\ \Lambda_2' &= \Lambda_2 + \Lambda_{fg_{\le U_1}} \star (\mu_f \cdot \eta)_{>V_1} \star f, \\ \Lambda_3' &= \Lambda_3 - (\mu_f \cdot \eta)_{>V_1} \star g, \\ \Lambda_4' &= \Lambda_4 + \Lambda_{fg_{>U_1}} \star (\mu_f \cdot \eta)_{>V_1} \star f.\end{aligned}$$

It remains to show that the sum of the three remainders is equal to zero, but this is true since, from (3.6)

$$\begin{aligned}&\Lambda_{fg_{\le U_1}} \star (\mu_f \cdot \eta)_{>V_1} \star f - (\mu_f \cdot \eta)_{>V_1} \star g + \Lambda_{fg_{>U_1}} \star (\mu_f \cdot \eta)_{>V_1} \star f \\ &= (\mu_f \cdot \eta)_{>V_1} \star (\Lambda_{fg_{\le U_1}} \star f - g + \Lambda_{fg_{>U_1}} \star f) = 0.\end{aligned}$$

□

4. Main proof

In the proof we denote with $\epsilon > 0$ any small positive constant that rises from the definitions of α_D and $\beta_D(k)$ as infima; at the end we will still indicate with ϵ the maximum of the constant previously considered. First we show, as R. C. Vaughan did in [6], that we can treat larger Q more easily than smaller Q.

4.1 The case $\mathbf{Q^2 > x}$. We only use the modified multiplicative large sieve (Theorem 3.2) with $M_1 = 1$, $f_1(1) = 1$, $M_2 = [x]$, $f_2(n) = \Lambda_{fg}(n)$. We obtain

$$\sum_{q \le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y \le x} \Big| \sum_{n \le y} \Lambda_{fg}(n)\chi(n) \Big| \ll (x^{\frac{1}{2}} Q + Q^2) \Big(\sum_{n \le x} |\Lambda_{fg}(n)|^2 \Big)^{\frac{1}{2}} \log x.$$

Using (H2) and the definition of $\mathscr{D}$

$$\begin{aligned}\sum_{q \le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y \le x} \Big| \sum_{n \le y} \Lambda_{fg}(n)\chi(n) \Big| &\ll (xQ + x^{\frac{1}{2}} Q^2) \log^{\frac{\alpha_{\Lambda_{fg}}}{2} + 1 + \epsilon} x \\ &\ll x^{\frac{1}{2}} Q^2 \log^{\frac{\alpha_{\Lambda_{fg}}}{2} + 1 + \epsilon} x\end{aligned}$$

since $Q^2 > x$.

From now on we can assume $Q^2 \leq x$. We set four parameters $U_0 = U_0(x,Q) \leq U_1 = U_1(x,Q)$, $V_1 = V_1(x,Q) < V_2 = V_2(x,Q)$. Recalling Lemma 3.5, for any Dirichlet character χ mod q we can write

$$\sum_{n\leq y} \Lambda_{fg}(n)\chi(n) = \sum_{i=1}^{4}\sum_{n\leq y} \Lambda_i'(n)\chi(n) = \sum_{i=1}^{4} S_i(y,\chi).$$

We prove the Theorem 2.1 by estimating each of the sums

$$S_i(x,Q) = \sum_{q\leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\leq x} |S_i(y,\chi)|, \quad 1 \leq i \leq 4.$$

4.2 The estimate for $\mathbf{S_1(x,Q)}$. Using hypothesis (H2) and definition (2.3) we obtain

$$|S_1(y,\chi)| = \Big|\sum_{n\leq \min\{U_1,y\}} \Lambda_{fg}(n)\chi(n)\Big| \leq \sum_{n\leq U_1} |\Lambda_{fg}(n)| \ll U_1 \log^{\beta_{\Lambda_{fg}}(0)+\epsilon} U_1,$$

and so

$$S_1(x,Q) = \sum_{q\leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\leq x} |S_1(y,\chi)| \ll U_1 Q^2 \log^{\beta_{\Lambda_{fg}}(0)+\epsilon} U_1.$$

4.3 The estimate for $\mathbf{S_2(x,Q)}$. We recall the definition

$$S_2(y,\chi) = -\sum_{n\leq y} \big(\Lambda_{fg_{\leq U_1}} \star (\mu_f \cdot \eta) \star f\big)(n)\chi(n),$$

we split this sum into two parts

$$S_2(y,\chi) = S_2'(y,\chi) + S_2''(y,\chi)$$

where

$$S_2'(y,\chi) = -\sum_{n\leq y} \big(\Lambda_{fg_{\leq U_0}} \star (\mu_f \cdot \eta) \star f\big)(n)\chi(n),$$

and

$$S_2''(y,\chi) = -\sum_{n\leq y} \big(\Lambda_{fg_{(U_0,U_1]}} \star (\mu_f \cdot \eta) \star f\big)(n)\chi(n).$$

For $S_2'(y,\chi)$, using (H4) and writing $n = abc$

$$\begin{aligned} |S_2'(y,\chi)| &= \Big|\sum_{a\leq U_0} \Lambda_{fg}(a)\chi(a) \sum_{b} \mu_f(b)\eta(b)\chi(b) \sum_{c\leq \frac{y}{ab}} f(c)\chi(c)\Big| \\ &\ll \sum_{a\leq U_0} |\Lambda_{fg}(a)| \sum_{b\leq V_2} |\mu_f(b)| \Big|\sum_{c\leq \frac{y}{ab}} f(c)\chi(c)\Big|, \end{aligned}$$

so that we can use hypothesis (H3) to estimate the innermost sum for non-principal primitive characters χ mod q. We get

$$\begin{aligned} |S_2'(y,\chi)| \ll\; & y^{\theta_f} q^{\frac{1}{2}} \log q \sum_{a\leq U_0} \frac{|\Lambda_{fg}(a)|}{a^{\theta_f}} \sum_{b\leq V_2} \frac{|\mu_f(b)|}{b^{\theta_f}} \\ & + y^{\gamma_f} \sum_{a\leq U_0} \frac{|\Lambda_{fg}(a)|}{a^{\gamma_f}} \sum_{b\leq V_2} \frac{|\mu_f(b)|}{b^{\gamma_f}}. \end{aligned}$$

Then, by using hypothesis (H2) and using four times definition (2.3), we obtain

$$\begin{aligned}|S_2'(y,\chi)| &\ll y^{\theta_f} V_2^{1-\theta_f} q^{\frac{1}{2}} \log^{\beta_{\mu_f}(\theta_f)+1+\epsilon}(V_2 q) \sum_{a\le U_0} \frac{|\Lambda_{fg}(a)|}{a^{\theta_f}} \\ &+ y^{\gamma_f} V_2^{1-\gamma_f} \log^{\beta_{\mu_f}(\gamma_f)+\epsilon} V_2 \sum_{a\le U_0} \frac{|\Lambda_{fg}(a)|}{a^{\gamma_f}} \\ &\ll y^{\theta_f}(U_0V_2)^{1-\theta_f} q^{\frac{1}{2}} \log^{\beta_{\mu_f}(\theta_f)+\beta_{\Lambda_{fg}}(\theta_f)+1+\epsilon}(U_0V_2q) \\ &+ y^{\gamma_f}(U_0V_2)^{1-\gamma_f} \log^{\beta_{\mu_f}(\gamma_f)+\beta_{\Lambda_{fg}}(\gamma_f)+\epsilon}(U_0V_2).\end{aligned}$$

Instead, for $\chi=\chi_0$ we have, using two times definition 2.3,

$$\begin{aligned}|S_2'(y,\chi_0)| &\le \sum_{a\le U_0} |\Lambda_{fg}(a)| \sum_{b\le V_2} |\mu_f(b)| \sum_{c\le \frac{y}{ab}} |f(c)| \\ &\ll y \log^{\beta_f(0)+\epsilon} y \sum_{a\le U_0} \frac{|\Lambda_{fg}(a)|}{a} \sum_{b\le V_2} \frac{|\mu_f(b)|}{b} \\ &\ll y \log^{\beta_f(0)+\beta_{\mu_f}(1)+\epsilon}(yV_2) \sum_{a\le U_0} \frac{|\Lambda_{fg}(a)|}{a} \\ &\ll y \log^{\beta_f(0)+\beta_{\mu_f}(1)+\beta_{\Lambda_{fg}}(1)+\epsilon}(y\,U_0V_2).\end{aligned}$$

This implies that

$$\begin{aligned}S_2'(x,Q) &= \sum_{q\le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\le x} |S_2'(y,\chi)| \\ &\ll x^{\theta_f}(U_0V_2)^{1-\theta_f} Q^{\frac{5}{2}} \log^{\beta_{\mu_f}(\theta_f)+\beta_{\Lambda_{fg}}(\theta_f)+1+\epsilon}(U_0V_2Q) \\ &+ x^{\gamma_f}(U_0V_2)^{1-\gamma_f} Q^2 \log^{\beta_{\mu_f}(\gamma_f)+\beta_{\Lambda_{fg}}(\gamma_f)+\epsilon}(U_0V_2) \\ &+ x \log^{\beta_f(0)+\beta_{\mu_f}(1)+\beta_{\Lambda_{fg}}(1)+\epsilon}(xU_0V_2).\end{aligned}$$

For $S_2''(y,\chi)$ we recall the definition

$$S_2''(x,Q) = \sum_{q\le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\le x} \Big| \sum_{n\le y} \big(\Lambda_{fg(U_0,U_1]} \star (\mu_f\cdot\eta) \star f\big)(n)\chi(n) \Big|.$$

We want to use Lemma 3.3. We choose $f_1=\Lambda_{fg}$, $f_2=(\mu_f\cdot\eta)\star f$, $N_1=U_0$ e $M_1=U_1$. From hypothesis (H4) we have $(\mu_f\cdot\eta)\star f \in \mathscr{D}$ with $\alpha_{(\mu_f\cdot\eta)\star f}=0$. Moreover we have stronger bounds than for other functions in $\mathscr{D}$, indeed we can include the denominator $1/\log(V_2/V_1)$ in (3.2) since this does not depend on the upper limit of each partial sums. Finally we obtain

$$S_2''(x,Q) \ll \Big(\big(x^{\frac{1}{2}}Q^2 + x\big) \log U_1 + x^{\frac{1}{2}} Q \Big(U_1^{\frac{1}{2}} + \frac{x^{\frac{1}{2}}}{U_0^{\frac{1}{2}}} \Big) \Big) \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+1+\epsilon} x}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})}$$

4.4 The estimate for $\mathbf{S_3(x,Q)}$. We recall the definition

$$S_3(x,Q) = \sum_{q\le Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\le x} \Big| \sum_{n\le y} \big((\mu_f\cdot\eta)\star g\big)(n)\chi(n) \Big|.$$

We define a step function $\mathscr{G}:\mathbb{R}\to\mathbb{R}$ by $\mathscr{G}(t)=g(1)$ if $t\le 1$ and $\mathscr{G}(t)=g(n)-g(n-1)$ if $n-1<t\le n$ for $n\ge 2$. Then we observe that $g(n)=\int_0^n \mathscr{G}(t)dt$ and that $\mathscr{G}$ is positive, since the function g is positive and increasing from (H1). We

write, by partial summation,

$$
\begin{aligned}
|S_3(y,\chi)| &= \Big|\sum\sum_{ab\le y} \mu_f(a)\eta(a)g(b)\chi(ab)\Big| \\
&= \Big|\sum_{a\le V_2} \mu_f(a)\eta(a)\chi(a) \sum_{b\le \frac{y}{a}} \chi(b)\int_0^b \mathscr{G}(t)dt\Big| \\
&= \Big|\sum_{a\le V_2} \mu_f(a)\eta(a)\chi(a) \int_0^{\frac{y}{b}} \sum_{t<b\le \frac{y}{a}} \chi(b)\,\mathscr{G}(t)\,dt\Big| \\
&\le \int_0^y \mathscr{G}(t) \sum_{a\le V_2} |\mu_f(a)\eta(a)| \Big|\sum_{t<b\le \frac{y}{a}} \chi(b)\Big|\,dt.
\end{aligned}
$$

We can use the Pólya-Vinogradov inequality to estimate the inner sum for non-principal characters

$$
|S_3(y,\chi)| \ll g(y)q^{\frac12}\log q \sum_{a\le V_2} |\mu_f(a)\eta(a)|.
$$

Moreover, using hypotheses (H2) and (H4) according with definition (2.3), we can write

$$
|S_3(y,\chi)| \ll g(y)V_2 q^{\frac12} \log^{\beta_{\mu_f}(0)+1+\epsilon}(V_2 q).
$$

For $\chi = \chi_0$, again using hypotheses (H2) and (H4) according with definition (2.3) we can write

$$
|S_3(y,\chi_0)| \ll g(y)y\sum_{a\le V_2} \frac{|\mu_f(a)|}{a} \ll g(y)y\log^{\beta_{\mu_f}(1)+\epsilon} V_2.
$$

We further obtain

$$
\begin{aligned}
S_3(x,Q) = \sum_{q\le Q}\frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\le x}|S_3(y,\chi)| &\ll g(x)V_2 Q^{\frac52}\log^{\beta_{\mu_f}(0)+1+\epsilon}(V_2 Q) \\
&\quad + g(x)x\log^{\beta_{\mu_f}(1)+\epsilon} V_2.
\end{aligned}
$$

4.5 The estimate for $\mathbf{S_4(x,Q)}$. We recall the definition

$$
S_4(x,Q) = \sum_{q\le Q}\frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y\le x}\Big|\sum_{n\le y} \big(\Lambda_{fg_{>U_1}} \star (\mu_f\cdot(1-\eta))\star f\big)(n)\chi(n)\Big|
$$

We notice that $(1-\eta) = (1-\eta)_{>V_1}$ from ((H4)) and clearly

$$
\Lambda_{fg_{>U_1}} \star \big((\mu_f\cdot(1-\eta))_{>V_1}\star f\big) = \Lambda_{fg_{>U_1}} \star \big((\mu_f\cdot(1-\eta))\star f\big)_{>V_1}.
$$

Moreover from (2.4) we have that

$$
\big((\mu_f\cdot(1-\eta))\star f\big)_{>V_1} = \big(\mathbb{1} - (\mu_f\cdot\eta)\star f\big)_{>V_1} = -\big((\mu_f\cdot\eta)\star f\big)_{>V_1}.
$$

So we now can use Remark 3.4 with $f_1 = \Lambda_{fg}, f_2 = -(\mu_f\cdot\eta)\star f, N_1 = U_1, N_2 = V_1$. In a similar way as we did for $S_2''(x,Q)$, we obtain

$$
S_4(x,Q) \ll \Big(\big(x^{\frac12}Q^2 + x\big)\log x + xQ\Big(\frac{1}{V_1^{\frac12}} + \frac{1}{U_1^{\frac12}}\Big)\Big)\frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+1+\epsilon} x}{\log^{\frac12}(\frac{V_2}{V_1})}.
$$

4.6 Completion of the proof. Putting these estimates together it holds that

$$\begin{aligned}
S_1(x,Q) &\ll U_1 Q^2 \log^{\beta_{\Lambda_{fg}}(0)+\epsilon} U_1,\\
S_2'(x,Q) &\ll x^{\theta_f}(U_0V_2)^{1-\theta_f} Q^{\frac{5}{2}} \log^{\beta_{\mu_f}(\theta_f)+\beta_{\Lambda_{fg}}(\theta_f)+1+\epsilon}(U_0V_2Q)\\
&\quad + x^{\gamma_f}(U_0V_2)^{1-\gamma_f} Q^2 \log^{\beta_{\mu_f}(\gamma_f)+\beta_{\Lambda_{fg}}(\gamma_f)+\epsilon}(U_0V_2)\\
&\quad + x \log^{\beta_f(0)+\beta_{\mu_f}(1)+\beta_{\Lambda_{fg}}(1)+\epsilon}(xU_0V_2),\\
S_2''(x,Q) &\ll \Big(\big(x^{\frac{1}{2}}Q^2 + x\big)\log U_1 + x^{\frac{1}{2}} Q\Big(U_1^{\frac{1}{2}} + \frac{x^{\frac{1}{2}}}{U_0^{\frac{1}{2}}}\Big)\Big) \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+1+\epsilon} x}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})},\\
S_3(x,Q) &\ll g(x) V_2 Q^{\frac{5}{2}} \log^{\beta_{\mu_f}(0)+1+\epsilon}(V_2Q) + g(x) x \log^{\beta_{\mu_f}(1)+\epsilon} V_2,\\
S_4(x,Q) &\ll \Big(\big(x^{\frac{1}{2}}Q^2 + x\big)\log x + xQ\Big(\frac{1}{V_1^{\frac{1}{2}}} + \frac{1}{U_1^{\frac{1}{2}}}\Big)\Big) \frac{\log^{\frac{\alpha_{\Lambda_{fg}}}{2}+1+\epsilon} x}{\log^{\frac{1}{2}}(\frac{V_2}{V_1})}.
\end{aligned}$$

This gives the claim. We must be careful with $g(x)$: in Corollary 2.2 we have chosen to incorporate it in L since in the benchmark case we have $g(x) = \log x$ but in general we have to know its growth and understand if it is better to integrate it in L or in M.

5. The choice of U_0, U_1, V_1, V_2

Since we have the trivial bounds (2.6) and (2.7) we would like to find four parameters such that $M = o(xQ^2)$. We also note that there is symmetry in M with U_1 and V_1, so we can always assume $U_1 = V_1$ and so the scale is $U_0 \le U_1 = V_1 < V_2$. Assuming that we can choose $U_0 \le U_1 = V_1 < V_2 \ll x$, with $V_2/U_1 \gg x^c$ for some $c > 0$ then $L \ll \log^{l+\epsilon} x$, where

$$\begin{aligned}
l = \max\Big\{&\beta_{\Lambda_{fg}}(0),\ \beta_{\mu_f}(\theta_f) + \beta_{\Lambda_{fg}}(\theta_f) + 1,\ \beta_{\mu_f}(\gamma_f) + \beta_{\Lambda_{fg}}(\gamma_f),\\
&\beta_f(0) + \beta_{\mu_f}(1) + \beta_{\Lambda_{fg}}(1),\ \frac{\alpha_{\Lambda_{fg}}+3}{2},\ \beta_{\mu_f}(0)+1,\ \beta_{\mu_f}(1)\Big\}.
\end{aligned}$$

In view of Lemma 3.1 we have the rough bound for l

$$\begin{aligned}
l &\le \max\Big\{\frac{\alpha_{\Lambda_{fg}}}{2},\ \frac{\alpha_{\mu_f}+\alpha_{\Lambda_{fg}}}{2} + 2\cdot\mathbb{1}(\theta_f) + 1,\ \frac{\alpha_{\mu_f}+\alpha_{\Lambda_{fg}}}{2} + 2\cdot\mathbb{1}(\gamma_f),\\
&\qquad \frac{\alpha_f+\alpha_{\mu_f}+\alpha_{\Lambda_{fg}}}{2} + 2,\ \frac{\alpha_{\Lambda_{fg}}+3}{2},\ \frac{\alpha_{\mu_f}}{2}+1\Big\}\\
&\le \max\Big\{\frac{\alpha_f+\alpha_{\mu_f}+\alpha_{\Lambda_{fg}}}{2} + 2,\ \frac{\alpha_{\mu_f}+\alpha_{\Lambda_{fg}}}{2} + 2\cdot\mathbb{1}(\theta_f) + 1\Big\}.
\end{aligned}$$

6. The classic case

In R. C. Vaughan's basic mean value Theorem we treat

$$f = 1, \quad g = \log, \quad \mu_f = \mu, \quad \Lambda_{fg} = \Lambda.$$

We have

$$\sum_{n\le x} 1 = x + O(1), \qquad \sum_{n\le x} \big|\Lambda(n)\big|^2 = x\log x + O(x),$$

$$\sum_{n\le x} \Lambda(n) = x + O\Big(\frac{x}{\log x}\Big), \qquad \sum_{n\le x} \frac{1}{n} = \log x + O(1),$$

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

In our notation we obtain

$$\begin{aligned} &\beta_\mu(0)=\beta_1(0)=0, && \alpha_\Lambda=1,\\ &\beta_\Lambda(0)=0, && \beta_\mu(1)=1,\\ &\beta_\Lambda(1)=1; && \end{aligned}$$

all these values clearly are minima. From Pólya-Vinogradov inequality we have

$$\theta_1=\gamma_1=0.$$

To satisfy (H4) we recall an estimate due to M. B. Barban and P. P. Vehov [1] related to Selberg's sieve (see S. Graham for a stronger result [3]). For each $1\le V_1<V_2$ it holds

$$\sum_{n=1}^{V}\Big|\big((\mu\cdot\eta)\star 1\big)(n)\Big|^2\ll\frac{V}{\log(\frac{V_2}{V_1})}, \tag{6.1}$$

where

$$\eta(b)=\begin{cases}1 & b\le V_1,\\ \frac{\log(\frac{V_2}{b})}{\log(\frac{V_2}{V_1})} & V_1<b\le V_2,\\ 0 & b>V_2.\end{cases} \tag{6.2}$$

As a Corollary of Theorem 2.1 we have the main result of [5].

Corollary 6.1. *For each $U_0=U_0(x,Q)\le U_1=U_1(x,Q)$, $V_1=V_1(x,Q)<V_2=V_2(x,Q)$ it holds*

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}\Lambda(n)\chi(n)\Big|\ll ML,$$

where M and L are

$$M=\max\Big\{U_1Q^2,\,(U_0V_2)Q^{\frac52},\,x,\,x^{\frac12}Q^2,\,\frac{xQ}{U_0^{\frac12}},\,x^{\frac12}U_1^{\frac12}Q,\,\frac{xQ}{V_1^{\frac12}}\Big\}$$

and

$$L=\max\Big\{\log(U_0V_2Q),\,\log^2(xU_0V_2),\,\frac{\log^{\frac52}(xU_1)}{\log^{\frac12}(\frac{V_2}{V_1})},\,\log^2(xV_2Q),\,\frac{\log^{\frac52}x}{\log^{\frac12}(\frac{V_2}{V_1})}\Big\}.$$

With this result A. Sedunova, in [5], obtained

Theorem 6.2. *For each $\epsilon>0$,*

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}\Lambda(n)\chi(n)\Big|\ll\big(x+x^{\frac{13}{14}+\epsilon}Q+x^{\frac12}Q^2\big)\log^2 x.$$

This follows taking for $Q\in[x^{3/7+\epsilon},x^{1/2}]$

$$U_0=x^{\frac47-\epsilon}Q^{-1},\quad U_1=V_1=x^{\frac47}Q^{-1},\quad V_2=x^{\frac47+\frac{5\epsilon}{2}}Q^{-1};$$

while for $Q\in[1,x^{3/7+\epsilon}]$

$$U_0=x^{\frac17-\epsilon},\quad U_1=V_1=x^{\frac17},\quad V_2=x^{\frac17+\frac{\epsilon}{2}}.$$

We remark that the exponent 13/14 is optimal here, i.e. searching for the minimal $A>0$ such that for each $\epsilon>0$ it holds

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\max_{y\le x}\Big|\sum_{n\le y}\Lambda(n)\chi(n)\Big|\ll\big(x+x^{A+\epsilon}Q+x^{\frac12}Q^2\big)\log^2 x$$

then one can show that only using Corollary 6.1 it cannot be taken $A<13/14$.

7. Application to the generalized von Mangoldt function

The generalized von Mangoldt function is defined as

$$\Lambda_k = \mu \star \log^k$$

for $k \in \mathbb{N}$. One can show the recursive relation

$$\Lambda_{k+1} = \Lambda_k \cdot \log + \Lambda \star \Lambda_k$$

and so, in particular, $\Lambda_k(n) \geq 0$. In [4] it is shown that

$$\sum_{n \leq x} \Lambda_k(n) \sim kx \log^{k-1} x. \tag{7.1}$$

From the Möbius inversion formula it holds

$$\log^k = \Lambda_k \star 1$$

and so $\Lambda_k(n) \leq (\log n)^k$. We can easily derive from this and (7.1) that

$$\sum_{n \leq x} \left|\Lambda_k(n)\right|^2 \ll_k x \log^{2k-1} x,$$

moreover, by partial summation and (7.1) we have

$$\sum_{n \leq x} \frac{\Lambda_k(n)}{n} = k \log^{k-1} x + o(\log^{k-1} x) + \int_1^x \frac{k \log^{k-1} t}{t} dt \sim \log^k x.$$

Finally, $\Lambda_k \in \mathscr{D}$ with $\beta_{\Lambda_k}(1) = k$, $\beta_{\Lambda_k}(0) = k-1$ and $\alpha_{\Lambda_k} \leq 2k-1$. We can use the main Theorem 2.1 with $f = 1$, $g = \log^k$ and then proceeding with the same choice of U and V as in [5] to obtain

Theorem 7.1. *For each $k \in \mathbb{N}$, $\epsilon > 0$ it holds*

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sideset{}{^*}\sum_{\chi \bmod q} \max_{y \leq x} \Big|\sum_{n \leq y} \Lambda_k(n)\chi(n)\Big| \ll_k \left(x + x^{\frac{13}{14}+\epsilon} Q + x^{\frac{1}{2}} Q^2\right) \log^{k+1} x.$$

This is clearly a generalization of Theorem 6.2.

8. Remark on hypothesis (H4)

We remark that in the classic case it holds something stronger than (6.1) as S. Graham has shown in [3]. We too can assume a stronger hypothesis than (H4).
(H4') *For each $1 \leq V_1 < V_2$ it holds*

$$\sum_{n=1}^{V} \big(\Gamma_1 \star f\big)(n)\big(\Gamma_2 \star f\big)(n) = V \log V_1 + O(V)$$

where

$$\Gamma_i(b) = \begin{cases} \mu_f(b) \log\left(\frac{V_i}{b}\right) & b \leq V_i, \\ 0 & b > V_i. \end{cases}$$

This implies (H4). Indeed we consider the same η as in (6.2), and observe that $\eta \cdot \mu_f = (\Gamma_2 - \Gamma_1)/\log(V_2/V_1)$, so we can write

$$\begin{aligned} \log^2\Big(\frac{V_2}{V_1}\Big) \sum_{n=1}^{V} \Big|\big((\mu_f \cdot \eta) \star f\big)(n)\Big|^2 &= \sum_{n=1}^{V} (\Gamma_1 \star f)^2(n) + \sum_{n=1}^{V} (\Gamma_2 \star f)^2(n) \\ &\quad - 2\sum_{n=1}^{V} (\Gamma_1 \star f)(n)(\Gamma_2 \star f)(n). \end{aligned}$$

Now we apply three times (H4') to obtain

$$\log^2\Big(\frac{V_2}{V_1}\Big)\sum_{n=1}^{V}\Big|\big((\mu_f\cdot\eta)\star f\big)(n)\Big|^2 = V\log V_1 + V\log V_2 - 2V\log V_1 + O(V)$$

and so (H4).

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