Signal Analysis Using Born–Jordan-Type Distributions

Signal Analysis using Born-Jordan-type Distributions

Elena Cordero, Maurice de Gosson, Monika Dörfler and Fabio Nicola

Abstract In this note we exhibit recent advances in signal analysis via time-frequency distributions. New members of the Cohen class, generalizing the Wigner distribution, reveal to be effective in damping artefacts of some signals. We will survey their main properties and drawbacks and present open problems related to such phenomena.

MSC: 42B10, 42B37

Key words: Time-frequency analysis, Wigner distribution, Born-Jordan distribution, B-Splines, Interferences, wave-front set, modulation spaces

1 Introduction

The Wigner distribution (Wigner transform, Wigner function or Wigner-Ville distribution) has a long tradition which started as a probability quasi-distribution in 1932 with Eugene Wigner's ground-breaking paper [30]. In 1948 it was reinvented by Jean Ville in [19] as a quadratic representation of the local time-frequency energy of

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a signal. This was the starting point for its numerous applications in signal analysis: from electrical engineering and communication theory to any field involving the problem of treating signals: seismology, biology, medicine etc.

Given two functions $f, g \in L^2(\mathbb{R}^d)$, their (cross-)Wigner distribution is defined to be

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i \omega y} f(x + \tfrac{1}{2}y)\overline{g(x - \tfrac{1}{2}y)}dy, \tag{1}$$

($\omega y = \omega \cdot y$ denoting the scalar product in $\mathbb{R}^d$). If $f = g$ we set $Wf := W(f,f)$, named the Wigner distribution of f. The quadratic nature of the Wigner distribution Wf causes the appearance of interferences between the distinct components of the signal. These interferences are artefacts and do not reflect true signal components, compare the idealized TF representation in Figure 1 below. This representation is ideal because it combines the beneficial properties of linear representations, namely no unwanted interferences, with those of the quadratic Wigner-distribution, namely no smearing in time-frequency. Roughly speaking, if a signal f is sum of two components f_1, f_2, then its Wigner distribution becomes

$$W(f_1 + f_2) = Wf_1 + Wf_2 + 2\mathcal{R}eW(f_1, f_2).$$

The cross-term $W(f_1, f_2)$ appearing above produces unwanted interferences, which are thus also called ghost terms: they don't correspond to actual signal components. This will be made clear in the following examples.

Recall the translation and modulation operators:

$$T_x f(y) = f(y - x), \quad M_\omega f(y) = e^{2\pi i y \omega} f(y), \quad x, \omega \in \mathbb{R}^d,$$

which combined are called time-frequency shifts:

$$\pi(z) f(y) = M_\omega T_x f(y) = e^{2\pi i y \omega} f(y - x), \quad z = (x, \omega).$$

We can show such unwanted phenomenum for a signal f that is the sum of four Gaussian atoms, that is, time-frequency shifts of the Gaussian function. For example, in dimension $d = 1$, consider the Gaussian function $\varphi(t) = e^{-\pi t^2}$ and the signal f that is the sum of the following 4 time-frequency shifts of φ:

$$f = \pi(20, 0.25)\varphi + \pi(40, 0.15)\varphi + \pi(40, 0.35)\varphi + \pi(60, 0.25)\varphi = \sum_{i=1}^{4} \varphi_i. \tag{2}$$

The plot below is an *idealized TF representation*, but does not correspond to anything computable from the signal. The Wigner Distribution Wf is represented in Figure 2 and we observe the following. Since

$$W(f,f) = \sum_{i=1}^{4} W\varphi_i + \sum_{i=1}^{4} \sum_{i \neq j} \mathcal{R}eW(\varphi_i, \varphi_j) \tag{3}$$

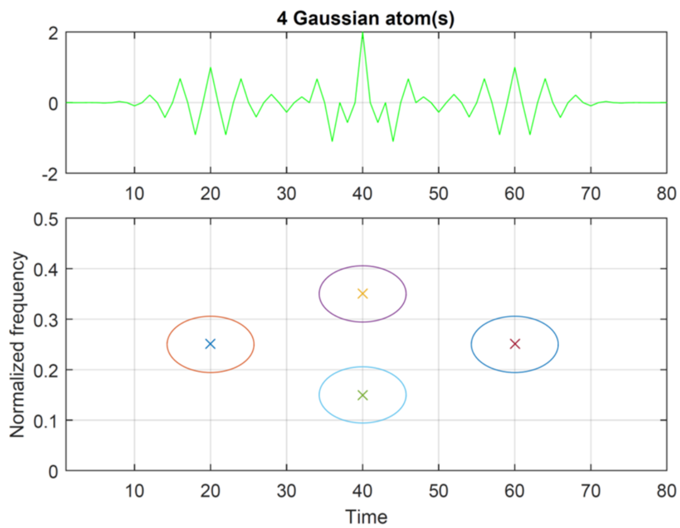


Fig. 1 An idealized TF representation of the sum of four Gaussian windows.

and since, with $\varphi_i(t) = \pi(x_i, \omega_i)\varphi$, the cross term $W(\varphi_i, \varphi_j)$ is given by

$$W(\varphi_i, \varphi_j)(x, \omega) = 2e^{[-\frac{\pi}{2}(x-\frac{x_i+x_j}{2})^2-2\pi(\omega-\frac{\omega_i+\omega_j}{2})^2]}e^{2\pi i[(\omega_i+\omega_j)x-(x_i-x_j)(\frac{\omega_i+\omega_j}{2})]}. \tag{4}$$

As depicted in Figure 2, we thus see six cross terms, of which two overlap in the center of the four Gaussian windows.

The four spots placed at the vertices of the rhombus are the real time-frequency content of the signal, whereas the other five spots represent the "ghost frequencies", as defined in [3], arising from the interferences of the four components of the signal.

To overcome this undesirable phenomenon, *reduced interference distributions* were proposed by Leon Cohen in [6], see also the textbooks [7, 18].

The idea underneath the introduction of the so-called Cohen class is to reduce the interferences by a smoothing operation obtained using a convolution product. Precisely, we define the Cohen class a follows.

Definition 1 A member of the Cohen class Qf is a quadratic time-frequency representation obtained by convolving the Wigner function Wf with a distribution $\theta \in \mathcal{S}'(\mathbb{R}^{2d})$ (called Cohen kernel), that is

$$Qf = Wf * \theta, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{5}$$

For $f \in \mathcal{S}(\mathbb{R}^d)$ it is easy to check that $Wf \in \mathcal{S}(\mathbb{R}^{2d})$ (see, e.g., [17]) so that Qf is a well-defined tempered distribution for every $f \in \mathcal{S}(\mathbb{R}^d)$.

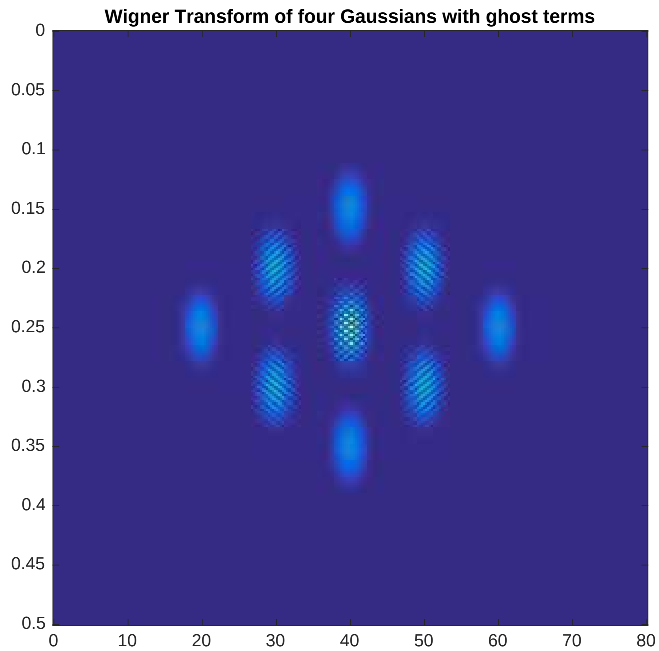


Fig. 2 Wigner distribution of the sum of four Gaussian windows.

A possible choice for the Cohen kernel is $\theta = \mathcal{F}_\sigma \Theta^1$, with $\mathcal{F}_\sigma \Theta^1$ being the symplectic Fourier transform of the function

$$\Theta^1(x,\omega) = \operatorname{sinc}(x\omega) = \begin{cases} \dfrac{\sin(\pi x\omega)}{\pi x\omega} & \text{for}\, x\omega \neq 0 \\ 1 & \text{for}\, x\omega = 0. \end{cases} \tag{6}$$

In this way we obtain the Born-Jordan (BJ) distribution:

$$Q^1 f = Wf * \mathcal{F}_\sigma(\Theta^1), \quad f \in L^2(\mathbb{R}^d), \tag{7}$$

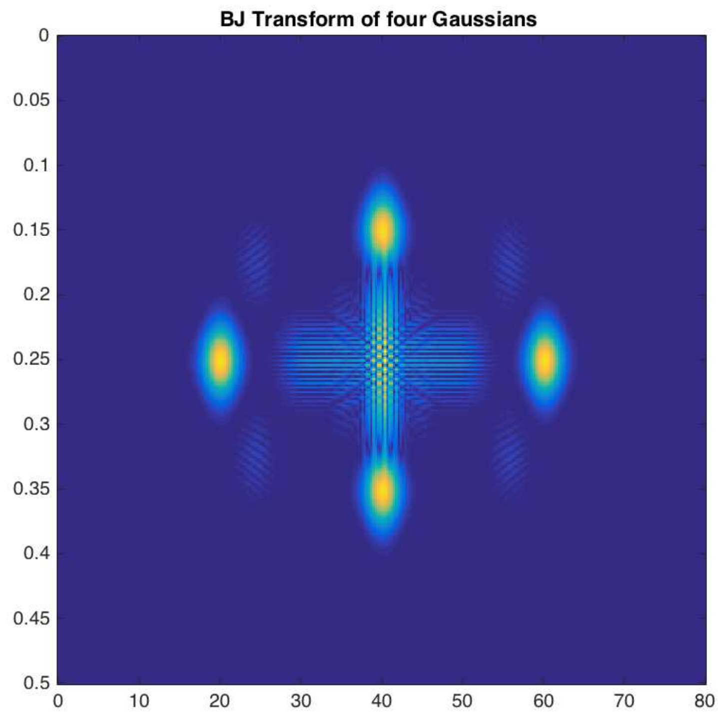
see [3, 6, 7, 8, 10, 18, 24] and the references therein.

Such distribution was first introduced in 1925 as a quantization rule by the phisicists M. Born and P. Jordan [5] and later widely employed by engineers for its *smoothing effects*, cf. the textbook [18].

Examples for tests and real-world signals show in the BJ distribution:

a) "ghost frequencies" (arising from the interferences of the several components which do not share the same time or frequency localization) are damped very well, see [3, 4].
b) The interferences arranged along the horizontal and vertical direction are substantially kept.
c) The noise is, on the whole, reduced.

The following picture represents the BJ distribution of the signal f in (4).



The smoothing effect of the BJ distribution can be seen quite clearly. But still there are artefacts that one would like to damp. This is the main purpose of the contribution [13], where new distributions from the Cohen class were proposed and studied. Here we present the most relevant issues of the contribution above, highlighting the mathematical explanation of the smoothing effects of the distributions.

Following the suggestions of Jean-Pierre Gazeau, in the manuscript [13] new interesting Cohen kernels were proposed using the B-spline functions B_n.

The sequence of B-splines $\{B_n\}_{n\in\mathbb{N}_+}$ is defined inductively as follows. The first B-Spline is

$$B_1(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t),$$

and the spline B_{n+1} is

$$B_{n+1}(t) = (B_n * B_1)(t) = \int_{\mathbb{R}} B_n(t-y)B_1(y)dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} B_n(t-y)dy. \quad (8)$$

The spline B_n is a piecewise polynomial of degree at most $n-1$, $n \in \mathbb{N}_+$, and satisfying $B_n \in C^{n-2}(\mathbb{R})$, $n \geq 2$. For the main properties we refer, e.g., to [16].

The *sinc* function is recaptured as $\operatorname{sinc}(\xi) = \mathcal{F}B_1(\xi)$ and by induction we infer

$$\operatorname{sinc}^n(\xi) = \mathcal{F}B_n(\xi), \quad n \in \mathbb{N}_+. \quad (9)$$

This suggests the following definition.

Definition 2 For $n \in \mathbb{N}$, the nth Born-Jordan kernel on $\mathbb{R}^{2d}$ is defined by

$$\Theta^n(x,\omega) = \operatorname{sinc}^n(x\omega), \quad (x,\omega) \in \mathbb{R}^{2d}. \quad (10)$$

The related Born-Jordan distribution of order n (BJDn) is

$$Q^n f = Wf * \mathcal{F}_\sigma(\Theta^n), \quad f \in L^2(\mathbb{R}^d). \quad (11)$$

The cross-BJDn is given by

$$Q^n(f,g) = W(f,g) * \mathcal{F}_\sigma(\Theta^n), \quad f,g \in L^2(\mathbb{R}^d). \quad (12)$$

We write $Q^n(f,f) = Q^n f$, for every $f \in L^2(\mathbb{R}^d)$.

For $n=0$, $\Theta^0 \equiv 1$ and $\mathcal{F}_\sigma(1) = \delta$, so that $Q^0 f = Wf$, the Wigner distribution of the signal f.

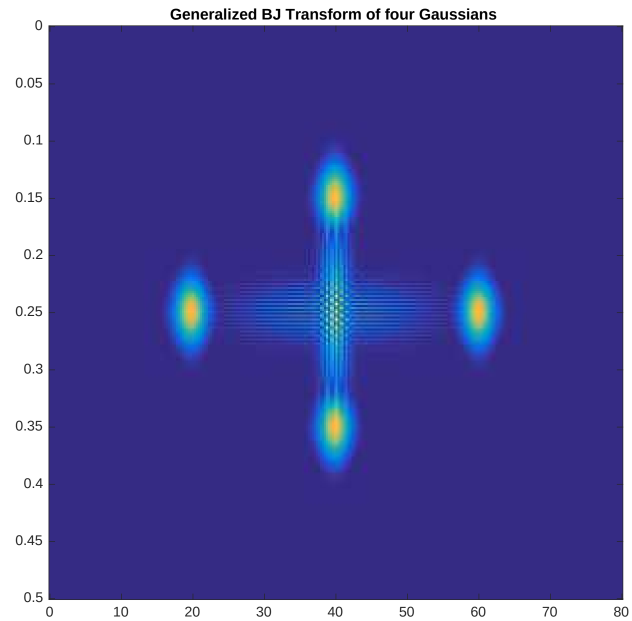
In the picture below we computed the Born-Jordan distribution of order 3 of the signal in (4).

Notice that this new class is a subset of the Cohen class, containing the Wigner and the classical BJ distribution. This subclass can be applied to signal processing since the mathematical explanations of their smoothing properties testifies the numerical evidences of dumping artefacts in many examples, and such reduction increases with n. We survey the different facets of this phenomenon, referring mainly to the theoretical results in [13], highlighted by new pictures in this note.

First, in Figures 3, 4 and 5 we show the Wigner, BJ, BJ3 distribution of a rotation of the original signal in (4). Figure 6 shows a comparison of the Wigner transform, the Born-Jordan transform and the fifth Born-Jordan transform of another sum of four time-frequency shifts of Gaussian functions. It is clearly visible, that the amount of cross-term suppression increases by applying higher-order smoothing.

This suggests that such distributions could be successfully applied in signal processing.

In the following sections we provide a rigorous mathematical explanation of the regularity and smoothness properties of Q^n; the notion of Fourier Lebesgue wave-front set will play the central role in showing the damping of interferences of Q^n



in comparison with the Wigner distribution. The use of function spaces from time-frequency analysis, combined with microlocal analysis techniques are the key tools of our proofs.

1.1 Notation

We use $x\omega = x \cdot \omega = x_1\omega_1 + \ldots + x_d\omega_d$ for the scalar product in $\mathbb{R}^d$. The brackets $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\mathbb{R}^d)$ or the duality pairing between Schwartz functions and temperate distributions (antilinear in the second argument). For functions f, g, we write $f \lesssim g$ if $f(x) \leq Cg(x)$ for every x and some constant $C > 0$, and similarly for $\gtrsim$. The notation $f \asymp g$ means $f \lesssim g$ and $f \gtrsim g$. We write $C_c^\infty(\mathbb{R}^d)$ for the class of smooth functions on $\mathbb{R}^d$ with compact support. The notation σ stands for the standard symplectic form on the phase space $\mathbb{R}^{2d} \equiv \mathbb{R}^d \times \mathbb{R}^d$; the phase space variable is denoted $z = (x, \omega)$ and the dual variable by $\zeta = (\zeta_1, \zeta_2)$. By definition $\sigma(z, \zeta) = Jz \cdot \zeta = \omega \cdot \zeta_1 - x \cdot \zeta_2$, where

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix}.$$

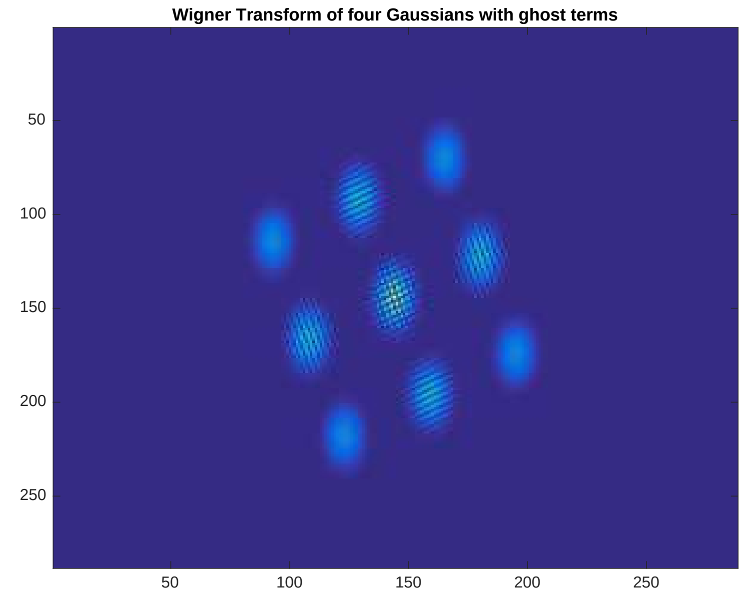


Fig. 3 The Wigner Transform shows 9 spots, only 4 spots represent the real signal, the others are ghost terms

The Fourier transform of a function f in $\mathbb{R}^d$ is

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x)\, dx,$$

and the symplectic Fourier transform of a function F in the phase space $\mathbb{R}^{2d}$ is

$$\mathcal{F}_\sigma F(\zeta) = \int_{\mathbb{R}^{2d}} e^{-2\pi i \sigma(\zeta, z)} F(z)\, dz.$$

The symplectic Fourier transform is an involution, i.e., $\mathcal{F}_\sigma(\mathcal{F}_\sigma F) = F$. Moreover, $\mathcal{F}_\sigma F(\zeta) = \mathcal{F}F(J\zeta)$.

Observe that $\Theta^n(J(\zeta_1, \zeta_2)) = \Theta^n(\zeta_1, \zeta_2)$ so that

$$\mathcal{F}_\sigma(\Theta^n) = \mathcal{F}(\Theta^n), \quad \forall n \in \mathbb{N}_+. \tag{13}$$

Fig. 4 The BJ Transform of the previous signal gives a better information than the Wigner one. Here one sees 4 spots that represent the time-frequency content of the signal though few little shades appear as interferences

2 Time-frequency and Microlocal Analysis Methods

In this section we recall the function spaces from time-frequency analysis and the wave-front set from microlocal analysis that play the key role in this study.

Modulation Spaces. Let us first recall another very popular time-frequency representation: *the short-time Fourier transform (STFT)*. Fix a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ (so-called *window*), then the short-time Fourier transform of $f \in cS'(\mathbb{R}^d)$ is

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(y)\, \overline{g(y-x)}\, e^{-2\pi i y\omega}\, dy, \quad (x,\omega) \in \mathbb{R}^{2d}. \tag{14}$$

Consider $1 \le p, q \le \infty$. The *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty \tag{15}$$

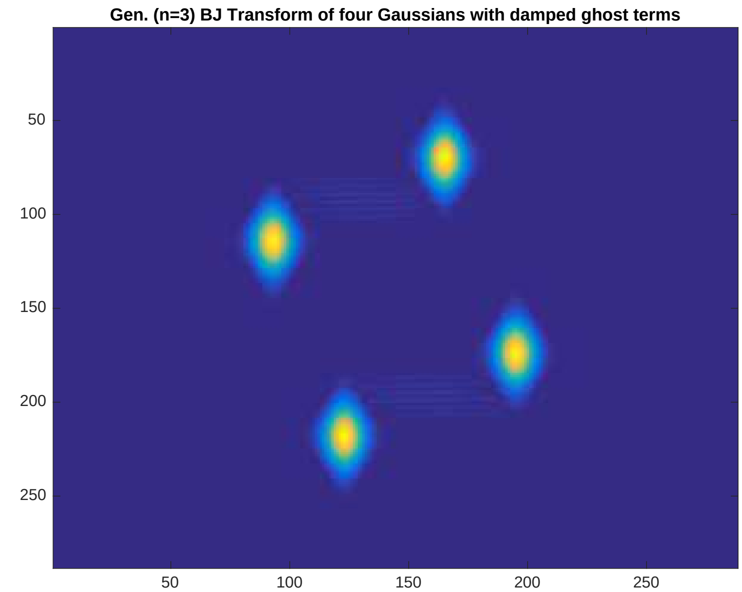


Fig. 5 The Generalized BJ Transform (order $n = 3$) of the same signal shows the best time-frequency content of it. The interferences are almost disappeared

(with obvious modifications for $p = \infty$ or $q = \infty$). We write $M^p(\mathbb{R}^d)$ instead of $M^{p,p}(\mathbb{R}^d)$. The modulation spaces are Banach spaces for any $1 \le p, q \le \infty$, and every non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ yields an equivalent norm in (15).

Modulation spaces were introduced in [21] and are now available in textbooks, see e.g., [23]. They include as special cases several function spaces arising in Harmonic Analysis. In particular for $p = q = 2$ we have

$$M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d),$$

whereas $M^1(\mathbb{R}^d)$ is the Feichtinger algebra $S_0(\mathbb{R}^d)$, cf. [22, 23].

In the notation $M^{p,q}$ the exponent p is a measure of decay at infinity (on average) in the scale of spaces ℓ^p, whereas the exponent q is a measure of smoothness in the scale $\mathcal{F}L^q$.

Other instances of modulation spaces, also known as *Wiener amalgam spaces*, are obtained by exchanging the order of integration in (15). Namely, for $p, q \in [1, \infty)$, the modulation space $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ is the subspace of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p \, d\omega \right)^{q/p} dx \right)^{1/q} < \infty$$

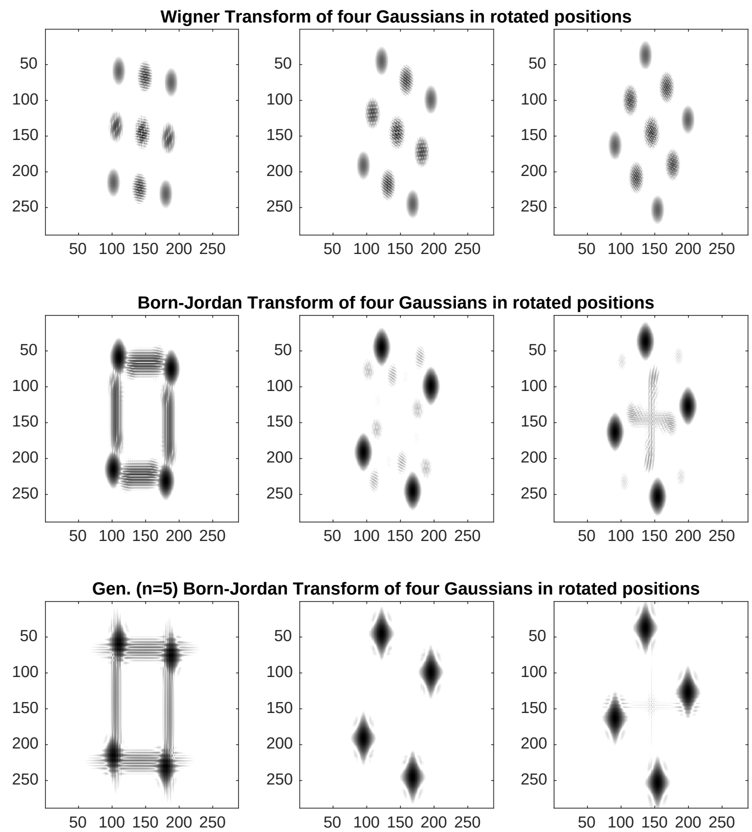


Fig. 6 Sum of four Gaussian functions in rotated positions: Comparison of Wigner distribution, Born-Jordan and generalised Born-Jordan distribution of order $n = 5$

(with obvious changes for $p = \infty$ or $q = \infty$). Using Parseval identity in (14), we infer the fundamental identity of time-frequency analysis

$$V_g f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega, -x),$$

hence

$$|V_g f(x,\omega)| = |V_{\hat{g}} \hat{f}(\omega, -x)| = |\mathcal{F}(\hat{f}\, T_\omega \overline{\hat{g}})(-x)|$$

so that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d} \|\hat{f}\, T_\omega \overline{\hat{g}}\|^q_{\mathcal{F}L^p}\, d\omega\right)^{1/q} = \|\hat{f}\|_{W(\mathcal{F}L^p, L^q)}.$$

Hence Wiener amalgam spaces can be viewed as the image under Fourier transform of modulation spaces: $\mathcal{F}(M^{p,q}) = W(\mathcal{F}L^p, L^q)$.

We will frequently use the following product property of Wiener amalgam spaces ([20, Theorem 1 (v)]): For $1 \le p, q \le \infty$,

$$\textit{if } f \in W(\mathcal{F}L^1, L^\infty) \textit{ and } g \in W(\mathcal{F}L^p, L^q) \textit{ then } fg \in W(\mathcal{F}L^p, L^q). \tag{16}$$

Taking $p = 1, q = \infty$, we obtain that $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ is an algebra under pointwise multiplication.

Proposition 1 *Let $1 \le p, q \le \infty$ and $A \in GL(d,\mathbb{R})$. Then, for every $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$,*

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p,L^q)} \le C|\det A|^{(1/p-1/q-1)}(\det(I + A^*A))^{1/2}\|f\|_{W(\mathcal{F}L^p,L^q)}. \tag{17}$$

In particular, for $A = \lambda I$, $\lambda > 0$,

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p,L^q)} \le C\lambda^{d\left(\frac{1}{p}-\frac{1}{q}-1\right)}(\lambda^2+1)^{d/2}\|f\|_{W(\mathcal{F}L^p,L^q)}. \tag{18}$$

In the sequel we shall use the following fact [12, Lemma 5.1].

Lemma 1 *Let $\chi \in C_c^\infty(\mathbb{R})$. Then the function $\chi(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

Wave-front set for Fourier-Lebesgue spaces For $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R}^d)$ is constituted by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H^s} := \|\widehat{f}(\omega)\langle\omega\rangle^s\|_{L^2} < \infty, \tag{19}$$

where $\langle\omega\rangle = (1 + ||^2)^{1/2}$. The H^s wave-front set allows to quantify the regularity of a function/distribution in the Sobolev scale, at any given point and direction. This is achieved by microlocalizing the definition of the H^s norm in (19) as follows (cf. [25, Chapter XIII]).

Definition 3 Given a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ its wave-front set $WF_{H^s}(f) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ is the set of points $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $\omega_0 \ne 0$, where the following condition is *not* satisfied: for some cut-off function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(x_0) \ne 0$ and some open conic neighborhood of $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of ω_0 we have

$$\|\mathcal{F}[\varphi f](\omega)\langle\omega\rangle^s\|_{L^2(\Gamma)} < \infty.$$

More generally one can start from the Fourier-Lebesgue spaces $\mathcal{F}L^q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, $1 \leq q \leq \infty$, which is the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the norm

$$\|f\|_{\mathcal{F}L^q_s(\mathbb{R}^d)} = \|\widehat{f}(\omega)\langle\omega\rangle^s\|_{L^q(\mathbb{R}^d)}, \tag{20}$$

is finite.

Arguing exactly as in Definition 3 with the space L^2 replaced by L^q, one can introduce the corresponding notion of wave-front set $WF_{\mathcal{F}L^q_s}(f)$.

Definition 4 Given $f \in \mathcal{S}'(\mathbb{R}^d)$ its wave-front set $WF_{\mathcal{F}L^q_s}(f) \subset \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ is the set of points $(x_0, \omega_0) \in \mathbb{R}^d \times \mathbb{R}^d$, $\omega_0 \neq 0$, where the following condition *is not satisfied*: for some cut-off function φ (i.e., φ is smooth and compactly supported on $\mathbb{R}^d$), with $\varphi(x_0) \neq 0$, and some open conic neighbourhood $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ of ω_0 it holds

$$\|\mathcal{F}[\varphi f](\omega)\langle\omega\rangle^s\|_{L^q(\Gamma)} < \infty. \tag{21}$$

Observe that $WF_{\mathcal{F}L^2_s}(f) = WF_{H^s}(f)$ is the standard H^s wave-front set in Definition 3.

For our purposes we recall some basic results about the action of constant coefficient linear partial differential operators on such wave-front set (cf. [26]). Given the operator

$$P = \sum_{|\alpha|\leq m} c_\alpha \partial^\alpha, \quad c_\alpha \in \mathbb{C},$$

for $1 \leq q \leq \infty$, $s \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$WF_{\mathcal{F}L^q_s}(Pf) \subset WF_{\mathcal{F}L^q_{s+m}}(f).$$

Consider now the inverse inclusion. We say that $\zeta \in \mathbb{R}^d$, $\zeta \neq 0$, is non characteristic for the operator P if

$$\sum_{|\alpha|=m} c_\alpha \zeta^\alpha \neq 0.$$

The following result is a microlocal version of the classical regularity result of elliptic operators (see [26, Corollary 1 (2)]):

Proposition 2 *Let $1 \leq q \leq \infty$, $s \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Let $z \in \mathbb{R}^d$ and assume that $\zeta \in \mathbb{R}^d \setminus \{0\}$ is non characteristic for P. Then, if $(z, \zeta) \notin WF_{\mathcal{F}L^q_s}(Pf)$, we have $(z, \zeta) \notin WF_{\mathcal{F}L^q_{s+m}}(f)$.*

3 Time-frequency Analysis of the nth Born-Jordan kernel

In this section we summarize the main results of the topic, obtained in the papers [11, 12, 13]. The Born-Jordan kernel Θ^1 in (6) satisfies

$$\Theta^1 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}),$$

cf. [12].

The previous property is true for any Θ^n, $n \in \mathbb{N}_+$, as shown below (see [13]).

Proposition 3 *For $n \in \mathbb{N}_+$, the function Θ^n in* (10) *belongs to the Wiener algebra $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

The properties of the kernels above yield the following result.

Theorem 1 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \leq p, q \leq \infty$. Then $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$, for every $n \in \mathbb{N}_+$.*

Proof We need to show that $Q^n f$ is in $M^{p,q}(\mathbb{R}^{2d})$. Taking the symplectic Fourier transform in (7) this is equivalent to

$$\Theta^n \mathcal{F}_\sigma(Wf) = \Theta^n Af \in W(\mathcal{F}L^p, L^q)$$

where $\mathcal{F}_\sigma(Wf) = Af$ is called the ambiguity function of f, see e.g., [23]. The claim is attained using the product property (16): by Proposition 3, the function $\Theta^n \in W(\mathcal{F}L^1, L^\infty)$ and by assumption $Wf \in M^{p,q}(\mathbb{R}^{2d})$ so that we infer $\mathcal{F}(Wf) \in W(\mathcal{F}L^p, L^q)$. Finally, $\mathcal{F}_\sigma(Wf)(\zeta) = \mathcal{F}(Wf)(J\zeta) \in W(\mathcal{F}L^p, L^q)$ by Proposition 1. □

The previous statement holds in greater generality and can be rephrased for members in the Cohen class as follows.

Theorem 2 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \leq p, q \leq \infty$ and the Cohen kernel (cf.* (5)*) $\theta \in M^{1,\infty}(\mathbb{R}^{2d})$. Then the corresponding distribution Qf is in $M^{p,q}(\mathbb{R}^{2d})$.*

Proof It is the consequence of the following convolution relation for modulation spaces in [9]:

$$M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d}),$$

for any $1 \leq p, q \leq \infty$. □

The chirp function $F(\zeta_1, \zeta_2) = e^{2\pi i \zeta_1 \zeta_2}$ enjoys the following property, see [12, 22].

Proposition 4 *The function $F(\zeta_1, \zeta_2) = e^{2\pi i \zeta_1 \zeta_2}$ belongs to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

By Proposition 4 and by the dilation properties for Wiener amalgam spaces in (17) we obtain

Corollary 1 *For $\zeta = (\zeta_1, \zeta_2)$, consider the function $F_J(\zeta) = F(J\zeta) = e^{-2\pi i \zeta_1 \zeta_2}$. Then $F_J \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.*

4 Smoothness of the Born-Jordan distribution of order *n*

In this section we survey the results in [13, Sec. 5], comparing the smoothness of the Born-Jordan distribution of order n with the Wigner distribution. To measure the smoothness of a signal f we use modulation spaces. They detect the concentration of functions and distributions on the time-frequency plane. Roughly speaking, a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $M^{p,q}(\mathbb{R}^d)$ if it decays at infinity like a function in $L^p(\mathbb{R}^d)$, whereas it displays a $\mathcal{F}L^q(\mathbb{R}^d)$-local regularity.

For the following global result we use the notation

$$\nabla_x \cdot \nabla_\omega := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j \partial \omega_j}. \tag{22}$$

Theorem 3 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{p,q}(\mathbb{R}^{2d})$ for some $1 \le p, q \le \infty$. Then, for any $n \in \mathbb{N}_+$,*

$$Q^n f \in M^{p,q}(\mathbb{R}^{2d})$$

and

$$(\nabla_x \cdot \nabla_\omega)^n Q^n f \in M^{p,q}(\mathbb{R}^{2d}). \tag{23}$$

Proof The claim $Q^n f \in M^{p,q}(\mathbb{R}^{2d})$ is proven in Theorem 1.

We now prove (23). Taking the symplectic Fourier transform we see that it is sufficient to prove that

$$(\zeta_1\zeta_2)^n \operatorname{sinc}^n(\zeta_1\zeta_2)\mathcal{F}_\sigma Wf = \frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q).$$

Using

$$\sin(\pi\zeta_1\zeta_2) = \frac{e^{\pi i\zeta_1\zeta_2} - e^{-\pi i\zeta_1\zeta_2}}{2i} \tag{24}$$

and applying Proposition 4, Corollary 1 and Proposition 1, with the scaling $\lambda = 1/\sqrt{2}$, we get $\sin(\pi\zeta_1\zeta_2) \in W(\mathcal{F}L^1, L^\infty)$.

Hence, for $n = 1$,

$$\frac{1}{\pi}\sin(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q)$$

by the product property (16). Assume now that, for a certain $n \in \mathbb{N}_+$,

$$\frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf \in W(\mathcal{F}L^p, L^q).$$

Then

$$\frac{1}{\pi^{n+1}}\sin^{n+1}(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf = \underbrace{\frac{1}{\pi}\sin(\pi\zeta_1\zeta_2)}_{\in W(\mathcal{F}L^1, L^\infty)} \cdot \underbrace{\frac{1}{\pi^n}\sin^n(\pi\zeta_1\zeta_2)\mathcal{F}_\sigma Wf}_{\in W(\mathcal{F}L^p, L^q)} \in W(\mathcal{F}L^p, L^q),$$

by (24) and the product property (16) again. By induction we attain the result.

We can now provide the mathematical explanation of the Q^n's smoothing effects:

Theorem 4 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a signal, with $Wf \in M^{\infty,q}(\mathbb{R}^{2d})$ for some $1 \leq q \leq \infty$. Let $(z,\zeta) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$, with $\zeta = (\zeta_1, \zeta_2)$ satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. Then*

$$(z,\zeta) \notin WF_{\mathcal{F}L^q_{2n}}(Q^n f).$$

Proof Consider $n \in \mathbb{N}_+$. We will apply Proposition 2 to the $2n$-th order operator P^n, where P is defined in (22). The non characteristic directions for P^n are given by the vectors $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^d \times \mathbb{R}^d$, satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. By (23) (with $p = \infty$) we have

$$WF_{\mathcal{F}L^q}(P^n Q^n f) = \emptyset,$$

because $\varphi F \in \mathcal{F}L^q$ if $\varphi \in C_c^\infty(\mathbb{R}^{2d})$ and $F \in M^{\infty,q}(\mathbb{R}^{2d})$ (with $F = P^n Q^n f$). This implies

$$(z,\zeta) \notin WF_{\mathcal{F}L^q}(P^n Q^n f), \quad \forall (z,\zeta) \text{ such that } \zeta = (\zeta_1,\zeta_2),\ \zeta_1 \cdot \zeta_2 \neq 0.$$

Since ζ is non characteristic for the operator P^n, by Proposition 2 we infer

$$(z,\zeta) \notin WF_{\mathcal{F}L^q_{2n}}(Q^n f)$$

for every $z \in \mathbb{R}^{2d}$. □

Roughly speaking, if the Wigner distribution Wf has local regularity $\mathcal{F}L^q$ and some control at infinity, then $Q^n f$ is smoother, possessing $s = 2n$ **additional derivatives**, at least in the directions $\zeta = (\zeta_1, \zeta_2)$ satisfying $\zeta_1 \cdot \zeta_2 \neq 0$. In dimension $d = 1$ this condition reduces to $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$. Hence this result explains the smoothing phenomenon of such distributions, which involves all the directions except those of the coordinates axes.

This is the reason why the interferences of two components which do not share the same time or frequency localization come out substantially reduced.

Observe that for $n = 1$ we recapture the damping phenomenon of the classical Born-Jordan distribution (cf. [12, Theorem 1.2]).

For signals in $L^2(\mathbb{R}^d)$, the previous result can be rephrased in terms of the classical Hörmander's wave-front set.

Corollary 2 *Let $f \in L^2(\mathbb{R}^d)$, so that $Wf \in L^2(\mathbb{R}^{2d})$. Let (z,ζ) be as in the statement of Theorem 4. Then $(z,\zeta) \notin WF_{H^{2n}}(Q^n f)$, that is the distribution $Q^n f$ has regularity H^{2n} at z and in the direction ζ.*

Proof We apply Theorem 4 with $q = 2$. In fact, for $f \in L^2(\mathbb{R}^d)$, Moyal's formula gives $Wf \in L^2(\mathbb{R}^{2d}) = M^{2,2}(\mathbb{R}^d) \subset M^{\infty,2}(\mathbb{R}^{2d})$, by inclusion relations for modulation spaces. Observe that the $\mathcal{F}L^2_{2n}$ wave-front set coincides with the H^{2n} wave-front set. □

What about smoothing effects in the directions $\zeta = (\zeta_1, \zeta_2)$: $\zeta_1 \cdot \zeta_2 = 0$?

It seems that the smoothing effects do not occur in these directions, as Fig. 1 shows. The mathematical explanation in terms of modulation spaces is below. Roughly speaking, if we assume that the distribution $Q^n f$ of a signal f has a local regularity $\mathcal{F}L^{q_1}$ better than the $\mathcal{F}L^{q_2}$-Wigner one, then necessarily it must hold $q_1 \geq q_2$. Hence the best we could expect is $q_1 = q_2$, that is, the same regularity.

Theorem 5 *Suppose that for some* $1 \leq p, q_1, q_2 \leq \infty$, $n \in \mathbb{N}_+$ *and* $C > 0$, *it occurs*

$$\|Q^n f\|_{M^{p,q_1}} \leq C\|Wf\|_{M^{p,q_2}}, \tag{25}$$

for every $f \in \mathcal{S}(\mathbb{R}^d)$. *Then* $q_1 \geq q_2$.

Proof The main steps are as follows. We will test the estimate (25) using rescaled Gaussian functions $f(x) = \varphi(\lambda x)$, with $\lambda > 0$ large parameter. Restricting to a neighbourhood of $\zeta_1 \cdot \zeta_2 = 0$, the constrain $q_1 \geq q_2$ must be satisfied.

An easy computation yields

$$W(\varphi(\lambda \cdot))(x, \omega) = 2^{d/2}\lambda^{-d}\varphi(\sqrt{2}\lambda x)\varphi(\sqrt{2}\lambda^{-1}\omega). \tag{26}$$

For every $1 \leq p, q \leq \infty$, the above formula gives

$$\|W(\varphi(\lambda \cdot))\|_{M^{p,q}} = 2^{d/2}\lambda^{-d}\|\varphi(\sqrt{2}\lambda \cdot)\|_{M^{p,q}}\|\varphi(\sqrt{2}\lambda^{-1} \cdot)\|_{M^{p,q}}.$$

By the dilation properties of Gaussians (first proved in [29, Lemma 1.8], see also [14, Lemma 3.2]):

$$\|W(\varphi(\lambda \cdot))\|_{M^{p,q}} \asymp \lambda^{-2d+d/q+d/p} \quad \text{as } \lambda \to +\infty. \tag{27}$$

We now study the $M^{p,q}$-norm of the BJDn $Q^n(\varphi(\lambda \cdot))$. It will be estimated from below obtaining the same expansion as in (27). In detail,

$$\|Q^n(\varphi(\lambda \cdot))\|_{M^{p,q}} = \|\mathcal{F}_\sigma(\Theta^n) * W(\varphi(\lambda \cdot))\|_{M^{p,q}}.$$

By taking the symplectic Fourier transform and using Lemma 1 and the product property (16) we have

$$\begin{aligned}\|\mathcal{F}_\sigma(\Theta^n) * W(\varphi(\lambda \cdot))\|_{M^{p,q}} &\asymp \|\Theta^n \mathcal{F}_\sigma[W(\varphi(\lambda \cdot))]\|_{W(\mathcal{F}L^p, L^q)} \\ &\gtrsim \|\Theta^n(\zeta_1, \zeta_2)\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda \cdot))]\|_{W(\mathcal{F}L^p, L^q)}\end{aligned}$$

for any $\chi \in C_c^\infty(\mathbb{R})$ and $n \in \mathbb{N}_+$. Choosing the function χ compactly supported in the interval $[-1/4, 1/4]$ and $\chi \equiv 1$ in the interval $[-1/8, 1/8]$ (the latter condition will be used later), we write

$$\chi(\zeta_1\zeta_2) = \chi(\zeta_1\zeta_2)\Theta^n(\zeta_1, \zeta_2)\Theta^{-n}(\zeta_1, \zeta_2)\tilde{\chi}(\zeta_1\zeta_2),$$

with $\tilde{\chi} \in C_c^\infty(\mathbb{R})$ supported in $[-1/2, 1/2]$ and $\tilde{\chi} = 1$ on $[-1/4, 1/4]$, therefore on the support of χ. Since by Lemma 1 the function $\Theta^{-n}(\zeta_1, \zeta_2)\tilde{\chi}(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1, L^\infty)$, by the product property the last expression can be estimated from below as

$$\|\chi(\zeta_1\zeta_2)\|_{W(\mathcal{F}L^p,L^q)} \gtrsim \|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)}.$$

Finally in the proof of [12, Theorem 1.4] it was shown

$$\|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi(\lambda\,\cdot))]\|_{W(\mathcal{F}L^p,L^q)} \gtrsim \lambda^{-2d+d/p+d/q} \quad \text{as } \lambda \to +\infty. \tag{28}$$

Comparing (28) with (27) we obtain the desired conclusion. □

5 Conclusion and Perspectives

The generalized Born-Jordan distributions presented in these notes produce an improvement in the damping of unwanted artefacts of some signals as the one represented in Fig. 1. For other pictures of signals where the smoothing effects are visible we refer to the original paper [13].

Let us underline that the emergence of interferences is a well-known drawback of any quadratic representation, the distributions BJDn are not immune to this phenomenon, as it can be seen in the following pictures, which are short extracts from real music signals. For comparison, we also used the Spectrogram, which is another member of the Cohen class (see [3] and references therein), given by

$$|V_f f(x,\omega)|^2, \quad f \in L^2(\mathbb{R}^d).$$

Many other alternative time-frequency distributions have been proposed for different practical purposes, we refer the interested reader to the textbooks [7, 18].

If we consider linear perturbations of the Wigner distribution, introduced and studied in [1, 2], then it turns out they do not provide effective damping of artefacts. In fact a negative answer concerning reduction of interferences is shown in [15].

Although it is clear there is no time-frequency distribution which is the best time-frequency representation to analyse any kind of signal, it is an open problem to find the right distribution for a certain class of signals. For signals which are sums of time-frequency shifts of Gaussians the representations BJDs work quite well, as shown in the present note. This suggests the following question:

Concerning the members of the Cohen class, what are the best possible kernels for damping artefacts?

Technical notes. The figures in these notes were produced using LTFAT (The Large Time-Frequency Analysis Toolbox), cf. [27].

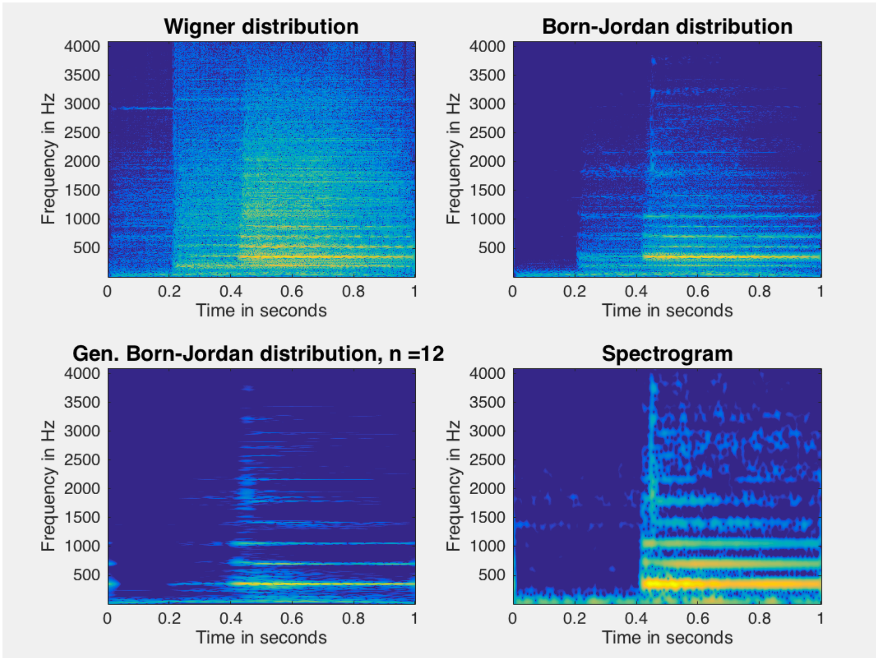


Fig. 7 A short extract of a musical signal

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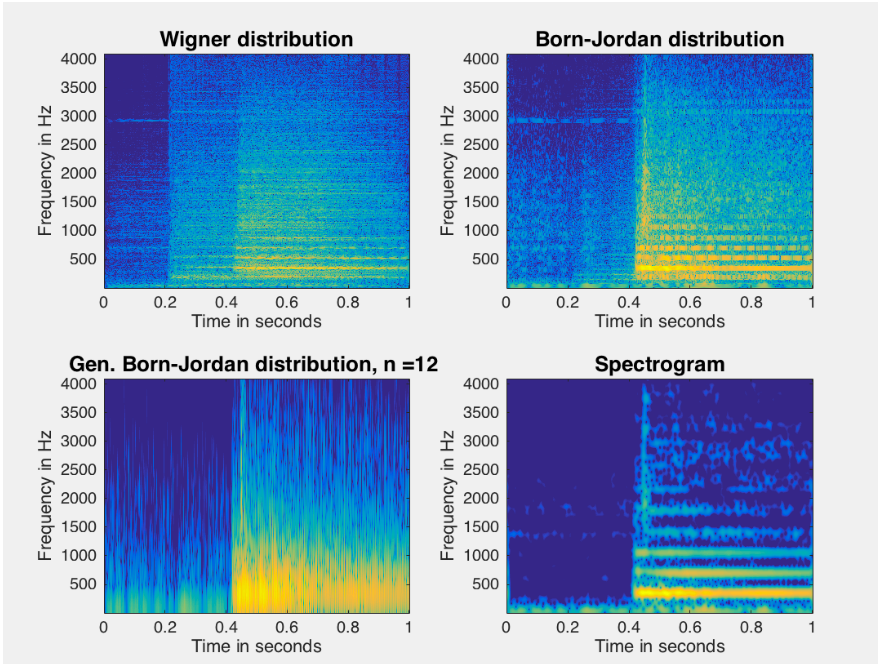

Fig. 8 A short extract of a musical signal

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