Structural Balance via Gradient Flows over Signed Graphs

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***Abstract*—Structural balance is a classic property of signed graphs satisfying Heider's seminal axioms. Mathematical sociologists have studied balance theory since its inception in the 1940s. Recent research has focused on the development of dynamic models explaining the emergence of structural balance. In this paper, we introduce a novel class of parsimonious dynamic models for structural balance based on an interpersonal influence process. Our proposed models are gradient flows of an energy function, called the dissonance function, which captures the cognitive dissonance arising from the violations of Heider's axioms. Thus, we build a new connection with the literature on energy landscape minimization. This gradient-flow characterization allows us to study the transient and asymptotic behaviors of our model. We provide mathematical and numerical results describing the critical points of the dissonance function.**

I. Introduction

1) Problem description and motivation: Signed graphs represent networked systems with interactions classified as positive or negative, e.g., cooperation or antagonism, promotion or inhibition, attraction or repulsion. Such graphs naturally arise in diverse fields, e.g., political science [14], communication studies [19] and biology [20]. In sociology [6], [9], they are used to represent friendly or antagonistic relationships, whereby signed edges may be interpreted as interpersonal sentiment appraisals. In the work by Heider [12], each individual appraises all other individuals either positively (friends, allies) or negatively (enemies, rivals). Heider postulated four famous axioms: (i) "the friend of a friend is a friend," (ii) "the enemy of a friend is an enemy," (iii) "the friend of an enemy is an enemy," and (iv) "the enemy of an enemy is a friend." Violations of these axioms lead to cognitive tensions and dissonances that the individuals strive to resolve; in this sense, Heider's axioms are consistent with the general theory of cognitive dissonance [8]. A signed network satisfying Heider's axioms is called *structurally balanced* and can have only two possible configurations: either all of its members have positive relationships with each other and become a unique faction, or there exist two factions in which members of the same faction are friends but enemies with every other member in the other faction. We refer to [6], [9] for textbook treatment and to [29] for a recent comprehensive survey.

Whereas Heider's theory describes the qualitative emergence of structural balance as the result of tension-resolving cognitive mechanisms, it does not provide a quantitative description of these mechanisms and dynamic models explaining the emergence of balance. The aim to fill this gap has given rise to the important research area of *dynamic structural balance*. The Kułakowski et al. [17] model postulates an influence process, whereby any individual i updates her appraisal of individual j based on what others positively or negatively think about j. The Traag et al. [27] model postulates a homophily process, whereby any individual i updates her appraisal of j according to how much she agrees with j on the appraisals of their common acquaintances. Both models explain convergence to structural balance under certain assumptions on the initial state (see below for more information). Remarkably, both models assume the existence of so-called *self-appraisals* (loops in the signed graph) that strongly influence the system dynamics. Self-appraisals can be interpreted as individuals' positive or negative opinions of themselves.

A second line of research, consistent with dissonance theory, has focused on formulating social balance via appropriate energy functions. The work [23] proposes an energy function for binary appraisal matrices with global minima that represent structurally stable configurations; it is argued that a dynamic structural balance model should aim to navigate through this energy landscape and look for its minima. Some models (e.g., [2], [3]) were designed precisely to achieve this task. The work [7] computes a distance to balance via a combinatorial optimization problem, inspired by Ising models.

The purpose of this paper is threefold. First, we aim to propose a more parsimonious model of the influence process establishing structural balance, that is, a model without self-appraisal weights. Our argument for dropping these variables is that balance theory axioms do not include self-appraisals, and the inclusion of such appraisals amounts to an additional assumption and introduces unnecessary complexities. Second, we aim to connect the literature on dynamic structural balance with the literature treating social balance as an optimization problem. Finally, we aim to emphasize through numerical simulations that our parsimonious model does not suffer from a key limitation present in the Kułakowski et al. model, namely that the Kułakowski et al. model cannot predict the emergence of structural balance from asymmetric initial configurations.

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2) Further comments on the state of the art: We now present a summary of the current literature on dynamic structural balance. Historically, the first models appeared in the physics community [2], [3], [25]. These models borrowed some concepts from statistical physics and had the particularity of assuming that the appraisals between individuals are binary valued (either $+1$ or -1). At the same time, they rely on hard-wired random mechanisms for the asynchronous updates of the appraisals that lack a sociological insightful interpretation.

Another type of proposed models is based on discrete- and continuous-time dynamical systems with real-valued appraisals. The seminal models of this kind are due to Kułakowski et al. [17] (later analyzed more formally by [22]) and Traag et al. [27]. Models with real-valued appraisals capture not only signs, but also magnitudes of positive or negative sentiments. All these models adopt synchronous updating and stipulate sociological meaningful rules for the updating of appraisals, based on either influence or homophily processes. The following facts are known about the Kułakowski et al. influence-based and the Traag et al. homophily-based models: the set of well-behaved initial conditions that lead the social network towards social balance for the first model is a subset of the set of normal matrices, while the second model can work under generic initial conditions. Similar results are obtained by [24] for two discrete-time models based on influence and homophily respectively: influence-based processes do not perform well under generic initial conditions (in contrast to the homophily-based processes). Finally, only the models proposed in [24] and a variation of the model by Kułakowski et al. proposed in the early work [17], have a bounded evolution of appraisals, whereas the others have finite escape time.

Recent work has also started to focus on dynamic models for other relevant configuration of signed graphs, e.g., configurations that satisfy only a subset of the four Heider's axioms. The work [10] provides a parsimonious model explaining the emergence of a generalized version of structural balance from any initial configuration; this model is based on an influence process of positive contagion whereby influence is accorded only to positively-appraised individuals. A second model in this area is proposed by [16]. Finally, there has been a third type of models that propose the emergence of structural balance or other generalized balance structures for undirected graphs from a game theoretical perspective [5], [21], [28].

3) Contributions: First of all, we contribute by proposing two new dynamic models that do not adopt the long-standing assumption of self-appraisals and describe the evolution of signed networks without self-loops. We argue that the introduction of self-weights is poorly justified and that a model without them is a more faithful representation of Heider's theory. The first model, called the *pure-influence model*, is a modification of the classic model by Kułakowski et al. which is obtained by eliminating self-appraisals (and thus reducing the system's dimension). Analysis of its convergence properties reduces to the analysis of our second model, called the *projected pure-influence model*, which arises as a projection of the first model onto the unit sphere. This second model has a self-standing interest, since it enjoys bounded evolution of the appraisals, while the first model shares the finite escape time property of the classic model by Kułakowski et al.

Our second contribution is to build a bridge between dynamic structural balance and structural balance as an optimization problem. We propose an energy function inspired by [23], namely the *dissonance function*, which measures the degree at which Heider's axioms are violated among the individuals of a social network. We show that this energy function has global minima that correspond to signed graphs satisfying structural balance in the case of real-valued appraisals (restricted on the unit sphere). Moreover, we show that our (projected) pure-influence model is the gradient system of the dissonance function in the case of undirected signed graphs, and hence the critical points of the dissonance function are the equilibria of our dynamical system. Thus, we establish a novel connection between dynamic structural balance and the characterization of structural balance as the minima of an energy function. Remarkably, our derivations show that this property of our models is enabled by the elimination of self-appraisals. Thus, the models contributed in this paper may be considered as both an interpersonal influence process and an extremum seeking dynamics for the dissonance function.

Our third and more detailed contribution is the mathematical analysis of the projected pure-influence model in the cases where the initial appraisal matrix is symmetric. In particular, we provide a complete characterization of the critical points of the dissonance function (i.e., the equilibrium points of the projected pure-influence model). This characterization relies upon a special submanifold of the Stiefel manifold and its properties. Along with the characterization of the critical points, we analyze their local stability properties and provide some results on convergence towards structural balance.

Our final contribution is a Monte Carlo numerical study of the convergence of our models to structural balance under generic initial conditions in both the symmetric and the asymmetric case. For the symmetric case, our numerical result is comparable to, but stronger than, what has already been proved for the Kułakowski et al. model: our models converge to structural balance under generic symmetric initial conditions. One key advantage of our models, as compared with those by Kułakowski et al., is that convergence to structural balance emerges under generic asymmetric initial conditions. Based on these numerical results, we formulate relevant conjectures.

4) Paper organization: Section II presents preliminary concepts. Section III presents our models and shows they are gradient flows. Section IV and Section V contain an analysis of equilibria and important convergence results, respectively. Section VI contains numerical results and conjectures. Finally, Section VII contains some concluding remarks.

II. Preliminaries

A. Signed weighted digraphs

Given an $n \times n$ matrix $X = (x_{ij})$ with entries taking values in $[-\infty, \infty]$, let $G(X)$ denote the signed directed graph where the directed edge $i \to j$ exists if and only if $x_{ij} \neq 0$, and x_{ij} represents its signed weight. The directed graph $G(X)$ is complete if X has no zero entries, except for the main diagonal. $G(X)$ has no self-loops if and only if X has zero diagonal entries. Let x_{i*} denote the ith row of the matrix X and x_{*i} the ith column of the matrix X. Let $\mathrm{sign}(X) = (\mathrm{sign}(x_{ij}))$, where $\mathrm{sign} : [-\infty, \infty] \to \{-1, 0, +1\}$ is as usual

$$\mathrm{sign}(x) = \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ +1, & \text{if } x > 0. \end{cases}$$

Given a sequence $a_1, \dots, a_n$, let $B = \mathrm{diag}(a_1, \dots, a_n)$ denote the diagonal $n \times n$ matrix (b_{ij}), where $b_{ii} = a_i$ and $b_{ij} = 0$ for $i \neq j$. For an $n \times n$ matrix X, define $\mathrm{diag}(X) = \mathrm{diag}(x_{11}, \dots, x_{nn})$. For a vector $v \in \mathbb{R}^n$, define $\mathrm{diag}(v) = \mathrm{diag}(v_1, \dots, v_n)$. Let $\mathbb{0}_n$ denote the $n \times 1$ vector of zeros, and $\mathbb{0}_{n\times n}$ the $n \times n$ matrix with zero entries.

Let $\succ$ and $\prec$ denote "entry-wise greater than" and "entry-wise less than," respectively.

A *triad* (if it exists) is a cycle between three nodes in $G(X)$. The *sign* of a triad is defined by the sign of the product of the weights composing a triad. For example, the triad $i \to j \to k \to i$ has sign $\mathrm{sign}(x_{ij}x_{jk}x_{ki})$.

A real-valued matrix Z is *irreducible* if its graph $G(Z)$ is strongly connected (a directed path between every two nodes exists) and *reducible* otherwise. If Z is reducible, a permutation matrix P exists such that the matrix

$$PZP^\top = \begin{bmatrix} Z_1 & * & \dots & * \\ 0 & Z_2 & \dots & * \\ \vdots & & & \\ 0 & & & Z_k \end{bmatrix}$$

is upper-triangular with irreducible blocks Z_i (some of them can be 1×1 matrices). If $Z = Z^\top$, the latter matrix is block-diagonal matrix $PZP^\top = \mathrm{diag}(Z_1, \dots, Z_k)$ and the graphs $G(Z_i)$ are the *connected components* of the graph $G(Z)$.

B. Sets of matrices and the Frobenius inner product

Given two matrices $A, B \in \mathbb{R}^{n\times n}$, their Frobenius inner product is defined by $\langle\langle A, B\rangle\rangle_F = \mathrm{trace}(B^\top A)$; the induced-norm is $\|A\|_F = \sqrt{\langle\langle A, A\rangle\rangle_F}$. Some important properties for the trace operator are: $\mathrm{trace}(A) = \mathrm{trace}(A^\top)$, $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, and, for all $d \in \mathbb{N}$, $\mathrm{trace}(A^d) = \sum_{i=1}^n \lambda_i^d$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A.

Let $\mathbb{R}^{n\times n}_{\text{zero-diag}}$ be the set of $n \times n$ real matrices with zero diagonal entries, and $\mathbb{R}^{n\times n}_{\text{zero-diag,symm}}$ be the set of symmetric matrices belonging to $\mathbb{R}^{n\times n}_{\text{zero-diag}}$. Let $\mathbb{S}^{n\times n}$ be the unit sphere in $\mathbb{R}^{n\times n}$, that is $A \in \mathbb{S}^{n\times n}$ if and only if $A \in \mathbb{R}^{n\times n}$ with $\|A\|_F = 1$. Similarly, we define the sets $\mathbb{S}^{n\times n}_{\text{zero-diag}} = \mathbb{R}^{n\times n}_{\text{zero-diag}} \cap \mathbb{S}^{n\times n}$ and $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}} = \mathbb{R}^{n\times n}_{\text{zero-diag,symm}} \cap \mathbb{S}^{n\times n}$.

Let $\mathbb{R}^{n\times n}_{\text{diag}}$ be the set of all real diagonal matrices and $\mathbb{R}^{n\times n}_{\text{sk-symm}}$ be the set of all skew-symmetric matrices. Then, we have the following orthogonal decomposition of $\mathbb{R}^{n\times n}$ equipped with the Frobenius inner product:

$$\mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}_{\text{sk-symm}} \oplus \mathbb{R}^{n\times n}_{\text{zero-diag,symm}} \oplus \mathbb{R}^{n\times n}_{\text{diag}}. \tag{1}$$

C. A review on structural balance

Throughout the paper we deal with social networks composed of $n \geq 3$ individuals, although the definition of structural balance (Definition II.3) is formally applicable to the case of degenerate networks with $n = 1$ or $n = 2$ nodes.

Definition II.1 (Appraisal matrix and network)**.** *We let the entry x_{ij} of the matrix $X \in \mathbb{R}^{n\times n}$ denote the appraisal (or qualitative evaluation) held by individual i of individual j. The sign of x_{ij} indicates if the relationship is positive $(+1)$, negative (-1) or of indifference (0). The magnitude of x_{ij} indicates the strength of the relationship. x_{ii} can be interpreted as i's self-appraisal. We call X the* appraisal matrix*, and $G(X)$ the* appraisal network.

Definition II.2 (Heider's axioms and social balance notions)**.** *The* Heider's axioms *are*
H1) A friend of a friend is a friend,
H2) An enemy of a friend is an enemy,
H3) A friend of an enemy is an enemy,
H4) An enemy of an enemy is a friend.
An appraisal network $G(X)$ is structurally balanced in Heider's sense*, if it is complete and satisfies axioms H1)-H4).*

Consider a complete appraisal network $G(X)$. We call a *faction* any group of agents whose members positively appraise each other. We say two factions are *antagonistic* if every representative from one faction negatively appraise every representative of the other faction. It can be shown [4], [11], [12] that Heider's structural balance condition for $G(X)$ with $n \geq 3$ nodes holds if and only if either the individuals constitute a single faction or can be partitioned into two antagonistic factions. The possession of the latter property may thus be considered as an alternative definition of structural balance (and is formally applicable to graphs without triads).

Definition II.3 (Structural balance)**.** *A complete appraisal network $G(X)$ is said to satisfy* structural balance*, if $G(X)$ is composed by one faction or two antagonistic factions; or, whenever $n \geq 3$, equivalently, that all triads are positive, i.e., $x_{ij}x_{jk}x_{ki} > 0$ for any different $i, j, k \in \{1, \dots, n\}$.*

Notice that a structurally balanced graph is always sign-symmetric: $\mathrm{sign}(x_{ij}) = \mathrm{sign}(x_{ji})$ for any $i \neq j$. For simplicity we will say that a matrix X *corresponds to* structural balance whenever $G(X)$ satisfies structural balance.

III. Proposed models and representation as gradient flows

In this section we propose our models defining them over the set of symmetric matrices. We postponed the general asymmetric setting to Section VI.

A. Pure-influence model

We propose our new dynamic model solely based on interpersonal appraisals.

Definition III.1 (Pure-influence model)**.** *The* pure-influence model *is a system of differential equations on the set of zero-diagonal matrices $\mathbb{R}^{n\times n}_{\text{zero-diag}}$ defined by*

$$\dot{x}_{ij} = \sum_{\substack{k=1 \\ k\neq i,j}}^n x_{ik}x_{kj}, \tag{2}$$

for any $i, j \in \{1, \dots, n\}$ and $i \neq j$. Here x_{ij}, $i \neq j$, are the off-diagonal entries of a zero-diagonal matrix $X \in \mathbb{R}^{n\times n}_{\text{zero-diag}}$. In equivalent matrix form, the previous equations read:

$$\dot{X} = X^2 - \operatorname{diag}(X^2), \qquad X(0) \in \mathbb{R}^{n\times n}_{\text{zero-diag}}. \tag{3}$$

We interpret X as the interpersonal appraisal matrix. While system (2) does not define the evolution of self-appraisals, the matrix reformulation (3) ensures $\operatorname{diag}(\dot{X}) = \mathbb{0}_{n\times n}$ and, since $X(0) \in \mathbb{R}^{n\times n}_{\text{zero-diag}}$ means $\operatorname{diag}(X(0)) = \mathbb{0}_{n\times n}$, we have $\operatorname{diag}(X(t)) = \mathbb{0}_{n\times n}$ for all positive times t.

Our model is a modification of the classical model proposed by Kułakowski et al. [17], where self-appraisals play a crucial role in the dynamics of the interpersonal appraisals.

Definition III.2 (Kułakowski et al. model)**.** *The* Kułakowski et al. model *is a system of differential equations on the state space $\mathbb{R}^{n\times n}$ defined by*

$$\dot{x}_{ij} = \sum_{k=1}^{n} x_{ik}x_{kj} = x_{ij}(x_{ii} + x_{jj}) + \sum_{\substack{k=1\\k\neq i,j}}^{n} x_{ik}x_{kj}, \tag{4a}$$

$$\dot{x}_{ii} = x_{ii}^2 + \sum_{\substack{k=1\\k\neq i}}^{n} x_{ik}x_{ki}, \tag{4b}$$

for any $i \neq j \in \{1, \dots, n\}$. In equivalent matrix form, the previous equations read: $\dot{X} = X^2$.

Remark III.1 (The problem with self-appraisals)**.** *The introduction of self-appraisals in model (4) is objectionable on several grounds. The first conceptual problem is that self-appraisals are not considered in any definition of structural balance in the social sciences. Heider's axioms in Definition II.2 do not take into account self-appraisals: social balance is a function of only interpersonal appraisals. Moreover, once self-appraisals are introduced, one needs to postulate why and how self-appraisals affect interpersonal appraisals, i.e., justify the choice of the first addendum for the right hand side of (4a). Finally, one needs to postulate how they evolve, i.e., justify the choice for the right hand side of (4b). In summary, the pure influence model (2) avoids these difficulties and stays closer to the foundations of structural balance, in which individuals are attending only to interpersonal appraisals. Even though $\dot{X} = X^2$ may appear mathematically simpler or more elegant than $\dot{X} = X^2 - \operatorname{diag}(X^2)$, we believe the latter model is actually more parsimonious, lower dimensional, and more faithful to Heiders' axioms.*

One easily notices the following important property of the pure-influence model (3): the right-hand side is an analytic function of X so that the equation enjoys (local) existence and uniqueness of the solutions. A second property is that, if $X(0) = X(0)^\top$, then $X(t) = X(t)^\top$ for all subsequent times. This implies that the pure-influence model is well defined over the set of symmetric (zero diagonal) matrices $\mathbb{R}^{n\times n}_{\text{zero-diag,symm}}$.

B. Dissonance function

We introduce and study the properties of a useful *dissonance function* that summarize the total amount of cognitive dissonances [8] among the members of a social network due to the lack of satisfaction of Heider's axioms. Recall that, according to Definition II.3, a triad $i \to j \to k \to i$ satisfies the axioms if and only if $x_{ij}x_{jk}x_{ki} > 0$.

Definition III.3 (Dissonance function)**.** *The* dissonance function $\mathcal{D} : \mathbb{R}^{n\times n}_{\text{zero-diag}} \to \mathbb{R}$ *is*

$$\mathcal{D}(X) = -\sum_{\substack{i,j,k=1\\ i\neq j, j\neq k, k\neq i}}^{n} x_{ij}x_{jk}x_{ki} = -\operatorname{trace}(X^3) = -\sum_{i=1}^{n} \lambda_i^3, \tag{5}$$

where $\{\lambda_i\}_{i=1}^n$ is the set of eigenvalues of X.

We plot $\mathcal{D}$ in a low-dimensional setting in Figure 1.

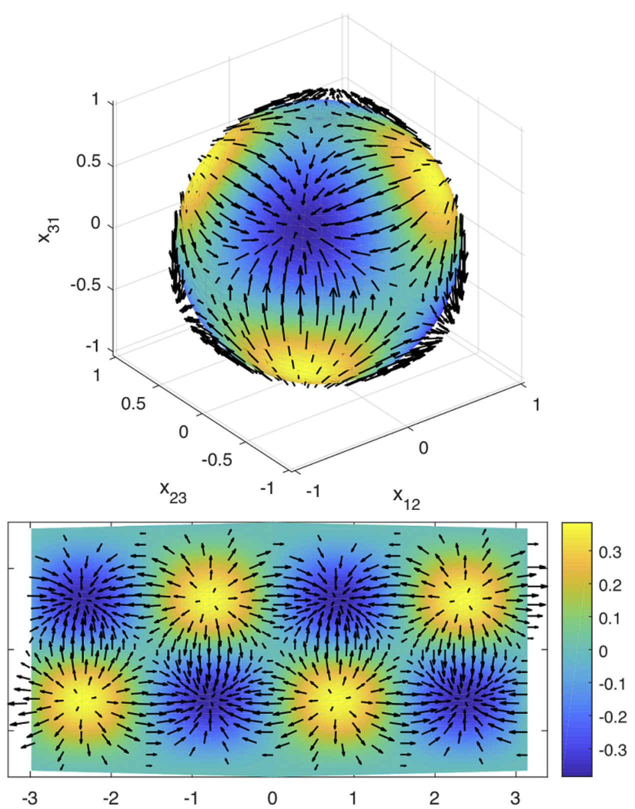


Fig. 1. For $n = 3$, an arbitrary symmetric unit-norm zero-diagonal matrix $X \in \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ is described by (x_{12}, x_{23}, x_{31}) with these coordinates living in the sphere $x_{12}^2 + x_{23}^2 + x_{31}^2 = 1$. In the upper figure, we plot this sphere with a heatmap, with dark blue being the lowest value and light yellow the largest value, according to the evaluation of the dissonance function $\mathcal{D}(X)$. The function has four global minima corresponding to the four possible configurations of $G(X)$ satisfying structural balance, and we can qualitatively appreciate the convergence of solution trajectories to these minima in the superimposed vector field on the sphere. The lower figure is a stereographic projection of the upper figure.

Energy landscapes in social balance theory are studied in [7], [23]. Our proposed dissonance function is the extension to $\mathbb{R}^{n\times n}_{\text{zero-diag}}$ of the energy function proposed by [23] for the setting of binary-valued symmetric appraisal matrices. For binary-valued appraisals, the global minima of $\mathcal{D}$ correspond to networks that satisfy structural balance, since all triads are positive (Definition II.3). Thus, $\mathcal{D}$ naturally measures to which extent Heider's axioms are violated in a complete graph.

Lemma III.2 (Properties of the dissonance function)**.** *Consider the dissonance function $\mathcal{D}$ and pick $X \in \mathbb{R}^{n\times n}_{\text{zero-diag}}$. Then*

(i) *$\mathcal{D}$ is analytic and attains its maximum and minimum values on any compact matrix subset of $\mathbb{R}^{n\times n}_{\text{zero-diag}}$,*
(ii) *if $G(X)$ satisfies structural balance, then $\mathcal{D}(X) < 0$,*
(iii) *$\mathcal{D}(X) = \mathcal{D}(X^\top)$,*
(iv) *$\mathcal{D}(X) = -\langle\!\langle X^2, X^\top\rangle\!\rangle_F$.*

Additionally, if $\|X\|_F = 1$, that is, $X \in \mathbb{S}^{n\times n}_{\text{zero-diag}}$, then

(v) *$-1 \le \mathcal{D}(X) \le 1$.*

Proof. Here we show only property (v), since the other properties follow easily from the definition of $\mathcal{D}$. We note:

$$\begin{aligned}\left\|X^2\right\|_F^2 &= \sum_{i,j=1}^n (X^2)_{ij}^2 = \sum_{i,j=1}^n (X_{i*}X_{*j})^2\\ &\le \sum_{i,j=1}^n \|X_{i*}\|_2^2\|X_{*j}\|_2^2 = \Big(\sum_{i=1}^n \|X_{i*}\|_2^2\Big)\Big(\sum_{j=1}^n \|X_{*j}\|_2^2\Big)\\ &= \Big(\sum\nolimits_{i,k=1}^n x_{ik}^2\Big)^2 = \|X\|_F^2 = 1.\end{aligned}$$

Now, note that the Frobenius norm on the set of matrices coincides with the Euclidean norm of a single vector obtained by stacking the column vectors of the matrix. Then, by the Cauchy-Schwarz inequality applied to the inner-product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_F$, it follows that: $|D(X)| = |\langle\!\langle X^2, X\rangle\!\rangle_F| \le \left\|X^2\right\|_F \|X\|_F \le (\|X\|_F)^3 \le 1$ when $\|X\|_F \le 1$. □

C. Transcription on the unit sphere and the projected pure-influence model

We start by noting a simple fact. Given a trajectory $X : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n\times n}_{\text{zero-diag}} \setminus \{\mathbb{0}_{n\times n}\}$, there exist unique trajectories $\eta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $Z : \mathbb{R}_{\ge 0} \to \mathbb{S}^{n\times n}_{\text{zero-diag}}$ such that $X(t) = \eta(t)\mathcal{Z}(t)$, where $\eta(t) = \|X(t)\|_F$ and $\mathcal{Z}(t) = X(t)/\|X(t)\|_F$.

Theorem III.3 (Transcription of the pure-influence model)**.** *The pure-influence model (2) with initial conditions in $\mathbb{R}^{n\times n}_{\text{zero-diag,symm}}$ can be expressed as the following system of differential equations:*

$$\begin{aligned}\dot{\mathcal{Z}} &= \eta\mathcal{P}_{\mathcal{Z}^\perp}(\mathcal{Z}^2 - \operatorname{diag}(\mathcal{Z}^2))\\ &= \eta(\mathcal{Z}^2 - \operatorname{diag}(\mathcal{Z}^2) + \mathcal{D}(\mathcal{Z})\mathcal{Z}), \qquad (6a)\\ \dot{\eta} &= -\mathcal{D}(\mathcal{Z})\eta^2, \qquad (6b)\end{aligned}$$

where $\eta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $\mathcal{Z} : \mathbb{R}_{\ge 0} \to \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$. Here $\mathcal{P}_{\mathcal{Z}^\perp}$ is the orthogonal projection onto $\operatorname{span}\{\mathcal{Z}\}^\perp$ in the vector space of square matrices with the Frobenius inner product.

Proof. Since $\dot{X} = \dot{\eta}\mathcal{Z} + \eta\dot{\mathcal{Z}}$ and $X^2 - \operatorname{diag}(X^2) = \eta^2\left(\mathcal{Z}^2 - \operatorname{diag}(\mathcal{Z}^2)\right)$, equation (3) can be written as

$$\dot{\eta}\mathcal{Z} + \eta\dot{\mathcal{Z}} = \eta^2\left(\mathcal{Z}^2 - \operatorname{diag}(\mathcal{Z}^2)\right). \qquad (7)$$

Differentiating the equality $\|\mathcal{Z}(t)\|_F^2 = \langle\!\langle \mathcal{Z}(t), \mathcal{Z}(t)\rangle\!\rangle_F = 1$, one shows that $\langle\!\langle \mathcal{Z}(t), \dot{\mathcal{Z}}(t)\rangle\!\rangle_F = 0$, that is, $\mathcal{Z}(t) \perp \dot{\mathcal{Z}}(t)$. Computing the Frobenius inner product with $\mathcal{Z}(t)$ on both sides of (7), equation (6b) is immediate:

$$\begin{aligned}\dot{\eta} &= \eta^2\langle\!\langle \mathcal{Z}(t), \mathcal{Z}^2(t) - \operatorname{diag}(\mathcal{Z}^2(t))\rangle\!\rangle_F\\ &= \eta^2\langle\!\langle \mathcal{Z}(t), \mathcal{Z}^2(t)\rangle\!\rangle_F = -\mathcal{D}(\mathcal{Z}(t))\eta^2.\end{aligned} \qquad (8)$$

where we have used the fact that $\mathcal{Z}(t)$ is symmetric, and that $\operatorname{diag}(\mathcal{Z}(t)) = \mathbb{0}_{n\times n}$ and hence $\langle\!\langle \mathcal{Z}(t), \operatorname{diag}(\mathcal{Z}^2(t))\rangle\!\rangle_F = \operatorname{trace}(\mathcal{Z}(t)^\top \operatorname{diag}(\mathcal{Z}^2(t))) = 0$. Substituting (8) into equation (7), one arrives at $\dot{\mathcal{Z}} = \eta\left(\mathcal{Z}^2 - \operatorname{diag}(\mathcal{Z}^2) + \mathcal{D}(\mathcal{Z})\right)$.

Given $Y \in \mathbb{R}^{n\times n}$, let $\mathcal{P}_{\mathcal{Z}}(Y) = \langle\!\langle Y, \mathcal{Z}\rangle\!\rangle_F\mathcal{Z}$, i.e., $\mathcal{P}_{\mathcal{Z}}$ is the orthogonal projection operator onto the linear space spanned by $\mathcal{Z}$; and let $\mathcal{P}_{\mathcal{Z}^\perp}(Y) = Y - \mathcal{P}_{\mathcal{Z}}(Y) = Y - \langle\!\langle Y, \mathcal{Z}\rangle\!\rangle_F\mathcal{Z}$ be the orthogonal projection onto the space perpendicular to the linear space spanned by $\mathcal{Z}$. Then, we observe that $\mathcal{P}_{\mathcal{Z}^\perp}(\mathcal{Z}) = 0$ and $\mathcal{P}_{\mathcal{Z}^\perp}(\mathcal{Z}) = \dot{\mathcal{Z}}$. Using these results, we apply $\mathcal{P}_{\mathcal{Z}^\perp}$ to both sides of (7) and obtain $\dot{\mathcal{Z}} = \eta\mathcal{P}_{\mathcal{Z}^\perp}(\mathcal{Z}^2 - \operatorname{diag}(\mathcal{Z}^2))$. This concludes the proof of equations (6). □

In what follows, we are primarily interested in the dynamics (6a), describing the behavior of the bounded component $\mathcal{Z}(t)$. From Lemma A.1 we observe that η is a time-scale change for (6a) and so, for our convenience, we get rid of it and obtain the following dynamical system on the unit sphere.

Definition III.4 (Projected pure-influence model)**.** *The* projected pure-influence model *is a system of differential equations on the manifold $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ defined by*

$$\dot{Z} = Z^2 - \operatorname{diag}(Z^2) + \mathcal{D}(Z)Z. \qquad (9)$$

Given a solution $Z(t)$ to (9) with initial condition $Z(0)$, Lemma A.1 in the Appendix shows that $Z(t)$ is a time-scaled version of a solution $\mathcal{Z}(t)$ to (6a) with initial condition $\mathcal{Z}(0) = Z(0)$, where η in (6b) can have any positive initial condition. Therefore, there is a solution $X(t)$ to (3) that is both a scaled and time-scaled version of $Z(t)$.

Similarly, projecting onto the unit sphere leads to a new model based on the Kułakowski et al. model.

Definition III.5 (Projected Kułakowski et al. model)**.** *The* projected Kułakowski et al. model *is a system of differential equations on the manifold $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ defined by*

$$\dot{Z}(t) = Z^2 + \mathcal{D}(Z)Z. \qquad (10)$$

D. Pure-influence is the gradient flow of the dissonance function

We now let $\operatorname{grad}\mathcal{D}$ denote the gradient vector field of the dissonance function $\mathcal{D}$ on the manifold $\mathbb{R}^{n\times n}_{\text{zero-diag}}$ equipped with the Riemannian metric tensor $\langle\!\langle\cdot,\cdot\rangle\!\rangle_F$. We also let $\mathcal{D}\big|_{\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}}$ denote the restriction of $\mathcal{D}$ onto the manifold $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$. We now present the first of our main results.

Theorem III.4 (The pure-influence models over symmetric matrices are gradient flows)**.** *Consider the pure-influence model (2) with $X(0) \in \mathbb{R}^{n\times n}_{\text{zero-diag,symm}}$ and the projected pure-influence model (9) with $Z(0) \in \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$. Then*

(i) *$t \mapsto X(t)$ remains in the set $\mathbb{R}^{n\times n}_{\text{zero-diag,symm}}$ and*

$$\dot{X} = -\tfrac{1}{3}\operatorname{grad}\mathcal{D}(X), \qquad (11)$$

(ii) *$t \mapsto Z(t)$ remains in the set $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ and*

$$\dot{Z} = -\tfrac{1}{3}\mathcal{P}_{Z^\perp}\big(\operatorname{grad}\mathcal{D}(Z)\big) = -\tfrac{1}{3}\operatorname{grad}\mathcal{D}\Big|_{\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}}(Z). \qquad (12)$$

In other words, the projected pure-influence model (9) is, modulo a constant factor, the gradient flow of the dissonance function $\mathcal{D}$ restricted to the manifold of zero-diagonal unit-norm symmetric matrices $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$.

Proof of Theorem III.4. The forward invariance of the set of symmetric matrices in both statements is immediate from the solution uniqueness. To prove equation (12), we adopt the slight abuse of notation $\operatorname{grad}\mathcal{D}(Z) = \operatorname{grad}\mathcal{D}\big|_{\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}}(Z)$. With this notation, $Z \mapsto \operatorname{grad}\mathcal{D}(Z)$ is [13, pages 15-17] the unique vector field on $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ such that

$$\frac{d}{dt}\mathcal{D}(Z(t)) = \langle\!\langle \operatorname{grad}\mathcal{D}(Z(t)), \dot{Z}(t)\rangle\!\rangle_F \tag{13}$$

for any differentiable $Z : [0,\infty) \to \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$. Here, both $\operatorname{grad}\mathcal{D}(Z(t))$ and $\dot{Z}(t)$ belong to the tangent space to the manifold $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$. Now, using the various properties of the trace inner product (e.g., $\dot{Z}(t) \perp Z(t)$), we compute

$$\begin{aligned}\dot{\mathcal{D}}(Z(t)) &= -(\operatorname{trace}(\dot{Z}(t)Z(t)Z(t)) + \operatorname{trace}(Z(t)\dot{Z}(t)Z(t)) \\ &\quad + \operatorname{trace}(Z(t)Z(t)\dot{Z}(t)) \\ &= -3\operatorname{trace}(\dot{Z}(t)Z^2(t)) = -3\langle\!\langle \dot{Z}(t), Z^2(t)\rangle\!\rangle_F \\ &= -3\langle\!\langle \dot{Z}(t), Z^2(t) - \operatorname{diag}(Z^2(t)) + \mathcal{D}(Z(t))Z(t)\rangle\!\rangle_F.\end{aligned}$$

Recalling that $Z^2 - \operatorname{diag}(Z^2) + \mathcal{D}(Z)Z \overset{(6a)}{=} P_{Z^\perp}(Z^2 - \operatorname{diag}(Z^2))$ belongs to the tangent space to the manifold $\mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ at the point $Z(t)$, one arrives at the equality

$$\operatorname{grad}\mathcal{D}(Z) = -3\big(Z^2 - \operatorname{diag}(Z^2) + \mathcal{D}(Z)Z\big).$$

This concludes the proof of statement (ii). Finally, equation (11) can be proved in a similar way. $\square$

IV. Classification of Symmetric Equilibria

We here give the complete classification of the symmetric equilibria in the projected pure-influence model (9); the classification of general asymmetric equilibria remains an open problem. Thanks to Theorem III.4, all symmetric equilibria of the projected pure-influence model are critical points of the dissonance function $\mathcal{D}$. We start with the equilibrium equation:

$$Z^2 + \mathcal{D}(Z)Z - \operatorname{diag}(Z^2) = \mathbb{0}_{n\times n}, \quad Z \in \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}. \tag{14}$$

Note that the equilibria Z^* with $\mathcal{D}(Z^*) = 0$ correspond to equilibria of the original system (3) $X(t) \equiv X^* = \eta(0)Z^*$, whereas the others with $\mathcal{D}(Z^*) \neq 0$ lead to

$$X(t) = \eta(t)Z^*, \quad \eta(t) = \frac{\eta(0)}{1 + t\eta(0)\mathcal{D}(Z^*)}$$

defined for $t \in [0, \frac{1}{\eta(0)\mathcal{D}(Z^*)})$ if $\mathcal{D}(Z^*) < 0$ (for which the solution is unbounded) or for $t \geq 0$ if $\mathcal{D}(Z^*) > 0$.

A. Normalized Stiefel matrices

To start with, we introduce a special important manifold of non-square matrices that we will use throughout the paper.

Definition IV.1 (Normalized Stiefel matrices)**.** *A matrix $V \in \mathbb{R}^{n\times k}$, for $k \leq n$, is* normalized Stiefel (nSt)*, if*

(i) the columns of V are pairwise orthogonal unit vectors, i.e., $V^\top V = I_k$;

(ii) the norm of each row is the same (obviously, it must be $\sqrt{k/n} \leq 1$): $\operatorname{diag}(VV^\top) = n^{-1}kI_n$.

Let $\mathrm{nSt}(n,k) \subseteq \mathbb{R}^{n\times k}$ denote the set of normalized Stiefel matrices.

In general, the rows of an nSt matrix *need not* be orthogonal. We recall from [15] the notion of *compact Stiefel manifold*, denoted by $\mathrm{St}(k,n) = \{X \in \mathbb{R}^{n\times k} \mid X^\top X = I_k\}$.

Lemma IV.1 (Characterization of nSt matrices)**.** *The set $\mathrm{nSt}(n,k)$, $k \leq n$, is a compact and analytic submanifold of $\mathbb{R}^{n\times k}$ of dimension $(k-1)n + 1 - k(k+1)/2$, and it is also a submanifold of the compact Stiefel manifold (and thus, $\mathrm{nSt}(n,k) \subseteq \mathrm{St}(k,n)$). Moreover,*

(i) $\mathrm{nSt}(n,n)$ is the set of orthogonal matrices,

(ii) for $k = 1$, the matrix V is nSt if and only if

$$V = \frac{1}{\sqrt{n}}\begin{bmatrix} s_1 \\ \vdots \\ s_n\end{bmatrix}, \tag{15}$$

for any numbers $s_i \in \{-1,+1\}$, $i \in \{1,\dots,n\}$,

(iii) for $k = 2$, the matrix V is nSt if and only if

$$V = \sqrt{\frac{2}{n}}\begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ \vdots & \vdots \\ \cos\alpha_n & \sin\alpha_n\end{bmatrix}, \tag{16}$$

for any set of angles $\alpha_1,\dots,\alpha_n$ satisfying

$$\sum_{m=1}^{n} e^{2\alpha_m\sqrt{-1}} = 0. \tag{17}$$

We postpone the proof of Lemma IV.1 to Appendix A. We remark that in the case of $n = k = 2$, the constraint (17) implies that $2\alpha_2 = \pi + 2\pi s + 2\alpha_1$, where $s \in \mathbb{Z}$, that is, $\alpha_2 = \pi/2 + \pi s + \alpha_1$ and $\cos\alpha_2 = (-1)^{s+1}\sin\alpha_1$, $\sin\alpha_2 = (-1)^s\cos\alpha_1$. Thus, the matrices in $\mathrm{nSt}(2,2)$ are orthogonal 2×2 matrices (representing proper or improper rotations):

$$V = \begin{bmatrix} \cos\alpha_1 & \sin\alpha_1 \\ -\varepsilon\sin\alpha_1 & \varepsilon\cos\alpha_1\end{bmatrix}, \quad \varepsilon \in \{-1,+1\}.$$

For a general k, it is difficult to give a closed-form description of all matrices from $\mathrm{nSt}(n,k)$. However, there are simple examples of matrices from $\mathrm{nSt}(n,k)$ in the case where $n = 2k$, including every matrix of the form

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} U_1 \\ U_2\end{bmatrix},$$

where U_i are orthogonal $k\times k$ matrices.

B. Technical results

The classification of equilibria relies on the following technical results that will be proved in Appendix A.

Lemma IV.2. *Suppose that $Z^2 - 2\alpha Z = \beta I_n$ for some symmetric $n\times n$ matrix Z with $\operatorname{diag}(Z) = \mathbb{0}_{n\times n}$ and scalars α, β. Then Z can be decomposed as*

$$Z = pVV^\top - qI_n = Z^\top \tag{18}$$

for some $V \in \mathrm{nSt}(n,k)$ $(1 \le k < n)$ and constants $p, q \ge 0$ such that $pk = qn$, $2\alpha = p - 2q$ and $\beta = q(p-q)$. Namely, $p = 2\sqrt{\alpha^2+\beta}$, $\quad q = \sqrt{\alpha^2+\beta} - \alpha$.

Corollary IV.3. *Given a matrix $Z = Z^\top$ with $\mathrm{diag}(Z) = \mathbb{0}_{n\times n}$, the matrix $Z^2 - 2\alpha Z$ is diagonal with s different eigenvalues $\beta_1 < \ldots < \beta_s$ of multiplicities $n_1, \ldots, n_s$ respectively ($n_1 + n_2 + \ldots + n_s = n$) if and only if there exists such a permutation matrix S that*

$$SZS^{-1} = \mathrm{diag}(Z_1, \ldots, Z_s),$$

where each Z_i is decomposed as (18) with parameters p_i, q_i, V_i, where $V_i \in \mathrm{nSt}(n_i, k_i)$ for some $k_i < n_i$ and

$$p_i = 2\sqrt{\alpha^2+\beta_i}, \quad q_i = \sqrt{\alpha^2+\beta_i} - \alpha. \tag{19}$$

Thus, for irreducible *$Z = Z^\top$ the matrix $Z^2 - 2\alpha Z$ is diagonal if and only if Z is decomposed as (18) with $p, q \ge 0$.*

C. Classification of irreducible symmetric equilibria

Theorem IV.4 (Irreducible equilibria for the projected pure-influence model)**.** *For the projected pure-influence model (9),*

(i) all irreducible symmetric equilibria are of the form

$$Z^* = pVV^\top - qI_n, \tag{20}$$

with $V \in \mathrm{nSt}(n,k)$, $1 \le k < n$, and

$$p = \sqrt{\frac{n}{k(n-k)}}, \quad q = \sqrt{\frac{k}{n(n-k)}}; \tag{21}$$

(ii) Z^* has k positive eigenvalues with value $p-q$ and $n-k$ negative eigenvalues with value $-q$;

(iii) the dissonance function satisfies

$$\mathcal{D}(Z^*) = -\frac{n-2k}{\sqrt{kn(n-k)}}, \tag{22}$$

and the right-hand side is monotonically increasing in $k \in \{1, \ldots, n-1\}$ (see Figure 2).

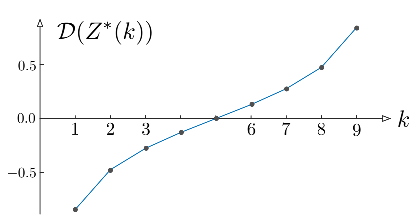


Fig. 2. For a network with size $n = 10$, the dissonance function $\mathcal{D}$ evaluated on all irreducible symmetric equilibria with $k \in \{1, \ldots, 9\}$ positive eigenvalues, according to equation (22).

Proof. We start by proving a technical statement. Pick $V \in \mathrm{nSt}(n,k)$, p, q real numbers and set $\theta = p - 2q$. Then, the matrix $Z = pVV^\top - qI_n = Z^\top$ satisfies the following properties:

(a) $Z^2 - \theta Z = q(p-q)I_n$, and thus $\mathrm{diag}(Z^2) = \theta\,\mathrm{diag}(Z) + q(p-q)I_n$;

(b) for any $p \neq 0$, the matrix Z has two eigenvalues $p-q$ and $(-q)$ whose multiplicities are k and $(n-k)$ respectively;

(c) the eigenspaces corresponding to $p-q$ and $-q$ are the image of V and the kernel of $V^\top$ respectively;

(d) $\mathrm{diag}(Z) = \mathbb{0}_{n\times n}$ if and only if $pk = qn$; in this situation, $\mathrm{trace}(Z^2) = q(p-q)n$ and $\mathcal{D}(Z) = -\mathrm{trace}(Z^2 Z^\top) = -\theta nq(p-q)$.

To prove (a), recall that $V^\top V = I_k$ and therefore

$$\begin{aligned} Z^2 &= p^2 VV^\top VV^\top + q^2 I_n - 2pqVV^\top = p\theta VV^\top + q^2 I_n \\ &= \theta Z + (pq - q^2)I_n. \end{aligned}$$

To prove (b) and (c), notice that for any vector $z = Vy$ one has $VV^\top z = V(V^\top V)y = Vy = z$, and thus $Zz = (p-q)z$. The space of such vectors is nothing else than the image of V and has dimension k (recall that the columns of V are orthogonal, and hence are linearly independent). If $V^\top z = 0$, then $Zz = -qz$, and the dimension of $\ker(V^\top)$ is $(n-k)$. Since $Z = Z^\top$ and $p - q \neq -q$ (except for the case where $p = q = 0$ and $Z = 0$, which is trivial), the two eigenspaces are orthogonal and their sum coincides with $\mathbb{R}^n$. Hence, there are no other eigenvalues. To prove (d), note first $p\,\mathrm{diag}(VV^\top) = (pk/n)I_n$, and thus $\mathrm{diag}(Z) = \mathbb{0}_{n\times n}$ if and only if $pk/n = q$. Using statement (a), one shows that in this situation $\mathrm{diag}(Z^2) = q(p-q)I_n$ and hence $\mathrm{trace}(Z^2) = q(p-q)n$. Thanks to (a), $Z^3 = \theta Z^2 + q(p-q)Z \Longrightarrow \mathrm{trace}(Z^3) = \theta\,\mathrm{trace}(Z^2) = \theta nq(p-q)$, which finishes the proof of (d).

Now, to prove the statement (i) of the theorem, let Z^* be an irreducible symmetric solution to equation (14). For $\alpha = -\mathcal{D}(Z^*)/2$, the matrix $(Z^*)^2 - 2\alpha Z^* = \mathrm{diag}(Z^{*2})$ is diagonal. Since Z^* is irreducible, it follows from Corollary IV.3 that Z^* can be decomposed as (20) with some $p, q \ge 0$. Then, from (a) and (d), it also follows that Z^* satisfies equation (14) if and only if $pk = qn$ (which comes from $\mathrm{diag}(Z^*) = \mathbb{0}_{n\times n}$) and $pq - q^2 = 1/n$ (which comes from $\mathrm{trace}(Z^{*2}) = 1$). This implies that $q = \sqrt{\frac{k}{n(n-k)}}$ and $p = \sqrt{\frac{n}{k(n-k)}}$.

Finally, statement (ii) follows from (b); and (iii) is obtained by substituting the values of p and q into the definition of the dissonance function (5) and noting that the smooth function $\kappa \mapsto -\frac{n-2\kappa}{\sqrt{n\kappa(n-\kappa)}}$ has positive derivative on $(0, n)$. □

D. Classification of reducible symmetric equilibria

The next theorem generalizes Theorem IV.4 and characterizes all symmetric equilibria for the projected pure-influence model and its proof can be found in Appendix A.

Theorem IV.5 (All equilibria for the projected pure-influence model)**.** *The matrix Z^* is an equilibrium (14) of the projected pure-influence model if and only if a permutation matrix S exists such that:*

(i) $SZ^*S^{-1} = \mathrm{diag}(Z_1^*, \ldots, Z_s^*)$, $s \ge 1$, $Z_i^* = Z_i^{*\top} \in \mathbb{R}^{n_i \times n_i}$;

(ii) the blocks Z_i^* admit representation (18): $Z_i^* = p_i V_i V_i^\top - q_i I_{n_i}$, where $p_i, q_i \ge 0$ and $V_i \in \mathrm{nSt}(n_i, k_i)$, $1 \le k_i < n_i$;

(iii) the sign $\varepsilon = \mathrm{sign}(n_i - 2k_i) \in \{-1, 0, 1\}$ is the same for all $i = 1, \ldots, s$ such that $Z_i^* \neq \mathbb{0}_{n_i \times n_i}$ and

(iv) each block $Z_i^* \neq \mathbb{0}_{n_i\times n_i}$ is irreducible and the corresponding coefficients p_i, q_i have the form

$$p_i = 2\sqrt{\alpha^2+\beta_i}, \quad q_i = \sqrt{\alpha^2+\beta_i} - \alpha, \tag{23}$$

where

a) for $\varepsilon \neq 0$, α and β_i are determined from

$$\alpha = \varepsilon \left(\sum_{i: Z_i \neq 0} \frac{4k_i n_i (n_i - k_i)}{(n_i - 2k_i)^2} \right)^{-1/2}, \qquad (24)$$
$$\beta_i = \alpha^2 \frac{4n_i k_i - 4k_i^2}{(n_i - 2k_i)^2};$$

b) for $\varepsilon = 0$, $\alpha = 0$, for all i, and β_i are chosen in such a way that $\sum_{i: Z_i \neq 0} \beta_i n_i = 1$.

Remark IV.6. *Let Z^* be a reducible equilibrium for the projected pure-influence model such that $G(Z^*)$ is composed of m (disconnected) subgraphs that satisfy structural balance. According to Definition II.3,* $G(Z^*)$ does not *satisfy structural balance since this definition requires $G(Z^*)$ to be complete.*

E. Structural balance and equilibria

We now characterize the equilibria corresponding to structural balance and how they minimize the dissonance function.

Corollary IV.7 (Balanced equilibria of the projected pure-influence model)**.** *For the projected pure-influence model (9), let $Z^* \in \mathbb{S}^{n\times n}_{\text{zero-diag}}$ be an equilibrium point with a single positive eigenvalue. Then,*

(i) after a relabelling of the agents, Z^* has the form

$$Z^* = \left[\begin{array}{c|c} Z' & \mathbb{0}_{n_1 \times (n-n_1)} \\ \hline \mathbb{0}_{(n-n_1)\times n_1} & \mathbb{0}_{(n-n_1)\times(n-n_1)} \end{array} \right] \qquad (25)$$

with $n_1 \leq n$ and

$$Z' = \frac{1}{\sqrt{n_1(n_1-1)}} (ss^\top - I_{n_1}), \qquad (26)$$

for some $s \in \{-1, +1\}^{n_1}$; and thus, for any fixed n_1, there are only $2^{n_1 - 1}$ different equilibria (with a single positive eigenvalue),

(ii) $G(Z')$ satisfies structural balance, with the binary vector s characterizing the distribution of the individuals in the single faction or in the two factions, and

(iii) if $G(Z^*)$ is a connected graph, then $G(Z^*)$ satisfies structural balance (being thus complete) and Z^* is a global minimizer to the optimization problem:

$$\begin{aligned} \underset{Z \in \mathbb{R}^{n\times n}}{\text{minimize}} \quad & \mathcal{D}(Z) \\ \text{subject to} \quad & Z \in \mathbb{S}^{n\times n}_{\text{zero-diag,symm}} \end{aligned}$$

and satisfies $\mathcal{D}(Z^*) = -\frac{n-2}{\sqrt{n(n-1)}}$.

Proof. Consider a permutation of indices from Theorem IV.5. Since Z^* has only one positive eigenvalue, it can have only one non-zero diagonal block $Z_i^* = Z'$. Statement (i) now follows from (20),(21) (with $k = 1, n = n_1$) and (15).

Regarding statement (ii), observe that for any different i, j and k,

$$z'_{ij} z'_{jk} z'_{ki} = \frac{(s_i s_j)(s_j s_k)(s_k s_i)}{(n_1(n_1-1))^{3/2}} = \frac{1}{(n_1(n_1-1))^{3/2}} > 0.$$

This inequality implies $\text{sign}(z'_{ij}) = \text{sign}(z'_{jk} z'_{ki})$ and thus we know that Z' satisfies structural balance. It is immediate to see that any i and j such that $s_i = s_j$ correspond to the same faction in the network $G(Z')$. This completes the proof for (ii).

Regarding statement (iii), we notice that the smooth function $\eta \mapsto -\frac{\eta-2}{\sqrt{\eta(\eta-1)}}$ has negative derivative for $\eta > 3/2$. Hence, the value of $\mathcal{D}(Z^*) = \mathcal{D}(Z') = -\frac{n_1-2}{\sqrt{n_1(n_1-1)}}$ at equilibrium (25) with one positive eigenvalue is minimal when $Z' = Z^*$ and $n_1 = n$, that is, the matrix is irreducible. Now, let us focus on the points that vanish the gradient of $\mathcal{D}$, i.e., the equilibria of the projected pure-influence model. Permuting the agents, we may confine ourselves to equilibria described in Theorem IV.5 that have s blocks of size n_i with $k_i < n_i$ positive eigenvalues, $i \in \{1, \dots, s\}$. To see why this is true, in the proof of Theorem IV.4 it was shown that $\mathcal{D}(Z_i^*) = -2\alpha n q_i (p_i - q_i) = -2\alpha\beta_i$. Next, if $\varepsilon = -1$, then $\alpha < 0$ and $D(Z^*) > 0$. If $\varepsilon = \text{sign}(n_i - 2k_i) = 0$ for all $Z_i^* \neq 0$, then $\mathcal{D}(Z^*) = \sum_i \mathcal{D}(Z_i^*) = 0$. As we know, the minimal value should be negative, so such equilibria cannot be global minimizers. Therefore, we may assume that $\varepsilon = 1$, that is, $k_i < n_i/2$ for all such i that $Z_i^* \neq 0$. Assume, without loss of generality, that $Z_1^*, \dots, Z_m^* \neq 0$ and $Z_{m+1}^*, \dots, Z_s^* = 0$. Denote $k_1 + \cdots + k_m = k'$ and $n_1 + \cdots + n_m = n' \leq n$. Note that the function $f(\xi) = \xi(1-\xi)/(1-2\xi)^2$ is convex on $(0, 1/2)$. Therefore, Jensen's inequality implies

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^m \frac{k_i n_i (n_i - k_i)}{(n_i - 2k_i)^2} &= \sum_{i=1}^m \frac{n_i}{n} f\left(\frac{k_i}{n_i}\right) \\ &\geq f\left(\sum_{i=1}^m \frac{k_i}{n'}\right) = f\left(\frac{k'}{n'}\right) = \frac{k'(n'-k')}{(n'-2k')^2}, \end{aligned}$$

and, in turn,

$$\mathcal{D}(Z^*) = -\left(\sum_{i=1}^m \frac{k_i n_i (n_i - k_i)}{(n_i - 2k_i)^2} \right)^{-1/2} \geq -\frac{n' - 2k'}{\sqrt{k' n' (n' - k')}}.$$

We know, however from Theorem IV.4 that the right-hand side is minimal when $k' = 1$, in which case the minimal value, as we have seen in the beginning in the proof, is achieved at $n' = n$. Hence, the irreducible equilibrium with one positive eigenvalue is the global minimizer of $\mathcal{D}^*$. □

Remark IV.8. *Let Z^* denote an equilibrium point with one positive eigenvalue. Then, $-Z^*$ has one negative eigenvalue and does not correspond to structural balance. All such $-Z^*$ correspond to isolated critical points of $\mathcal{D}$.*

F. Examples of equilibria with two positive eigenvalues

Let Z^* be any equilibrium of the projected pure-influence model parameterized by $\text{nSt}(n, 2)$ matrices, so that it has two positive eigenvalues. Let us assume first that it is irreducible. Then, another class of equilibria is found using the parametrization (16). It can be easily shown that

$$Z^* = \sqrt{\frac{2}{n(n-2)}} (\theta_{ij})_{i,j=1}^n, \quad \theta_{ij} = \begin{cases} 0, \ i = j \\ \cos(\alpha_i - \alpha_j), i \neq j. \end{cases}$$

Here the angles α_i should satisfy the relation (17). Interestingly, many of such matrices do not correspond to structural balance. Consider, for example, the case where the unit vectors in (17) constitute a regular n-gon: $\alpha_i = \frac{\pi(i-1)}{n}$, $i = 1, \ldots, n$. For any pair $i, j > i$ the entry z_{ij} is negative if $(j-i) > n/2$, positive if $j - i < n/2$ and zero if $j - i = n/2$ (possible only for even n). If n is odd, the graph is complete, otherwise, the pairs of nodes $(i, i + n/2)$ for $i = 1, \ldots, n/2$ are not connected. For example, in the smallest dimension $n = 3$, by setting $\alpha_1 = 0$, $\alpha_2 = \pi/3$ and $2\pi/3$, we obtain the equilibrium

$$Z^* = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & +1 & -1 \\ +1 & 0 & +1 \\ -1 & +1 & 0 \end{bmatrix}$$

which does not correspond to structural balance. Actually, in the case where $n = 3$ or $n \geq 5$, the graph always contains imbalanced triads. For instance, for $n \geq 3$ being odd the nodes $i = 1$, $j = (n-1)/2$ and $\ell = (n+3)/2$ always constitute such a triad: $z_{i\ell} < 0$, whereas $z_{ij}, z_{j\ell} \geq 0$. For an even number $n \geq 6$, one may take $i = 1$, $j = n/2$, $\ell = n/2 + 2$. In the case $n = 4$, the equilibrium Z^* corresponds to an incomplete cyclic graph such that $\mathcal{D}(Z^*) = 0$:

$$Z^* = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

For the reducible matrix case, since Z^* has two positive eigenvalues, $G(Z^*)$ contains two disconnected subgraphs that satisfy structural balance with possibly other isolated nodes.

V. Convergence to Balanced Equilibria and Stability Analysis

We now provide convergence results for our models towards equilibria that correspond to structural balance. We present a supporting lemma and then our main theorem.

Lemma V.1. *Assume that the solution of* (2) *satisfies* $x_{i*}(t_0) = \mathbb{0}_{1\times n}$ *at some* $t_0 \geq 0$, *that is, in the graph* $G(X(t_0))$ *node* i *does not communicate to any other node. Then,* $x_{i*}(t) \equiv \mathbb{0}_{1\times n}$ *for any* $t \geq 0$. *The same holds for the solutions of* (9).

Proof. Since the right-hand sides of (2) and (9) are analytic, any solution is a real-analytic function of time. Assuming that $x_{ij}(t_0) = 0$ for all j, one finds that $\dot{x}_{ij}(t_0) = 0$. Differentiating (2), it is easy to show that $\ddot{x}_{ij}(t_0) = 0$, and so on, $x_{ij}^{(m)}(t_0) = 0$ for any $m \geq 1$. In view of analyticity, one has $x_{ij}(t) \equiv 0$ for any t. Similarly, $z_{ij}(t_0) = 0\,\forall j$ entails that $z_{ij}(t) \equiv 0$ for any solution of (9). ☐

Theorem V.2 (Convergence results and dynamical properties)**.** *Consider the pure-influence model* (2) *with an initial condition* $X(0) \in \mathbb{R}^{n\times n}_{\text{zero-diag,symm}}$ *and the projected pure-influence model* (9) *with initial condition* $Z(0) = \frac{X(0)}{\|X(0)\|_F}$. *Then,*

(i) the solution $Z(t)$ *converges to a single critical point of the dissonance function* $\mathcal{D}$;

(ii) the number of negative eigenvalues of $Z(t)$ *is non-decreasing.*

Moreover, if $X(0)$ *has one positive eigenvalue, then*

(iii) $\lim_{t\to+\infty} Z(t) = Z^*$, *where* Z^* *is as in* (26), *so that* $G(Z(t))$ *or one of its connected components (while the rest of nodes are isolated) reaches structural balance in finite time;*

(iv) $X(t)$ *achieves the same sign structure as* Z^* *in finite time;*

(v) nonzero entries of $X(t)$ *diverge to infinity in finite time.*

Proof. For convenience, throughout this proof, let us denote $W(t) = \frac{X(t)}{\|X(t)\|_F}$, i.e., $X(t) = \eta(t)W(t)$ with $\eta(t)$ evolving according to (6a) and $W(t)$ evolving according to (6b). From the construction of the transcription of the pure-influence model in Theorem III.3, we have that $\eta(t) = \|X(t)\|_F$ and so $\eta(t) > 0$ for all well-defined $t \geq 0$. Moreover, Lemma A.1 let us conclude that $W(t) = Z(\int_0^t \eta(s)ds)$ for all $t \geq 0$, and thus the solution $X(t)$ is well defined.

To prove (i), recall that (9) is a gradient flow dynamics of the analytic function $\mathcal{D}$, and the trajectory $Z(t)$ stays on a compact manifold and, in particular, is bounded. The classical result of Łojasiewicz [1] implies convergence of the trajectory to a single fixed point.

To prove (ii), we enumerate the eigenvalues of $Z(t)$ in the descending order $\lambda_1(t) \geq \lambda_2(t) \ldots \geq \lambda_n(t)$ and consider the corresponding orthonormal bases of eigenvectors $v_i(t)$. Since $Z_i(t)v_i(t) = \lambda_i(t)v_i(t)$ and $v_i(t)^\top v_i(t) = 1$, we obtain $\dot{Z}v_i + Z\dot{v}_i = \dot{\lambda}_i v_i + \lambda_i \dot{v}_i$ and $\dot{v}_i(t)^\top v_i(t) = 0$. Therefore,

$$\dot{\lambda}_i = v_i^\top \dot{Z} v_i + v_i^\top Z \dot{v}_i = v_i^\top \dot{Z} v_i + \lambda_i v_i^\top \dot{v}_i = v_i^\top \dot{Z} v_i,$$

entailing the following differential equation

$$\dot{\lambda}_i = \lambda_i^2 + \mathcal{D}(Z)\lambda_i - v_i^\top \operatorname{diag}(Z^2) v_i. \tag{27}$$

Notice that all diagonal entries of $\operatorname{diag}(Z^2)$ are nonnegative. Now, due to Lemma V.1, if the ith row of X was initially the zero vector, then it will continue being the same for all times and also for Z; and, moreover, $\operatorname{diag}(Z^2)_{ii} = 0$ and there exists a zero eigenvalue with its associated eigenvector having zero entries in all the positions of the entries where $\operatorname{diag}(Z^2)$ are positive. Then, it immediately follows from (27) that if $\lambda_i(0) = 0$ due to $Z(0)$ having a row being the zero vector $\mathbb{0}_{1\times n}$, then $\dot{\lambda}_i = 0$.

Now, let $\mathcal{N}$ be the set of indices i such that $\operatorname{diag}(Z^2)_{ii} > 0$. Thus, for any $i \in \mathcal{N}$, if λ_i crosses the real axis at time t^*, i.e., $\lambda(t^*) = 0$, then

$$\dot{\lambda}_i(t^*) = -(v_i(t^*))^\top \operatorname{diag}\left(Z^2(t^*)\right) v_i(t^*) < 0. \tag{28}$$

Therefore, if $\lambda_i(t_0) \leq 0$ for some $t_0 \geq 0$, then $\lambda_i(t) \leq 0$ for all $t \geq t_0$. This finishes the proof for (ii).

Notice that since $\operatorname{trace}(Z(t)) = 0$ and $Z(t) = Z(t)^\top \neq \mathbb{0}_{n\times n}$, then $Z(t)$ has at least one positive eigenvalue. Then, equation (28) implies that

$$\Lambda := \{Z \in \mathbb{S}^{n\times n}_{\text{zero-diag,symm}} \mid Z \text{ has only one positive eigenvalue}\}$$

is forward invariant and, in particular, the limit $Z^* = \lim_{t\to\infty} Z(t)$ (existing in view of statement (i)) belongs to Λ. Since Z^* is a critical point of $\mathcal{D}$ (or, in view of Theorem III.4, the equilibrium of (9)), it has the structure described by Corollary IV.7.

By continuity of the flow $Z(t)$, there is a finite time τ such that $G(Z(t))$ has the same sign structure as $G(Z^*)$ for all $t \geq \tau$. This finishes the proof for (iii).

Now we prove the last two statements of the theorem. Knowing the convergence result from (iii), Lemma A.1 tells us that introducing the term η as in the transcribed system (6a) to the projected pure-influence model has the simple effect of altering the convergence rate properties for $Z(t)$. Therefore, there always exist a finite time $\tau^* \geq 0$ such that, for any $t \geq \tau^*$, $W(t)$ satisfies the sign properties of statement (iii) regarding structural balance. Moreover, the fact that $X(t) = \eta(t)W(t)$ and $\eta(t) \geq 0$ by construction, immediately implies (iv). Now, let $g(t) := -\mathcal{D}(W(t))$, and notice that $g(t)$ is a strictly positive continuous function for all (well-defined) $t \geq \tau^*$. Now, from equation (6b), we have the system $\dot{\eta}(t) = g(t)\eta^2(t)$, with solution $\eta(t) = \frac{\eta(\tau)}{1-\eta(\tau)\int_\tau^t g(s)ds}$ for $t \geq \tau$. Then, since $\int_\tau^t g(s)ds$ is a monotonic strictly increasing function on $t \geq \tau$, we have that $\eta(t) \to +\infty$ as $t \to t^*$, where $t^* > \tau^*$ is some finite time such that $\int_\tau^{t^*} g(s)ds = \frac{1}{\eta(\tau)}$ (note that $t^* > \tau^*$ holds from the relationship $W(t) = Z(\int_0^t \eta(s)ds)$). Then, we conclude that the solution $\eta(t)$ and the entries of $X(t)$ diverge in some finite time t^*, which proves (v). $\square$

Corollary V.3. *Consider the same conditions as in Theorem V.2, i.e., the projected pure-influence model with initial condition $Z(0) \in \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ having one positive eigenvalue. If $\mathcal{D}(Z(0)) < -\frac{n-3}{\sqrt{(n-1)(n-2)}}$, then $G(Z(t))$ eventually reaches structural balance.*

The previous theorem immediately implies that the set of irreducible equilibria with a single positive eigenvalue is (locally) asymptotically stable. We present further results on the stability of equilibria.

Lemma V.4 (Further results on stability of the equilibria)**.** *Consider a symmetric equilibrium point Z^* for the projected pure-influence model* (9)*. Without loss of generality, assume that Z^* has no row equal to the zero vector*[1]*. If $\mathcal{D}(Z^*) \geq 0$, then Z^* is an unstable equilibrium point and does not correspond to structural balance.*

Proof. Write the analytic projected influence system (9) as $\dot{Z} = f(Z) := Z^2 - \text{diag}(Z^2) + \mathcal{D}(Z)Z$, thereby defining $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, and compute

$$\frac{\partial f_{ij}(Z)}{\partial z_{ij}} = \mathcal{D}(Z) + \frac{\partial \mathcal{D}(Z)}{\partial z_{ij}} z_{ij},$$
$$\frac{\partial \mathcal{D}(Z^*)}{\partial z_{ij}} = -3 \sum\nolimits_{\substack{k=1 \\ k\neq i,j}}^{n} z^*_{ik} z^*_{kj}.$$

Now, the Jacobian of f, denoted by Df, is a $(n^2-n)\times(n^2-n)$ matrix (since we do not consider self-appraisals). Let $Df(Z^*)$ be the Jacobian evaluated at Z^* and let $\{\lambda_i\}_{i=1}^{n^2-n}$ be the set of its eigenvalues. Then, we compute

$$\begin{aligned}\sum_{i=1}^{n^2-n} \lambda_i &= \text{trace}(Df(Z^*)) = \sum_{i=1}^{n} \sum\nolimits_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial f_{ij}(Z^*)}{\partial z_{ij}} \\ &= (n^2-n)\mathcal{D}(Z^*) + 3\mathcal{D}(Z^*) = (n^2-n+3)\mathcal{D}(Z^*).\end{aligned}$$

Since $n^2 - n + 3 > 0$ for $n \geq 3$, we draw the following conclusions for $\mathcal{D}(Z^*) \geq 0$: (i) $Df(Z^*)$ contains at least one positive eigenvalue and so the equilibrium point Z^* is unstable; (ii) at least one triad in $G(Z^*)$ is unbalanced and so Z^* does not correspond to structural balance. $\square$

[1] If Z^* had a row equal to the zero vector, then, in the lemma statement, we would replace n by $n_1 < n$, where n_1 is the number of rows of Z^* that are not equal to the zero vector.

VI. Simulation results and conjectures

The generic convergence of trajectories to the minima of $\mathcal{D}$ (or, equivalently, the convergence from almost all initial conditions) is an open problem. However, we present strong numerical evidence that support such claim. We first remark that, from the proof of Theorem III.3, the projected pure-influence model (9) can be generalized over any asymmetric matrix in $\mathbb{S}^{n\times n}_{\text{zero-diag}}$ by replacing $\mathcal{D}(Z)$ by $-\text{trace}(Z^\top Z^2)$ and this is the model we will refer throughout this section.

A *generic asymmetric initial condition* $X(0)$ for the pure-influence model (2) is a matrix that is generated with each entry independently sampled from a uniform distribution with support $[-100, 100]$, and its diagonal entries set to zero. A *generic symmetric initial condition* is similarly constructed by only sampling the upper triangular entries of the matrix. For the projected pure-influence model, we say $Z(0) = \frac{X(0)}{\|X(0)\|_F}$ is a (non-)symmetric generic initial condition depending on how $X(0)$ was generated. We immediately see from the proof of Theorem V.2, that $Z(t)$ converges to social balance if and only if $X(t)$ converges to social balance. Indeed, given that $X(t)$ diverges at some finite time $\bar{t}$, we have $Z(\infty) = \frac{X(\bar{t}^-)}{\|X(\bar{t}^-)\|_F}$.

For a fixed network size n, we use a Monte Carlo method [26] to estimate the probability p of the event "under a generic asymmetric initial condition $Z(0)$, $Z(t)$ converges to structural balance in finite time". We estimate p by performing N independent simulations (i.e., each simulation generates a new independent initial condition) and obtaining the proportion $\hat{p}_N$, also known as the empirical probability, of times that the simulation indeed had $Z(t)$ converging to structural balance in finite time. For any accuracy $1 - \epsilon \in (0, 1)$ and confidence level $1 - \eta \in (0, 1)$ we have that $|\hat{p}_N - p| < \epsilon$ with probability greater than $1 - \eta$ if the Chernoff bound $N \geq \frac{1}{2\epsilon^2} \log \frac{2}{\eta}$ is satisfied. For $\epsilon = \eta = 0.01$, the bound is satisfied by $N = 27000$. We performed the $N = 27000$ independent simulations with $n \in \{5, 6\}$, and found that $\hat{p}_N = 1$. Our observations let us conclude that *for generic asymmetric initial condition $Z(0)$ and $n \in \{5, 6\}$, with 99% confidence level, there is at least 0.99 probability that $Z(t)$ converges to structural balance in finite time.*

Similarly, we performed the same Monte Carlo analysis for generic symmetric initial conditions with $n \in \{3, 5, 6, 15\}$, and found for that $\hat{p}_N = 1$ for all n. Therefore, we conclude that *for any symmetric generic initial condition $Z(0)$ and $n \in \{3, 5, 6, 15\}$, with 99% confidence level, there is at least 0.99* *probability that $Z(t)$ converges to structural balance in finite time.*

We report three more observations and then state a resulting conjecture. First, remarkably, we found that all of our simulations (for any type of random initial condition) that converged to structural balance in finite time, did it by converging to an equilibrium point having only one positive eigenvalue inside the set of scale-symmetric matrices, which is a superset of the set of symmetric matrices (see Appendix B). Second, we did not perform experiments for larger sizes of n due to computational constraints. Third, unfortunately, for $n = 3$, we did find randomly-generated asymmetric initial conditions whose numerically-computed solutions do not converge to structural balance.

Conjecture 1 (Convergence from generic initial conditions)**.** *Consider the pure-influence model* (2) *with some initial condition $X(0)$, and the projected pure-influence model* (9) *with initial condition $Z(0) = \frac{X(0)}{\|X(0)\|_F}$. Then,*

(i) under generic asymmetric initial conditions, $\lim_{t\to+\infty} Z(t) = Z^*$ for a sufficiently large n,

(ii) under generic symmetric initial conditions, $\lim_{t\to+\infty} Z(t) = Z^*$ for any n,

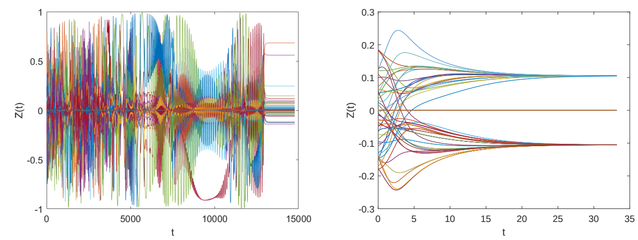
where Z^* is scale-symmetric (and particularly symmetric for (ii)) corresponding to structural balance. Then, $Z(t)$ reaches structural balance in finite time. Moreover, $X(t)$ reaches structural balance in finite time with same sign structure as Z^*, and also diverges in finite time.

Similarly, we performed the same simulation analysis for the Kułakowski et al. model (4), which converges to structural balance if and only if the projected Kułakowski model (10) does. To generate a generic initial condition for this system, we generated an $n \times n$ matrix with each entry independently sampled from a uniform distribution with support $[-100, 100]$, and then divide it by its Frobenius norm. We performed $N = 27000$ independent simulations with $n \in \{5, 6\}$, and found that *for generic initial condition $Z(0)$ and $n = 5$, only 16.94% converged to structural balance, and for $n = 6$, only 11.50% converged to structural balance.*

Also, for $n = 3$, not all simulations converged to structural balance. We remark that not all of the networks for which the system converged and did not satisfy structural balance were complete, some of them were networks with only self-loops, e.g., Figure 5(a). Similarly, we performed the same Monte Carlo analysis for symmetric initial conditions with $n \in \{3, 5, 6, 15\}$. Our results show that *for symmetric generic initial condition, $Z(0)$ did not always converge to structural balance for $n = 3$, but, for $n \in \{5, 6, 15\}$, with 99% confidence level, there is at least 0.99 probability that $Z(t)$ converges to structural balance in finite time.*

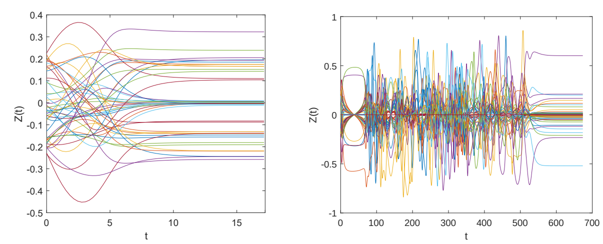
These Monte Carlo results are expected, since it has been formally proved that the Kułakowski et al. model converges to structural balance only under generic symmetric initial conditions as $n \to \infty$ [22] and negative results for asymmetric conditions are given by [27].

See Figure 3 for a comparison of trajectories of the pure-influence model in both generic and symmetric generic initial conditions. Figure 4 shows a comparison between our projected pure-influence model, which does not consider self-appraisals, and the projected influence model, which considers self-appraisals. Note how not considering self-appraisals drastically changes the convergence time as well as the dynamic behavior of the interpersonal appraisals.



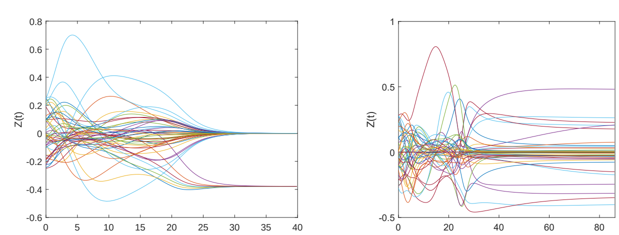
(a) Projected pure-influence model (9) with generic asymmetric initial condition (b) Projected pure-influence model (9) with generic symmetric initial condition

Fig. 3. Convergence to structural balance for a network of size $n = 10$. We plot the evolution of all the entries of $Z(t)$.



(a) Projected influence model (10) with generic asymmetric initial condition (b) Projected pure-influence model (9) with generic asymmetric initial condition

Fig. 4. Convergence comparison for a network of size $n = 7$ (a) with and (b) without the consideration of self-appraisals. We first generated an $n \times n$ random matrix W with each entry independently sampled from a uniform distribution with support $[-100, 100]$. Then, for (a), we normalize this matrix to have unit Frobenius norm and used it as the initial condition. For (b), we set the diagonal entries of W to zero and then normalize it to have unit Frobenius norm and use it as the initial condition. In this example, (a) did not converged to structural balance, whereas (b) did. We plot the evolution of all the entries of the appraisal matrix.



(a) Projected influence model (10) with generic asymmetric initial condition (b) Projected pure-influence model (9) with generic asymmetric initial condition

Fig. 5. Convergence comparison for a network of size $n = 7$ (a) with and (b) without the consideration of self-appraisals. The setting is the same one as in Figure 4, but with a different random initial condition. (a) converged to a network with only diagonal negative entries (all interpersonal appraisals go to zero), whereas (b) converged to structural balance.

VII. Conclusion

We propose two new dynamic structural balance models that incorporates more psychologically plausible assumptions than previous models in the literature, based on a modification by a model proposed by Kułakowski et al. We have established important convergence properties for these models and also that, most importantly, they correspond to gradient systems over an energy function that characterizes the violations of Heider's axioms for the symmetric case. We also expanded our results to a set of asymmetric matrices called scale-symmetric. Numerical results illustrates that, under generic initial conditions, our models converges to structural balance (for sufficiently large n) and thus have better convergence properties than the previous model by Kułakowski et al.

As future work, we propose to further study the general case of asymmetric (and non-scale-symmetric) equilibria and the convergence properties of our models under arbitrary initial conditions. For example, numerical simulations of the projected pure-influence model from generic (asymmetric) initial conditions illustrate how this system features transient chaos before converging towards an equilibrium. Future work will focus on models with a more sociologically justified transient behavior. Finally, one could study the removal of the self-appraisals in other dynamical structural balance models, like the homophily-based model by Traag el al. [27].

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Appendix A
Supporting Results and Proofs

Lemma A.1. *Let $x(t)$ be the solution to $\dot{x} = f(x)$ from initial condition $x(0)$, with f being a continuously differentiable vector field. Let η be a positive continuous scalar function. Then, $z(t)$ is the solution to $\dot{z} = \eta(t)f(z)$ with initial condition $z(0) = x(0)$ if and only if $z(t) = x(\int_0^t \eta(s)ds)$.*

Proof. Consider the time transformation $\bar{t}(t) = \int_0^t \eta(s)ds$, which is well-defined since it is continuous and monotonically increasing on t (recall that $\eta(s) > 0$ for $s \in [0, t]$), with $\bar{t} = 0$ if and only if $t = 0$. Now, from the chain rule, it follows that

$$\frac{dz}{dt} = \frac{dx(\bar{t})}{d\bar{t}}\frac{d\bar{t}}{dt} = f(z)\eta(t), \quad z(0) = x(0).$$

This finishes proof of the "if" part. The "only if" part follows from the uniqueness theorem. □

Proof of Lemma IV.1. First, to prove that the set $\mathrm{nSt}(n,k)$, $k \le n$ is a submanifold of the compact Stiefel manifold, define the smooth map $\Phi : \mathrm{St}(n,k) \to \mathbb{R}^n$ by $X \mapsto (\|X_{i*}\|_2^2, \ldots, \|X_{n*}\|_2^2)^\top$, where X_{i*} is the ith row of X. Then, we have that $\mathrm{nSt}(n,k) = \Phi^{-1}((k/n,\ldots,k/n)^\top)$ and it is easy to prove the mapping Φ has constant rank n. Thus, we use the Constant-Rank Level Set Theorem [18] to conclude our claim. The properties of compactness and analyticity are immediate from the definition of the set $\mathrm{nSt}(n,k)$, $k \le n$.

Now, notice that conditions ((i)) and ((ii)) from Definition IV.1 impose, in total, $\frac{k(k+1)}{2} + n$ constraints on kn independent variables, however, these constraints are linearly dependent: one of them can be removed (for instance, if one requires condition (i) from Definition IV.1, then suffices to constrain only sums of $n-1$ rows, whereas the remaining sum automatically equals k/n)). Whenever $k \le n$ and $n \ge 3$, one has $\frac{k(k+1)}{2} + n - 1 < kn$, which implies that the set $\mathrm{nSt}(n,k)$ has the dimension $(k-1)n + 1 - k(k+1)/2$.

Statements (i) and (ii) are immediate. Now regarding (iii), it is obvious that each row has norm $\sqrt{k/n}$ if and only if V can be written as (16). Notice now the columns are unit vectors if and only if $\sum_{m=1}^n \cos^2\alpha_i = n/2 = \sum_{m=1}^n \sin^2\alpha_i$, which in turn holds if and only if $\sum_m \cos 2\alpha_m = 2\sum_m \cos^2\alpha_m - n = 0$. Similarly, the columns are orthogonal if and only if $\sum_{m=1}^n \cos\alpha_i \sin\alpha_i = 0 = \frac{1}{2}\sum_m \sin 2\alpha_m$. These two constraints are equivalent to (17). □

Proof of Lemma IV.2. The case where $\alpha = \beta = 0$ is trivial: $Z = 0$ and it obviously can be decomposed as in (18) with $p = q = 0$. Notice that every eigenvalue of $Z = Z^\top$ corresponds to the eigenvalue $\lambda^2 - 2\alpha\lambda$ of $Z^2 - 2\alpha Z$, and hence $\lambda^2 - 2\alpha\lambda - \beta = 0$. Therefore, $\alpha^2 + \beta \ge 0$ (otherwise, eigenvalues of Z would be complex). Furthermore, $\alpha^2 + \beta \ne 0$ (otherwise, $\lambda = \alpha$ would be the only eigenvalue of Z of multiplicity n, and one would have $\mathrm{trace}(Z) = \alpha n$, entailing that $\alpha = \beta = 0$). Denoting $\Delta = \sqrt{\alpha^2 + \beta}$, the matrix Z has two different eigenvalues $\alpha + \Delta$ and $\alpha - \Delta$, denote their multiplicities by k and $n-k$. Then $(\alpha+\Delta)k + (\alpha-\Delta)(n-k) = 0$. Denoting $q = \Delta - \alpha$ and $p = 2\Delta > 0$, one has $(p-q)k - q(n-k) = 0$ or, equivalently, $pk = qn$ thus, $q > 0$.

Consider the orthonormal eigenvectors $v_1,\ldots,v_k$, corresponding to the eigenvalue $p - q = \alpha + \Delta$ and orthonormal eigenvectors $w_1,\ldots,w_{n-k}$, corresponding to $-q = \alpha - \Delta$. The sequence $v_1,\ldots,v_k,w_1,\ldots,w_{n-k}$ constitutes an orthonormal basis of eigenvectors for the operator Z. Stacking the columns v_i and w_i, one obtains $n \times k$ and $n \times (n-k)$ matrices $V = (v_1,\ldots,v_k)$, $W = (w_1,\ldots,w_{n-k})$. The matrix $[V,W]$ is orthonormal and diagonalizes Z, that is, $Z[V,W] = [V,W]\begin{bmatrix}(p-q)I_{FIXHERE} & 0\\ 0 & -qI_{FIXHERE}\end{bmatrix}$ and thus $Z = (p-q)VV^\top - qWW^\top$. Since $VV^\top + WW^\top = I_n$, Z is decomposed as (18). It remains to notice that $V^\top V = I_k$ by definition of the orthonormal basis and $\mathrm{diag}(VV^\top) = (q/p)I_n = (k/n)I_n$ since, by (18), $\mathrm{diag}(Z) = \mathbb{0}_{n\times n}$. To finish the proof, notice that $p - 2q = 2\alpha$ and $\beta = \Delta^2 - \alpha^2 = (\Delta-\alpha)(\Delta+\alpha) = q(p-q)$. □

Proof of Corollary IV.3. Let $f(z) = z^2 - 2\alpha z$, $z \in \mathbb{C}$. It suffices to show that, if $f(Z) = \mathrm{diag}(\beta_1 I_{n_1},\ldots,\beta_s I_{n_s})$, then $Z = \mathrm{diag}(Z_1,\ldots,Z_s)$, where $f(Z_i) = \beta_i I_{n_i}$. This statement will be proved for any analytic function $f(z)$. It is well known that the spectrum of $f(Z)$ consists of all points $f(\lambda)$, where λ is an eigenvalue of Z. Consider the set of eigenvalues of Z that belong to $f^{-1}(\beta_i)$ and let $\mathcal{X}_i$ be the sum of corresponding eigenspaces. Then $\mathcal{X}_i$ is invariant under the operator Z, and $\mathbb{R}^n = \oplus_{i=1}^s \mathcal{X}_i$ (the sum is orthogonal). Also, $f(Z)x = \beta_i x$ for any $x \in \mathcal{X}_i$. For any basis vector $e_r = (0,\ldots,1,\ldots,0)^\top$ consider the decomposition $e_r = \oplus_{i=1}^s e_r^i$, $e_r^i \in \mathcal{X}_i$. Then $Ze_r = \oplus_{i=1}^s Ze_r^i$, $Ze_r^i \in \mathcal{X}_i$ and $f(Z)e_r = \oplus_{i=1}^s f(Z)e_r^i = \oplus_{i=1}^s \beta_i e_r^i$. Suppose that $1 \le r \le n_1$. Then $f(Z)e_r = \beta_1 e_r$. Since $\beta_1,\ldots,\beta_s$ are pairwise different, we have $e_r = e_r^1$ and $e_r^2 = \ldots = e_r^s = 0$. Similarly, for $n_1 + n_2 + \ldots + n_{j-1} + 1 \le r \le n_1 + n_2 + \ldots + n_{j-1} + n_j$ one has $e_r = e_r^j$ $(j = 2,\ldots,s)$.

In other words, each $\mathcal{X}_i$ contains n_i basis vectors e_r, where $n_1 + n_2 + \ldots + n_{i-1} + 1 \le r \le n_1 + n_2 + \ldots + n_{i-1} + n_i$ and thus $\dim \mathcal{X}_i \ge n_i$. Recalling that $n_1 + \ldots + n_s = n$, one shows that $\dim \mathcal{X}_i = n_i \,\forall i$ and thus $\mathcal{X}_i$ is spanned by the corresponding basis vectors. Since $\mathcal{X}_i$ is invariant under Z, $Z = \mathrm{diag}(Z_1,\ldots,Z_s)$, where the block Z_i has dimension $n_i \times n_i$. Obviously, $f(Z_i) = \beta_i I_{n_i}$. The statement of Corollary is now immediate from Lemma IV.2. □

Proof of Theorem IV.5. We prove the necessity first. Denoting $2\alpha = -\mathcal{D}(Z)$. By assumption, $Z^2 - 2\alpha Z$ is diagonal. Statements (i) and (ii) follow from Corollary IV.3, entailing also that p_i, q_i can be represented as (23) with some β_i. Since $Z_i^2 = 2\alpha Z_i + \beta_i I_{n_i}$ and $\mathrm{diag}(Z_i) = \mathbb{0}_{n_i\times n_i}$, one has $\mathrm{trace}\, Z_i^2 = \beta_i n_i$, therefore

$$\sum\nolimits_{i=1}^s \beta_i n_i = \mathrm{trace}(Z^2) = 1. \tag{29}$$

Recall also that for each i one has $p_i k_i = q_i n_i$ or, equivalently,

$$\frac{2k_i}{n_i} = \frac{\sqrt{\alpha^2+\beta_i} - \alpha}{\sqrt{\alpha^2+\beta_i}} = 1 - \frac{\alpha}{\sqrt{\alpha^2+\beta_i}} \quad \forall i : p_i, q_i \ne 0.$$

(if $\alpha = 0$, one always has $p_i, q_i \ne 0$, otherwise it is possible that $\beta_i = 0$ and then $Z_i = 0$). This implies condition 3 ($\varepsilon = \mathrm{sign}\,\alpha$) and allows to determine α, β_i. In the case where $\varepsilon \ne 0$ notice that $n_i - 2k_i \ne 0$ for any i such that $Z_i \ne 0$. Thus

$$\frac{\beta_i + \alpha^2}{\alpha^2} = \frac{n_i^2}{(n_i - 2k_i)^2} \iff \beta_i = \alpha^2\frac{4n_ik_i - 4k_i^2}{(n_i-2k_i)^2}.$$

In view of (29), one obtains that

$$\alpha = \varepsilon\left(\sum_{i:Z_i\ne\mathbb{0}_{n_i\times n_i}} \frac{4k_in_i(n_i-k_i)}{(n_i-2k_i)^2}\right)^{-1/2},$$

which entails (24). In the case of $\alpha = 0$, one has $p_i = 2\sqrt{\beta_i}$, $q_i = \sqrt{\beta_i}$ for any i, and (29) implies that $\sum_i q_i^2 n_i = 1$. This finishes the proof of statement (iv).

The proof of sufficiency is similar. For any i such that $Z_i \ne 0$, the coefficients p_i, q_i have the form (23) (if $\varepsilon \ne 0$, this is implied by (iv)a, otherwise we choose $\alpha = 0$ and $\beta_i = q_i^2 = p_i^2/4$). Therefore, we have $Z_i^2 - 2\alpha Z_i = \beta_i Z_i$ and, in particular, $Z^2 - 2\alpha Z$ is diagonal. A straightforward computation shows that $p_i k_i = q_i n_i$ and thus $\mathrm{diag}(Z_i) = \mathbb{0}_{n_i\times n_i}\,\forall i$, in particular, $\operatorname{diag}(Z) = \mathbb{0}_{n\times n}$. Also, $\operatorname{diag}(Z_i^2) = \beta_i I_{n_i}$, and statement (iv) now implies that $\operatorname{trace} Z^2 = 1$. It remains to notice that $Z_i^3 = 2\alpha Z_i^2 + \beta_i Z_i$, and hence $\operatorname{trace}(Z_i^3) = 2\alpha\beta_i n_i$. Hence, $\mathcal{D}(Z) = -\operatorname{trace}(Z^3) = -2\alpha$, $Z^2 + \mathcal{D}(Z)Z$ is a diagonal matrix, and Z is an equilibrium (14). □

Appendix B
Scale-symmetric matrices

We now generalize our results for symmetric appraisal networks to a class of asymmetric matrices. We define the sets of *scale-symmetric* matrices

$$\begin{aligned}
\mathbb{R}^{n\times n}_{\text{zero-diag,dss}} &= \{A \in \mathbb{R}^{n\times n}_{\text{zero-diag}} \mid \text{there exists } \gamma \succ \mathbb{0}_n \text{ such that} \\
&\qquad A\operatorname{diag}(\gamma) = (A\operatorname{diag}(\gamma))^\top\},\\
\mathbb{S}^{n\times n}_{\text{zero-diag,dss}} &= \mathbb{S}^{n\times n}_{\text{zero-diag}} \cap \mathbb{R}^{n\times n}_{\text{zero-diag,dss}}.
\end{aligned}$$

Note that $\mathbb{S}^{n\times n}_{\text{zero-diag,dss}} \supset \mathbb{S}^{n\times n}_{\text{zero-diag,symm}}$ and

$$\mathbb{S}^{n\times n}_{\text{zero-diag,dss}} = \bigcup_{\gamma\succ\mathbb{0}_n} \mathbb{S}^{n\times n}_{\text{zero-diag,dss}}(\gamma),$$

$$\mathbb{S}^{n\times n}_{\text{zero-diag,dss}}(\gamma) = \{A \in \mathbb{S}^{n\times n}_{\text{zero-diag}} \mid A\operatorname{diag}(\gamma) = (A\operatorname{diag}(\gamma))^\top\}.$$

.

Lemma B.1. *Consider any $\gamma \succ \mathbb{0}_n$ and some matrix $A \in \mathbb{R}^{n\times n}$ such that $A\operatorname{diag}(\gamma) = \operatorname{diag}(\gamma)A^\top$. Then,*

(i) A has real eigenvalues and it is diagonalizable,

(ii) $\operatorname{trace}(A^2) = 0$ if and only if $A = 0$.

Proof. Since $A\operatorname{diag}(\gamma)$ is symmetric, then $A' = \operatorname{diag}(\gamma)^{-1/2}A\operatorname{diag}(\gamma)^{1/2}$ is also symmetric and thus has real eigenvalues and its eigenvectors form an orthogonal basis. Now, let (λ, v) be an eigenpair for A'. Then, by defining $u = \operatorname{diag}(\gamma)^{1/2}v$, we observe that $Au = \lambda u$, and so $(\lambda, \operatorname{diag}(\gamma)v)$ is an eigenpair for A. Hence the eigenvectors of A form a basis, and thus A is diagonizable. This proves (i).

Observe that $A = \operatorname{diag}(\gamma)A^\top\operatorname{diag}(\gamma)^{-1}$. Then, $\operatorname{trace}(A^2) = \operatorname{trace}(A\operatorname{diag}(\gamma)A^\top\operatorname{diag}(\gamma)^{-1})$. From simple algebraic operations, it can be found that $\operatorname{trace}(A^2) = \sum_{i=1}^n\sum_{j=1}^n \frac{\gamma_j}{\gamma_i}a_{ij}^2$. Since $\frac{\gamma_i}{\gamma_j} > 0$, $\operatorname{trace}(A^2) = 0$ if and only if $A = 0$. This proves (ii). □

In view of Lemma B.1, a matrix A is scale-symmetric if and only if $A = D^{-1}A_sD$, where $D > 0$ is a positive diagonal matrix (in Lemma B.1, $D = \operatorname{diag}(\gamma^{-1/2})$ for some $\gamma \succ \mathbb{0}_n$) and A_s a symmetric matrix.

Recall the invariance property of the pure-influence model (2): if $X(0) = X(0)^\top$, then $X(t) = X(t)^\top$ for all $t > 0$. We are now ready to provide a more general version of this property: If $D > 0$ is a diagonal matrix and $X(t)$ is a solution, then $DX(t)D^{-1}$ is also a solution. For this reason, if $X(0) = DX_s(0)D^{-1}$ is a scale-symmetric matrix with some $X_s(0) = X_s(0)^\top$, then the solution $X(t) = DX_s(t)D^{-1}$ is scale-symmetric. A similar result holds for the projected pure-influence model (9). Indeed, all of the theoretical results obtained in this paper for symmetric appraisal matrices, can be generalized to scale-symmetric appraisal matrices. For example, if $X(0) \in \mathbb{R}^{n\times n}_{\text{zero-diag,dss}}$ ($Z(0) \in \mathbb{S}^{n\times n}_{\text{zero-diag,dss}}$) then $t \mapsto \mathcal{D}(X(t))$ ($t \mapsto \mathcal{D}(Z(t))$) is monotonically nondecreasing in $\mathbb{R}^{n\times n}_{\text{zero-diag,dss}}$ ($\mathbb{S}^{n\times n}_{\text{zero-diag,dss}}$).

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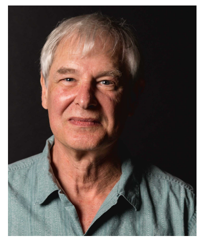

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