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Abstract. The paper is devoted to asymptotic behavior of synchronization systems, i.e. Lur'e–type systems with periodic nonlinearities and infinite sets of equilibrum. This class of systems can not be efficiently investigated by standard Lyapunov functions. That is why for synchronization systems several new methods have been elaborated in the framework of Lyapunov direct method. Two of them: the method of periodic Lyapunov functions and the nonlocal reduction method, proved to be rather efficient. In this paper we combine these two methods and the Kalman-Yakubovich-Popov lemma to obtain new frequency–algebraic criteria ensuring Lagrange stability and the convergence of solutions.

1. Introduction

In this paper we give further development for stability investigation of Lurie–type systems of indirect control with periodic nonlinearities. This class of systems involves damped pendulums, electric motors, power generators, vibrational units, synchronization circuits (phase and frequency locked loops). It also involves certain biological systems. Such systems are often called synchronization systems. They are featured by denumerable sets of equlibria (both Lyapunov stable and unstable ones). The desired asymptotic behavior for synchronization systems is the convergence of every solution to a certain equlibrium. This type of stability is often mentioned as gradient–like behavior.

The qualitative investigation of synchronization systems started almost a hundred years ago in paper [1] where the second order equation

$$\ddot{\sigma} + a\dot{\sigma} + \varphi(\sigma) = 0 \quad (a > 0) \tag{1}$$

with a periodic function $\varphi(\sigma)$ was considered. In case $\varphi(\sigma) = \sin\sigma - \beta$ equation (1) is a well-known equation of the viscously damped pendulum. It may also serve as mathematical model for synchronous machine. The paper [1] was succeeded by a series of published works [2–6] It turned out that there exists a bifurcation value a_{cr} such that for $a > a_{cr}$ every solution of (1) converges. Later in [7] the stability regions for a second order synchronization system were established by qualitative and numerical methods.

Of course, the Lyapunov second method was used for synchronization systems of low order but only for Lyapunov stability of isolated equlibria. It was clear that standard Lyapunov functions destined for systems with single equlibrium are of no good for gradient–like ones.

For synchronization systems special tools were elaborated within the framework of Lyapunov direct method. The most efficient proved to be the method of periodic Lyapunov functions proposed in [8] for synchronization systems of the third order. It used the Lurie-Postnikov function modified by special technique (Bakaev–Guzh procedure). In [9] this method was generalized and combined with Kalman-Yakubovich-Popov (KYP) lemma, so that efficiently verified stability conditions could be obtained.

In parallel to the method of periodic Lyapunov functions, in [9] another method was proposed. It exploited for high order system the information about a low-order synchronization system with known asymptotic properties. It was called the method of nonlocal reduction. Trajectories of low-order system (reduction system) were involved into Lyapunov function.

In this paper the ideas of these two methods and the KYP-lemma are combined. Investigation of Lagrange stability started in [10] is supplemented with investigation of gradient–like behavior. As a result, new criteria of the convergence of solutions for Lurie-type synchronization system are obtained.

2. The problem setup

Consider the system

$$\begin{aligned} \tfrac{dz(t)}{dt} &= Az(t) + b\varphi(\sigma(t)), \\ \tfrac{d\sigma(t)}{dt} &= c^* z(t) + \rho\varphi(\sigma(t)). \end{aligned} \tag{2}$$

Here $A \in \mathbb{R}^{m\times m}$, $b, c \in \mathbb{R}^m$, $\rho \in \mathbb{R}$, $z : \mathbb{R}_+ \to \mathbb{R}^m$,
$\sigma : \mathbb{R}_+ \to \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, the symbol (*) is used for the Hermitian conjugation.

We suppose that the pair (A, b) is controllable, the pair (A, c) is observable and matrix A is Hurwitz. The function φ is C^1-smooth, Δ–periodic and

$$\varphi^2(\sigma) + \varphi'(\sigma)^2 \neq 0. \tag{3}$$

It has two zeros: $0 \le \sigma_1 < \sigma_2 < \Delta$ with

$$\varphi'(\sigma_1) > 0,\ \varphi'(\sigma_2) < 0. \tag{4}$$

We suppose for the definiteness that

$$\int_0^\Delta \varphi(\sigma)d\sigma \le 0. \tag{5}$$

Notice that if $(z(t), \sigma(t))^T$ is a solution of (2) then $(z(t), \sigma(t)+\Delta k)^T$ $(k \in Z)$ is also a solution of (2). So (2) has a cylindric phase space.

System (2) has a denumerable set of Lyapunov stable and Lyapunov unstable equlibria $\Lambda = \{(0, \sigma_j + \Delta k)^T : j = 1, 2, k \in Z\}$. Its asymptotic behavior is described by two types of stability: Lagrange stability (every solution is bounded) and gradient–like behavior (every solution converges). Notice that gradient-like behavior does not guarantee Lyapunov stability of a specific equlibrium.

Lagrange stability is the basic property of synchronization systems. If a Lagrange stable system is monostable (every bounded solution converges), it is gradient-like.

3. Frequency–algebraic criteria for Lagrange stability of synchronization systems

In this section we demonstrate two various frequency-algebraic criteria of Lagrange stability based on different modifications of Bakaev–Guzh procedure.

Consider the equation (1). It is equivalent to the system

$$\begin{aligned} &\dot{z} = -az - \varphi(\sigma) \quad (a > 0), \\ &\dot{\sigma} = z. \end{aligned} \tag{6}$$

which has been exhaustively investigated (see for example [11, pp. 185-201] and the bibliography there). System (6) has Lyapunov stable equlibria $(0, \sigma_1 + \Delta k)$ and saddle–points $(0, \sigma_2 + \Delta k)$. Here $k \in Z$.

Proposition 1 *[11, pp. 185-201]. For any $\varphi(\sigma)$ there exists a bifurcational value a_{cr} such that for $a > a_{cr}$ every solution of (6) converges to some equilibrium and for $a \le a_{cr}$ the system (6) has both converging solutions and solutions with $z(t) = \dot{\sigma}(t) \ge \varepsilon > 0$.*

The phase portrait of (6) in case $a > a_{cr}$ ($\sigma_1 = 0, (0,0)^T$ is a stable focus) is shown in Fig. 1.

System (6) is associated with a first-order equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma} + aF(\sigma) + \varphi(\sigma) = 0 \quad (F(\sigma) = \dot{\sigma} = z). \tag{7}$$

Two separatrices z_1 and z_2 "going into" the saddle point $(0, \sigma_2)$ (see Fig. 1) "merge" and form a solution $F_0(\sigma)$ of (7). Consider solutions

$$F_k(\sigma) = F_0(\sigma + \Delta k) \quad (k \in Z). \tag{8}$$

Proposition 2 *[9, 11, pp. 185-201]. If $a > a_{cr}$, then the solutions $F_k(\sigma)$ have the following properties:*

$$\begin{aligned} &P1)\; F_k(\sigma_2 + \Delta k) = 0; \\ &P2)\; F_k(\sigma) \ne 0 \quad \mathit{for} \quad \sigma \ne \sigma_2 + \Delta k; \\ &P3)\; F_k(\sigma) \to \pm\infty \quad \mathit{as} \quad \sigma \to \mp\infty. \end{aligned} \tag{9}$$

We are going to use the solutions $F_k(\sigma)$ in Lyapunov-type functions.

Our argument combines new Lyapunov-type functions and KYP–lemma. So we need the transfer function of (2) from φ to $-\dot{\sigma}$:

$$K(p) = -\rho + c^*(A - pI_m)^{-1}b \quad (p \in \mathbb{C}), \tag{10}$$

where I_m is an $m \times m$ - unit matrix.

Introduce the constants

$$\mu_1 \triangleq \inf_{\sigma\in[0,\Delta)} \varphi'(\sigma),\; \mu_2 \triangleq \sup_{\sigma\in[0,\Delta)} \varphi'(\sigma) \quad (\mu_1, \mu_2 < 0), \tag{11}$$

$$\nu_0 \triangleq \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta |\varphi(\sigma)|\, d\sigma}. \tag{12}$$

$$\lambda_0 \triangleq \min_{i=1,\dots,m} |Re\, \lambda_i| \quad (i = 1, \dots, m), \tag{13}$$

where λ_i is an eigenvalue of the matrix A.

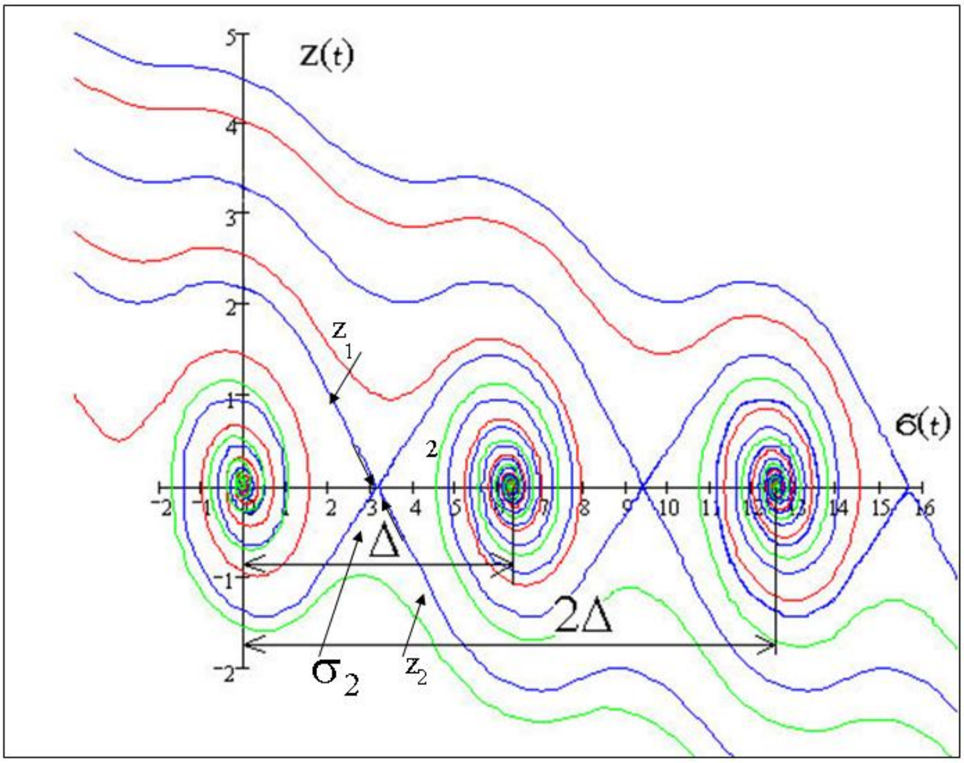

Figure 1. A phase portrait of (6) for $a > a_{cr}$

Theorem 1 . *Suppose there exist positive* $\tau, \varepsilon, \delta$, $\lambda \in (0, \lambda_0)$, $\alpha_1 \le \mu_1, \alpha_2 \le \mu_2$ *such that the following conditions are satisfied:*
1) for all $\omega \ge 0$ *the inequality is valid:*

$$\begin{gathered}\pi(\omega, \lambda) \stackrel{\Delta}{=} Re\{K(i\omega - \lambda) - \tau(K(i\omega - \lambda)+ \\ +\alpha_1^{-1}(i\omega - \lambda))^*(K(i\omega - \lambda) + \alpha_2^{-1}(i\omega - \lambda))\} - \varepsilon|K(i\omega - \lambda)|^2 \ge \delta;\end{gathered} \tag{14}$$

2) for $\varkappa_1 \in [0, 1]$ *the quadratic form*

$$P(x, y, z) = \lambda x^2 + \varepsilon y^2 + \delta z^2 + (1 - \varkappa_1)\nu_0 yz + a_{cr}\sqrt{\varkappa_1} xy \tag{15}$$

is positive definite.
Then system (2) *is Lagrange stable.*

Proof. Consider the system

$$\frac{dy}{dt} = Qy(t) + L\eta(t), \; \frac{d\sigma}{dt} = D^*y(t). \tag{16}$$

where $y(t) = (z(t), \varphi(\sigma(t)))^T$, $\eta(t) = \frac{d}{dt}\varphi(\sigma(t))$,

$$Q = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D = \begin{bmatrix} c^* \\ \rho \end{bmatrix}. \tag{17}$$

Introduce the quadratic form

$$\begin{aligned} G(y,\eta) = 2y^*H((Q+\lambda I_{m+1})y + L\eta) + \delta(L^*y)^2 + \varepsilon(D^*y)^2 + \\ +y^*LD^*y + \tau(D^*y - \alpha_1^{-1}\eta)(D^*y - \alpha_2^{-1}\eta) \qquad (y \in \mathbb{R}^{m+1}, \eta \in \mathbb{R}). \end{aligned} \tag{18}$$

By KYP lemma, the inequality (14) guaranties that there exists a matrix $H = H^*$ [9] such that

$$G(y,\eta) \leq 0, \qquad \forall y \in \mathbb{R}^{m+1}, \eta \in \mathbb{R}. \tag{19}$$

Let

$$H = \begin{bmatrix} H_0 & h \\ h^* & \alpha \end{bmatrix} \quad (H_0 \in \mathbb{R}^{m\times m}, h \in \mathbb{R}^m, \alpha \in \mathbb{R}). \tag{20}$$

Then for $\bar{y} = (z,0)^T$ it is true that

$$G(\bar{y},0) = 2z^*H_0(A+\lambda I_m)z + (\varepsilon+\tau)(c^*z)^2 \ \forall z \in \mathbb{R}^m. \tag{21}$$

Since the pair $(A+\lambda I_m, b)$ is controllable, the matrix H_0 is positive definite [12]. So if $\varphi(\sigma(\bar{t})) = 0$, one has

$$y^*(\bar{t})Hy(\bar{t}) > 0, \quad z(\bar{t}) \neq 0. \tag{22}$$

Consider the condition 2) of the Theorem. It implies that

$$\varepsilon > \frac{a_{cr}^2\varkappa_1}{4\lambda} + \frac{(1-\varkappa_1)^2\nu_0^2}{4\delta}. \tag{23}$$

Let $\varepsilon = \varepsilon_1 + \varepsilon_2$, where

$$\varepsilon_2 \triangleq \frac{(1-\varkappa_1)^2\nu_0^2}{4\delta}. \tag{24}$$

Then

$$\varepsilon_1 > \frac{a_{cr}^2\varkappa_1}{4\lambda}. \tag{25}$$

System

$$\begin{aligned} \dot{z} &= -2\sqrt{\lambda\varepsilon_1}z - \varkappa_1\varphi(\sigma), \\ \dot{\sigma} &= z. \end{aligned} \tag{26}$$

by linear change of variable t can be transformed to the system (6) with $a = 2\sqrt{\frac{\lambda\varepsilon_1}{\varkappa_1}}$. So the equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma} + 2\sqrt{\lambda\varepsilon_1}F(\sigma) + \varkappa_1\varphi(\sigma) = 0 \tag{27}$$

in virtue of (25) has the solutions $F_k(\sigma)$ with the properties P1, P2, P3.

Introduce the Lyapunov functions

$$V_k(t) = y^*(t)Hy(t) - \frac{1}{2}F_k^2(\sigma(t)) + \varkappa_2\int_{\sigma_2}^{\sigma(t)} \Psi_0(\zeta)d\zeta. \tag{28}$$

Here, H is from (19), $\varkappa_2 = 1 - \varkappa_1$,

$$\Psi_0(\zeta) \triangleq \varphi(\zeta) - \nu_0|\varphi(\zeta)|. \tag{29}$$

It is obvious that

$$\int_0^{\Delta} \Psi_0(\zeta)d\zeta = 0. \tag{30}$$

In virtue of system (16)

$$\dot{V}_k(t)+2\lambda V_k(t)=2y^*(t)H((Q+\lambda I_{m+1})y(t)+L\dot{\varphi}(\sigma(t)))-\\ -F_k'(\sigma(t))F_k(\sigma(t))\dot{\sigma}(t)+\varkappa_2\Psi_0(\sigma(t))\dot{\sigma}(t)-\lambda F_k^2(\sigma(t))+2\lambda\varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi_0(\zeta)d\zeta. \tag{31}$$

Now we can apply the inequality (19)

$$\dot{V}_k(t)+2\lambda V_k(t)\le-\varepsilon\dot{\sigma}^2(t)-\delta\varphi^2(\sigma(t))-\dot{\sigma}(t)\varphi(\sigma(t))+2\sqrt{\lambda\varepsilon_1}F_k(\sigma(t))\dot{\sigma}(t)+\\ +\varkappa_1\varphi(\sigma(t))\dot{\sigma}(t)+\varkappa_2\varphi(\sigma(t))\dot{\sigma}(t)-\varkappa_2\nu_0|\varphi(\sigma(t))|\dot{\sigma}(t)-\lambda F_k^2(\sigma(t))+2\lambda\varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi_0(\zeta)d\zeta. \tag{32}$$

It follows from (5) and (30) that

$$\int_{\sigma_2}^{\sigma(t)}\Psi_0(\zeta)d\zeta\le 0,\quad\forall\sigma. \tag{33}$$

Then

$$\dot{V}_k(t)+\lambda V_k(t)\le-(\varepsilon_2\dot{\sigma}^2(t)+\delta\varphi^2(\sigma(t))+\varkappa_2\nu_0|\varphi(\sigma(t))|\dot{\sigma}(t))-\\ -(\varepsilon_1\dot{\sigma}^2(t)-2\sqrt{\lambda\varepsilon_1}F_k(\sigma(t))\dot{\sigma}(t)+\lambda F_k^2(\sigma(t))),\quad\forall t\ge 0, \tag{34}$$

whence in virtue of (24)

$$\dot{V}_k(t)+2\lambda V_k(t)\le 0\quad\forall t\ge 0. \tag{35}$$

Hence

$$V_k(t)e^{2\lambda t}\le V_k(0),\quad\forall t\ge 0,\forall k\in Z. \tag{36}$$

Notice that

$$V_k(0)=y^*(0)Hy(0)-\tfrac{1}{2}F_k^2(\sigma(0))+\varkappa_2\int_{\sigma_2}^{\sigma(0)}\Psi(\zeta)d\zeta. \tag{37}$$

The property P3 of $F_k(\sigma)$ implies that one can always choose a natural $k_0\in\mathbb{N}$ in such a way that $V_{\pm k_0}(0)<0$. Then

$$V_{\pm k_0}(t)<0,\quad\forall t\ge 0. \tag{38}$$

Suppose $\bar{t}$ is such that

$$\sigma(\bar{t})=\sigma_2+\Delta K,\quad K\in Z. \tag{39}$$

Then $\varphi(\sigma(\bar{t}))=0$,

$$\int_{\sigma_2}^{\sigma(\bar{t})}\Psi_0(\zeta)d\zeta=0 \tag{40}$$

and it follows from (22) that

$$y^*(\bar{t})Hy(\bar{t})=z^*(\bar{t})H_0z(\bar{t})\ge 0. \tag{41}$$

The inequality (38) implies that

$$F_{\pm k_0}^2(\sigma(\bar{t}))\ne 0. \tag{42}$$

Hence, for any $z(0),\sigma(0)$ there exist a $k_0\in\mathbb{N}$ such that

$$\sigma_2-\Delta k_0<\sigma(t)<\sigma_2+\Delta k_0,\quad\forall t\ge 0. \tag{43}$$

Theorem 1 is proved. ■

Introduce the function

$$\Phi(\sigma) = \sqrt{(1-\alpha_1^{-1}\varphi'(\sigma))(1-\alpha_2^{-1}\varphi'(\sigma))}, \tag{44}$$

with $\alpha_1 \leq \mu_1, \alpha_2 \geq \mu_2$ and the constant

$$\nu = \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta \Phi(\sigma)\,|\varphi(\sigma)|\,d\sigma}. \tag{45}$$

Theorem 2 . *Suppose there exist $\lambda \in (0,\lambda_0), \varkappa, \varepsilon, \tau, \delta > 0, \alpha_1 = -\alpha_2$ such that the following conditions are satisfied:*
1) for all $\omega \geq 0$ the inequality (14) *is true:*
2)

$$4\lambda\varepsilon > a_{cr}^2(1 - \frac{2\sqrt{\tau\delta}}{|\nu|}); \tag{46}$$

3) $\max\Phi(\sigma) \leq |\nu|^{-1}$.
Then (2) *is Lagrange stable.*

Proof. Proceeding from (46), choose a $\varkappa_1 \in (0,1)$ such that

$$\frac{4\lambda\varepsilon}{a_{cr}^2} > \varkappa_1 \tag{47}$$

and

$$\varkappa_2 \triangleq 1 - \varkappa_1 < \frac{2\sqrt{\tau\delta}}{|\nu|}. \tag{48}$$

Henceforth our argument is analogous with the proof of Theorem 1. Introduce Lyapunov-type functions

$$v_k(t) = y^*(t)Hy(t) - \tfrac{1}{2}F_k^2(\sigma(t)) + \varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi(\zeta)d\zeta. \tag{49}$$

Here, H is from (19), $F_k(\zeta)\,(k \in Z)$ is a solution of (27) with properties P1–P3, and

$$\Psi(\zeta) \triangleq \varphi(\zeta) - \nu|\varphi(\zeta)|\Phi(\zeta). \tag{50}$$

Notice that

$$\int_0^\Delta \Psi(\zeta)d\zeta = 0. \tag{51}$$

In virtue of (16) we have

$$\begin{aligned}\dot v_k(t) + 2\lambda v_k(t) &= 2y^*(t)H[(Q+\lambda I_{m+1})y(t) + L\varphi(\sigma(t))] - F_k'(\sigma(t))F_k(\sigma(t))\dot\sigma(t)+\\ &+\varkappa_2\Psi(\sigma(t))\dot\sigma(t) - \lambda F_k^2(\sigma(t)) + 2\lambda\varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi(\zeta)d\zeta.\end{aligned} \tag{52}$$

It follows from (27) and (50) that

$$\begin{aligned}\dot v_k(t) + 2\lambda v_k(t) &= 2y^*(t)H[(Q+\lambda I_{m+1}y(t)) + L\varphi(\sigma(t))] + 2\sqrt{\lambda\varepsilon}F_k(\sigma(t))\dot\sigma(t)+\\ &+\varkappa_1\varphi(\sigma(t))\dot\sigma(t) - \varkappa_2\nu|\varphi(\sigma(t))|\Phi(\sigma(t))\dot\sigma(t) + 2\lambda\varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi(\zeta)d\zeta - \lambda F_k^2(\sigma(t)).\end{aligned} \tag{53}$$

Then from (19) we have

$$\dot{v}_k(t) + 2\lambda v_k(t) \le -\varepsilon\dot{\sigma}^2(t) - \delta\varphi^2(\sigma(t)) - \tau\dot{\sigma}^2(t)\Phi^2(\sigma(t)) + 2\sqrt{\lambda\varepsilon}F_k(\sigma(t))\dot{\sigma}(t) - \\ -\lambda F_k^2(\sigma(t)) + \lambda\varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi(\zeta)d\zeta - \nu\varkappa_2|\varphi(\sigma(t))|\Phi(\sigma(t))\dot{\sigma}(t), \tag{54}$$

whence

$$\dot{v}_k(t) + 2\lambda v_k(t) \le [-\delta\varphi^2(\sigma(t)) - \tau(\dot{\sigma}(t)\Phi(\sigma(t)))^2 - \\ -\nu\varkappa_2|\varphi(\sigma(t))|\Phi(\sigma(t))\dot{\sigma}(t)] + 2\lambda\varkappa_2\int_{\sigma_2}^{\sigma(t)}\Psi(\zeta)d\zeta. \tag{55}$$

The inequality (48) implies that the first summand in the right-hand part of (55) is a negative definite quadratic form of $|\varphi(\sigma(t))|$ and $\Phi(\sigma(t))\dot{\sigma}(t)$. From (4) and (51) we have

$$\int_{\sigma_2}^{\sigma(t)}\Psi(\zeta)d\zeta < 0. \tag{56}$$

Thus,

$$\dot{v}_k(t) + 2\lambda v_k(t) \le 0. \tag{57}$$

Hence,

$$v_k(t)e^{2\lambda t} \le v_k(0). \tag{58}$$

Since

$$v_k(0) = y^*(0)Hy(0) - \tfrac{1}{2}F_k^2(\sigma(0)) + \varkappa_2\int_{\sigma_2}^{\sigma(0)}\Psi(\zeta)d\zeta, \tag{59}$$

one can always choose a natural $k_0 \in \mathbb{N}$ in such a way that $v_{\pm k_0}(0) < 0$. Then

$$v_{\pm k_0}(t) < 0, \quad \forall t \ge 0. \tag{60}$$

Henceforth one can repeat the proof of Theorem 1 beginning with (38). ■

4. Gradient–like behavior

Theorem 3 . *Suppose there exist positive* $\tau, \varepsilon, \delta$, $\lambda \in (0, \lambda_0)$, $\alpha_1 \le \mu_1, \alpha_2 \le \mu_2$ *such that*

$$\alpha_1^{-1}\alpha_2^{-1} = 0, \tag{61}$$

$$(\alpha_1^{-1} + \alpha_2^{-1})\rho \le 0 \tag{62}$$

and all the conditions of Theorem 1 or Theorem 2 are satisfied. Then the following relations are true:

$$\lim_{t\to+\infty} z(t) = 0, \tag{63}$$

$$\lim_{t\to+\infty} \sigma(t) = q, \quad \varphi(q) = 0. \tag{64}$$

Proof. Consider separately a Lyapunov function $W(t) = y^*(t)Hy(t)$. In virtue of system (16) one has

$$\frac{dW}{dt} + 2\lambda W(t) = 2y^*(t)H[(Q + \lambda I_{m+1})y(t) + L\dot{\varphi}(\sigma(t))]. \tag{65}$$

Since the frequency–domain inequality (14) is fulfilled, there exists a matrix $H = H^*$ such that the inequality (19) is valid, whence

$$\frac{dW}{dt} + 2\lambda W(t) + \varepsilon\dot{\sigma}^2(t) + \dot{\sigma}(t)\varphi(\sigma(t)) + \delta\varphi^2(\sigma(t)) \le 0. \tag{66}$$

It follows from (61) that the quadratic form $G(y, \eta)$ is linear with respect to η. Since $G(y, \eta)$ is nonnegative for all y, η, we conclude that

$$2HL = \tau(\alpha_1^{-1} + \alpha_2^{-1})D. \tag{67}$$

Taking into account (17), we get

$$2h = \tau(\alpha_1^{-1} + \alpha_2^{-1})c,\ 2\alpha = \tau(\alpha_1^{-1} + \alpha_2^{-1})\rho. \tag{68}$$

Then

$$W(t) = z^*(t)H_0z(t) + \tau(\alpha_1^{-1} + \alpha_2^{-1})c^*z(t)\varphi(\sigma(t)) + \tfrac{1}{2}(\alpha_1^{-1} + \alpha_2^{-1})\tau\rho\varphi^2(\sigma(t)). \tag{69}$$

The relations (66) and (69) together with the equations (68) imply that

$$\begin{gathered}\frac{dW}{dt} + (2\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1}) + 1)\dot{\sigma}(t)\varphi(\sigma(t)) - \lambda\tau(\alpha_1^{-1} + \alpha_2^{-1})\rho\varphi^2(\sigma(t)) + \\ +2\lambda z^*(t)H_0z(t) + \varepsilon\dot{\sigma}^2(t) + \delta\varphi^2(\sigma(t)) \le 0,\end{gathered} \tag{70}$$

whence

$$\begin{gathered}\frac{d}{dt}\{W(t) + (2\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1}) + 1)\int_{\sigma(0)}^{\sigma(t)} \varphi(\zeta)d\zeta\} + \varepsilon\dot{\sigma}^2(t) + \delta\varphi^2(\sigma(t)) \le \\ \le \lambda\tau(\alpha_1^{-1} + \alpha_2^{-1})\rho\varphi^2(\sigma(t)) - 2\lambda z^*(t)H_0z(t).\end{gathered} \tag{71}$$

Notice that since H_0 is positive definite and the inequality (62) is true, the right-hand part of (71) is negative.

Then one can deduce from (71) that

$$\varepsilon\int_0^t \dot{\sigma}^2(\tau)d\tau + \delta\int_0^t \varphi^2(\sigma(\tau))d\tau \le W(0) - W(t) - (2\lambda\tau(\alpha_1^{-1} + \alpha_2^{-1}) + 1)\int_{\sigma(0)}^{\sigma(t)} \varphi(\zeta)d\zeta. \tag{72}$$

All the conditions of Theorem 1 or Theorem 2 are fulfilled here. So every solution of system (2) is bounded for $t \in \mathbb{R}_+$. It follows that the right part of (72) is also bounded, for $t \in \mathbb{R}_+$. Thus

$$\dot{\sigma}(t), \varphi(\sigma(t)) \in L_2(0, +\infty). \tag{73}$$

It is easy to establish that the relations (63), (64) follow from (73) [11]. Theorem 3 is proved. ■

Example. Consider a phase-locked loop (PLL) with

$$K(p) = T\frac{Tsp + 1}{Tp + 1} \quad (T > 0,\ s \in (0, 1)) \tag{74}$$

and

$$\varphi(\sigma) = \sin\sigma - \beta \quad (\beta \in (0, 1)). \tag{75}$$

We choose $\alpha_1^{-1} = 0, \alpha_2^{-1} = 1$. Then the frequency domain inequality (14) takes the form

$$\begin{aligned}&\omega^2T^2(-(\varepsilon + \tau)T^2s^2 - \delta + (1 + \tau\lambda)Ts + \tau(1 - s)) + (T(1 + \tau\lambda)(1 - Ts\lambda)(1 - T\lambda) - \\ &-\delta(1 - T\lambda)^2 - (\varepsilon + \tau)T^2(1 - Ts\lambda)^2 > 0 \quad (\lambda < \tfrac{1}{T},\ \omega \ge 0).\end{aligned} \tag{76}$$

Consider the case of small T^2 and choose $\varkappa_1 = 1$. Then from (15) and (23) we have

$$4\lambda\varepsilon > a_{cr}^2 \tag{77}$$

Let $\lambda = (2T)^{-1}$. It is clear that (76) is valid for δ and τ small enough and $4\lambda\varepsilon < (T^2(1-0.5s))^{-1}$. We can borrow the value of a_{cr}^2 from [11, Fig.2.16.7]. For example, if $\beta \le 0.98$, we have $a_{cr}^{-2} > 0.8$. It follows that Theorem 1 and Theorem 3 give for capture value β the estimate 0.98 if $T^2 < 0.8$ (we choose $4\lambda\varepsilon = 1.25$). Meanwhile the Bakaev–Guzh technique gives for $s = 0.2$ and small T^2 the estimate $\beta = 0.93$ [13].

5. Conclusion

In this paper two novel Lyapunov–type functions for systems of indirect control with periodic nonlinearities are constructed . With the help of the Kalman–Yakubovich–Popov lemma, new frequency-algebraic criteria ensuring the gradient–like behavior are obtained.

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