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THE FREDHOLM FACTORIZATION IN PRESENCE OF PENETRABLE RECTANGULAR RODS

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Abstract

The Fredholm factorization of the Wiener Hopf equations is extended to study planar stratified regions in presence of impenetrable and/or penetrable rectangular cylinders. The general theory requires to consider several aspects of the Wiener Hopf technique. In order to illustrate it with an example, in this paper we limit the analysis to the diffraction by a dielectric rectangular rod lying on a PEC plane

1. The Wiener Hopf Equations

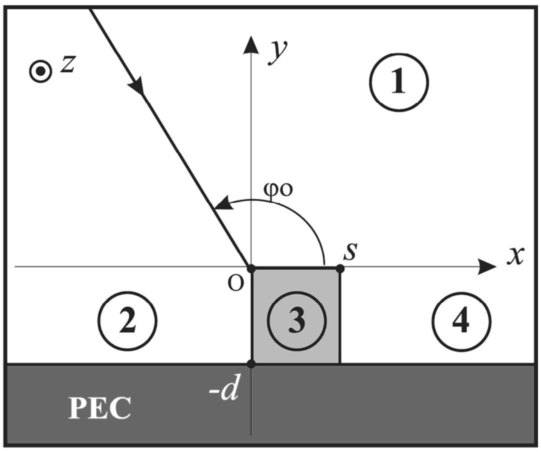

Figure 1. The diffraction by a dielectric rectangular rod lying on a PEC plane

The geometry presented in Fig.1 consists of four regions: region 1 (y>0), region 2 (x<0, -d<y<0), region 3 (0<x<s, -d<y<0), region 4 (x>s, -d<y<0).
We consider not magnetic material i.e. the magnetic permeability is assumed the same of the free space for all regions. Regions 1,2,4 are filled by free space, while the region 3 is filled by dielectric material with relative permittivity ε_r.
For simplicity the source is an Ez-polarized plane wave having incidence angle φ_o. Consequently the not vanishing field components are $E_z(x,y)$, $H_x(x,y)$ and $H_y(x,y)$.
According to the Wiener Hopf (WH) technique [1] we can write suitable WH equations in the four regions.

They are:

Region 1

$$-I_{\pi+}(-\eta)+I_o(\eta)+e^{j\eta s}I_+(\eta)= \\ =Y_c(\eta)[V_{\pi+}(-\eta)+V_o(\eta)+e^{j\eta s}V_+(\eta)] \tag{1}$$

$$-I_{\pi+}(\eta)+I_o(-\eta)+e^{-j\eta s}I_+(-\eta)= \\ =Y_c(\eta)[V_{\pi+}(\eta)+V_o(-\eta)+e^{j\eta s}V_+(-\eta)] \tag{2}$$

$$-e^{j\eta s}I_{\pi+}(\eta)+I_{\pi o}(\eta)+I_+(-\eta)= \\ =Y_c(\eta)[e^{j\eta s}V_{\pi+}(\eta)+V_{\pi o}(\eta)+V_+(-\eta)] \tag{3}$$

$$-e^{-j\eta s}I_{\pi+}(\eta)+I_{\pi o}(-\eta)+I_+(\eta)= \\ =Y_c(\eta)[e^{-j\eta s}V_{\pi+}(-\eta)+V_{\pi o}(-\eta)+V_+(\eta)] \tag{4}$$

Region 2

$$-p_o(\tau)+\eta\, p_1(\tau)+I_{\pi+}(\eta)=Y_d(\eta)V_{\pi+}(\eta) \tag{5}$$

Region 3

$$-\eta e^{j\eta s}q_1(\tau_3)-e^{j\eta s}q_o(\tau_3)+\eta\, p_1(\tau_3)+p_o(\tau_3)-I_o(\eta)= \\ =Y_{d3}(\eta)V_o(\eta) \tag{6}$$

$$\eta q_1(\tau_3)-q_o(\tau_3)-\eta e^{j\eta s}p_1(\tau_3)+e^{j\eta s}p_o(\tau_3)-I_{\pi o}(\eta)= \\ =Y_{d3}(\eta)V_{\pi o}(\eta) \tag{7}$$

Region 4

$$q_o(\tau)+\alpha q_1(\tau)-I_+(\alpha)=Y_d(\alpha)V_+(\alpha) \tag{8}$$

In the above equations several spectral quantities are defined in terms of bilateral/unilateral/finite interval Fourier transforms:

$$V_+(\eta)=e^{-j\eta s}\int_s^\infty E_z(x,0)e^{j\eta x}dx \;,\; V_{\pi+}(\eta)=\int_{-\infty}^0 E_z(x,0)e^{-j\eta x}dx$$

$$I_+(\eta)=e^{-j\eta s}\int_s^\infty H_x(x,0)e^{j\eta x}dx \quad I_{\pi+}(\eta)=-\int_{-\infty}^0 H_x(x,0)e^{-j\eta x}dx$$

$$V_o(\eta)=\int_0^s E_z(x,0)e^{j\eta x}dx \;,\; V_{\pi o}(\eta)=e^{j\eta x}\int_0^s E_z(x,0)e^{-j\eta x}dx$$

$$I_o(\eta)=\int_0^s H_x(x,0)e^{j\eta x}dx \quad I_{\pi o}(\eta)=e^{j\eta x}\int_0^s H_x(x,0)e^{-j\eta x}dx$$

In eqs (1)-(8) we have reported also the following quantities

$Y_c(\eta)=\frac{\sqrt{k^2-\eta^2}}{k\,Z_o}$, $Y_d(\eta)=-j\frac{\sqrt{k^2-\eta^2}}{k\,Z_o}\cot\left[\sqrt{k^2-\eta^2}\,d\right]$

$Y_{d3}(\eta)=-j\frac{\sqrt{\varepsilon_r k^2-\eta^2}}{k\,Z_o}\cot\left[\sqrt{\varepsilon_r k^2-\eta^2}\,d\right]$,

$\tau=\sqrt{k^2-\eta^2}$, $\tau_3=\sqrt{\varepsilon_r k^2-\eta^2}$

with k and Z_o are respectively the propagation constant and the impedance of the free space.

We remark that while Eqs.(1-4) are complete WH equations, Eqs. (5-8) are incomplete [1] for the presence of the four even unknowns meromorphic function $p_o(\tau)$, $p_1(\tau)$, $q_o(\tau)$,$q_1(\tau)$. A Mittag-Leffler expansion of them yields:

$$p_o(\tau)=-\sum_{n=1}^{\infty}\frac{A_n}{\eta^2-\alpha_n^2},\quad p_o(\tau_3)=-\sum_{n=1}^{\infty}\frac{A_n}{\eta^2-\chi_n^2},$$

$$p_1(\tau)=-\sum_{n=1}^{\infty}\frac{B_n}{\eta^2-\alpha_n^2},\quad p_1(\tau_3)=-\sum_{n=1}^{\infty}\frac{B_n}{\eta^2-\chi_n^2}$$

$$q_o(\tau)=-\sum_{n=1}^{\infty}\frac{C_n}{\eta^2-\alpha_n^2},\quad q_o(\tau_3)=-\sum_{n=1}^{\infty}\frac{C_n}{\eta^2-\chi_n^2}$$

$$q_1(\tau)=-\sum_{n=1}^{\infty}\frac{D_n}{\eta^2-\alpha_n^2},\quad q_1(\tau_3)=-\sum_{n=1}^{\infty}\frac{D_n}{\eta^2-\chi_n^2}$$

where

$$\alpha_n=\sqrt{k^2-(\frac{n\pi}{d})^2},\quad \chi_n=\sqrt{\varepsilon_r k^2-(\frac{n\pi}{d})^2}$$

In order to complete the WH equations (5-8) we must express the four set of the constants A_n,B_n,C_n,D_n in terms of the spectral unknowns and in particular we have selected the four plus spectral unknowns $V_+(\eta)$, $V_{\pi+}(\eta)$, $V_o(\eta)$, $V_{\pi o}(\eta)$. Taking into account that the residues on the poles $-\alpha_n$ in the first members of (5) and (8) must be equal to the residues of the second members we get

$$\alpha_n\frac{B_n}{-2\alpha_{nn}}+\frac{A_n}{-2\alpha_n}=-\frac{jn^2\pi^2}{d^3kZ_o\alpha_{nn}}V_{\pi+}(-\alpha_n)\tag{9}$$

$$-\alpha_n\frac{D_n}{-2\alpha_{nn}}+\frac{C_n}{-2\alpha_n}=-\frac{jn^2\pi^2}{d^3kZ_o\alpha_{nn}}V_{+}(-\alpha_n)\tag{10}$$

Similar considerations can be derived for the poles $-\chi_n$ in equations (6) and (7) by yielding:

$$-\chi_n e^{-j\chi_n s}\frac{D_n}{-2\chi_n}+e^{-j\chi_n s}\frac{C_n}{-2\chi_n}+\chi_n\frac{B_n}{-2\chi_n}-\frac{A_n}{-2\chi_n}=\\=-\frac{jn^2\pi^2}{d^3kZ_o\chi_n}V_o(-\chi_n)\tag{11}$$

$$\chi_n\frac{D_n}{-2\chi_n}+\frac{C_n}{-2\chi_n}-\chi_n e^{-j\chi_n s}\frac{B_n}{-2\chi_n}-e^{-j\chi_n s}\frac{A_n}{-2\chi_n}=\\=-\frac{jn^2\pi^2}{d^3kZ_o\chi_n}V_{\pi o}(-\chi_n)\tag{12}$$

Solving equations (9-12) we complete the equations (5-8) because we can express A_n, B_n, C_n, D_n or $p_o(\alpha), p_1(\alpha), q_o(\alpha), q_1(\alpha)$ in terms of $V_+(-\alpha_n), V_{\pi+}(-\alpha_n), V_o(-\chi_n), V_{\pi o}(-\chi_n)$.

For reasons of space the solution of the system (9-12) is not reported here.

2. The Fredholm Equations of the Problem.

The application of the Fredholm factorization [2] is based on five steps:

1) Deduction of the WH equations of the problem
2) Reduction of the WH equations to Fredholm integral equations (FIE)
3) Solution of the Fredholm integral equations
4) Analytical continuation of the numerical solution of the FIE
5) Evaluation of the physical field components if present: reflected and refracted plane waves, diffracted fields, surface waves, lateral waves, leaky waves, mode excitations, near field and so on.

In section 1 we have accomplished the step 1. In this section we will introduce the final Fredholm equations starting from the WH equations (1-8). To get these equations we remember that while the classical factorization separates the minus unknowns from the plus unknowns by working on all the equations of the WH system at the same time, the Fredholm factorization simply eliminates time after time the minus unknown present in any single equation. This elimination is accomplished through a Cauchy decomposition [1-8] In region 1, corresponding to eqs. (1-4), we get four integral equations that can be interpreted as network model of Norton kind. In fact they explicitly relate the currents $I_+(\eta)$, $I_{\pi+}(\eta)$, $I_o(\eta)$ $I_{\pi o}(\eta)$ to the four voltages $V_+(\eta)$, $V_{\pi+}(\eta)$, $V_o(\eta)$ $V_{\pi o}(\eta)$. The use of network formalism allows to order and systematize the mathematical procedure avoiding redundancy [9].

In a similar way, in regions 2-4, the Cauchy decomposition applied to (5-8) yields four integral equations that can be interpreted as network model of Norton kind. This representation is alternative to the previous one, since it relates the currents $I_+(\eta)$, $I_{\pi+}(\eta)$, $I_{\pi o}(\eta)$ to the four voltages $V_+(\eta)$, $V_{\pi+}(\eta)$, $V_o(\eta)$ $V_{\pi o}(\eta)$. We observe that this last representation involves the fours set of the constants A_n,B_n,C_n,D_n. We remember that these unknowns are related to the spectra $V_+(\eta)$, $V_{\pi+}(\eta)$, $V_o(\eta)$, $V_{\pi o}(\eta)$.

Using the two alternative representations described above, by substitution, we can eliminate the currents $I_+(\eta), I_{\pi+}(\eta), I_o(\eta), I_{\pi o}(\eta)$. This eliminations yield a system of Fredholm integral equations of order four having as unknowns the spectra $V_+(\eta), V_{\pi+}(\eta), V_o(\eta), V_{\pi o}(\eta)$. Further details to get the solutions and the numerical implementation will be presented at the conference for the problem described in Fig. 1.

4. Comparison with Other Methods in Special Cases

We have successfully compared the results provided by the Fredholm integral equations of this paper with those available in literature in several geometries. In particular we have considered the problem of the diffraction by a conducting strip and the rectangular impenetrable cylinder. The usual WH technique applied to these problems [1,10-13] yields modified WH equations that have been faced by using the Jones method [1,10-14]. We remember that also this method reduces the factorization to Fredholm integral equations. Improvements on this strategy have been reported in [13,14] to better deal with large rods. However Jones's method requires a preliminary exact factorization of some scalars that in in the method reported in this paper is not required. Moreover apparently the Jones's method has not been applied up to now to study penetrable cylindrical rods.

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