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# Leonov's nonlocal reduction technique for nonlinear integro-differential equations ⋆

**Vera B. Smirnova** *,** **Anton V. Proskurnikov** ***,****

* *Saint-Petersburg State University of Architecture and Civil Engineering, Saint-Petersburg, Russia*
** *Saint-Petersburg State University, Saint-Petersburg, Russia*
*** *Politecnico di Torino, Turin, Italy*
**** *Institute for Problems of Mechanical Engineering of the Russian Academy of Sciences (IPME RAS), Saint-Petersburg, Russia*

**Abstract:**
Starting from pioneering works by Lur'e, Popov and Zames, global stability theory for nonlinear control systems has been primarily focused on systems with only one equilibrium. Global stability criteria for other kinds of attractors (such as e.g. infinite sets of equilibria) are not well studied and typically require special tools, primarily based on the Lyapunov method. Analysis of stability becomes especially complicated for infinite-dimensional dynamical systems with multiple equilibria, e.g. systems described by delay or more general convolutionary equations. In this paper, we propose novel stability criteria for infinite-dimensional systems with periodic nonlinearities, which have infinite sets of equilibria and describe dynamics of phase-locked loops and other synchronization circuits. Our method combines Leonov's nonlocal reduction technique with the idea of Popov's "integral indices" and allows to obtain new frequency-domain conditions, ensuring the convergence of every solution to one of the equilibria points.



*Keywords:* Nonlinear dynamics, stability, frequency-domain methods, phase-locked loop

## 1. INTRODUCTION

In spite of enormous progress in nonlinear control (Khalil, 1996; Kokotović and Arcak, 2001; Fradkov et al., 1999), the results on global stability and stabilization prevailing in the literature are primarily concerned with stability of the single equilibrium. Among systems with multiple equilibria, the most studied are discontinuous systems arising e.g. in sliding mode control (Gelig et al., 2004; Utkin, 1992) and systems with cylindric phase space (Leonov et al., 1996a; Kudrewicz and Wasowicz, 2007) that arise as feedback interconnections of linear time-invariant systems and periodic nonlinearities. Such systems describe pendulum-like mechanical systems, electrical machines (Stoker, 1950) and synchronization circuits such as phase-locked loops (PLL) (Margaris, 2004; Best, 2003; Leonov et al., 2015). The global stability (called also gradient-like behavior) of a system with non-unique equilibrium is typically understood as the convergence of every solution to one of the equilibria points. Most of the existing results on global stability are confined to ordinary differential equations and exploit specially designed Lyapunov functions. The direct extension of these results to infinite-dimensional systems such as e.g. semigroup equations (Gil, 1998) remains a non-trivial open problem.

At the same time, infinite-dimensional systems with periodic nonlinearities can be examined by the techniques stemming from the Popov's method of "a priori integral indices" (Popov, 1973; Rasvan, 2006), later developed into the method of integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997; Yakubovich, 2000, 2002). Compared to the usual absolute stability criteria (Megretski and Rantzer, 1997; Yakubovich, 2002), the relevant results have two principal differences. Since the system has no globally stable equilibrium, the stability with respect to the full vector of state and input variables is replaced by the stability with respect to some output (which serves as a nonlinear "distance" to the set of equilibria). Besides this, special IQCs have to be designed exploiting the periodicity of the nonlinear feedback. Notice that in PLLs this periodic nonlinearity, typically, has nonzero integral over period, since it depends on the constant deviation between the natural frequencies of the reference and controlled oscillators (Leonov et al., 2015). The IQCs are derived by either using Leonov's nonlocal reduction principle (Leonov et al., 1996b, 1992) (and involve trajectories of a specially chosen comparison systems) or special decomposition of the nonlinearity, known as the "Bakaev-Guzh technique" (Leonov et al., 1996b, 1992).

In this paper, we combine the two aforementioned techniques and derive new stability criteria for systems with periodic nonlinearities. For the reader's convenience, we do not rely on the general IQC method and give direct proofs of all results, stemming from the ideas of (Popov, 1973).

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## 2. PRELIMINARIES AND NOTATION

Following (Smirnova and Proskurnikov, 2019), we consider an integro-differential equation

$$\dot{\sigma}(t) = b(t) + \rho\varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\,d\tau, \quad (1)$$

Here $h \geq 0$ ; $\varphi : \mathbb{R} \to \mathbb{R}$; $\gamma, b : [0, +\infty) \to \mathbb{R}$, The solution of (1) is uniquely determined by the initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t) \in \mathbb{C}[-h, 0]. \quad (2)$$

We adopt the following assumptions.

**Assumption 1:** The functions $b(\cdot)$, $\gamma(\cdot)$ are continuous and decay exponentially as $t \to \infty$, in particular, $b(t)e^{rt}$, $\gamma(t)e^{rt}$ are $L_2$-summable for some $r > 0$.

**Assumption 2:** The function $\varphi(\cdot)$ is $\mathbb{C}^1$-smooth, $\Delta$–periodic $\varphi(\sigma) = \varphi(\sigma+\Delta)$ and has two simple zeros $\sigma_1 < \sigma_2$ in the interval $[0, \Delta)$ with $\varphi'(\sigma_1) > 0$, $\varphi'(\sigma_2) < 0$, so that

$$\varphi(\sigma) > 0\ \forall\sigma \in (\sigma_1, \sigma_2), \quad \varphi(\sigma) < 0\ \forall\sigma \in (\sigma_2, \sigma_1 + \Delta) \quad (3)$$

Without loss of generality, we assume that

$$\int_0^\Delta \varphi(\zeta)\,d\zeta \leq 0. \quad (4)$$

(otherwise, the signs of $\varphi, \gamma, \rho$ can be flipped).

Equation (1) is a special case of Lur'e system obtained by interconnection of the integro-differential linear system and the periodic nonlinearity. As discussed in (Smirnova and Proskurnikov, 2019), system (1) arises in many applications, but the main motivation is the dynamics of PLLs. For PLLs, $\sigma(t)$ stands for the phase error (deviation between the phases of the reference and the controlled oscillators) and the convergence of all solutions to equilibria corresponds to phase locking (Leonov et al., 2015).

The frequency-domain stability conditions formulated below will use the transfer function of the linear part from the input $\xi = \varphi(\sigma)$ to $(-\dot{\sigma})$, defined as

$$K(p) = -\rho e^{-ph} + \int_0^\infty \gamma(t)e^{-pt}\,dt \quad (p \in \mathbb{C}). \quad (5)$$

**Definition 1.** System (1) is said to be *globally stable* (Leonov et al., 2015) or *gradient-like* (Leonov, 2006) if every solution converges to one of the equilibria

$$\dot{\sigma}(t) \xrightarrow[t\to\infty]{} 0, \quad \sigma(t) \xrightarrow[t\to\infty]{} \sigma_{eq}, \quad \varphi(\sigma_{eq}) = 0. \quad (6)$$

The system is said to be *Lagrange* stable if the solution $\sigma(t)$ is bounded for every initial condition $\sigma^0$.

Notice that global stability in the sense of Definition 1 does not guarantee Lyapunov stability of a specific equilibrium. As shown by the simplest example of pendulum, the system usually has both stable and unstable equilibria. It may seem that the Lagrange stability is a much weaker property than convergence of all solutions. Nevertheless, it is often possible to prove the *dichotomy* property of the system (Leonov et al., 1996b), ensuring that every solution either converges or grows unbounded. For dichotomic systems, Lagrange and global stability are equivalent. For this reason, our method is based on proving Lagrange stability as a preliminary step, allowing to establish convergence.

Our method will use two techniques, previously introduced in the literature. The first trick is the *Bakaev-Guzh procedure* (Leonov et al., 1992), which is based on the following decomposition of nonlinearity

$$\varphi(\sigma) = \Psi(\sigma) + \nu|\varphi(\sigma)|, \quad \nu \triangleq \frac{\int_0^\Delta \varphi(\zeta)d\zeta}{\int_0^\Delta |\varphi(\zeta)|\,d\zeta}. \quad (7)$$

In view of (4) and (7), we have $\nu \in (-1, 0)$, therefore, $\Psi(\sigma) = \varphi(\sigma) - \nu|\varphi(\sigma)|$ has the same sign as $\varphi(\sigma)$ (and thus $\Psi(\sigma) = 0 \Leftrightarrow \varphi(\sigma) = 0$). From (7) and (3), one has

$$\int_0^\Delta \Psi(\sigma)d\sigma = 0, \quad (8)$$

$$\Psi(\sigma) \geq 0 \text{ if } \sigma \in (\sigma_1, \sigma_2), \quad (9)$$

$$\Psi(\sigma) \leq 0 \text{ if } \sigma \in (\sigma_2, \sigma_1 + \Delta). \quad (10)$$

The second technique is the Leonov's nonlocal reduction method (Leonov, 1984), exploiting the properties of a special *comparison system*. As a comparison system, we use the model of a viscously damped pendulum

$$\begin{aligned} \dot{z} &= -az - \varphi(\sigma) \quad (a > 0), \\ \dot{\sigma} &= z, \end{aligned} \quad (11)$$

which has been exhaustively investigated, see (Leonov et al., 1996b, 1992) and references therein. Equation (11) has Lyapunov stable equilibria $(0, \sigma_1 + k\Delta)$ and saddle-point equilibria $(0, \sigma_2 + k\Delta)$ (here $k = 0, \pm1, \ldots$).

**Proposition 1.** (Leonov et al., 1996b) *There exists a value $a_{cr}$ such that if $a > a_{cr}$ every solution of (11) converges to some equilibrium.*

**Proposition 2.** (Leonov et al., 1996b)*If $a > a_{cr}$ the first order equation*

$$F(\sigma)\frac{dF}{d\sigma} + aF(\sigma) + \varphi(\sigma) = 0 \quad (F(\sigma) = \dot{\sigma} = z), \quad (12)$$

*associated with (11), has a solution $F_0(\sigma)$ such that*

$$\begin{aligned} F_0(\sigma_2) = 0, \quad F_0(\sigma) \neq 0 \quad \forall\sigma \neq \sigma_2, \\ F_0(\sigma) \xrightarrow[\sigma\to\mp\infty]{} \pm\infty. \end{aligned} \quad (13)$$

The solution $F_0(\sigma)$ is produced by two separatrices which meet at the point $(0, \sigma_2)$ (Fig. 1).

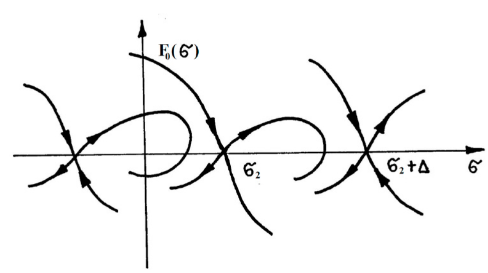


Fig. 1. The separatrices of a saddle and the solution $F_0(\sigma)$

## 3. STABILITY CRITERIA

We now formulate our first main result, establishing a sufficient condition for Lagrange stability.

*Theorem 1.* Suppose there exist positive $\varepsilon, \delta, \varkappa \in [0, 1]$ and $\lambda \in (0, \frac{r}{2})$ satisfying the following two conditions:
1) the frequency-domain condition

$$ReK(\imath\omega - \lambda) - \varepsilon|K(\imath\omega - \lambda)|^2 \geq \delta \quad \forall\omega \geq 0; \quad (14)$$

2)the quadratic form $W$ is positive definite, where

$$W(x, y, z) \triangleq \lambda x^2 + \varepsilon y^2 + \delta z^2 + a_{cr}\sqrt{\varkappa}xy + (1-\varkappa)\nu yz. \quad (15)$$

Then system (1) is Lagrange stable.

**Proof.** The proof consists of three parts.

*A. Popov's method of a priori integral indices (Rasvan, 2006)*

For an arbitrary solution $\sigma(t)$ of (1) and an arbitrary $T > 0$ we introduce the functions

$$\eta(t) \triangleq \varphi(\sigma(t)), \quad \eta_T(t) \triangleq \begin{cases} \eta(t), & t \in [0, T], \\ 0, & t \notin [0, T], \end{cases}$$

$$\zeta_T(t) \triangleq \rho\eta_T(t-h) - \int_0^t \gamma(t-h)\eta(\tau)\,d\tau. \tag{16}$$

Henceforth we denote

$$[f]^\mu(t) = f(t)e^{\mu t}, \quad \rho_0 = \rho e^{\lambda h}. \tag{17}$$

It can be shown that

$$[\zeta_T]^\lambda(t) = \rho_0[\eta_T]^\lambda(t-h) - \int_0^t [\gamma]^\lambda(t-\tau)[\eta]^\lambda(\tau)\,d\tau,$$
$$[\zeta_T]^\lambda(t) = [\dot\sigma]^\lambda(t) - [b]^\lambda(t) \quad \forall t \in [0, T] \tag{18}$$
$$[\eta_T]^\lambda, [\zeta_T]^\lambda \in L_2[0, +\infty) \bigcap L_1[0, +\infty).$$

In view of the Plancherel theorem, the integral functional

$$I_T \triangleq \int_0^\infty \{[\eta_T]^\lambda(t)[\zeta_T]^\lambda(t) + \varepsilon([\zeta_T]^\lambda(t))^2 + \delta([\eta_T]^\lambda)^2\}\,dt$$

can be expressed via the Fourier transforms

$$I_T = \frac{1}{2\pi}\int_{-\infty}^\infty \{\mathfrak{F}^*([\zeta_T]^\lambda)(\imath\omega)\mathfrak{F}([\eta_T]^\lambda(\imath\omega) + \tag{19}$$
$$+\varepsilon|\mathfrak{F}([\zeta_T]^\lambda)(\imath\omega)|^2 + \delta|\mathfrak{F}([\eta_T]^\lambda)(\imath\omega)|^2\}\,d\omega,$$

where $\mathfrak{F}(f)(\imath\omega)$ stands for Fourier–transform of function $f$ and $(^*)$ is the complex conjugation. Since

$$\mathfrak{F}([\zeta_T]^\lambda)(\imath\omega) = -K(\imath\omega - \lambda)\mathfrak{F}([\eta_T]^\lambda)(\imath\omega) \tag{20}$$

the frequency-domain condition (14) implies that

$$I_T \le 0, \quad \forall T > 0. \tag{21}$$

Since $\eta_T(t) = 0\,\forall t > T$, it is obvious that

$$I_T > \int_0^T \{[\eta_T]^\lambda[\zeta_T]^\lambda + \varepsilon([\zeta_T]^\lambda)^2 + \delta([\eta_T]^\lambda)^2\}\,dt \overset{(18)}{=} \tag{22}$$
$$= I_{1T} + I_{2T},$$

$$I_{1T} \triangleq \int_0^T \{\eta(t)\dot\sigma(t) + \varepsilon\dot\sigma^2(t) + \delta\eta^2(t)\}e^{2\lambda t}\,dt, \tag{23}$$
$$I_{2T} \triangleq \int_0^T \{-\eta(t)b(t) - 2\varepsilon b(t)\dot\sigma(t) + \varepsilon b^2(t)\}e^{2\lambda t}\,dt.$$

It follows from Assumption 1 that

$$|I_{2T}| < C_1 \quad \forall T > 0 \tag{24}$$

where $C_1$ depends only on the nonlinearity $\varphi(\cdot)$ and the initial condition. Then we have from (24) and (22) that

$$I_{1T} < C_1, \quad \forall T > 0. \tag{25}$$

The condition 2) of Theorem 1 implies that

$$4\lambda\varepsilon\delta > (1-\varkappa)^2\nu^2\lambda + a_{cr}^2\varkappa\delta. \tag{26}$$

Denoting

$$\varepsilon_2 \triangleq \frac{(1-\varkappa)^2}{4\delta}\nu^2, \quad \varepsilon_1 \triangleq \varepsilon - \varepsilon_2, \tag{27}$$

one obtains from (26) that

$$4\lambda\varepsilon_1 > a_{cr}^2\varkappa. \tag{28}$$

According to (27) one has

$$I_{1T} = J_{1T} + J_{2T}, \tag{29}$$

$$J_{1T} \triangleq \int_0^T \{(1-\varkappa)\varphi(\sigma(t))\dot\sigma(t) + \varepsilon_2\dot\sigma^2(t) + \tag{30}$$
$$+\delta\varphi^2(\sigma(t))\}e^{2\lambda t}\,dt,$$
$$J_{2T} \triangleq \int_0^T \{\varkappa\varphi(\sigma(t))\dot\sigma(t) + \varepsilon_1\dot\sigma^2(t)\}e^{2\lambda t}\,dt.$$

*B. Bakaev–Guzh technique*

Using the decomposition (7), we can now estimate the function $J_{1T}$. Note that

$$J_{1T} = \int_0^T \{(1-\varkappa)\nu|\varphi(\sigma(t))|\dot\sigma(t) + \varepsilon_2\dot\sigma^2(t) +$$
$$+\delta\varphi^2(\sigma(t))\}e^{2\lambda t}\,dt + (1-\varkappa)\int_0^T \Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt. \tag{31}$$

The first addend in the right–hand part of (31) is positive definite thanks to (27). We rewrite the second addend using Bonnet's mean value theorem (Hobson, 1909). Since the function $e^{2\lambda t}$ is increasing on $[0, T]$, there exists $T_0 \in [0, T]$ satisfying the condition

$$\int_0^T \Psi(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt = e^{2\lambda T}\int_{T_0}^T \Psi(\sigma(t))\dot\sigma(t)\,dt = \tag{32}$$
$$= e^{2\lambda T}\int_{\sigma(T_0)}^{\sigma(T)} \Psi(\zeta)\,d\zeta.$$

Then in virtue of the properties (8), (9), (10) one has that

$$\int_{\sigma(T_0)}^{\sigma(T)} \Psi(\zeta)\,d\zeta \ge 0 \tag{33}$$

for any $\sigma(T_0)$ if $\sigma(T) = \sigma_2 + k\Delta$ with $k \in \mathbb{Z}$.

It follows then from (25) and (29) that

$$J_{2T} \le C_1, \text{ if } \sigma(T) = \sigma_2 + k\Delta\ (k \in \mathbb{Z}). \tag{34}$$

*C. Non-local reduction technique*

Consider the equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma} + 2\sqrt{\frac{\lambda\varepsilon_1}{\varkappa}}F(\sigma) + \varphi(\sigma) = 0. \tag{35}$$

It follows from (28) and Proposition 2 that (35) has a solution $F_0(\sigma)$ with the properties (13). Note that $\hat F_0 = \sqrt{\frac{\varkappa}{2}}F_0$ is a solution of the equation

$$\hat F_0(\sigma)\hat F_0'(\sigma) + \sqrt{2\lambda\varepsilon_1}\hat F_0 + \frac{\varkappa}{2}\varphi(\sigma) = 0. \tag{36}$$

Since $\varphi$ is $\Delta$–periodic, (36) has a set of solutions

$$\hat F_k(\sigma) \triangleq \hat F_0(\sigma - k\Delta) \quad (k \in \mathbb{Z}), \tag{37}$$

which are featured by the following properties:

$$\hat F_k(\sigma_2 + k\Delta) = 0, \tag{38}$$
$$\hat F_k(\sigma) \ne 0 \text{ for } \sigma \ne \sigma_2 + k\Delta, \tag{39}$$
$$\hat F_k(\sigma) \to \pm\infty \text{ as } \sigma \to \mp\infty. \tag{40}$$

After some computations, one obtains that

$$J_{2T} = \int_0^T \{\varkappa\varphi(\sigma(t))\dot\sigma(t) + \varepsilon_1\dot\sigma^2(t) + \tag{41}$$
$$+2\hat F_k'(\sigma(t))\hat F_k(\sigma(t))\dot\sigma(t) + 2\lambda\hat F_k^2(\sigma(t))\}e^{2\lambda t}\,dt -$$
$$-\hat F_k^2(\sigma(T))e^{2\lambda T} + \hat F_k^2(\sigma(0)),$$

and therefore

$$J_{2T}=\int_0^T\{G(\dot\sigma(t),\varphi(\sigma(t)),\hat F_k(\sigma(t))\hat F_k'(\sigma(t)))-\\ -\frac{1}{4\varepsilon_1}\Big(\varkappa\varphi(\sigma(t))+2\hat F_k(\sigma(t))\hat F_k'(\sigma(t))\Big)^2+\\ +2\lambda\hat F_k^2(\sigma(t))\}e^{2\lambda t}\,dt-\hat F_k^2(\sigma(T))e^{2\lambda T}+\hat F_k^2(\sigma(0)), \tag{42}$$

where $G$ stands for the quadratic form

$$G(x,y,z)=\left(\sqrt{\varepsilon_1}x+\frac{\varkappa}{2\sqrt{\varepsilon_1}}y+\frac{1}{\sqrt{\varepsilon_1}}z\right)^2\geq 0. \tag{43}$$

Then

$$J_{2T}\geq\int_0^T\Big(\sqrt{2\lambda}\hat F_k(\sigma(t))-\frac{\varkappa}{2\sqrt{\varepsilon_1}}\varphi(\sigma(t))-\\ -\frac{1}{\sqrt{\varepsilon_1}}\hat F_k'(\sigma(t))\hat F_k(\sigma(t))\Big)\Big(\sqrt{2\lambda}\hat F_k(\sigma(t))+\\ \frac{1}{\sqrt{\varepsilon_1}}\hat F_k(\sigma(t))\hat F_k'(\sigma(t))+\frac{\varkappa}{2\sqrt{\varepsilon_1}}\varphi(\sigma(t))\Big)e^{2\lambda t}\,dt-\\ -\hat F_k^2(\sigma(T))e^{2\lambda T}+\hat F_k^2(\sigma(0)). \tag{44}$$

Since $\hat F_k$ is a solution of (36) the first summand in right–hand part of (44) is equal to zero. Then it follows from (44) and (34) that

$$\hat F_k^2(\sigma(t))e^{2\lambda t}\geq\hat F_k^2(\sigma(0))-C_1 \text{ for } \sigma(t)=\sigma_2+l\Delta\ (l,k\in\mathbb{Z}). \tag{45}$$

Let us choose the number $k_0\in N$ so large that

$$\sigma_2-k_0\Delta<\sigma(0)<\sigma_2+k_0\Delta, \tag{46}$$

and

$$\hat F_{\pm k_0}^2(\sigma(0))>C_1. \tag{47}$$

Such number $k_0$ exists in virtue of the properties (38), (39), (40). The inequality (47) implies that

$$\hat F_{\pm k_0}^2(\sigma_2+l\Delta)\neq 0\quad (l\in\mathbb{Z}). \tag{48}$$

So $\sigma(t)$ can not reach the values of $\sigma_2\pm k_0\Delta$:

$$\sigma_2-k_0\Delta<\sigma(t)<\sigma_2+k_0\Delta. \tag{49}$$

Theorem 1 is proved. ■

The conditions of Lagrange stability can be further refined, taking into account the slope restriction on the nonlinearities. Let

$$\mu_1\triangleq\inf_{\sigma\in[0,\Delta)}\varphi'(\sigma);\\ \mu_2\triangleq\sup_{\sigma\in[0,\Delta)}\varphi'(\sigma) \tag{50}$$

It is clear that $\mu_1<0<\mu_2$.

*Theorem 2.* Suppose there exist positive $\varepsilon,\delta,\tau,\varkappa\in[0,1]$, $\lambda\in[0,\frac{r}{2})$, $\alpha_1\leq\mu_1,\alpha_2\geq\mu_2$ such that for $\omega\geq 0$ the frequency–domain inequality

$$\pi(\omega,\lambda)\triangleq Re\{K(\imath\omega-\lambda)-\tau(K(\imath\omega-\lambda)+\\ +\alpha_1^{-1}(\imath\omega-\lambda))^*(K(\imath\omega-\lambda)+\alpha_2^{-1}(\imath\omega-\lambda))\}-\\ -\varepsilon|K(\imath\omega-\lambda)|^2\geq\delta \tag{51}$$

is valid and the quadratic form $W(x,y,z)$ defined by (15) is positive definite. Then the system (1) is Lagrange stable.

**Proof.** This proof retraces the proof of Theorem 1. It also consists of three parts, in each part a certain method being applied. Since algebraic restriction on the parameters ($W(x,y,z)>0$) is the same for both theorems, parts B and C from the proof of Theorem 1 remain the same here. But as the frequency inequality (51) essentially differs from inequality (14) the part A in this proof is more complicated than in the proof of Theorem 1.

Let $\sigma(t)$ be an arbitrary solution of (1), $\eta(t)=\varphi(\sigma(t))$. Determine the function

$$v(t)\triangleq\begin{cases}0, & \text{if } t<0,\\ t, & \text{if } t\in[0,1],\\ 1, & \text{if } t>1.\end{cases} \tag{52}$$

For $T>1$ introduce the functions

$$\eta_T^1(t)\triangleq\begin{cases}v(t)\eta(t), & \text{if } t\leq T,\\ 0, & \text{if } t>T;\end{cases} \tag{53}$$

$$\zeta_T^1(t)\triangleq\rho\eta_T^1(t-h)-\int_0^t\gamma(t-\tau)\eta_T^1(\tau)\,d\tau; \tag{54}$$

$$\sigma_0(t)=b(t)-\int_0^t\gamma(t-\tau)(1-v(\tau))\eta(\tau)\,d\tau-\\ -\rho(v(t-h)-1)\eta(t-h). \tag{55}$$

Taking into account (17) we have:

$$[\zeta_T^1]^\lambda(t)=[\dot\sigma]^\lambda(t)-[\sigma_0]^\lambda(t)\quad\text{for } t\in[0,T], \tag{56}$$

with

$$[\zeta_T^1]^\lambda(t)=\rho_0[\eta_T^1]^\lambda(t-h)-\int_0^t[\gamma]^\lambda(t-\tau)[\eta_T^1]^\lambda(\tau)\,d\tau. \tag{57}$$

Notice that for any$T>1$

$$[\zeta_T^1]^\lambda,[\eta_T^1]^\lambda\in L_2[0,+\infty)\bigcap L_1[0,+\infty) \tag{58}$$

Denote the set of all $\sigma_2+k\Delta$ ($k\in\mathbb{Z}$) by S. Let

$$\Sigma\triangleq\{T:T>1,\,\sigma(T)\in S\}. \tag{59}$$

If $\Sigma$ is bounded then the function $\sigma(t)$ is bounded as well.

Suppose that $\Sigma$ is not bounded. Then determine the function

$$\xi_{T,\lambda}(t)\triangleq\frac{d}{dt}([\eta_T^1]^\lambda)-\lambda[\eta_T^1]^\lambda\quad(T\in\Sigma,t\neq 0,T) \tag{60}$$

and consider the functionals

$$R_T\triangleq\int_0^\infty\{[\eta_T^1]^\lambda[\zeta_T^1]^\lambda+\varepsilon([\zeta_T^1]^\lambda)^2+\delta([\eta_T^1]^\lambda)^2+\\ +\tau([\zeta_T^1]^\lambda-\alpha_1^{-1}\xi_{T,\lambda})([\zeta_T^1]^\lambda-\alpha_2^{-1}\xi_{T,\lambda})\}\,dt\quad(T\in\Sigma). \tag{61}$$

Notice that

$$\mathfrak{F}([\zeta_T^1]^\lambda)(\imath\omega)=-K(\imath\omega-\lambda)\mathfrak{F}([\eta_T^1]^\lambda)(\imath\omega), \tag{62}$$

$$\mathfrak{F}(\frac{d}{dt}[\eta_T^1]^\lambda)=\imath\omega\mathfrak{F}[\eta_T^1]^\lambda(\imath\omega). \tag{63}$$

Then in virtue of Plancherel theorem one has

$$R_T=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Big(Re\{K(\imath\omega-\lambda)-\tau(K(\imath\omega-\lambda)+\\ +\alpha_1^{-1}(\imath\omega-\lambda))^*(K(\imath\omega-\lambda)+\alpha_2^{-1}(\imath\omega-\lambda))\}-\\ -\varepsilon|K(\imath\omega-\lambda)|^2-\delta\Big)|\mathfrak{F}([\eta_T^1]^\lambda)|^2\,dt. \tag{64}$$

It follows from (51) that

$$R_T\leq 0\quad(T\in\Sigma). \tag{65}$$

Substituting in (61) the function $[\zeta_T^1]^\lambda(t)$ computed by (56) and taking into account the definition (53) we have

$$R_T\geq I_{1T}+I_{3T}+I_{4T}\quad(T\in\Sigma). \tag{66}$$

where $I_{1T}$ is defined by (23),

$$I_{3T} \triangleq \int_0^T (\dot\sigma(t) - \alpha_1^{-1}\dot\varphi(\sigma(t)))(\dot\sigma(t) - \alpha_2^{-1}\dot\varphi(\sigma(t)))e^{2\lambda t}\,dt = \\ = \int_0^T (1-\alpha_1^{-1}\varphi'(\sigma(t)))(1-\alpha_2^{-1}\varphi'(\sigma(t)))\dot\sigma^2(t))e^{2\lambda t}\,dt, \tag{67}$$

and the functional $I_{4T}$ is bounded by a constant which does not depend on $T$:

$$|I_{4T}| \le C_2, \quad \forall T > 0. \tag{68}$$

Notice that

$$\alpha_1 \le \frac{d\varphi(\sigma)}{d\sigma} \le \alpha_2, \tag{69}$$

whence

$$I_{3T} \ge 0. \tag{70}$$

It follows from (65), (66), (68), (70) that

$$I_{1T} \le C_2 \quad (T \in \Sigma), \tag{71}$$

where $C_2$ does not depend on $T$. The formula (71) is just alike the formula (25). So beginning from (71) all the argument of the proof of Theorem 1 can be repeated for this theorem. ■

Modifying the Bakaev-Guzh decomposition, one can derive a modified version of Theorem 2 as follows. Denote

$$\Phi(\sigma) \triangleq \sqrt{(1-\alpha_1^{-1}\varphi'(\sigma))(1-\alpha_2^{-1}\varphi'(\sigma))}, \tag{72}$$

with $\alpha_1 \le \mu_1, \alpha_2 \ge \mu_2$ and the constant

$$\nu_0 = \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta |\varphi(\sigma)|\,\Phi(\sigma)\,d\sigma}. \tag{73}$$

*Theorem 3.* Suppose there exist $\lambda \in (0, \frac{r}{2})$, $\varepsilon, \tau, \delta > 0, \alpha_1 = -\alpha_2$ satisfying the following conditions:
1) the frequency-domain condition (51) holds $\forall \omega \ge 0$;
2) the parameters satisfy the inequalities

$$4\lambda\varepsilon > a_{cr}^2\left(1 - \frac{2\sqrt{\tau\delta}}{|\nu_0|}\right), \quad |\nu_0| < 1. \tag{74}$$

Then system (1) is Lagrange stable.

**Proof.** We use here the proof of Theorem 2 starting from the inequality (66), which can be rewritten as follows

$$R_T \ge I_{5T} + I_{4T} \quad (T \in \Sigma), \tag{75}$$

where

$$I_{5T} = \int_0^T \{\delta(\varphi(\sigma(t)))^2 + \varepsilon\dot\sigma^2(t) + \dot\sigma(t)\varphi(\sigma(t)) + \\ +\tau\dot\sigma^2(t)\Phi^2(\sigma(t)\}e^{2\lambda t}\,dt. \tag{76}$$

It follows from (65) and (68) that

$$I_{5T} \le C_2 \quad (T \in \Sigma). \tag{77}$$

Let us choose $\varkappa \in (0,1)$ such that

$$\frac{4\lambda\varepsilon}{a_{cr}^2} > \varkappa > 1 - \frac{2\sqrt{\tau\delta}}{|\nu_0|}. \tag{78}$$

Then

$$1 - \varkappa < \frac{2\sqrt{\tau\delta}}{|\nu_0|}. \tag{79}$$

Let

$$I_{5T} = J_{1T} + J_{2T}, \tag{80}$$

where

$$J_{1T} \triangleq \int_0^T \{(1-\varkappa)\varphi(\sigma(t))\dot\sigma(t) + \delta\varphi^2(\sigma(t)) + \\ +\tau(\dot\sigma(t)\Phi(\sigma(t)))^2)\}e^{2\lambda t}\,dt. \tag{81}$$

$$J_{2T} \triangleq \int_0^T \{\varkappa\varphi(\sigma(t))\dot\sigma(t) + \varepsilon\dot\sigma^2(t)\}e^{2\lambda t}\,dt, \tag{82}$$

Apply Bakaev–Guzh procedure to functional $J_{1T}$. For the purpose introduce the function

$$\Psi_1(\sigma) \triangleq \varphi(\sigma) - \nu_0|\varphi(\sigma)|\Phi(\sigma). \tag{83}$$

Then

$$J_{1T} \triangleq \int_0^T \{(1-\varkappa)\nu_0|\varphi(\sigma(t))|\Phi(\sigma(t))\dot\sigma(t) + \\ +\tau(\dot\sigma(t)\Phi(\sigma(t)))^2 + \delta\varphi^2(\sigma(t))\}e^{2\lambda t}\,dt + \\ +(1-\varkappa)\int_0^T \Psi_1(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt \tag{84}$$

The first summand in right–hand part of (84) is positive definite in virtue of (79). For the second addend we have

$$\int_0^T \Psi_1(\sigma(t))\dot\sigma(t)e^{2\lambda t}\,dt = e^{2\lambda T}\int_{\sigma(T_0)}^{\sigma(T)} \Psi_1(\zeta)\,d(\zeta) \tag{85}$$

where $T_0 \in [0,T]$. Notice that

$$\int_0^\Delta \Psi_1(\zeta)\,d\zeta = 0, \tag{86}$$

$\Psi_1(\sigma_1) = \Psi_1(\sigma_2) = 0$, and $\Psi_1(\zeta)\varphi(\zeta) \ge 0$. So

$$\int_{\sigma(T_0)}^{\sigma(T)} \Psi_1(\zeta)\,d\zeta \ge 0 \quad (T \in \Sigma). \tag{87}$$

It follows then from (77), (80), and (87) that

$$J_{2T} \le C_2 \quad (T \in \Sigma). \tag{88}$$

Consider the equation

$$F(\sigma)\frac{dF(\sigma)}{d\sigma} + 2\sqrt{\frac{\lambda\varepsilon}{\varkappa}}F(\sigma) + \varphi(\sigma) = 0. \tag{89}$$

Since (78) and (89) are valid we can repeat all the argument of part C of Theorem 1 and thus finish the proof of Theorem 3. ■

Each of Theorems 1-3 can be transformed into the criterion of global stability by adding another frequency-domain inequality, being similar to (51), in view of the following *dichotomy criterion.*

*Theorem 4.* (Leonov et al., 1996b) Suppose there exist $\varepsilon, \tau, \delta > 0, \alpha_1 \le \mu_1, \alpha_2 \ge \mu_2$ such that

$$\pi(\omega, 0) \ge \delta, \quad \forall \omega \ge 0. \tag{90}$$

Then any bounded solution of (1) converges (6).

The requirement of Theorem 4 is usually fulfilled if (51) is valid for all $\omega$ and certain $\varepsilon, \tau, \lambda, \delta, \alpha_1, \alpha_2$.

**Example.** Consider a phase–locked loop (PLL) with the proportional integrating filter:

$$K(p) = T\frac{Tmp+1}{Tp+1} \quad (m \in (0,1)) \tag{91}$$

and the sine-shaped nonlinearity

$$\varphi(\sigma) = \sin(\sigma) - \beta \quad (\beta \in (0,1)). \tag{92}$$

Let us apply Theorem 3. We have

$$|\nu_0| = \frac{2\pi\beta}{4\beta + \pi - 2\arcsin\beta + 2\beta\sqrt{1-\beta^2}} \tag{93}$$

Condition $|\nu_0| < 1$ is valid for $\beta \le 0.8$.

For $\delta, \varepsilon, \tau > 0, \lambda \in (0, 0.5T^{-1})$ and $|\alpha_1| = \alpha_2 = 1$ the frequency–domain inequality (51) takes the form

$$\pi(\omega,\lambda) \stackrel{\Delta}{=} \tau(\omega^2+\lambda^2) + ReK(\imath\omega-\lambda) - \\ -(\varepsilon+\tau)|K(\imath\omega-\lambda)|^2 \geq \delta, \quad \forall\omega\in\mathbb{R}. \tag{94}$$

It is not difficult to make certain that if

$$\tau+\varepsilon = \frac{1}{T(m+1)},\ \delta = \frac{mT}{(1+m)} - \delta_0\ (0<\delta_0<\frac{mT}{(1+m)}), \tag{95}$$

the inequality (94) is true for all $\omega \geq 0$, $\lambda$ being chosen small enough, depending on $\delta_0$ and $T$.

Notice that

$$2\sqrt{\tau\delta} = \frac{2\sqrt{m}}{(1+m)} - \varepsilon', \tag{96}$$

with $\varepsilon'$ depending on $\varepsilon$ and $\delta_0$ which can also be chosen small enough.

Consider the case of $m = 0.2, \beta = 0.5$. In this case

$$\frac{2\sqrt{m}}{(1+m)|\nu_0|} > 1.1. \tag{97}$$

So for $\varepsilon, \lambda, \delta_0$ small enough the requirements of Theorem 3 are satisfied for all $T > 0$.

Thus for $m = 0.2$ the capture capacity $\beta$ is no less than 0.5 for all $T > 0$. This estimate is rather good for $T^2 > 20$, since the genuine value of the capture capacity is no greater than 0.6 (Leonov et al., 1996b, p. 232, fig.2.16.7).

Meanwhile it follows from (Gelig et al., 2004) that the NRT applied without the help of Bakaev–Guzh procedure gives for $T^2 > 5$ the estimate $\beta < 0.5$, and more than that the estimate for $\beta$ diminishes as the value of $T^2$ increases. For instance, if $T^2 > 100$ NRT gives $\beta < 0.1$.

On the other hand, it can be shown that for $T < 0.5$ Theorem 2 (with $\varkappa = 1$) guaranties the capture capacity $\beta = 1$, whereas the Bakaev–Guzh procedure gives a more conservative estimate $\beta = 0.93$ (Smirnova and Proskurnikov, 2019). Hence, the combination of the two methods appears to be more efficient than any single method.

## 4. CONCLUSION

In this paper, we study asymptotic behavior of synchronization (pendulum-like) systems with distributed parameters. We combine two methods previously used for stability analysis, namely, Leonov's method of nonlocal reduction and Bakaev–Guzh procedure, introducing a novel class of Lyapunov–type functions. Using Popov's method of a priori integral indices, we derive new frequency–algebraic criteria for the convergence of solutions. In future works, we are going to extend the nonlocal reduction method to high-order integro-differential equations.

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