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A GENERAL METHOD TO CONSTRUCT INVARIANT PDES ON HOMOGENEOUS MANIFOLDS

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Abstract. Let $M = G/H$ be an $(n+1)$-dimensional homogeneous manifold and $J^k(n, M) =: J^k$ be the manifold of k-jets of hypersurfaces of M. The Lie group G acts naturally on each J^k. A G–invariant partial differential equation of order k for hypersurfaces of M (i.e., with n independent variables and 1 dependent one) is defined as a G–invariant hypersurface $\mathcal{E} \subset J^k$. We describe a general method for constructing such invariant partial differential equations for $k \geq 2$. The problem reduces to the description of hypersurfaces, in a certain vector space, which are invariant with respect to the linear action of the stability subgroup $H^{(k-1)}$ of the $(k-1)$–prolonged action of G. We apply this approach to describe invariant partial differential equations for hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$ and in the conformal space $\mathbb{S}^{n+1}$. Our method works under some mild assumptions on the action of G, namely: A1) the group G must have an open orbit in J^{k-1}, and A2) the stabilizer $H^{(k-1)} \subset G$ of the fibre $J^k \to J^{k-1}$ must factorize via the group of translations of the fibre itself.

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1. Introduction

1.1. **Starting point.** In this paper we continue a research program started by us in [1] and, from a slightly different point of view, by D. The in [8], that is developing tailor–made geometric and algebraic methods to explicitly construct partial differential equations (PDEs, for short) that admit a given group of symmetries. The problem itself is rather old and classical: its origins date back to the works of Lie, Darboux, Cartan and others. In what follows, we consider the global problem of describing G–invariant PDEs, where G is a Lie group acting transitively on an $(n+1)$–dimensional homogeneous manifold $J^0 = M = G/H$: such manifold is interpreted as the space of n independent variables and a dependent one. We propose a method to construct such invariant PDEs: our approach admits also a local reformulation in terms of the Lie algebra $\mathfrak{g}$ of the group G and, as such, it can be regarded as a method to construct $\mathfrak{g}$–invariant PDEs. One of the key tool of our analysis is going to be the affine structure of the bundles $\pi_{\ell,\ell-1} : J^\ell(n, M) \longrightarrow J^{\ell-1}(n, M)$ of the spaces of ℓ–jets of n–dimensional embedded submanifolds of M (that is, hypersurfaces of M), for $\ell \geq 2$. By contrast, in the aforementioned paper [1], the authors started from a homogeneous $(2n+1)$–dimensional *contact* manifold for a complex simple Lie group and looked for invariant hypersurfaces, in the corresponding Lagrangian Grassmanian, whose algebraic degree (measured via the Plücker embedding) is minimal: only in some special cases such a minimal degree is attained by the so–called Lagrangian Chow transform of the sub–adjoint varieties, which, in general, display a very high degree; the same output of the Lagrangian Chow transform can be obtained by the original techniques, based on Jordan algebras instead, developed by D. The in [8].

In the classical language of symmetries of PDEs, when a group G acts on M we say that it acts by *point transformations*, whereas when G acts on the contact manifold $J^1(n, M)$ we speak of *contact transformations* instead (see, e.g., [2]); it is well known that not all contact manifolds are the projectivized cotangent bundle of a manifold and, even when they are, not all contact transformations can be obtained by lifting point ones. More precisely, in [1] the departing point is a homogeneous contact manifold with respect to a *complex* simple Lie group of contact transformations, whereas in the present work we deal with *real* Lie groups acting on $M = J^0(n, M)$ and satisfying some mild assumptions, see Section 1.3 below.

1.2. **Preliminary definitions.** Throughout this paper $M = G/H$ will be an $(n+1)$–dimensional homogeneous manifold and $S \subset M$ an embedded hypersurface of M, unless otherwise specified. Locally, in an appropriate local chart

$$(u, \boldsymbol{x}) = (u, x^1, \dots, x^n) \qquad (1)$$

of M, the hypersurface S can be described by an equation $u = f(\boldsymbol{x}) = f(x^1, \dots, x^n)$, where f is a smooth function of the variables $x^1, \dots, x^n$, that we refer to as the *independent* variables, to distinguish them from

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the remaining coordinate u, that is the *dependent* one.[1] We say that such a chart is *admissible* for S or, equivalently, that the hypersurface S is (locally) admissible for the chart $(u,\boldsymbol{x})$. We denote by $S_f = S$ the graph of f:

$$S_f := \{\big(f(\boldsymbol{x})\,,\boldsymbol{x}\big)\} = \{u = f(\boldsymbol{x})\}\,.$$

Given two hypersurfaces S_1 and S_2 through a common point $\boldsymbol{p}$, one can always choose a chart $(u,\boldsymbol{x})$ near $\boldsymbol{p}$ that is admissible for both: $S_1 = S_{f_1}$, $S_2 = S_{f_2}$. This paves the ground for the following definition: even if it is intrinsically geometric, we rather give it in a coordinate–wise form to better fit the general approach of the paper; standard techniques allow to show its independence of the choice of coordinates.

Definition 1.1. Two hypersurfaces S_{f_1}, S_{f_2} through a common point $\boldsymbol{p}=(u,\boldsymbol{x})$ are called *ℓ–equivalent* at $\boldsymbol{p}$ if the Taylor expansions of f_1 and f_2 coincide at $\boldsymbol{x}$ up to order ℓ. The class of ℓ–equivalent hypersurfaces to a given hypersurface S at the point $\boldsymbol{p}$ is denoted by $[S]^\ell_{\boldsymbol{p}}$. The union

$$J^\ell(n,M) := \bigcup_{\boldsymbol{p}\in M}\{[S]^\ell_{\boldsymbol{p}} \mid S \text{ is a hypersurface of } M \text{ passing through } \boldsymbol{p}\}$$

of all these equivalence classes is the *space of ℓ–jets of hypersurfaces* $J^\ell(n,M)$ of M.

Note that $J^1(n,M) = \mathrm{Gr}_n(TM) = \mathbb{P}T^*M$. From now on, when there is no risk of confusion, we let

$$J^\ell := J^\ell(n,M)\,.$$

The natural projections

$$\pi_{\ell,m} : J^\ell \longrightarrow J^m\,,\quad [S]^\ell_{\boldsymbol{p}} \longmapsto [S]^m_{\boldsymbol{p}}\,,\quad \ell > m\,,$$

define a tower of bundles

$$\cdots \longrightarrow J^\ell \longrightarrow J^{\ell-1} \longrightarrow \cdots \longrightarrow J^1 = \mathbb{P}T^*M \longrightarrow J^0 = M\,,$$

that turn out to be affine for $\ell \geq 2$. For any $a^m \in J^m$, the fiber of $\pi_{\ell,m}$ over a^m will be indicated by the symbol

$$J^\ell_{a^m} := \pi^{-1}_{\ell,m}(a^m)\,.$$

1.3. **Assumptions on the Lie group G.** In what follows, unless otherwise specified, o is a fixed point of $M = G/H$ (an "origin") and o^ℓ a point of J^ℓ, so that $M = G\cdot o$. This allows us to consider the fibre $J^\ell_{o^{\ell-1}}$ as a vector space with the origin o^ℓ playing the role of zero vector. The group G acts naturally on each ℓ–jet space J^ℓ:

$$G \ni g : o^\ell = [S]^\ell_o \in J^\ell \to [g(S)]^\ell_{g(o)} \in J^\ell,\ o \in S\,.$$

We observe that such an action preserves the affine structure of the fibres of $\pi_{\ell,\ell-1}$ for $\ell \geq 2$. We denote by $H^{(\ell)}$ the stability subgroup G_{o^ℓ} in G of the point o^ℓ:

$$H^{(\ell)} := G_{o^\ell}\,.$$

In this context, G–invariant PDEs (of order k) are submanifolds of J^k such that $G\cdot\mathcal{E} = \mathcal{E}$.

In order to formulate our method for constructing these G–invariant PDEs, we have to slightly restrict the class of groups G under consideration; more precisely, we are going to assume that there exists a point $o^k \in J^k$, with $k \geq 2$, projecting to $o^{k-1} \in J^{k-1}$ such that:

(A1) the orbit

$$\check{J}^{k-1} := G\cdot o^{k-1} = G/H^{(k-1)} \subset J^{k-1}$$

through o^{k-1} is open;

(A2) the orbit

$$W^k := \tau(H^{(k-1)})\cdot o^k \subset J^k_{o^{k-1}} \tag{2}$$

through o^k, where

$$\tau : H^{(k-1)} \to \mathrm{Aff}(J^k_{o^{k-1}}) \tag{3}$$

is the natural affine action of the stability subgroup $H^{(k-1)} = G_{o^{k-1}}$ on the fibre $J^k_{o^{k-1}}$, is a vector space and the group of translation of W^k, that we denote by T_{W^k}, is contained in $\tau(H^{(k-1)})$.

Remark 1.1. Assumption (A2) implies that there is a point $o^k \in J^k$ over the point o^{k-1} such that the restriction of the affine bundle $\pi_{k,k-1} : J^k \to J^{k-1}$ to the orbit $G\cdot o^k$ is an affine subbundle of $\pi_{k,k-1}$ (over the base $\check{J}^{k-1}$).

[1]A reader who is familiar with the standard literature about jet spaces may have noticed that we reversed the order of $\boldsymbol{x}$ and u: this choice will be more convenient for us as the coordinate u will play the role of the "0^{th} coordinate".

1.4. **A method for obtaining G–invariant PDEs.** Our main concern is the problem of finding G–invariant PDEs $\mathcal{E} \subset J^k$: if G satisfies the mild assumptions explained in Section 1.3 above, the affine structure of the natural bundle $\pi_{k,k-1} : J^k \to J^{k-1}$, for $k \geq 2$, will allow us to recast it as the problem of describing submanifolds of the fibre $J^k_{o^{k-1}}$ that are invariant under the affine action of the stability subgroup $H^{(k-1)}$ in G of o^{k-1}. Moreover, in our main Theorem 3.1, we reduce this last problem to describe G–invariant submanifolds of the quotient vector space

$$V^k := J^k_{o^{k-1}}/W^k \,. \tag{4}$$

Since the action of G on V^k is linear, the above problem becomes a standard problem in the theory of invariants of a linear Lie group, which is much simpler than the initial one—that is, the problem of describing the invariants for a non–linear action of the Lie group G on the manifold J^k. As an application of the main Theorem 3.1, we solve this problem in the case when the homogeneous manifold $M = G/H$ defines either the Euclidean or the conformal geometry (in the sense of F. Klein). We stress that the approach we propose does not rely on machine–aided computations and at the same time sheds light on some geometric properties of the G–invariant PDEs.

1.5. **Structure of the paper.** In Section 2 we recall some basic definitions concerning the geometry of the spaces $J^\ell = J^\ell(n, M)$ of ℓ–order jets of hypersurfaces of an $(n+1)$–dimensional manifold M, as well as of their subbundles, that are systems of PDEs in one unknown variable.

In Section 3, under the assumptions (A1) and (A2) of Section 1.3, we prove the main Theorem 3.1, which reduces the construction of G–invariant PDEs $\mathcal{E} \subset J^k$ to the description of hypersurfaces of V^k that are invariant with respect to the linear action of the stability group $H^{(k-1)}$.

In Section 4 we carefully examine the case when $M = \mathbb{E}^{n+1}$ is the $(n+1)$–dimensional Euclidean space, considered as the homogeneous space $\mathbb{E}^{n+1} = \mathsf{SE}(n+1)/\,\mathsf{SO}(n+1)$ of the group of orientation–preserving motions. It would be sensible to stress that the main purpose of discussing here the Euclidean case is that of testing the results of Section 3 on a particularly simple and well–known ground.

In Section 5 we pass to the case when M is the conformal sphere $\mathbb{S}^{n+1}$, that is a homogeneous space of the special orthogonal group $\mathsf{SO}(1, n+2)$, called also the *Möbius group*: we obtain the conformally invariant PDEs in terms of invariants of a certain space of traceless quadratic forms. For instance, in Section 5.4.1, in the case of two independent variables, we see that the unique $\mathsf{SO}(1,4)$–invariant PDE is expressed in terms of the Fubini's conformally invariant first fundamental form.

We would like to underline that the invariant PDEs we found are expressed as the zero set of a function of the invariants (of a certain order) of the considered group. This, of course, does not guarantee that the so–obtained PDE is a scalar one, as the zero set of a real–valued function is not always a codimension–one submanifold. In fact, this happens for the conformal group, as described in .

A key clarification is in order. The main output of the applicative Sections 4 and 5 are invariant polynomials: their zero sets will then provide us with the G–invariant PDEs, understood as hypersurfaces, that were predicted by the main theoretical result, Theorem 3.1, but only in the case when the aforementioned zero set is a submanifold of codimension one (see on this concern the example treated in Section 5.4.1). In this perspective, in Sections 4 and 5 we produce more invariant objects than those given by Theorem 3.1, and indeed the corresponding Theorem 4.1 and Theorem 5.1 state that among the zero sets there are the PDEs anticipated by Theorem 3.1—they do not claim that these zero sets account for all such PDEs. Such a discrepancy disappears in the complex case, and this is the main reason why in the already cited work [1] the authors worked with complex Lie groups form the outset.

2. The affine structure of the bundles of jet spaces

2.1. **Jets of hypersurfaces of M.** The space J^ℓ has a natural structure of smooth manifold: one way to see this is to extend the local coordinate system (1) on M to a coordinate system

$$(u, \boldsymbol{x}, \dots, u_i, \dots, u_{ij}, \dots, u_{i_1\cdots i_l}, \dots) = (u, x^1, \dots, x^n, \dots, u_i, \dots, u_{ij}, \dots, u_{i_1\cdots i_l}, \dots) \tag{5}$$

on J^ℓ, where each coordinate function[2] $u_{i_1\cdots i_k}$, with $k \leq \ell$, is unambiguously defined by the rule

$$u_{i_1\cdots i_k}\left([S_f]^\ell_{\boldsymbol{p}}\right) = \partial^k_{i_1\cdots i_k} f(\boldsymbol{x})\,, \quad \boldsymbol{p} = (u, \boldsymbol{x})\,, \quad k \leq \ell\,. \tag{6}$$

In formula (6) above the symbol ∂_i denotes the partial derivative ∂_{x^i}, for $i = 1, \dots, n$; we recall that the hypersurface $S = S_f$ is the graph of the function $u = f(\boldsymbol{x})$ and, as such, it is admissible for the chart $(u, \boldsymbol{x})$.

[2] The $u_{i_1\cdots i_k}$'s are symmetric in the lower indices.

The ℓ–lift of S is defined by

$$S^{(\ell)} := \{[S]^{\ell}_{\boldsymbol{p}} \mid \boldsymbol{p} \in S\} . \tag{7}$$

It is an n–dimensional submanifold of J^ℓ. If $S = S_f$ is the graph of $u = f(\boldsymbol{x})$, then $S_f^{(\ell)}$ can be naturally parametrized as follows:[3]

$$\left(u = f(\boldsymbol{x}), \boldsymbol{x}, \dots u_i = \frac{\partial f}{\partial x^i}(\boldsymbol{x}), \dots u_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}), \dots\right) . \tag{8}$$

Remark 2.1. In the case M is a fibre bundle $\pi : M \to B$ with n–dimensional fibres, one can define the space of ℓ–jets $J^l\pi$ of π as the space of ℓ–jets of the graphs of local sections of π. The space $J^\ell\pi$ is an open dense subset of $J^\ell = J^\ell(n, M)$. In the case that $\pi : \mathbb{R} \times N \to N$ is the trivial bundle, $\dim N = n$, then $J^\ell\pi$ coincides with the space of ℓ–jets of functions on N.

2.2. **The tautological bundle and the higher order contact distribution on J^ℓ.** In this section, to not overload the notation, we denote a point $[S]^\ell_{\boldsymbol{p}} \in J^\ell$ by a^ℓ. The next lemma is well known.

Lemma 2.1. *Any point $a^\ell = [S]^\ell_{\boldsymbol{p}} \in J^\ell$ canonically defines the n–dimensional subspace*

$$T_{a^{\ell-1}} S^{(\ell-1)} \subset T_{a^{\ell-1}} J^{\ell-1} , \quad a^{\ell-1} = \pi_{\ell,\ell-1}(a^\ell) . \tag{9}$$

Definition 2.1. The *tautological* rank–n vector bundle $\mathcal{T}^\ell \subset \pi^*_{\ell,\ell-1}(TJ^{\ell-1})$ is the bundle over J^ℓ whose fiber over the point a^ℓ is given by (9), i.e.,

$$\mathcal{T}^\ell = \left\{(a^\ell, v) \in J^\ell \times TJ^{\ell-1} \mid v \in T_{a^{\ell-1}} S^{(\ell-1)}\right\} .$$

The (truncated) total derivatives

$$D_i^{(\ell)} := \partial_{x^i} + \sum_{k=1}^{\ell} \sum_{j_1 \le \cdots \le j_{k-1}} u_{j_1 \dots j_{k-1}\, i}\, \partial_{u_{j_1 \dots j_{k-1}}} , \quad i = 1 \dots n , \tag{10}$$

constitute a local basis of the bundle $\mathcal{T}^\ell$.

By considering the preimage of the tautological bundle on J^ℓ via the differential $d\pi_{\ell,\ell-1}$ of the canonical projection $\pi_{\ell,\ell-1}$, we get a distribution on J^ℓ, denoted by $\mathcal{C}^\ell$:

$$\mathcal{C}^\ell := (d\pi_{\ell,\ell-1})^{-1} \mathcal{T}^\ell .$$

Definition 2.2. $\mathcal{C}^\ell$ is called the ℓ^{th} *order contact structure* or *Cartan distribution* (on J^ℓ).

Above formula, applied to a particular point a^ℓ of J^ℓ that projects on $a^{\ell-1} \in J^{\ell-1}$, reads

$$\mathcal{C}^\ell_{a^\ell} = (d\pi_{\ell,\ell-1})^{-1}(T_{a^{\ell-1}} S^{(\ell-1)}) = \mathcal{T}^\ell_{a^\ell} \oplus T^v_{a^\ell} J^\ell , \tag{11}$$

where $T^v J^\ell := \ker(d\pi_{\ell,\ell-1})$ is the *vertical subbundle* of TJ^ℓ.

Distribution $\mathcal{C}^\ell$ can be considered as a "higher order contact structure" [5, 6] since, for $\ell = 1$, if (u, x^i, u_i) is a chart on J^1, then

$$\mathcal{C} := \mathcal{C}^1 = \ker(\theta) , \quad \text{where} \quad \theta = du - u_i dx^i \tag{12}$$

is a contact form. For $\ell > 1$, the planes of the ℓ^{th} order contact structure $\mathcal{C}^\ell$ are the kernels of the following system of 1–forms (Pfaff system):

$$\theta = du - u_i dx^i , \ \theta_{i_1 \dots i_k} = du_{i_1 \dots i_k} - u_{i_1 \dots i_k h} dx^h, \ \ k < \ell, \ i = 1, \dots, n .$$

2.3. **The affine structure of the bundles $J^\ell \to J^{\ell-1}$ for $\ell \geq 2$.** In this section we describe the affine structure of the bundles $\pi_{\ell,\ell-1} : J^\ell \to J^{\ell-1}$, $\ell \geq 2$. In order to state Proposition 2.1 below, we need to introduce yet another bundle over J^1, tightly related with the tautological bundle defined before: according to Definition 2.1, the tautological bundle $\mathcal{T} := \mathcal{T}^1$ is the vector bundle over J^1 defined by

$$\mathcal{T}_{[S]^1_{\boldsymbol{p}}} := \mathcal{T}^1_{[S]^1_{\boldsymbol{p}}} = T_{\boldsymbol{p}} S .$$

Definition 2.3. The *normal bundle* $\mathcal{N}$ is the following line bundle over J^1:

$$\mathcal{N}_{[S]^1_{\boldsymbol{p}}} := N_{\boldsymbol{p}} S := T_{\boldsymbol{p}} M / T_{\boldsymbol{p}} S .$$

[3]We stress once again that a switch has occurred between the first and the second entry, with respect to a more standard literature.

To simplify notations, we denote by ∂_u the equivalence class $\partial_u \bmod \mathcal{T}$.

Next Lemma is well known (see for instance [5, 7]): it describes the vertical subbundle $T^v J^\ell$ of J^ℓ in terms of the bundles $\mathcal{T}$ and $\mathcal{N}$.

Lemma 2.2. *We have that*

$$T^v J^\ell \simeq \pi^*_{\ell,1}(S^\ell \mathcal{T}^* \otimes \mathcal{N}) \,.$$

In local coordinates (5)*, the above isomorphism gives (up to a constant) the bijection*

$$\frac{\partial}{\partial u_{i_1 \cdots i_\ell}} \longleftrightarrow dx^{i_1} \odot \cdots \odot dx^{i_\ell} \otimes \partial_u \,, \tag{13}$$

where $\odot$ is the symmetric product.

The next proposition is also well known ([5, 7]) and it is crucial for our purposes.

Proposition 2.1. *For $\ell \geq 2$ the bundles $J^\ell \to J^{\ell-1}$ are affine bundles modeled by the vector bundles $\pi^*_{\ell-1,1}(S^\ell \mathcal{T}^* \otimes \mathcal{N})$. In particular, once a chart $(u, \boldsymbol{x})$ has been fixed, the choice of a point $[S]^\ell_{\boldsymbol{p}}$ (the origin) defines the identification $J^\ell_{[S]^{\ell-1}_{\boldsymbol{p}}}$ with $S^\ell T^*_{\boldsymbol{p}} S$.*

Below we give a sketch of the proof based on the action of a generic element v of $\left(\pi^*_{\ell-1,1}(S^\ell \mathcal{T}^* \otimes \mathcal{N})\right)_{[S]^{\ell-1}_{\boldsymbol{p}}}$ on the fibre $J^\ell_{[S]^{\ell-1}_{\boldsymbol{p}}}$ (for more details see for instance [7]). Without loss of generality, we can choose $\boldsymbol{p} = o = (0, \boldsymbol{0})$ and a chart $(u, \boldsymbol{x})$ admissible by S, so that $S = \{u = f(\boldsymbol{x})\} = S_f$. The aforementioned element can be written as

$$v = v_{i_1 \cdots i_\ell}\, dx^{i_1} \odot \cdots \odot dx^{i_\ell} \otimes \partial_u \in S^\ell(T^*_{\boldsymbol{p}} S) \otimes \mathcal{N}_{[S]^1_{\boldsymbol{p}}}$$

(see Lemma 2.2); its action on $[S]^\ell_{\boldsymbol{p}} = [S_f]^\ell_{\boldsymbol{p}}$ is given by $v : [S_f]^\ell_{\boldsymbol{p}} \to [S_g]^\ell_{\boldsymbol{p}}$ where

$$g(\boldsymbol{x}) = f(\boldsymbol{x}) + \frac{1}{\ell!} v_{i_1 \cdots i_\ell} x^{i_1} \cdot \cdots \cdot x^{i_\ell} \,,$$

i.e., the ℓ–order derivatives $f_{i_1 \cdots i_\ell}$ are sent to $g_{i_1 \cdots i_\ell} = f_{i_1 \cdots i_\ell} + v_{i_1 \cdots i_\ell}$.

Warning 2.1. From now on, the symmetric product $dx^i \odot dx^j$ (resp. $\partial_{x^i} \odot \partial_{x^j}$) will be denoted simply by $dx^i dx^j$ (resp. $\partial_{x^i} \partial_{x^j}$).

3. A general construction of G–invariant PDEs on a homogeneous manifold $M = G/H$

As above, let $M = G/H = G \cdot o$, $o \in M$, be an $(n+1)$–dimensional homogeneous manifold. Recall (see Section 1.3) that G acts on each jet space $J^\ell = J^\ell(n, M)$. We recall the following definitions.

Definition 3.1. A *system* of m PDEs of order k is an m–codimensional submanifold $\mathcal{E} \subset J^k$. A *solution* of the system $\mathcal{E}$ is a hypersurface $S \subset M$ such that $[S]^k_{\boldsymbol{p}} \in \mathcal{E}$ for all $\boldsymbol{p} \in S$. The system $\mathcal{E}$ is called *G–invariant* if $G \cdot \mathcal{E} = \mathcal{E}$.

The aim of this section is to reduce the problem of describing G–invariant scalar PDEs (i.e., when $m = 1$) of order k, that is, G–invariant hypersurfaces in J^k, to the problem of describing hypersurfaces in a certain vector space, invariant under the linear action of the stability subgroup $H^{(k-1)}$ of the point $o^{k-1} \in J^{k-1}$.

Below we give some definitions together with some preliminary lemmas that are important to state the main Theorem 3.1.

Definition 3.2. A homogeneous manifold $M = G/H$ is called *k–admissible* for $k \geq 2$ if assumptions (A1) and (A2) of Section 1.3 are satisfied.

Definition 3.3. A hypersurface $S \subset M$ through the point o that is homogeneous with respect to a subgroup of G for which $o^{k-1} = [S]^{k-1}_o$ and $o^k = [S]^k_o$ satisfy (A1) and (A2) of Section 1.3 is called a *fiducial hypersurface.*

Remark 3.1. It is worth stressing that the main theoretical result, Theorem 3.1 below, does not require the existence of a fiducial hypersurface, whereas its applications to the particular cases discussed later in Sections 4 and 5, become geometrically more trasparent thanks to an obvious choice of a fiducial hypersurface; in particular, the latter allows constructing a preferred coordinate system in which the so–obtained invariant PDEs look particularly simple. The authors did not deepen the problem of existence of a fiducial hypersurface for *all* k–admissible homogeneous manifolds $M = G/H$.

Below we state an elementary lemma concerning affine subgroups that are semidirect product of their linear (canonically associated) group and a translational one.

Let V be a vector space, treated as an affine space with origin o. Let H be a subgroup of $\mathrm{Aff}(V) = V \rtimes \mathsf{GL}(V)$. Assume that $W := H \cdot o$ is a vector subspace of V and that the corresponding group of translations T_W is a subgroup of H. Then

$$H = T_W \rtimes L_H\,, \tag{14}$$

where L_H is the linear subgroup of the stabilizer of the origin o. Since T_W is a normal subgroup, we have that $L_H \cdot W = W$. Denote by U a subspace complementary to W, that is

$$V = W \oplus U\,, \tag{15}$$

so that the natural projection $p : V \to V/W$ defines an identification $p|_U : U \to V/W$. In view of (14), an element $h \in H$ can be uniquely presented as

$$h = T_{w(h)} \cdot L_h \in H\,, \tag{16}$$

so that its induced action on V/W corresponds to the linear action given by $h : u \to \overline{L}_h \cdot u$, where L_h, in terms of decomposition (15), is

$$L_h = \begin{pmatrix} * & * \\ 0 & \overline{L}_h \end{pmatrix}. \tag{17}$$

Lemma 3.1. *Let H as in* (14) *and V as in* (15)*. Then there exists a* 1–1 *correspondence between $\overline{L}_H$–invariant hypersurfaces $\overline{\Sigma} \subset U = V/W$ and (cylindrical) H–invariant hypersurfaces $\Sigma = W + \overline{\Sigma}$ in V.*

Proof. Let $\overline{\Sigma} \subset U$ be a $\overline{L}_H$–invariant hypersurface. Then $\Sigma = W + \overline{\Sigma}$ is a H–invariant hypersurface of V since, for each $w + u \in \Sigma = W + \overline{\Sigma}$, in view of (16) and (17),

$$h(w+u) = L_h(T_{w(h)}(w+u)) = L_h(w + w(h) + u) = L_h(w + w(h)) + \overline{L}_h(u) \in W + \overline{\Sigma}\,.$$

Conversely, if $\Sigma \subset V = W \oplus U$ is an H–invariant hypersurface, then $T_W \cdot \Sigma = W + \Sigma \subset \Sigma$, i.e., Σ is a cylindrical hypersurface and the quotient $\overline{\Sigma} = \Sigma \cap U$ is an $\overline{L}_H$–invariant hypersurface in $U = V/W$. □

Recall now that, if S is a fiducial hypersurface in the sense of Definition 3.3, then

$$o^\ell := [S]^\ell_o \in J^\ell$$

plays the role of the origin in J^ℓ. Furthermore, we have the following identification (see Proposition 2.1):

$$J^\ell_{o^{\ell-1}} = S^\ell(T^*_o S) \otimes N_o S\,.$$

If we represent the fiducial hypersurface S as a graph S_f, then we can write (see again Proposition 2.1):

$$J^\ell_{o^{\ell-1}} = S^\ell(T^*_o S_f)\,. \tag{18}$$

From now on, we will use this identification.

For any $h \in H^{(\ell-1)}$ we may decompose the affine transformation $\tau(h) \in \mathrm{Aff}(J^\ell_{o^{\ell-1}})$ (see (3)) into the product of its linear part $A_h \in \mathsf{GL}(S^\ell(T^*_o S_f))$, which is the stabilizer of the point o^ℓ, and the translation T_h along the vector $\tau(h)(o^\ell)$, i.e.,

$$\tau(h) = T_h \cdot A_h\,.$$

Now we assume that the homogeneous manifold $M = G/H$ is k–admissible. Thus, condition (A2) shows that

$$W^k = \tau(H^{(k-1)}) \cdot o^k = \{T_h \cdot o^k,\ h \in H^{(k-1)}\}$$

is a vector subspace of $J^k_{o^{k-1}}$ and, furthermore, that any element $\tau(h) \in \tau(H^{(k-1)})$ can be decomposed as follows (see (16)):

$$\tau(h) = T_{w(h)} \cdot L_h\,, \quad T_{w(h)}\,,\ L_h \in \tau(H^{(k-1)})\,.$$

Hence,

$$\tau(H^{(k-1)}) = T_{W^k} \rtimes L_{H^{(k-1)}}\,, \tag{19}$$

where $L_{H^{(k-1)}}$ is the stabilizer of o^k. Applying Lemma 3.1 to the affine subgroup $\tau(H^{(k-1)}) \subset \mathrm{Aff}(J^k_{o^{k-1}})$ we get the following corollary.

Corollary 3.1. *Let $M = G/H$ be a k–admissible homogeneous manifold. Then there exists a 1–1 correspondence between $L_{H^{(k-1)}}$–invariant hypersurfaces $\overline{\Sigma} \subset J^k_{o^{k-1}}/W^k$ and (cylindrical) $\tau(H^{(k-1)})$–invariant hypersurfaces $\Sigma = p^{-1}(\overline{\Sigma}) \subset J^k_{o^{k-1}}$, where*

$$p : J^k_{o^{k-1}} \to J^k_{o^{k-1}}/W^k \tag{20}$$

is the natural projection.

Lemma 3.2. *Let $\pi : P \longrightarrow B$ be a bundle. Assume that a Lie group G of automorphisms of π, such that $B = G/H$, acts transitively on B, where H is the stabilizer of a point $o \in B$. Then:*

i) *any H–invariant function F on $P_o := \pi^{-1}(o)$ extends to a G–invariant function $\widehat{F}$ on P (where $\widehat{F}(gy) = F(y)$ for $y \in P_o$ and $g \in G$), and this is a 1–1 correspondence;*
ii) *any H–invariant hypersurface Σ of the fiber P_o extends to a G–invariant hypersurface $\mathcal{E}_\Sigma := G \cdot \Sigma$ of P, and this is a 1–1 correspondence.*

Proof. The stabilizer H acts on P_o and we may identify π with the homogeneous bundle $\pi : G \times_H P_o \to B = G/H$ associated with the principal bundle $G \to G/H$ and the action of H on P_o. Recall that $G \times_H P_o$ is the orbit space of the manifold $G \times P_o$ with respect to the action of H, given by

$$H \ni h : (g, y) \mapsto (gh^{-1}, hy) .$$

i) The restriction to P_o of a G–invariant function F on $G \times_H P_o$ is identified with a left–invariant function on $G \times P_o$, which is also H–invariant, that is a function $F(g, y)$ such that $F(g'g, y) = F(gh^{-1}, hy) = F(g, y)$ for all $g, g' \in G$, $h \in H$, $y \in P_o$. Such a function is identified with an H–invariant function on P_o.
ii) An H–invariant hypersurface $\Sigma \subset P_o$ defines a $(G \times H)$–invariant hypersurface $G \times \Sigma \subset G \times P_o$. It projects onto the G–invariant hypersurface $\mathcal{E}_\Sigma = G \cdot \Sigma$ in $P = G \times_H P_o$. □

Corollary 3.1, together with Lemma 3.2, applied to the bundle $\pi_{k,k-1}$, implies the following theorem, which is the main result of this section.

Theorem 3.1. *Let $M = G/H$ be a k–admissible homogeneous manifold (see Definition 3.2). Then there is a natural 1–1 correspondence between $L_{H^{(k-1)}}$– invariant hypersurfaces $\overline{\Sigma}$ (see also (19)) of $J^k_{o^{k-1}}/W^k$ and G–invariant hypersurfaces $\mathcal{E}_{\overline{\Sigma}} := \mathcal{E}_{p^{-1}(\overline{\Sigma})} = G \cdot p^{-1}(\overline{\Sigma})$ of $J^k = J^k(n, M)$, where p is the natural projection (20).*

In view of the discussions we did so far, taking into account Theorem 3.1, we get the following strategy for constructing G–invariant PDEs imposed on the hypersurfaces of a k–admissible homogeneous manifold $M = G/H$:

(1) calculate the orbit $W^k = \tau(H^{(k-1)}) \cdot o^k$ and decompose $\tau(H^{(k-1)})$ accordingly to (19);
(2) describe $L_{H^{(k-1)}}$–invariant hypersurfaces $\overline{\Sigma} \subset V^k = J^k_{o^{k-1}}/W^k$;
(3) write down the G–invariant equations $\mathcal{E}_{\overline{\Sigma}} = G \cdot p^{-1}(\overline{\Sigma})$ in coordinates (5).

In the next sections we implement this strategy for the Euclidean and the conformal space.

Remark 3.2. It may be worth noticing that, in order to have the above strategy work, only the assumptions (A1) and (A2) on the origins o^k and o^{k-1} are strictly necessary. The fiducial hypersurface (see Definition 3.3) is just a way to introduce the aforementioned origins in a more tangible geometric way: indeed, they reflect the presence of a "reference object" to which we compare the jet of a generic hypersurface. The hypothesis of existence of such a fiducial hypersurface, which we even require to be homogeneous with respect to a subgroup of G, is by no means restrictive: we will see below that in both the Euclidean and the conformal case, such a hypersurface clearly exists.

4. Invariant PDEs for hypersurfaces of $\mathbb{E}^{n+1}$

In this section we assume that $M = \mathbb{E}^{n+1} = G/H = \mathsf{SE}(n+1)/\,\mathsf{SO}(n+1)$ is the Euclidean $(n+1)$–dimensional space, considered as homogeneous space of the group

$$G = \mathsf{SE}(n+1) = \mathbb{R}^{n+1} \rtimes \mathsf{SO}(n+1)$$

of orientation–preserving motions. We will find $\mathsf{SE}(n+1)$–invariant PDEs following the approach described in the previous section.

4.1. **Geometry of the jet spaces $J^\ell(n,\mathbb{E}^{n+1})$ for $\ell=1,2$ in terms of $G=\mathsf{SE}(n+1)$.** In this section we give a description of the jet spaces $J^\ell=J^\ell(n,\mathbb{E}^{n+1})$ in terms of the Lie group action of $\mathsf{SE}(n+1)$ and prove that the Euclidean homogeneous space $\mathbb{E}^{n+1}=G/H$ is a 2–admissible manifold.

Denote by $(u,\boldsymbol{x})=(u,x^1,\dots,x^n)$ the Euclidean coordinates associated to an orthonormal frame $\{e_0,e_1,\dots,e_n\}$ at a point $o\in\mathbb{E}^{n+1}$ and identify $\mathbb{E}^{n+1}$ with the arithmetic vector space $\mathbb{R}^{n+1}$ of coordinates. In particular, $o=(0,0,\dots,0)\in\mathbb{R}^{n+1}$. Using the standard euclidean metric, we identify covectors with vectors. We fix the hyperplane

$$S_0=\langle e_1,\dots,e_n\rangle$$

through the origin o. In the coordinates $(u,\boldsymbol{x})$, S_0 is defined by the function $u=f(\boldsymbol{x})=0$ and all the coordinates $u_{i_1\cdots i_\ell}$ of its lift $S_0^{(l)}$ (see (7)–(8)) are identically zero. Below we will show that S_0 is a fiducial hypersurface according to Definition 3.3: to this end we denote by $o^k=[S_0]^k_o$ the distinguished point of J^k over the origin $o\in\mathbb{E}^{n+1}$. In particular $o^1\in J^1$, considered as a tangent vector space, will be denoted by

$$V:=T_oS_0=\langle e_1,\dots,e_n\rangle\,. \tag{21}$$

Proposition 4.1. *The homogeneous space $\mathbb{E}^{n+1}=G/H$ is 2–admissible. More precisely, $J^1=G\cdot o^1=G/H^{(1)}$, where $H^{(1)}=\mathsf{O}(n)$ is the subgroup of $H=\mathsf{SO}(n+1)$ which preserves the vector e_0 up to the sign. The hyperplane S_0 is a fiducial hypersurface.*

Proof. The stability subgroup of the origin $o\in\mathbb{E}^{n+1}$ is $\mathsf{SO}(n+1)$. It acts transitively on the Grassmannian $\mathrm{Gr}_n(T_o\mathbb{E}^{n+1})\simeq\mathbb{P}T^*_o\mathbb{E}^{n+1}$, which is the fiber of the bundle $J^1=\mathbb{P}T^*\mathbb{E}^{n+1}\to J^0=\mathbb{E}^{n+1}$ over the point o. The stability subgroup of the point $o^1=[S_0]^1_o$ is the subgroup $H^{(1)}=\mathsf{O}(n)$ of $H=\mathsf{SO}(n+1)$: it preserves $e_0\in V^\perp$ up to the sign. Hence $J^1=G/H^{(1)}$ and the condition (A1) of Section 1.3 is satisfied.
Let $S_f=\{u=f(\boldsymbol{x})\}$ be a hypersurface through the origin o with unit normal vector e_0 at o, so that $[S_f]^1_o=V$. Then the second jet $[S_f]^2_0$ has coordinates $\boldsymbol{x}=0$, $u=0$, $u_i=\frac{\partial f}{\partial x^i}(0)=0$, $u_{ij}=\frac{\partial^2 f}{\partial x^i\partial x^j}(0)$ (see also the notation (6)) and it is identified with the second fundamental form $\beta=u_{ij}dx^idx^j\in S^2V^*$ of the hypersurface S_f at o. Recalling that the fiber $J^2_{o^1}$ with the origin o^2 is identified with the space S^2V^* (see (18)), the natural action τ of $\mathsf{O}(n)$ on $J^2_{o^1}\simeq S^2V^*$ is:

$$\tau(B):\beta\in S^2V^*\mapsto B^t\beta B\in S^2V^*\,,\quad B\in H^{(1)}=\mathsf{O}(n)\,.$$

In particular, this shows that G has no open orbits in J^2. Furthermore, since $\tau(H^{(1)})$ is a linear group, its translational part W^2 is trivial and the condition (A2) of Section 1.3 is satisfied, so that $\mathbb{E}^{n+1}$ is a 2–admissible homogeneous space and S_0 a fiducial hypersurface. □

Note that, in this case,

$$V^2=J^2_{o^1}/W^2=J^2_{o^1}=S^2V^*\,. \tag{22}$$

Corollary 4.1. *The stability subgroup $H^{(2)}$ of the point $o^2=[S_f]^2_o$ of a hypersurface S_f through the point o with $[S_f]^1_o=o^1$ is the subgroup of $H^{(1)}=\mathsf{O}(n)$ that preserves the second fundamental form β of S_f, that is*

$$H^{(2)}=\{B\in\mathsf{O}(n)\mid\tau(B)(\beta)=\beta\}\,.$$

4.2. **Construction of $\mathsf{SE}(n+1)$–invariant PDEs.** In the considered case, the construction of $\mathsf{SE}(n+1)$–invariant PDEs, in view of Theorem 3.1, reduces to the description of $\tau(\mathsf{O}(n))\simeq\mathsf{O}(n)$–invariant hypersurfaces in $J^2_{o^1}=S^2V^*\simeq S^2\mathbb{R}^n$ (see (21) and the identification (18)).

Denote by $k_1,\dots,k_n$ the eigenvalues of the shape operator $A=g^{-1}\circ\beta$ (principal curvatures), where g is the restriction to (21) of the euclidean metric of $\mathbb{E}^{n+1}$. Any $\mathsf{O}(n)$–invariant polynomial on S^2V^* is a polynomial $F(\sigma_1,\dots,\sigma_n)$ where σ_i, $i=1,\dots,n$, are the elementary symmetric functions of the principal curvatures $k_1,\dots,k_n$ or, equivalently, of the symmetric functions

$$\tau_m=\mathrm{trace}\,(A^m)\,.$$

An invariant polynomial $F=F(\tau_1,\dots,\tau_n)$ defines the $\mathsf{O}(n)$–invariant algebraic hypersurface

$$\Sigma=\overline{\Sigma}=\{F=0\}\subset S^2V^* \tag{23}$$

(see also (22)). The associated hypersurface $\mathcal{E}=\mathcal{E}_\Sigma=\mathsf{SE}(n+1)\cdot\Sigma\subset J^2$ is an $\mathsf{SE}(n+1)$–invariant hypersurface, that defines a second order PDEs polynomial in the second derivatives. Solutions to PDE $\mathcal{E}_\Sigma$ are hypersurface $S\subset M$ whose shape operator $A_{\boldsymbol{p}}$ satisfies, $\forall\,\boldsymbol{p}\in S$, the equation

$$F(\tau_{1\boldsymbol{p}},\dots,\tau_{n\boldsymbol{p}})=F\left(\mathrm{trace}(A_{\boldsymbol{p}}),\mathrm{trace}(A^2_{\boldsymbol{p}}),\dots,\mathrm{trace}(A^n_{\boldsymbol{p}})\right)=0\,. \tag{24}$$

Remark 4.1. More generally, any function $F(\tau_1,\dots,\tau_n)$ defines the $\mathsf{SE}(n+1)$–invariant PDE $F=0$.

Finally, Theorem 3.1 implies the following one.

Theorem 4.1. *Any second order* $\mathsf{SE}(n+1)$*–invariant PDE for hypersurfaces* $S_f = \{u = f(\boldsymbol{x})\}$ *in* $\mathbb{E}^{n+1}$ *which is a polynomial in the second order derivatives of* f *is of the form* (24)*, where* F *is a polynomial of* n *variables.*

4.2.1. *Description of* $\mathsf{SE}(n+1)$*–invariant PDEs in local coordinates.* Below we write down, in coordinates (5), $\mathsf{SE}(n+1)$–invariant PDEs $\mathcal{E} \subset J^2 = J^2(n, \mathbb{E}^{n+1})$ for a hypersurface $S_f = \{u = f(\boldsymbol{x})\}$. The first fundamental form g of S_f is given by

$$g = g_{ij} dx^i dx^j = \big(\delta_{ij} + u_i u_j\big) dx^i dx^j \,,$$

so that

$$g^{-1} = g^{ij} \partial_{x^i} \partial_{x^j} = \left(\frac{\det(g)\delta_{ij} - u_i u_j}{\det(g)}\right) \partial_{x^i} \partial_{x^j} \,, \quad \det(g) = 1 + \sum_{h=1}^{n} u_i^2 \,.$$

The second fundamental form β of S_f is given by

$$\beta = \beta_{ij} dx^i dx^j = \frac{u_{ij}}{\sqrt{\det(g)}} dx^i dx^j \,.$$

Thus, the trace of the shape operator $A = g^{-1} \circ \beta$ of the hypersurface S_f is given by

$$\text{trace}(A) = \text{trace}(g^{-1} \circ \beta) = \sum_{i,j=1}^{n} \frac{\big(\det(g)\delta_{ij} - u_i u_j\big) u_{ij}}{\det(g)^{\frac{3}{2}}} \,, \tag{25}$$

so that one can easily obtain the local expression of a $\mathsf{SE}(n+1)$–invariant PDE in view of Theorem 4.1. For instance, in the case of $n = 2$, i.e., of 2 independent variables, $\tau_1 = \text{trace}(A)$ and the equation $\tau_1 = 0$, in view of (25), gives the classical equation for minimal hypersurfaces:

$$(1 + u_2^2) u_{11} - 2 u_1 u_2 u_{12} + (1 + u_1^2) u_{22} = 0 \,.$$

Also, note that the classical Monge-Ampère equation is obtained as follows:

$$\frac{1}{2}\Big(\big(\text{trace}(A)\big)^2 - \text{trace}(A^2)\Big) = \frac{u_{11} u_{22} - u_{12}^2}{1 + u_1^2 + u_2^2} = 0 \,.$$

5. Invariant PDEs for hypersurfaces of $\mathbb{S}^{n+1}$

In this section, we describe second order G–invariant PDEs in the case when $M = G/H := \mathbb{S}^{n+1} = \mathsf{SO}(1, n+2)/\mathsf{Sim}(\mathbb{E}^{n+1})$ is the conformal sphere, that is the sphere $\mathbb{S}^{n+1}$ endowed with the conformal class $[g]$ of the standard metric g, considered as a homogeneous manifold of the conformal group $G = \mathsf{SO}(1, n+2)$, called also the Möbius (or Lorentz) group. We use the standard model of the conformal sphere as the projectivisation of the light cone in the Minkowski vector space $\mathbb{R}^{1,n+2}$. The stabilizer of the point $o = \mathbb{R}p$, where p is an isotropic line, is isomorphic to the group $\mathsf{Sim}(\mathbb{E}^{n+1})$ of similarities of the Euclidean space $\mathbb{E}^{n+1}$.

5.1. Geometry of the conformal sphere.

5.1.1. *The standard decomposition of the Minkowski space* $W = \mathbb{R}^{1,n+2}$ *and of the Möbius group* $G = \mathsf{SO}(W)$. Let $W = \mathbb{R}^{1,n+2}$ be the pseudo–Euclidean vector space with an orthonormal basis

$$\{p, e_0, e_1, \dots, e_n, q\} \,, \tag{26}$$

where p and q are isotropic vectors. With respect to the basis (26), we have the decomposition

$$W = \mathbb{R}p \oplus E \oplus \mathbb{R}q = \mathbb{R}p \oplus (\mathbb{R}e_0 \oplus E^{e_0}) \oplus \mathbb{R}q \,, \tag{27}$$

where

$$E^0 := \langle e_1, \dots, e_n \rangle \,.$$

We shall denote by g_W the Minkowski metric on W.

Remark 5.1. The Euclidean subspace E is the orthogonal complement of the hyperbolic plane $\mathbb{R}p \oplus \mathbb{R}q$: therefore, it is not determined by the isotropic vector p, but the canonical projection $\pi : E \to \overline{E} := p^\perp / \mathbb{R}p$ is an isometry of E onto the factor space $\overline{E}$, equipped with the induced Euclidean metric. We denote by

$$\overline{E} = \mathbb{R}\bar{e}_0 \oplus \overline{E}^{e_0}$$

the orthogonal decomposition of $\overline{E}$: it is the projection through π of the orthogonal decomposition $E = \mathbb{R}e_0 \oplus E^{e_0}$.

Decomposition (27) can be regarded as a depth–one gradation of the linear space W, which induces the following gradation of the Lie algebra $\mathfrak{g} = \mathfrak{so}(W) = \bigwedge^2 W$ of the Möbius group $G = \mathsf{SO}(W)$:

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} = \underbrace{(\mathbb{R}q \wedge E)}_{\text{deg}=-1} \oplus \underbrace{(\mathbb{R}(p \wedge q) \oplus \mathfrak{so}(E))}_{\text{deg}=0} \oplus \underbrace{(\mathbb{R}p \wedge E)}_{\text{deg}=+1} . \tag{28}$$

The Lie algebra gradation (28) integrates to a (local, i.e., defined in some open dense domain) decomposition of the Möbius group

$$\mathsf{SO}(W) \overset{\text{loc.}}{=} G^{-1} \cdot G^0 \cdot G^{+1}, \tag{29}$$

where

$$\begin{aligned} G^0 &= \mathsf{CO}(E)\,, \\ G^{-1} &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ -\xi & \mathrm{id} & 0 \\ -\frac{1}{2}\|\xi\|^2 & \xi^t & 1 \end{pmatrix} \mid \xi \in E \right\} \simeq E\,, \\ G^{+1} &= \left\{ \begin{pmatrix} 1 & \xi^t & -\frac{1}{2}\|\xi\|^2 \\ 0 & \mathrm{id} & -\xi \\ 0 & 0 & 1 \end{pmatrix} \mid \xi \in E \right\} \simeq E\,. \end{aligned}$$

Note that the groups $G^{\pm 1}$ are isomorphic to the vector group $E = \mathbb{R}^{n+1}$ and that we always consider vectors as column–matrices . One can check directly that $\mathsf{SO}(E) \cdot G^{+1}$ (resp., $\mathsf{SO}(E) \cdot G^{-1}$) is the stabilizer G_p (resp., G_q) of the point p (resp., q) in G.

5.1.2. *The conformal sphere as projectivised light cone* $\mathbb{P}W_0$ *in* W. The isotropic cone

$$W_0 = \{0 \neq w \in W \mid w^2 = 0\}$$

(the set of non–zero isotropic vectors in W) is a homogeneous manifold of the Möbius group $G = \mathsf{SO}(W)$: $W_0 = G/G_p$. The *conformal sphere* $\mathbb{S}^{n+1}$ is defined as the projectivization of the isotropic cone:

$$\mathbb{S}^{n+1} := \mathbb{P}W_0\,.$$

The Möbius group G acts transitively on $\mathbb{S}^{n+1}$. We consider the isotropic line

$$o := \ell_0 := [p]\,,$$

where p is as in (27). Then $\mathbb{S}^{n+1}$ is identified with the homogeneous space

$$\mathbb{S}^{n+1} = G/H = G/G_{[p]}\,,$$

where $H = G_{[p]} = G_0 \cdot G^{+1}$ is the stabilizer of the origin $o = [p]$:

$$G_{[p]} = \left\{ \begin{pmatrix} a & \eta^t & -\frac{a}{2}\|\eta\|^2 \\ 0 & B & -a\eta \\ 0 & 0 & \frac{1}{a} \end{pmatrix}\,, \quad B \in \mathsf{SO}(E)\,,\ \eta \in E\,,\ a \in \mathbb{R} \right\}.$$

In other words,

$$H = G_{[p]} = \mathsf{Sim}(E) = G^{+1} \rtimes \mathsf{CO}(E) = E \rtimes \mathsf{CO}(E) \tag{30}$$

is isomorphic to the *group of similarities* $\mathsf{Sim}(E)$ of the Euclidean space E.

The Lie algebra $\mathfrak{h} = \mathfrak{g}_{[p]}$ of $G_{[p]}$ is the stability (parabolic) subalgebra

$$\mathfrak{h} = \mathfrak{g}_{[p]} = \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} = (\mathbb{R}(p \wedge q) \oplus \mathfrak{so}(E)) \oplus (\mathbb{R}p \wedge E)\,,$$

given by the non–negative part of (28).

5.1.3. *The isotropy group* $j(H)$*, the tangent bundle* $T\mathbb{S}^{n+1}$ *and the bundle* $J^1 = J^1(n, \mathbb{S}^{n+1})$ *of the conformal sphere.* The tangent space to $\mathbb{S}^{n+1}$ at $o = \mathbb{R}p$ is given by

$$T_o\mathbb{S}^{n+1} = \overline{E} = p^\perp/\mathbb{R}p \simeq \mathfrak{g}/\mathfrak{g}_{[p]} = \mathfrak{g}^{-1} = \mathbb{R}q \wedge E\,. \tag{31}$$

The isotropy representation

$$j : H = G^{+1} \rtimes \mathsf{CO}(E) \to \mathsf{GL}(\overline{E}) \tag{32}$$

in the tangent space (31) has kernel G^{+1} and it reduces to the standard action of the linear conformal group $j(H) = \mathsf{CO}(\overline{E}) = \mathbb{R}^+ \times \mathsf{SO}(\overline{E})$ on $\overline{E} = \langle \overline{e}_0, \overline{e}_1, \dots, \overline{e}_n \rangle$. We have the natural identification

$$T\mathbb{S}^{n+1} = G \times_{j(H)} \overline{E} \longrightarrow \mathbb{S}^{n+1} = G/H$$

of the tangent bundle $T\mathbb{S}^{n+1}$ with the homogeneous vector bundle $G\times_{j(H)}\overline{E}$ over G/H, defined by the isotropy action (32).

Denote by $[g_o]$ the $\mathsf{CO}(E)$–invariant conformal metric on the tangent space (31); it defines a G–invariant conformal metric g on the sphere $\mathbb{S}^{n+1}$.

To simplify notation, we set

$$V=\overline{E}^{e_0}\,,$$

so that it is possible to identify V with E^{e_0} (see above Remark 5.1). The hyperplane $V\subset\overline{E}=T_o\mathbb{S}^{n+1}$ is a point of the space $J^1=J^1(n,\mathbb{S}^{n+1})$. We shall denote such point also by o^1.

Recall that the fiber J^1_o of the bundle J^1 at the point $o=[p]$ is identified with the Grassmannian $\mathrm{Gr}_n(\overline{E})$ of hyperplanes of $\overline{E}=T_o\mathbb{S}^{n+1}$ and then with the projective space $\mathbb{P}\overline{E}^*$. The isotropy group $j(H)$ acts transitively on this space and the stability subgroup $j(H)_{o^1}$ of $o^1=V$ is $\mathsf{CO}(V)=\mathbb{R}^+\times\mathsf{O}(V)$. Thus, we get the following proposition.

Proposition 5.1. *The Möbius group G acts transitively on $J^1=J^1(n,\mathbb{S}^{n+1})$ with the stabilizer of the point $o^1=V$ given by $H^{(1)}=G^{+1}\rtimes\mathsf{CO}(V)$. In particular, $J^1=G/H^{(1)}$.*

Corollary 5.1. *In terms of Lie algebras, the isotropy action of $\mathfrak{h}=\mathfrak{g}^0\oplus\mathfrak{g}^{+1}$ on $T_o\mathbb{S}^{n+1}=\overline{E}=\mathbb{R}\overline{e}_0\oplus V$ satisfies*

$$\ker j=\mathfrak{g}^{+1}=\mathbb{R}p\wedge E\,,\ j(\mathfrak{h})=j(\mathfrak{g}^0)=\mathfrak{co}(\overline{E})\,.$$

The stability subalgebra of the point $o^1=V$ in $\mathfrak{h}$ is

$$\mathfrak{h}^{(1)}=\mathbb{R}p\wedge E\oplus\mathbb{R}p\wedge q\oplus\mathfrak{so}(V)=(\mathfrak{so}(V)\oplus V)\oplus(\mathbb{R}p\wedge q\oplus\mathbb{R}e_0\wedge p)\tag{33}$$

and, moreover,

$$j(\mathfrak{h}^{(1)})=j(p\wedge q)\oplus j(\mathfrak{so}(V))\,,$$

where

$$j(p\wedge q)=-\operatorname{id}$$

and

$$j(\mathfrak{so}(V))e_0=0,\ j\big(\mathfrak{so}(E^{e_0})\big)|_V=\mathfrak{so}(V).$$

5.2. **Standard coordinates of the conformal sphere $\mathbb{S}^{n+1}$.** To describe the fiber $J^2_{o^1}$, we define an appropriate coordinate system in $\mathbb{S}^{n+1}$. Let us consider the system of coordinates

$$(\lambda,u,\boldsymbol{x},s):=(\lambda,u,x^1,\dots,x^n,s)\tag{34}$$

in W associated to the basis (26), such that $W\ni w=\lambda p+ue_0+\sum x^ie_i+sq$. We set

$$x=\sum_{i=1}^n x^ie_i\,,\quad \|x\|^2=\sum_{i=1}^n(x^i)^2\,.$$

Coordinates (34) are homogeneous coordinates of the projective space $\mathbb{P}W$. Taking $\lambda=1$, we consider (u,x^i,s) as associated local affine coordinates in $\mathbb{P}W$. Then the conformal sphere $\mathbb{S}^{n+1}=\mathbb{P}W_0$ has local coordinates (u,x^i) such that

$$w=p+ue_0+x+s(x)q\in\mathbb{S}^{n+1}\,,$$

where

$$s(u,x)=-\frac{1}{2}(u^2+\|x\|^2)\,.\tag{35}$$

We call such coordinates the *standard coordinates of the conformal sphere.* They depend on an isotropic vector $p\in\mathbb{R}p=o$, on the lift $E\subset W$ of the tangent space $\overline{E}=p^\perp/\mathbb{R}p$ and on an orthogonal decomposition $E=\mathbb{R}e_0\oplus E^{e_0}$. Then the isotropic vector q is defined as the vector $q\in E^\perp$ with $p\cdot q=1$.

5.3. **Hyperspheres in $\mathbb{S}^{n+1}$ with fixed 1–jet $o^1=V$.** Below we prove that the set of the hyperspheres S of $\mathbb{S}^{n+1}$ through the point o and with given tangent space $T_oS=o^1=V$ forms a 1–parametric family which is an orbit of the stability group $H^{(1)}$ of the point o^1. To see this we calculate the second jet $[S]^2_o$ of a hypersphere S in local coordinates.

Let $e\in W$ be a unit spacelike vector, i.e., $e\cdot e=1$, and let $W^e=e^\perp$ be the orthogonal hyperplane to e.

Definition 5.1. The projectivization $S^e=\mathbb{P}W^e_0$ of the isotropic cone W^e_0 is a hypersurface of the conformal sphere $\mathbb{S}^{n+1}=\mathbb{P}W_0$, which is called a *hypersphere.*

The vector e defines an orientation of S^e. Two hyperspheres (respectively, oriented hyperspheres) S^e, $S^{e'}$ coincides if and only e and e' coincides up to sign (respectively, $e = e'$). Hence, the set of oriented hyperspheres is identified with the anti de Sitter space $W_1 = \{e \in W,\ e\cdot e = 1\}$ of unit vectors, which is the homogeneous space $W_1 = G/G_e = \mathsf{SO}(1, n+2)/\,\mathsf{SO}(1, n+1)$ of the Möbius group G. An element $g \in G$ acts on a hypersphere S^e by

$$gS^e = S^{ge},\ g \in G.$$

Remark 5.2. Let us consider the basis (26). Let

$$W = \mathbb{R}p \oplus E \oplus \mathbb{R}q = \mathbb{R}p \oplus (\mathbb{R}e_0 \oplus E^{e_0}) \oplus \mathbb{R}q$$

be the associated decomposition, and recall that

$$T_o\mathbb{S}^{n+1} = \overline{E} := p^{\perp}/\mathbb{R}p = (\mathbb{R}p \oplus E)/\mathbb{R}p \simeq E\,.$$

Note that S^{e_0} is an oriented hypersphere through the point $o = [p]$ with the tangent space

$$T_oS^{e_0} = o^1 := V = (\mathbb{R}p \oplus E^{e_0})/\mathbb{R}p\,.$$

Lemma 5.1. *The set of the oriented hyperspheres S^e through the point o with a given tangent space $T_oS^e = V$ forms the 1–parametric family S^{e_0-tp}, $t \in \mathbb{R}$, which is an orbit of the action τ of the stability subgroup $H^{(1)} = G^{+1} \rtimes \mathsf{CO}(\overline{E}^{e_0})$ of the point o^1 on the fiber $J^2_{o^1}$. More precisely, the 1–parametric subgroup*

$$A^{e_0}_t := \exp t(e_0 \wedge p)$$

of $G^{+1} = \ker j$ acts transitively on the set of the hyperspheres of $\mathbb{S}^{n+1}$ through the point o and with fixed tangent space $o^1 = V$ and transforms S^{e_0} into S^{e_0-tp}.

Proof. Let S^e be an oriented hypersphere through the point $o = [p]$. Then T_oS^e is a hyperplane in $T_o\mathbb{S}^{n+1} = \overline{E} = (\mathbb{R}p \oplus \mathbb{R}e_0 \oplus E^{e_0})/\mathbb{R}p$, orthogonal to e. If $T_oS^e = V = (\mathbb{R}p \oplus E^{e_0})/\mathbb{R}p$, then the unit vector $e = \pm e_0 + \mu p$, for some $\mu \in \mathbb{R}$. If the hyperspheres S^e and S^{e_0} have the same orientation, then $e = e_0 + \mu p$. Since the group G acts transitively both on the set of hyperspheres (which is isomorphic to the anti de Sitter space of unit space–like vectors) and on J^1, the stability group $H^{(1)}$ of the point o^1 acts transitively on the set of hyperspheres with fixed 1–jet o^1. We describe the action of the 1–parametric subgroup $A^{e_0}_t = \exp t(e_0 \wedge p)$ on S^{e_0}. Since $e_0 \wedge p$ acts by $q \to e_0 \to -p \to 0\,, E^{e_0} \to 0$, we get $A^{e_0}_t = \mathrm{id} + te_0 \wedge p + \frac{1}{2}t^2(e_0 \wedge p)^2$. Thus, we obtain the following formula for the action of $A^{e_0}_t$:

$$A^{e_0}_t : \begin{cases} p \to p \\ e_0 \to e_0 - tp \\ x \to x \\ q \to q + te_0 - \frac{1}{2}t^2p \end{cases} \tag{36}$$

This shows that $A^{e_0}_t(S^{e_0}) = S^{e_0-tp}$. ☐

5.3.1. *The affine action τ of the stability subgroup $H^{(1)}$ on the fiber $J^2_{o^1}$ and $\mathsf{SO}(1, n+2)$–invariant PDEs.* In view of (33), we write the group $H^{(1)}$ as the direct product of the conformal group $V \rtimes \mathsf{CO}(V)$ of V and a 1–dimensional central subgroup A^{e_0} generated by the Lie algebra $\mathbb{R}e_0 \wedge p$:

$$H^{(1)} = (V \rtimes \mathsf{CO}(V)) \times A^{e_0}\,, \tag{37}$$

where

$$A^{e_0} = \exp \mathbb{R}e_0 \wedge p = \{A^{e_0}_t\,,\ t \in \mathbb{R}\}\,.$$

We will see that the conformal group $V \rtimes \mathsf{CO}(V)$ acts in the natural way on the space S^2V^* as a linear group while the central subgroup A^{e_0} acts via parallel translations in the direction of $g \in S^2V^*$, where g the restriction of the Minkowski metric g_W to $E^{e_0} = V$:

$$g := (g_W)\big|_V\,. \tag{38}$$

In order to show that S^{e_0} is a fiducial hypersurface, we set

$$o^\ell := [S^{e_0}]^\ell_o$$

and we compute the 2–jet $[S^{e_0+\mu p}]^2_o$ at o of the hypersphere $S^{e_0+\mu p}$. The action of each element of the stability subgroup $H^{(1)} = G^{+1} \rtimes \mathsf{CO}(V) \subset \mathsf{SO}(W)$ on a hypersurface

$$S_f = \{w = p + f(x)e_0 + x + s(x)q\,,\ x \in E^{e_0}\}\,, \quad [S_f]^1_o = o^1\,, \tag{39}$$

where $s(x) := s(f(x), x)$ is given by (35), can be easily described. Actually, we only need to know the action of the elements $B \in \mathsf{CO}(V)$ and $A^{e_0}_t \in G^{+1} = \ker j$. Since each element $B \in \mathsf{CO}(V)$ is a linear transformation which acts trivially on p and q, we have the following Lemma.

Lemma 5.2. *Let S_f as in (39) be a hypersurface such that $f(0) = 0$, $u_i = \frac{\partial f}{\partial x^i}(0) = 0$, so that $[S_f]^1_o = o^1 = V$. Let*

$$[S_f]^2_o = u_{ij}dx^i dx^j =: \beta \in S^2 T^*_o S_f = S^2 V^*\,, \quad u_{ij} = \tfrac{\partial^2 f}{\partial x^i \partial x^j}(0)\,. \tag{40}$$

Then an element $B \in \mathsf{CO}(V)$ transforms the hypersurface S_f into $B(S_f) = S_{B^*f}$, where $(B^*f)(x) = f(B(x))$. In particular, B acts on the 2–jet β in the standard way

$$(B^*\beta)(x,x) = \beta(B(x), B(x))\,,\ x \in V\,.$$

Lemma 5.3. *The element $A^v_t := \exp tv \wedge p$ acts trivially on the fiber $J^2_{o^1} = S^2V^*$, for any $v \in V$.*

Proof. It is a straightforward computation based on the formula

$$A^v_t = \mathrm{id} + tv \wedge p - \frac{1}{2}t^2\|v\|^2 p \otimes p\,,$$

where $p \otimes p$ is meant as an endomorphism via the metric (38).

□

In view of (36) and (39), we have the following description of the action of $A^{e_0}_t$ on the hypersurface S_f with parametric equation $u = f(x)$.

Lemma 5.4. *In the hypotheses of Lemma 5.2, there exists a $\widetilde{f}_t$ such that*

$$A^{e_0}_t(S_f) = S_{\widetilde{f}_t} = \left\{ \left(1 - f(x)t - \frac{1}{2}s(x)t^2\right)p + (f(x) + ts(x))e_0 + x + s(x)q \right\}.$$

In particular, the action of $A^{e_0}_t$ on $[S_f]^2_o = \beta$ is

$$A^{e_0}_t([S_f]^2_o) = [S_{\widetilde{f}_t}]^2_o = [S_f]^2_o - \frac{1}{2}tg = \beta - \frac{1}{2}tg\,,$$

where g is given by (38).

Proof. It is a straightforward computation based on the construction of the desired $\widetilde{f}_t$:

$$\widetilde{f}_t(\widetilde{x}) = \frac{f(x) + ts(x)}{1 - f(x)t - \frac{1}{2}s(x)t^2}\,, \quad \widetilde{x} = \frac{x}{1 - f(x)t - \frac{1}{2}s(x)t^2}\,.$$

□

In view of the above lemmas and recalling that $J^2_{o^1} = S^2V^*$, we see that the action $\tau : H^{(1)} \to \mathrm{Aff}(S^2V^*)$ is given by:

$$\begin{aligned} \tau(V) &= \mathrm{Id}\,, \\ \tau(B)(\beta) &= \beta(B(\cdot), B(\cdot))\,, \quad B \in \mathsf{CO}(V)\,, \\ \tau(A^{e_0}_t)(\beta) &= \beta - \frac{1}{2}tg\,. \end{aligned}$$

The next corollary follows from the fact that the hypersphere $S^{e_0} = \{p + x + s(x)q\}$ is defined by the equation $u(x) = 0$.

Corollary 5.2. *The 1–parametric family $S_t := S^{e_0 - tp} = A^{e_0}_t(S^{e_0})$ of hyperspheres of $\mathbb{S}^{n+1}$ with tangent space $o^1 = V$ has 2–jets $[S_t]^2_o = -\frac{1}{2}tg$. Hence, the orbit $H^{(1)} \cdot o^2 = \{A^{e_0}_t(o^2) = -\frac{1}{2}tg\,, t \in \mathbb{R}\}$ and the hypersphere S^{e_0} is a fiducial hypersurface.*

Corollary 5.3. *The central subgroup A^{e_0} of the group $H^{(1)}$ (see (37)) acts on the fiber $J^2_{o^1} = S^2V^*$ as parallel translation along the line $\mathbb{R}g$ and the normal subgroup $V \rtimes \mathsf{CO}(V)$ acts by linear transformation* $\ker(\tau) = V$ *in a natural way. In particular the conformal sphere $\mathbb{S}^{n+1}$ is a 2–admissible homogeneous manifold.*

5.4. **Construction of $\mathsf{SO}(1, n+2)$–invariant PDEs.** Now we are ready to give a construction of all $\mathsf{SO}(1, n+2)$–invariant second order PDEs for hypersurfaces in $\mathbb{S}^{n+1}$.

Like in the Euclidean case, Theorem 3.1 reduces the description of such PDEs to the description of $\mathsf{CO}(V)$–invariant hypersurfaces $\overline{\Sigma} \subset S^2_0(V^*)$, where $S^2_0(V^*)$ is the space of trace–free quadratic forms on the tangent space $V = T_oS^{e_0} = o^1$. According to our strategy, we have to construct a quotient of the affine space $J^2_{o^1}$ where the action of the group $H^{(1)}$ becomes linear. Recall that the second jet $[S_f]^2_o$ of a hypersurface S_f with $[S_f]^1_o = o^1$ is represented by the quadratic form β, see (40).

Definition 5.2. The traceless part $A_\circ$ of the shape operator $A = g^{-1} \circ \beta$, where g is as in (38) and β as in (40) is called the *conformal shape operator of S_f* at the point o.

Let us observe that the conformal shape operator $A_{\circ a}$ of a hypersurface S_f of M is well defined at any point $a \in S_f$ and it depends only upon $[S_f]^2_a$. Moreover, the action of the group G_{a^1} on $A_{\circ a}$ reduces to the standard action of the group $\mathsf{CO}(V)$ on the space of traceless forms $S^2_0 V^*$ (see also [4]). Finally, let us recall that the relative invariants of $S^2_0 V^*$ with respect to the group $\mathsf{CO}(V)$ are homogeneous polynomials

$$F = F(\sigma_2^\circ, \dots, \sigma_n^\circ)\,,\ \deg(\sigma_h^\circ) = h\,,$$

where the $\sigma_i^\circ = \sigma_i^\circ(k_1, \dots, k_n)$'s are the elementary symmetric functions of the eigenvalues of the conformal shape operator $A_\circ$, or, equivalently,

$$F = F(\tau_2^\circ, \dots, \tau_n^\circ)\,,\ \tau_h^\circ := \mathrm{trace}(A_\circ^h)\,.$$

Such an invariant polynomial defines the $\mathsf{CO}(V)$–invariant algebraic hypersurface

$$\overline{\Sigma} = \{F = 0\} \subset S^2_0 V^*\,. \tag{41}$$

The associated hypersurface $\mathcal{E} = \mathcal{E}_{\overline{\Sigma}} = \mathsf{SO}(1, n+2) \cdot \Sigma \subset J^2$ is an $\mathsf{SO}(1, n+2)$–invariant hypersurface, that defines a second order PDEs polynomial in the second derivatives. Solutions to PDE $\mathcal{E}_{\overline{\Sigma}}$ are hypersurface $S \subset M$ whose conformal shape operator $A_{\circ \boldsymbol{p}}$ satisfies, $\forall\, \boldsymbol{p} \in S$, the equation

$$F(\tau^\circ_{2\boldsymbol{p}}, \dots, \tau^\circ_{n\boldsymbol{p}}) = F\left(\mathrm{trace}(A^2_{\circ\boldsymbol{p}}), \dots, \mathrm{trace}(A^n_{\circ\boldsymbol{p}})\right) = 0\,. \tag{42}$$

In analogy with the Euclidean case, we can once again conclude that Theorem 3.1 implies the following one.

Theorem 5.1. *Any second order $\mathsf{SO}(1, n+2)$–invariant PDE for hypersurfaces of $\mathbb{S}^{n+1}$ which is a polynomial in the second–order derivatives is of the form (42), where F is a homogeneous polynomial, $\deg(\tau_h^\circ) = h$.*

5.4.1. *Description of $\mathsf{SO}(1, n+2)$–invariant PDEs in local coordinates.* In the present section one can use the local description of g^{-1} and β contained in Section 4.2.1

Let $A = g^{-1} \circ \beta$ the shape operator of a hypersurface $S_f = \{u = f(\boldsymbol{x})\}$, where g and β are, respectively, its first and the second fundamental form. The traceless second fundamental form $\beta_\circ$ and the conformal shape operator of a hypersurface $S_f = \{u = f(\boldsymbol{x})\}$ is

$$\beta_\circ = \beta - \frac{1}{n}\,\mathrm{trace}(A) g = \beta - Hg\,, \quad A_\circ = g^{-1} \circ \beta_\circ = A - \frac{\mathrm{trace}(A)}{n}\mathrm{Id} = A - H\,\mathrm{Id}\,,$$

where H is the mean curvature.

For $n = 2$ the only relative conformal invariant is

$$\det(A_\circ) \left(= -\frac{1}{2}\,\mathrm{trace}(A_\circ^2)\right) = \det(A) - \frac{1}{4}\,\mathrm{trace}(A)^2 = K - H^2\,,$$

where K is the Gaussian curvature. We underline that the quantity $H^2 - K$ is the coefficient of the Fubini first conformally invariant fundamental form.[4] Also, $H^2 - K = \frac{1}{4}(k_1 - k_2)^2$, k_i being the principal curvatures, so that $\mathcal{E} := \{H^2 - K = 0\}$ describes points having the same principal curvatures, i.e., umbilical ones. A deeper analysis shows that $\mathcal{E}$ is a *system* of two PDEs. Indeed, examining $\mathcal{E}$ over the point o^1, i.e., with $u_1 = u_2 = 0$, one obtains a sum of squares. To see this it is enough to recall that

$$\begin{aligned} H &= \frac{1}{2}\frac{(u_2^2 + 1)u_{11} - 2u_1u_2 + (u_1^2 + 1)u_{22}}{(u_1^2 + u_2^2 + 1)^{\frac{3}{2}}}\,, \\ K &= \frac{u_{11}u_{11} - u_{12}^2}{(u_1^2 + u_2^2 + 1)^2}\,, \end{aligned}$$

and then to replace the values $u_1 = 0$, $u_2 = 0$ in

$$H^2 - K = \frac{1}{4}\frac{((u_2^2 + 1)u_{11} - 2u_1u_2 + (u_1^2 + 1)u_{22})^2}{(u_1^2 + u_2^2 + 1)^3} - \frac{u_{11}u_{22} - u_{12}^2}{(u_1^2 + u_2^2 + 1)^2}\,,$$

which yelds immediately

$$(H^2 - K)_{o^1} = \left(\frac{1}{2}u_{11} - \frac{1}{2}u_{22}\right)^2 + u_{12}^2\,.$$

By invariance, it follows that also the whole PDE $\mathcal{E}$ is a subset of codimension 2.

[4]In the work [3] three of us (JG, GM and GM) have clarified the role of the conformally invariant fundamental form in the theory of PDEs of Monge–Ampère type.

For $n=3$, taking also into account that the characteristic polynomial of $A_\circ$ is $\sum_{i=0}^n \sigma^\circ_{n-i}\lambda^i$ and that of A is $\sum_{i=0}^n \sigma_{n-i}\lambda^i$, we have that

$$\begin{aligned}
\sigma^\circ_3 \quad &= \det(A_\circ) = \frac{1}{3}\operatorname{trace}(A_\circ^3) = \frac{2}{27}(\operatorname{trace}(A))^3 + \frac{1}{3}\operatorname{trace}(A)\sigma_2 + \det(A) = \\
&= 2H^3 + H\sigma_2 + K\,, \\
\sigma^\circ_2 \quad &= \frac{1}{2}\operatorname{trace}(A_\circ^2) = \frac{1}{3}(\operatorname{trace}(A))^2 + \sigma_2 = 3H^2 + \sigma_2\,.
\end{aligned}$$

Thus, for instance,

$$2H^3 + H\sigma_2 + K = 0\,, \quad 3H^2 + \sigma_2 = 0\,, \quad (2H^3 + H\sigma_2 + K)^2 + (3H^2 + \sigma_2)^3 = 0$$

are $\mathsf{SO}(1,5)$–invariant PDEs.

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