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Reducing the Dimensionality of 6-D MoM Integrals Applying Twice the Divergence Theorem

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***Abstract*—In this paper we propose a scheme for evaluating the 6-D interaction integrals appearing in volume integral equation solved with the Method of Moments and tetrahedral elements. We treat as a whole the double volume integral, applying the divergence theorem first on the source domain and then on the test domain. With the proper variable transformation and reordering, the 6-D integrals are expressed as two radial integrals plus four linear integrals over the source and observation domain planes.**

***Index Terms*—integral equations, moment methods, numerical analysis, singular integrals.**

I. Introduction

Volume integral equation (VIE) techniques are essential in helping us to obtain accurate solutions of electromagnetic (EM) problems using the method of moments (MoM). VIE are particularly useful in cases involving inhomogeneous materials. However, the difficult-to-compute the singular potential integrals present in MoM system matrices has hampered the rigorous solution of radiation and scattering problems needed to develop numerical codes able to adequately model and predict the EM behavior of analyzed problems. Recently Bleszynski et al. presented a method allowing an analytical conversion of expressions for matrix elements of the tensor and vector Green functions from 6-D volumetric to 4-D surface integrals with nonsingular integrands [1].

In this paper we extend the applicability of the method developed for 4-D reaction integrals in [2], [3] to the evaluation of double volumetric integrals on source and test domains of the 6-D reaction integrals. The proposed method for 4-D reaction integrals consists in the double application of the divergence theorem, reducing the 4-D integrals to two radial integrals plus two contour integrals over the source and observation domain boundaries. Analogously, for the volumetric case the theorem is applied twice reducing the double volume integrals to two radial integrals plus two surface integrals over the source and observation domain boundaries. The divergence theorem is applied directly in the physical domains for both the source and observation point integrals. In this sense, the scheme is quite general, i.e., is not limited to well-shaped elements nor to ad-hoc treatments of self-, edge-, or vertex-adjacent geometries. Unlike the surface case, in the volumetric case, points in the source or test element plane do not need to be imaged in the plane of the other element.

Additionally, we introduce a parameterization that will allow us to reduce the two 4-D remaining integral to four 1-D integrals that can be evaluated analytically.

II. Formulation

The aim of this paper is to perform an accurate and efficient evaluation of 6-D integrals of the form

$$I_{V,V'} = \int_V \int_{V'} F(\mathbf{r},\mathbf{r}')\, dV' dV, \qquad (1)$$

where $F(\mathbf{r},\mathbf{r}')$ typically takes the form

$$F(\mathbf{r},\mathbf{r}') = t(\mathbf{r}) g(\mathbf{r},\mathbf{r}') b(\mathbf{r}'), \qquad (2)$$

being $t(\mathbf{r})$ either a scalar or a vector component of a test basis function, and $b(\mathbf{r}')$ a similarly defined (source) basis function, $g(\mathbf{r},\mathbf{r}')$ is either a scalar, or a vector or dyadic Green's function, and V and V' are the volumetric domains of tetrahedral test and source basis functions, respectively.

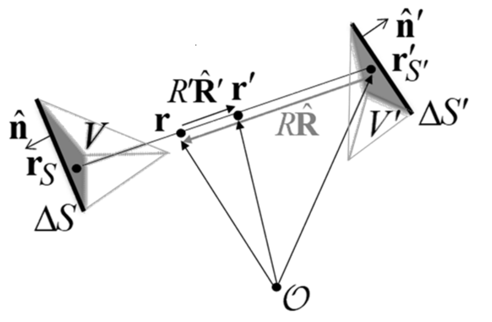

Fig. 1. The orientation of a pair of tetrahedral elements in space and some geometry definitions.

Applying the twice the divergence theorem as described in [4], [5], the integral (1) can be written as

$$\oint_{S'}\oint_{S}\frac{(\hat{\mathbf{n}}\cdot\hat{\mathbf{R}})(\hat{\mathbf{n}'}\cdot\hat{\mathbf{R}}')}{{R_{SS'}}^2}\int_0^{R_{SS'}}\int_0^{R} F(\mathbf{r},\mathbf{r}')R'^2\,dR'\,dR\,dS\,dS', \qquad (3)$$

where $\mathbf{r}'_{S'}$ is a point on the boundary S' of V', and $\hat{\mathbf{n}}'$ is the external normal to the boundary surface of the tetrahedral volume, and $\mathbf{r}' = \mathbf{r} + R'\hat{\mathbf{R}}'$, $\hat{\mathbf{R}}' = (\mathbf{r}' - \mathbf{r})/R'$, $0 \leq R' \leq R \equiv |\mathbf{r} - \mathbf{r}'_{S'}|$, $R' = |\mathbf{r}' - \mathbf{r}|$. S is the boundary of V, $\hat{\mathbf{n}}$ is the outward pointing normal to S, $\mathbf{r} = \mathbf{r}'_{S'} + R\hat{\mathbf{R}}$, $\hat{\mathbf{R}} = -\hat{\mathbf{R}}' = (\mathbf{r}_S - \mathbf{r}_{\mathbf{S}'})/R_{SS'}$, $0 \leq R \leq R_{SS'}$, $R_{SS'} = |\mathbf{r}_S - \mathbf{r}'_{S'}|$, and $\mathbf{r}_S$ is a point on S. All these definitions can be easily seen in Fig. 1. The interchange of integration order is permitted by the independence of the observation and source coordinate variables and their associated domains. The resulting representation has features in common with those of [6], with two inner radial integrals and two outer integrals over source and test element surfaces. The radial integrals can be performed in closed form for both the dynamic and static forms of the free space Greens function $G\left(|\mathbf{r} - \mathbf{r}'|^{-1}\right)$ and $\nabla G\left(|\mathbf{r} - \mathbf{r}'|^{-1}\right)$, with or without polynomial vector bases.

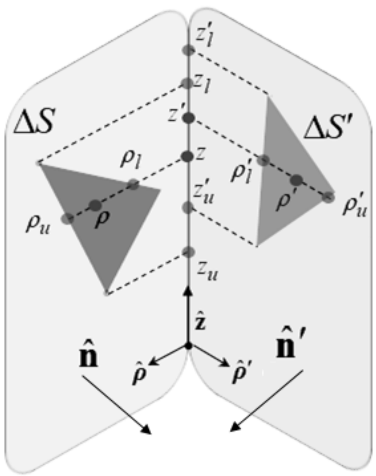

Fig. 2. Geometry definitions for integrating over a line segment pair.

To analyze the two outer surface integrals, we apply variable transformations for both surfaces. If we consider

$$\mathcal{F}(\mathbf{r}, \mathbf{r}') = \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{R}})(\hat{\mathbf{n}}' \cdot \hat{\mathbf{R}}')}{{R_{SS'}}^2} \int_0^{R_{SS'}} \int_0^{R} F(\mathbf{r}, \mathbf{r}') {R'}^2 dR' dR\,, \quad (4)$$

the contribution to the boundary integral (3) of a single face pair, ΔS and $\Delta S'$, can be written as

$$\int\limits_{\Delta S'} \int\limits_{\Delta S} \mathcal{F}(\mathbf{r}, \mathbf{r}') dS\, dS' = \int_z \int_\rho \int_{z'} \int_{\rho'} \mathcal{F}(z, \rho, z', \rho') d\rho' dz' d\rho dz, \quad (5)$$

where the surfaces ΔS and $\Delta S'$ are parameterized using the intersection line of these two surfaces, $\hat{\mathbf{z}}$ is directed along the line of intersection, and the vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\rho}}'$ are orthogonal to the intersection line and the normals to the planes that contain the surfaces S and S' respectively, as seen in Fig. 2. Due to the independence of the observation and source coordinate variables and their associated domains, we can reorder the integral (5) as

$$\int\limits_{\Delta S'} \int\limits_{\Delta S} \mathcal{F}(\mathbf{r}, \mathbf{r}') dS\, dS' = \int_z \int_{z'} \int_\rho \int_{\rho'} \mathcal{F}(z, \rho, z', \rho') d\rho' d\rho dz' dz. \quad (6)$$

To further smooth the integral we apply a double variable transformation for ρ and ρ' as

$$u = \sqrt{1 - \cos\beta} \cdot \frac{\rho + \rho'}{\sqrt{2}}, \quad v = \sqrt{1 + \cos\beta} \cdot \frac{\rho - \rho'}{\sqrt{2}}, \quad (7)$$

where β is the angle between the two planes containing the observation and source surface integration domains. In order to help us implement some further transformations to further smooth integrands of (6), we consider now that the inner function of the radial integral (4) is constant (i.e., the static kernel, $F(\mathbf{r}, \mathbf{r}') = 1/R$). Using these variable transformations and considering the static kernel we can write the two integrals in ρ and ρ' as

$$\begin{aligned} &\int_\rho \int_{\rho'} \mathcal{F}(z, \rho, z', \rho') d\rho' d\rho = \\ &\int_\rho \int_{\rho'} \frac{(\hat{\mathbf{n}} \cdot \hat{\mathbf{R}})(\hat{\mathbf{n}}' \cdot \hat{\mathbf{R}}')}{{R_{SS'}}^2} \int_0^{R_{SS'}} \int_0^{R} R' dR' dR\,, d\rho' d\rho = \\ &\left(\frac{1 + \cos\beta}{2}\right) \left(\int_v \int_u \frac{u^2}{R} du dv - \int_v \int_u \frac{v^2}{R} du dv \right) dz' dz, \end{aligned} \quad (8)$$

that can be integrated analytically. The distance R simplifies to

$$R = \sqrt{u^2 + v^2 + \Delta z^2} \quad (9)$$

where $\Delta z = z - z'$. The above analysis allows us to identify transformations that accelerate the numerical integration of the paired surface integrals.

III. Preliminary Numerical Results

In order to analyze the accuracy of the proposed scheme for evaluating the 6-D reaction integral we examine the static potential in the Method of Moments discretization of the Electric Field Integral Equation (EFIE) analyzing the convergence behavior of the integral (6). We consider two tetrahedra with a common vertex, as shown in Fig. 3 (inset). Only the two outer integrals, over z and z', are calculated numerically. The other integrals are evaluated analytically. A Gauss-Legendre (GL) quadrature scheme is compared to a reference result obtained using the GL scheme with the highest number of points we have available for this schema (150 points).

In Fig. 3, the convergence of the integral over z and z' is investigated. Fig. 3 shows the number of correct significant digits obtained increasing the number of sample points for these linear integrals. As can be seen, the method can reach machine precision accuracy increasing the number of points.

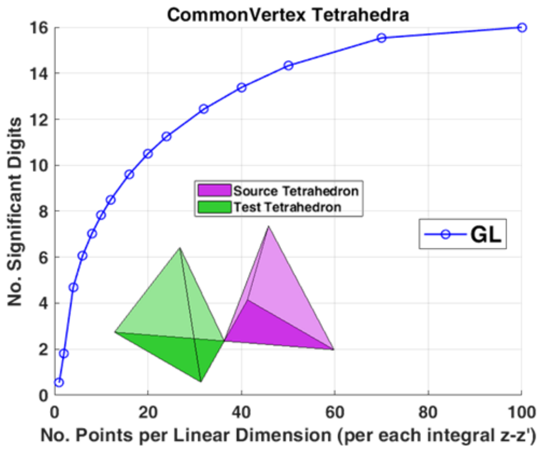

Fig. 3. Near-field convergence of surface integrals. Inset: Orientation of a pair of tetrahedral elements in space.

IV. Conclusion

The proposed scheme is based on two applications of the divergence theorem with an appropriate integration reordering and a variable transformation. For tetrahedral elements, the 6-D integrals are expressed as two radial integrals plus four linear integrals over source and observation face pairs. The method is numerically validated for static kernels arising in the EFIE and similar formulations.

The next step in this research activity will be to examine the possibility of using other transformations to further smooth the resulting integrands and hence accelerate their convergence.

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