A characterization of complex space forms via Laplace operators

A CHARACTERIZATION OF COMPLEX SPACE FORMS VIA LAPLACE OPERATORS

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Abstract. Inspired by the work of Z. Lu and G. Tian [6], in this paper we address the problem of studying those Kähler manifolds satisfying the Δ-property, i.e. such that on a neighborhood of each of its points the k-th power of the Kähler Laplacian is a polynomial function of the complex Euclidean Laplacian, for all positive integer k (see below for its definition). We prove two results: 1. if a Kähler manifold satisfies the Δ-property then its curvature tensor is parallel; 2. if an Hermitian symmetric space of classical type satisfies the Δ-property then it is a complex space form (namely it has constant holomorphic sectional curvature). In view of these results we believe that if a complete and simply-connected Kähler manifold satisfies the Δ-property then it is a complex space form.

Contents

1. Introduction and statement of the main results

Let Δ be the Kähler Laplacian on an n-dimensional Kähler manifold (M, g) i.e., in local coordinates $z = (z_1, \dots, z_n)$,

$$\Delta = \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_j \partial \bar{z}_i},$$

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where $g^{i\bar{j}}$ denotes the inverse matrix of the Kähler metric. We define the complex Euclidean Laplacian with respect to z as the differential operator

$$\Delta_c^z = \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.$$

A key point in Lu and Tian's proof of the local rigidity theorem ([6] Theor. 1.2) supporting their conjecture about the characterization of the Fubini-Study metric g_{FS} on $\mathbb{C}\mathrm{P}^n$ through the vanishing of the log-term of the universal bundle, consists in a special relation between Kähler and complex Euclidean Laplacians that occurs on $(\mathbb{C}\mathrm{P}^n, g_{FS})$. More precisely, Lu and Tian have shown the following:

Theorem 1.1 ([6] Prop. 6.1)**.** *In the center x_0 of the affine coordinate system z of $(\mathbb{C}\mathrm{P}^n, g_{FS})$, every smooth function ϕ defined in a neighborhood of x_0 fulfills the following equations for every positive integer k*

$$\Delta^k \phi(0) = p_k(\Delta_c^z)\phi(0), \tag{1}$$

where p_k is a monic polynomial of degree k with real coefficients (and consequently constant term equal to zero).

Since complex projective spaces are homogeneous manifolds, the theorem actually states that for $M = (\mathbb{C}\mathrm{P}^n, g_{FS})$ the following property holds:

(Δ-property) *For any arbitrary point $x \in M$ there exists a coordinate system z centered at x such that* (1) *holds for any positive integer k, being p_k a monic polynomial of degree k independent of x with real coefficients.*

The Δ-property is trivially verified also for $\mathbb{C}^n$ endowed with the flat Kähler metric g_0 and for the complex hyperbolic space $\mathbb{C}H^n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < 1\}$ endowed with the Kähler metric g_{hyp} whose associated Kähler form is $\omega_{hyp} = -\sqrt{-1}\partial\bar{\partial}\log(1 - \sum_{i=1}^n |z_i|^2)$. This immediately follows again by homogeneity and from the following:

Proposition 1.2. *Condition* (1) *is satisfied for any positive integer* k *in the center of a radial metric*[1].

This proposition generalizes Theorem 1.1. Its proof (which follows the same outlines of Lu and Tian's result) will be given in the next Section.

By the above considerations it is then natural to try to classify those Kähler manifolds satisfying the Δ-property. In this direction we have the following two theorems.

Theorem 1.3. *Let* (M, g) *be a Kähler manifold which satisfies the* Δ*-property. Then its curvature tensor is parallel.*

Theorem 1.4. *An Hermitian symmetric space of classical type satisfying the* Δ*-property is a complex space form.*

If Theorem 1.4 could be extended also to the case of Hermitian symmetric spaces of *exceptional types* then the complex space forms should be characterized by the Δ-property as expressed by the following:

Conjecture. *The only complete and simply-connected Kähler manifolds satisfying the* Δ*-property are the complex space forms.*

The following two sections are devoted to the proofs of Proposition 1.2, Theorem 1.3 and Theorem 1.4.

2. Proofs of Proposition 1.2 and Theorem 1.3

Proof of Proposition 1.2. Let $\Phi(|z|^2)$ be a Kähler potential of a radial Kähler metric g. Then the inverse matrix of g reads locally as

$$g^{i\bar{j}} = \frac{1}{\Phi'}\left(\delta_{ij} - \frac{\Phi''}{\Phi' + |z|^2\Phi''} z_j \bar{z}_i\right),$$

where Φ' and Φ'' represent the first and the second derivative of Φ with respect to $t = |z|^2$.

We immediately see that (1) is verified at point $z = 0$ for $k = 1$ (up to rescaling z). Now, let us assume by induction that (1) holds true

[1] Namely a Kähler metric admitting a Kähler potential which depends only on the sum $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$ of the moduli of a local coordinates system z.

for some k and let us prove it for $k+1$. By the inductive assumption one has

$$\Delta^{k+1}\phi(0) = \sum_{l=0}^{k} a_{k,l}(\Delta_c^z)^l(\Delta\phi)(0) =$$

$$= \sum_{l=0}^{k} a_{k,l}(\Delta_c^z)^l \left(\frac{1}{\Phi'} \sum_{i,j} \left(\delta_{ij} - \frac{\Phi''}{\Phi' + |z|^2\Phi''} z_j \bar{z}_i \right) \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_i} \right)(0). \quad (2)$$

In order to prove that

$$\Delta^{k+1}\phi(0) = \sum_{l=0}^{k+1} a_{k+1,l}(\Delta_c^z)^l \phi(0) \quad (3)$$

it is enough to show this for $\phi = |z^P|^2$ and $\phi = z^P \bar{z}^Q$ with $P \neq Q$, where we are denoting $z^P = z_1^{P_1} \cdots z_n^{P_n}$ for $P = (P_1, \ldots, P_n) \in \mathbb{N}^n$, since every smooth ϕ decomposes as a series of such monomials.

Let us first consider the case $\phi = z^P \bar{z}^Q$, $P \neq Q$. On the one hand, one has $(\Delta_c^z)^l \phi(0) = 0$; on the other hand, by using (2), it is easy to see that $\Delta^{k+1}\phi(0) = 0$, and then (3) is trivially true for any choice of the coefficients $a_{k+1,l}$.

Now, consider $\phi = |z^P|^2$ and let $|P| := P_1 + \ldots + P_n = p$. In this case, one has that (3) is true if and only if $\Delta^{k+1}\phi(0) = p!P!a_{k+1,p}$. By using (2), one has

$$\Delta^{k+1}\phi(0) = \sum_{l=0}^{k} a_{k,l}(\Delta_c^z)^l \left(\sum_i \frac{P_i^2 |z^{P-e_i}|^2}{\Phi'} - \sum_{i,j} \frac{\Phi'' P_i P_j |z^P|^2}{\Phi'(\Phi' + |z|^2\Phi'')} \right) \Big|_0,$$

where we are denoting by e_i the vectors of the canonical basis of $\mathbb{R}^n$. By taking into account Leibniz's rule for the derivatives of a product of functions, we can state that for any $P \in \mathbb{N}^n$ there exists a constant $C_{p,l}^{\psi}$ depending only on p, l and the radial smooth function ψ such that

$$(\Delta_c^z)^l \left(|z^P|^2 \psi(|z|^2) \right) \Big|_0 = C_{p,l}^{\psi} (\Delta_c^z)^p |z^P|^2 \Big|_0 = C_{p,l}^{\psi} p! P!.$$

Furthermore, if $p > l$, then $C_{p,l}^{\psi} = 0$ independently of ψ and $C_{h,h}^{\Phi'^{-1}} = 1$ for every positive integer h because we rescaled local coordinates z so that $\Phi'(0) = 1$. After a straightforward computation, we get the following relation which determines the polynomial p_{k+1}:

$$a_{k+1,p} = a_{k,p-1} + \sum_{l=p}^{k} a_{k,l} \left(C_{p-1,l}^{\Phi'^{-1}} - p^2 C_{p,l}^{\overline{\frac{\Phi''}{\Phi'(\Phi'+|z|^2\Phi'')}}} \right).$$

□

The aim of the rest of the section is to prove Theorem 1.3, i.e. that if a Kähler manifold satisfies the Δ-property then the covariant derivatives of its Riemann tensor R vanish identically.

We begin by showing the following result (which will be used in the proofs of both Theorem 1.3 and Theorem 1.4).

Theorem 2.1. *A Kähler manifold (M, g) is Einstein if and only if for each point $x \in M$ there exist local coordinates z centered at x such that (1) is satisfied for $k = 1, 2$.*

Proof. Let $x \in M$ and z be a holomorphic normal coordinate system on M centered at x. Clearly, $\Delta\phi(0) = \Delta_c^z\phi(0)$ and (1) is satisfied for $k = 1$. Now, if (M, g) is a Einstein manifold, in local coordinates we have[2]

$$\lambda g_{i\bar{j}} = \mathrm{Ric}_{i\bar{j}} = g^{k\bar{h}} \left(-\partial_{k\bar{h}} g_{i\bar{j}} + g^{p\bar{q}} \partial_k g_{i\bar{q}} \partial_{\bar{h}} g_{p\bar{j}} \right). \tag{4}$$

Hence, if we evaluate the previous equation at x, we get

$$\sum_h \partial_{h\bar{h}} g^{i\bar{j}}(0) = \lambda \delta^{ij}. \tag{5}$$

By (5), we get

$$\Delta^2\phi(0) = g^{h\bar{k}} \partial_{k\bar{h}} \big(g^{i\bar{j}} \partial_{j\bar{i}} \phi \big) \Big|_0 = \Big((\Delta_c^z)^2 + \lambda \Delta_c^z \Big) \phi(0) \tag{6}$$

that is (1) is satisfied also for $k = 2$.

Conversely, let us now suppose that for each $x \in M$ there exists a local coordinate system w with respect to which (1) is fulfilled for $k = 1, 2$. By comparing in both sides of (1) for $k = 2$ the third order

[2]We are going to use the notation ∂_i to denote $\frac{\partial}{\partial z_i}$ and a similar notation for higher order derivatives. We are also going to use Einstein's summation convention for repeated indices.

derivative's coefficient, we get

$$\frac{\partial g_w^{i\bar{j}}}{\partial w_k}(0) + \frac{\partial g_w^{k\bar{j}}}{\partial w_i}(0) = 0$$

for every index i, j and k, where we denote by $g_w^{i\bar{j}}$ the $(i,\bar{j})$ entry of the inverse matrix of $g\left(\frac{\partial}{\partial w_\alpha}, \frac{\partial}{\partial \bar{w}_\beta}\right)$. Let z be a holomorphic normal coordinate system around the same point of w. Therefore, the previous equation implies

$$\frac{\partial z_\gamma}{\partial w_k}\frac{\partial}{\partial z_\gamma}\left(\overline{\frac{\partial w_i}{\partial z_\alpha}} g_z^{\alpha\bar{\beta}} \frac{\partial w_j}{\partial z_\beta}\right)\Big|_0 + \frac{\partial z_\gamma}{\partial w_i}\frac{\partial}{\partial z_\gamma}\left(\overline{\frac{\partial w_k}{\partial z_\alpha}} g_z^{\alpha\bar{\beta}} \frac{\partial w_j}{\partial z_\beta}\right)\Big|_0 = 0.$$

Since (1) for $k = 1$ reads as

$$\Delta_c^z \phi(0) = \Delta\phi(0) = \Delta_c^w \phi(0) = \sum_{i,j,\alpha,\beta} \frac{\partial z_\alpha}{\partial w_i}\overline{\frac{\partial z_\beta}{\partial w_i}} \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}\Big|_0,$$

$\frac{\partial z_\alpha}{\partial w_i}\Big|_0$ needs to be a unitary matrix. Hence, we get

$$\frac{\partial^2 w_j}{\partial z_\alpha \partial z_\beta}(0) = 0$$

for every index j, α and β. By considering that local coordinates w satisfy (6), hence they need to satisfy (5), we have

$$\lambda\delta^{ij} = \sum_h \frac{\partial^2 g_w^{i\bar{j}}}{\partial w_h \partial \bar{w}_h}\Big|_0 = \sum_h \frac{\partial z_\gamma}{\partial w_h}\frac{\partial w_j}{\partial z_\beta}\frac{\partial^2 g_z^{\alpha\bar{\beta}}}{\partial z_\gamma \partial \bar{z}_\delta}\overline{\frac{\partial w_i}{\partial z_\alpha}}\,\overline{\frac{\partial z_\delta}{\partial w_h}}\Big|_0 = \sum_h \frac{\partial^2 g_z^{i\bar{j}}}{\partial z_h \partial z_h}\Big|_0.$$

This means that $\mathrm{Ric}_{i\bar{j}} = \lambda g_{i\bar{j}}$ at x, and then by the arbitrariness of x we conclude that g is a Kähler-Einstein metric. The theorem is proved. □

Remark 1. Notice that, by combining Theorem 2.1 with the uniformization theorem, one gets a proof of our conjecture for complex dimension $n = 1$.

Now we are finally ready to prove Theorem 1.3.

Proof of Theorem 1.3. For any $x \in M$, let w be a local coordinate system centered at x with respect to which (1) is satisfied for any positive integer k. In particular, by comparing the fifth order derivative's coefficients of both sides of (1) for $k=4$, we get that

$$\left(\partial_{h\bar{k}l}g_w^{i\bar{j}}+\partial_{i\bar{k}l}g_w^{h\bar{j}}+\partial_{i\bar{k}h}g_w^{l\bar{j}}+\partial_{h\bar{j}l}g_w^{i\bar{k}}+\partial_{i\bar{j}l}g_w^{h\bar{k}}+\partial_{i\bar{j}h}g_w^{l\bar{k}}\right)(0)=0 \quad (7)$$

for every choice of indexes i, j, h, k and l.

Let z be a holomorphic normal coordinate system around the same point of w. By taking into account that in the proof of Theorem 2.1 we have shown that every second order derivative of the holomorphic change of coordinates sending z to w vanishes at $z=0$ and $\frac{\partial z_\beta}{\partial w_j}\Big|_0$ is a unitary matrix, we get

$$\sum_{\alpha,\beta}\frac{\partial^3 g_z^{\alpha\bar{\beta}}}{\partial z_\gamma\partial\bar{z}_\delta\partial z_\epsilon}\left(\overline{\frac{\partial z_\beta}{\partial w_j}}\,\overline{\frac{\partial z_\delta}{\partial w_k}}+\overline{\frac{\partial z_\beta}{\partial w_k}}\,\overline{\frac{\partial z_\delta}{\partial w_j}}\right)\left(\frac{\partial z_\alpha}{\partial w_i}\frac{\partial z_\gamma}{\partial w_h}\frac{\partial z_\epsilon}{\partial w_l}+\right.$$

$$\left.+\frac{\partial z_\alpha}{\partial w_h}\frac{\partial z_\gamma}{\partial w_i}\frac{\partial z_\epsilon}{\partial w_l}+\frac{\partial z_\alpha}{\partial w_l}\frac{\partial z_\gamma}{\partial w_i}\frac{\partial z_\epsilon}{\partial w_h}\right)\Big|_0=0.$$

Therefore, for every index α, β, γ, δ and ϵ, a relation similar to (7) holds true also with respect to holomorphic normal coordinates z:

$$\left(\partial_{\gamma\bar{\delta}\epsilon}g_z^{\alpha\bar{\beta}}+\partial_{\alpha\bar{\delta}\epsilon}g_z^{\gamma\bar{\beta}}+\partial_{\alpha\bar{\delta}\gamma}g_z^{l\bar{\beta}}+\partial_{\gamma\bar{\beta}\epsilon}g_z^{\alpha\bar{\delta}}+\partial_{\alpha\bar{\beta}\epsilon}g_z^{\gamma\bar{\delta}}+\partial_{\alpha\bar{\beta}\gamma}g_z^{\epsilon\bar{\delta}}\right)(0)=0.$$

Since

$$\frac{\partial^3 g^{\alpha\bar{\beta}}}{\partial z_\gamma\partial\bar{z}_\delta\partial z_\epsilon}(0)=-\frac{\partial^3 g_{\alpha\bar{\beta}}}{\partial z_\gamma\partial\bar{z}_\delta\partial z_\epsilon}(0),$$

we get

$$\frac{\partial^3 g_{\alpha\bar{\beta}}}{\partial z_\gamma\partial\bar{z}_\delta\partial z_\epsilon}(0)=0$$

for every index α, β, γ, δ and ϵ. It follows that the covariant derivatives of the Riemann tensor must vanish identically as desired. □

Remark 2. Notice that in fact we have proved the stronger statement that if for any $x\in M$ there exists a local coordinate system centered at x with respect to which (1) is satisfied for $k=1,\dots,4$, then curvature tensor of M is parallel.

3. Proof of Theorem 1.4

The first step in the proof of our second main result consists in characterizing complex projective spaces among Hermitian symmetric spaces of compact type[3] by means of the relations between Kähler and complex Euclidean Laplacians (1). Before proving such theorem, we need a technical lemma.

We have to recall that Alekseevsky and Perelomov described explicitly in [1] holomorphic coordinates for every flag manifold G/K which turns out to be an orbit of the adjoint action of a classical compact semisimple Lie group G on its Lie algebra $\mathfrak{g}$. Throughout the paper we are going to call these coordinates *Alekseevsky-Perelomov coordinates*. Moreover, we will write $\mathbb{C}\mathrm{P}^1_r$ to denote the product of r complex projective spaces $\mathbb{C}\mathrm{P}^1$ equipped with the product metric $g^r_{FS} = g_{FS} \oplus \ldots \oplus g_{FS}$.

Lemma 3.1. *Let M be an irreducible classical n-dimensional HSSCT of rank r endowed the (unique up to rescaling) Kähler-Einstein metric. Then there exists a local coordinate system w such that the Kähler immersion's equations of $\left(\mathbb{C}\mathrm{P}^1_r, g^r_{FS}\right)$ into M read as*

$$\begin{cases} w_i = z_i & \textit{for } i = 1, \ldots, r \\ w_i = 0 & \textit{for } i = r+1, \ldots, n \end{cases}$$

where z are affine coordinates on $\mathbb{C}\mathrm{P}^1_r$.

Proof. We are going to show explicitly in the following case by case analysis how the isometric embedding of $\left(\mathbb{C}\mathrm{P}^1_r, g^r_{FS}\right)$ into an irreducible classical HSSCT of rank r reads with respect to Alekseevsky-Perelomov coordinates. The desired local coordinate system w is obtained in a obvious way from the Alekseevsky-Perelomov one.

Case 1: $SU(N)/S\big(U(k) \times U(N-k)\big)$.
The Alekseevsky-Perelomov coordinates are given by the entries w_{ij} of a complex $(N-k) \times k$ matrix W and the potential of the Kähler-Einstein metric (up to a constant) with respect to these coordinates reads as

$$\log\det\left(I_k + {}^T\bar{W}W\right).$$

We can easily prove that a Kähler immersion of $\mathbb{C}P^1_m$, where $m = \min\{k, N-k\}$, is given by sending affine coordinates $z = (z_1, \ldots, z_m)$

[3]From now on HSSCT.

to the matrix $W(z) \in M_{N-k,k}(\mathbb{C})$ defined by

$$W(z)_{ij} = z_i \delta_{ij}.$$

Case 2: $SO(2N)/U(N)$.

The Alekseevsky-Perelomov coordinates are given by the entries w_{ij} of a skew-symmetric complex $N \times N$ matrix W and the potential of the Kähler-Einstein metric (up to a constant) with respect to these coordinates reads as

$$\log\det\left(I_N + {}^T\bar{W}W\right).$$

We can easily prove that a Kähler immersion of $\mathbb{C}P^1_{\left[\frac{N}{2}\right]}$ is given by sending affine coordinates $z = (z_1, \dots, z_{\left[\frac{N}{2}\right]})$ to the matrix

$$W(z) = \begin{pmatrix} 0 & z_1 & 0 & 0 & \cdots \\ -z_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & z_2 & \\ 0 & 0 & -z_2 & 0 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}.$$

Case 3: $Sp(N)/U(N)$.

The Alekseevsky-Perelomov coordinates are given by the entries w_{ij} of a symmetric $N \times N$ complex matrix W and the potential the Kähler-Einstein metric (up to a constant) in these coordinates reads as

$$\log\det\left(I_N + {}^T\bar{W}W\right).$$

We can easily prove that a Kähler immersion of $\mathbb{C}P^1_N$ is given by sending affine coordinates $z = (z_1, \dots, z_N)$ to the matrix

$$W(z)_{ij} = z_i \delta_{ij}.$$

Case 4a: $SO(2N)/\big(SO(2N-2) \times SO(2)\big)$, with $N \geq 4$.

The Alekseevsky-Perelomov coordinates are $(v_2, \dots, v_N, v'_2, \dots, v'_N) \in \mathbb{C}^{2N-2}$ and the potential the Kähler-Einstein metric (up to a constant) in these coordinates reads as

$$\log\left(1+\sum_{j=2}^{N}|v_j|^2+\sum_{j=2}^{N}|v_j'|^2+4\left|\sum_{j=2}^{N}v_jv_j'\right|^2\right).$$

We can easily prove that a Kähler immersion of $\mathbb{C}P^1\times\mathbb{C}P^1$ is given by setting

$$v_2=\frac{z_1}{\sqrt{2}}, v_3=\frac{z_2}{\sqrt{2}}, v_4=\ldots=v_N=0,$$
$$v_2'=0, v_3'=\frac{z_1}{\sqrt{2}}, v_4'=\frac{z_2}{\sqrt{2}}, v_5'=\ldots=v_N'=0,$$

where (z_1,z_2) are affine coordinates on $\mathbb{C}P^1\times\mathbb{C}P^1$.

Case 4b: $SO(2N+1)/\big(SO(2N-1)\times SO(2)\big)$, with $N\geq 4$.
The Alekseevsky-Perelomov coordinates are $(v_2,\ldots,v_N,v_2',\ldots,v_N',u)\in\mathbb{C}^{2N-1}$. The potential the Kähler-Einstein metric (up to a constant) in these coordinates reads as

$$\log\left(1+\sum_{j=2}^{N}|v_j|^2+\sum_{j=2}^{N}|v_j'|^2+|u|^2+4\left|\sum_{j=2}^{N}v_jv_j' \ -u^2\right|^2\right)$$

and a Kähler immersion of $\mathbb{C}P^1\times\mathbb{C}P^1$ is given by setting

$$v_2=\frac{z_1}{\sqrt{2}}, v_3=\frac{z_2}{\sqrt{2}}, v_4=\ldots=v_N=0,$$
$$v_2'=0, v_3'=\frac{z_1}{\sqrt{2}}, v_4'=\frac{z_2}{\sqrt{2}}, v_5'=\ldots=v_N'=u=0,$$

where (z_1,z_2) are affine coordinates on $\mathbb{C}P^1\times\mathbb{C}P^1$. □

We are now in a position to characterize complex projective spaces among irreducible HSSCT.

Theorem 3.2. *Complex projective spaces are the unique classical irreducible HSSCT which satisfy the Δ-property.*

Proof. Let M be an n-dimensional HSSCT endowed with the Kähler-Einstein metric g. We denote by λ the Einstein constant. Let $\tilde{z}$ be a holomorphic normal coordinate system.

By combining (4), (5) and (6) above, we get that every smooth function ϕ defined in a neighborhood V of the center of $\tilde{z}$ fulfills the following

$$\Delta^3\phi(0) = \Big((\Delta_c^{\tilde z})^3 + 3\lambda(\Delta_c^{\tilde z})^2 + \lambda^2\Delta_c^{\tilde z}\Big)\phi(0) + 2\sum_{l,h=1}^{n}\partial_{l\bar h} g^{i\bar j}\partial_{jh\bar l \bar i}\phi\,\Big|_0 +$$

$$+\sum_{l,h=1}^{n}\partial_{lh} g^{i\bar j}\partial_{j\bar h\bar l\bar i}\phi\,\Big|_0 + \sum_{l,h=1}^{n}\partial_{\overline{lh}} g^{i\bar j}\partial_{jhl\bar i}\phi\,\Big|_0 + \sum_{l,h=1}^{n}\partial_{lh\overline{lh}} g^{i\bar j}\partial_{j\bar i}\phi\,\Big|_0, \quad (8)$$

where we use that by differentiating (4) and evaluating in the center of $\tilde z$, the coefficients of the third order derivatives of ϕ vanish.

If z are affine coordinates on $(\mathbb{C}\mathrm{P}^1_r, g^r_{FS})$, where r is equal to the rank of M, by taking into account Lemma 3.1 and by considering that Alekseevsky-Perelomov coordinates w are normal up to rescaling by suitable constants that we call μ_i (see [4] Theor. 1), we can compute

$$\Delta^3\big(|z_1|^4\big)\Big|_0 = \frac{3\lambda}{(\mu_1)^2}\big(\Delta_c^w\big)^2(|z_1|^4)\Big|_0 + \frac{8}{(\mu_1)^2}\frac{\partial^2 g^{1\bar 1}}{\partial w_1\partial\overline{w}_1}\Big|_0 = \frac{12\lambda+16}{(\mu_1)^2}. \quad (9)$$

Furthermore, if $r\neq 1$, namely M is different from a complex projective space, we also compute

$$\Delta^3\big(|z_1z_2|^2\big)\Big|_0 = \frac{3\lambda}{\mu_1\mu_2}\big(\Delta_c^w\big)^2(|z_1z_2|^2)\Big|_0 +$$

$$+4\left(\frac{1}{(\mu_1)^2}\frac{\partial^2 g^{2\bar 2}}{\partial w_1\partial\overline{w}_1} + \frac{1}{(\mu_2)^2}\frac{\partial^2 g^{1\bar 1}}{\partial w_2\partial\overline{w}_2} + \frac{1}{\mu_1\mu_2}\frac{\partial^2 g^{1\bar 2}}{\partial w_2\partial\overline{w}_1} + \frac{1}{\mu_1\mu_2}\frac{\partial^2 g^{2\bar 1}}{\partial w_1\partial\overline{w}_2}\right) = \frac{6\lambda}{\mu_1\mu_2}. \quad (10)$$

If M has rank greater than 1, let us assume by contradiction that the Δ-property is valid, in particular around each point of M there exists a local coordinate system with respect to which (1) is satisfied for $k=1,2,3$. Let us denote such coordinate system by $f=(f_1,\dots,f_n)$.

Since we have shown in Theorem 2.1 that every second order derivative of the holomorphic change of coordinates sending f to $\tilde z$ vanish at $f=0$, we get

$$\Delta^3\phi(0) = \Big((\Delta_c^f)^3 + \sum_{i=1}^{2}a_i(\Delta_c^f)^i\Big)\phi(0) =$$

$$= \Big(\sum_{i=1}^{2}a_i(\Delta_c^{\tilde z})^i\Big)\phi\Big|_0 + \sum_{i_1,i_2,i_3,\alpha,\beta}\frac{\partial^3\tilde z_\alpha}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}\overline{\frac{\partial^3\tilde z_\beta}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}}\frac{\partial^2\phi}{\partial\tilde z_\alpha\partial\bar{\tilde z}_\beta}\Big|_0 +$$

$$+(\Delta_c^{\tilde z})^3\phi\Big|_0+\sum_{\substack{i_1,i_2,i_3\\ \alpha_1,\dots,\alpha_4}}\frac{\partial^3\tilde z_{\alpha_4}}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}\prod_{l=1}^3\overline{\frac{\partial\tilde z_{\alpha_l}}{\partial f_{i_l}}}\frac{\partial^4\phi}{\partial\bar{\tilde z}_{\alpha_1}\partial\bar{\tilde z}_{\alpha_2}\partial\bar{\tilde z}_{\alpha_3}\partial\tilde z_{\alpha_4}}\Big|_0+$$

$$+\sum_{\substack{i_1,i_2,i_3\\ \alpha_1,\dots,\alpha_4}}\overline{\frac{\partial^3\tilde z_{\alpha_4}}{\partial f_{i_1}\partial f_{i_2}\partial f_{i_3}}}\prod_{l=1}^3\frac{\partial\tilde z_{\alpha_l}}{\partial f_{i_l}}\frac{\partial^4\phi}{\partial\tilde z_{\alpha_1}\partial\tilde z_{\alpha_2}\partial\tilde z_{\alpha_3}\partial\bar{\tilde z}_{\alpha_4}}\Big|_0.$$

The previous formula implies the relation

$$\Delta^3\big(|z_1|^4\big)(0)=2\frac{\mu_2}{\mu_1}\Delta^3\big(|z_1z_2|^2\big)(0),$$

therefore we have a contradiction from the comparison with (9) and (10). $\square$

Remark 3. Notice that we have proved the stronger statement that the complex projective spaces are the unique classical irreducible HSSCT such that around any point there exists a local coordinate system with respect to which (1) is satisfied for $k=1,2,3$.

Remark 4. Theorem 3.2 proves also that the Δ-property cannot be satisfied in any not irreducible classical HSSCT, because they contain an embedded $(\mathbb{C}\mathrm{P}^1_2, g^2_{FS})$ whose Kähler immersion's equations locally reads with respect to holomorphic normal coordinates as in Lemma 3.1.

Finally, we can prove our second main result.

Proof of Theorem 1.4. Every Hermitian symmetric space[4] can be decomposed as a Kähler product

$$(\mathbb{C}^n,g_0)\times(C_1,g_1)\times\ldots\times(C_h,g_h)\times(N_1,\hat g_1)\times\ldots\times(N_l,\hat g_l),$$

where $(\mathbb{C}^n,g_0)$ is the flat Euclidean space, (C_i,g_i) are irreducible HSSCT and $(N_i,\hat g_i)$ are irreducible HSS of noncompact type[5].

By Theorem 2.1, a HSS where (1) is fulfilled for $k=1,2$, is the flat Euclidean space otherwise it is a Kähler product of HSS of either compact or noncompact type. Hence, we are going to prove our statement

[4] From now on HSS.

[5] Namely N_i is a bounded symmetric domains with a multiple of the Bergman metric denoted by $\hat g_i$.

by characterizing hyperbolic spaces among classical HSS of noncompact type in analogy with what we have done for projective spaces in Theorem 3.2.

Let z be the restriction of Euclidean coordinates to a classical bounded symmetric domain $(N, \hat{g})$ such that $\hat{g}$ is a Kähler-Einstein metric. Let $(N^*, \hat{g}^*)$ be its compact dual. We can think z as the restriction of Alekseevsky-Perelomov coordinates of N^* to N. Furthermore a Kähler potential Φ for the metric $\hat{g}$ is given by

$$\Phi(z, \bar{z}) = -\Phi^*(z, -\bar{z})_{|N} \tag{11}$$

where $\Phi^*(z, \bar{z})$ is a Kähler potential for $\hat{g}^*$ (see [2] and [3] for details).

By (11) and (8), we get

$$\Delta^3_N\big(|z_i z_j|^2\big)(0) = -\Delta^3_{N^*}\big(|z_i z_j|^2\big)(0)$$

for every $1 \leq i, j \leq \dim(N)$. Hence, if N and N^* are not irreducible or else if they are irreducible but they have rank different from 1, namely N is not a hyperbolic space, (1) for $k = 3$ cannot be satisfied as proved in Theorem 3.2. □

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