BMO Spaces on Weighted Homogeneous Trees

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Abstract. We consider an infinite homogeneous tree $\mathcal{V}$ endowed with the usual metric d defined on graphs and a weighted measure μ. The metric measure space $(\mathcal{V}, d, \mu)$ is nondoubling and of exponential growth, hence the classical theory of Hardy and BMO spaces does not apply in this setting. We introduce a space $BMO(\mu)$ on $(\mathcal{V}, d, \mu)$ and investigate some of its properties. We prove in particular that $BMO(\mu)$ can be identified with the dual of a Hardy space $H^1(\mu)$ introduced in a previous work and we investigate the sharp maximal function related with $BMO(\mu)$.

Dedicated to Guido Weiss on the occasion of his 90th birthday

1. Introduction

The classical space of functions of bounded mean oscillation BMO was introduced in the Euclidean setting by John and Nirenberg [11]. It is defined as the set of locally integrable functions f such that

$$\sup_B \frac{1}{|B|} \int_B |f - f_B| \, \mathrm{d}x < \infty \,, \tag{1}$$

where the supremum is taken over all Euclidean balls and f_B denotes the average of f on B. A celebrated result of Fefferman and Stein [7] identifies BMO with the dual of the classical Hardy space H^1.

Various extensions of such theory have been considered in the literature. The first extension was developed on spaces of homogeneous type in the sense of Coifman and Weiss [5, 6, 15, 16]. These are metric measure spaces (X, d, μ) where the doubling condition is satisfied, i.e., there exists a constant C such that

$$\mu\big(B(x, 2r)\big) \leqslant C\, \mu\big(B(x, r)\big) \qquad \forall x \in X\,, \quad \forall r > 0, \tag{2}$$

where $B(x, r)$ denotes the ball centred at x of radius r in the metric d. In this setting functions in $BMO(\mu)$ satisfy the analogue of condition (1), where metric balls are involved. Subsequently extensions of the theory of Hardy and BMO spaces have been considered in the literature on various metric measure spaces which do not satisfy the doubling condition (2). Due to the lack of the doubling condition, it is less clear which is a suitable condition to define a BMO space which enjoys all the properties of the classical one, and in particular it is not clear which subsets of the space can be chosen to replace balls in condition (1).

The literature on this subject is huge and we shall only cite here some contributions. In particular, various results on this subject have been obtained on nondoubling Riemannian manifolds [3, 14, 17] and on Lie groups of exponential growth [13, 18]. A few results have been obtained in the discrete setting of infinite graphs [4, 9].

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The goal of the present paper is to develop a theory for a BMO space on a homogeneous tree $\mathcal{V}$ endowed with the usual metric d defined on a graph and a weighted measure μ (see Section 2 for the details), such that $(\mathcal{V}, d, \mu)$ is a nondoubling space. Such weighted homogeneous trees were first studied by Hebisch and Steger [10], who developed a Calderón–Zygmund theory on them. In particular, they proved that there exists a family of appropriate sets in $\mathcal{V}$, which are called Calderón–Zygmund sets, which replace the family of balls in the classical Calderón–Zygmund theory. In [1] we introduced an atomic Hardy space $H^1(\mu)$ on $(\mathcal{V}, d, \mu)$, where atoms are functions supported in Calderón–Zygmund sets, with vanishing integral and satisfying a certain size condition.

We shall define here a space of functions of bounded mean oscillation $BMO(\mu)$ adapted to this setting, for which the oscillation in the analogue of condition (1) is measured on Calderón–Zygmund sets. Then we show that $BMO(\mu)$ can be identified with the dual of the Hardy space $H^1(\mu)$. More precisely, we introduce a family of spaces $BMO_q(\mu)$, with $q \in [1, \infty)$, for which the integrability condition (1) is expressed in terms of an L^q-norm, and show that all such spaces coincide. As a consequence, we find the real interpolation spaces between $L^r(\mu)$ and $BMO(\mu)$, for $r \in [1, \infty)$. It would be interesting to find the complex interpolation space between $L^r(\mu)$ and $BMO(\mu)$, as well. To work in this direction, we introduce and study the sharp maximal function associated with Calderón–Zygmund sets, and show that the L^p-norm of a function is controlled by the L^p-norm of a variant of its sharp maximal function (see Theorem 20).

The paper is organized as follows. In Section 2 we introduce weighted homogeneous trees, we recall the definition of Calderón–Zygmund sets and study some of their geometric properties. In Section 3 we recall the definition of the Hardy space $H^1(\mu)$, we define the space $BMO(\mu)$ and prove the duality between these two spaces. As a consequence, we deduce a real interpolation result and a boundedness result for integral operators whose kernel satisfy a suitable Hörmander condition. The last section is devoted to some inequalities involving the sharp maximal function defined in terms of Calderón–Zygmund sets.

Positive constants are denoted by C; these may differ from one line to another, and may depend on any quantifiers written, implicitly or explicitly, before the relevant formula.

2. Weighted homogeneous trees and Calderón–Zygmund sets

In this section we first introduce the infinite homogeneous tree and we define a distance and a measure on it.

Definition 1. *An infinite homogeneous tree of order $m+1$ is a graph $T = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices and $\mathcal{E}$ denotes the set of edges, with the following properties:*

(i) *T is connected and acyclic;*
(ii) *each vertex has exactly $m+1$ neighbours.*

On $\mathcal{V}$ we define the distance $d(x, y)$ between two vertices x and y as the length of the shortest path between x and y. We also fix a doubly-infinite geodesic g in T, that is a connected subset $g \subset \mathcal{V}$ such that

(i) for each element $v \in g$ there are exactly two neighbours of v in g;
(ii) for every couple (u, v) of elements in g, the shortest path joining u and v is contained in g.

We define a mapping $N : g \to \mathbb{Z}$ such that

$$|N(x) - N(y)| = d(x, y) \qquad \forall x, y \in g\,. \tag{3}$$

This corresponds to the choice of an origin $o \in g$ (the only vertex for which $N(o) = 0$) and an orientation for g; in this way we obtain a numeration of the vertices in g. We define the level function $\ell : \mathcal{V} \to \mathbb{Z}$ as

$$\ell(x) = N(x') - d(x, x'),$$

where x' is the only vertex in g such that $d(x, x') = \min\{d(x, z) : z \in g\}$. For $x, y \in \mathcal{V}$ we say that *y lies above x* if

$$\ell(x) = \ell(y) - d(x, y).$$

In this case we also say that *x lies below y*.

Let μ be the measure on $\mathcal{V}$ such that for each function $f : \mathcal{V} \to \mathbb{C}$

$$\int_{\mathcal{V}} f \, \mathrm{d}\mu = \sum_{x \in \mathcal{V}} f(x) m^{\ell(x)}. \tag{4}$$

Then μ is a weighted counting measure such that the weight of a vertex depends only on its level, and the weight associated to a certain level is given by q times the weight of the level immediately underneath. In particular, it can be proved [1] that for every $x_0 \in \mathcal{V}$ and $r > 0$ the measure of the ball centred at x_0 of radius r is $\mu\big(B(x_0, r)\big) = m^{\ell(x_0)} \frac{m^{r+1} + m^r - 2}{m-1}$. Hence, the metric measure space $(\mathcal{V}, d, \mu)$ is of exponential growth and nondoubling.

To develop a Calderón–Zygmund theory on this nondoubling setting, it is useful to introduce suitable subsets of $\mathcal{V}$, called trapezoids. These sets were first defined in [10]. We shall recall below their definition and their properties.

Definition 2. *We call trapezoid a set of vertices $S \subset \mathcal{V}$ for which there exist $x_S \in \mathcal{V}$ and $a, b \in \mathbb{R}_+$ such that*

$$S = \{x \in \mathcal{V} : x \text{ lies below } x_S, a \leqslant \ell(x_S) - \ell(x) < b\}. \tag{5}$$

In the following we will refer to x_S as the root node of the trapezoid. Among all trapezoids we are mostly interested in those where a and b are related by particular conditions, as specified in the following definitions.

Definition 3. *A trapezoid $R \subset \mathcal{V}$ is an admissible trapezoid if and only if one of the following occurs:*

(i) *$R = \{x_R\}$ with $x_R \in \mathcal{V}$, that is R consists of a single vertex;*
(ii) *$\exists x_R \in \mathcal{V}$, $\exists h(R) \in \mathbb{N}^+$ such that*

$$R = \{x \in \mathcal{V} : x \text{ lies below } x_R, h(R) \leqslant \ell(x_R) - \ell(x) < 2h(R)\}.$$

R is called degenerate in case (i) and non-degenerate in case (ii).

We set $h(R) = 1$ in the degenerate case. In both cases, $h(R)$ can be interpreted as the height of the admissible trapezoid, which coincides with the number of levels spanned by R. We call *width* of the admissible trapezoid R the quantity $w(R) = m^{\ell(x_R)}$. We have that

$$\mu(R) = h(R) m^{\ell(x_R)} = h(R) w(R). \tag{6}$$

We now introduce the family of Calderón–Zygmund sets. They are trapezoids, even if not of admissible type (except for the degenerate case); they consist of suitable enlargements of admissible trapezoids, constructed according to the following definition.

Definition 4. *Given a non-degenerate admissible trapezoid R, the envelope of R is the set*

$$\tilde{R} = \left\{ x \in \mathcal{V} : x \text{ lies below } x_R \,, \frac{h(R)}{2} \leqslant \ell(x_R) - \ell(x) < 4h(R) \right\} ; \tag{7}$$

we set $h(\tilde{R}) = h(R)$. The envelope of a non-degenerate admissible trapezoid is also called a non-degenerate Calderón–Zygmund set. Given a degenerate admissible trapezoid R, the envelope of R is the set $\tilde{R} = R$. We denote by $\mathcal{R}$ the family of all the Calderón–Zygmund sets.

We refer the reader to [1] for the properties of such sets, in particular see [1, Propositions 2, 4] for the proof of the following result.

Proposition 5. *Let R be an admissible trapezoid and $\tilde{R}$ its envelope. Then*

(i) $\mu(\tilde{R}) \leqslant 4\mu(R)$;
(ii) *for all $z \in \tilde{R}$, we have $\tilde{R} \subset B(z, 8h(\tilde{R}))$.*

For any Calderón–Zygmund set $\tilde{R}$ we define an enlargement of it, whose measure is comparable with its measure. This can be thought as a substitute for the doubling condition in this setting.

Definition 6. *Given a Calderón–Zygmund set $\tilde{R}$, we define the set*

$$\tilde{R}^* = \left\{ x \in \mathcal{V} : d(x, \tilde{R}) < h(\tilde{R})/4 \right\} . \tag{8}$$

It is easy to see that there exists a positive constant C such that for every Calderón–Zygmund set $\tilde{R}$

$$\mu(\tilde{R}^*) \leqslant C\mu(\tilde{R}) . \tag{9}$$

In the following proposition we construct a covering of $\mathcal{V}$ made by an increasing family of Calderón–Zygmund sets.

Proposition 7. *There exists a family of Calderón–Zygmund sets $\{\tilde{Q}_n\}_{n=0}^{\infty}$ such that $\tilde{Q}_n \subset \tilde{Q}_{n+1}$ and $\bigcup_n \tilde{Q}_n = \mathcal{V}$.*

Proof. Consider the family $\{\tilde{Q}_n\}_{n=0}^{\infty}$ where

- $\tilde{Q}_0$ is the Calderón–Zygmund set with root node $x_0 = o$ and height $h_0 = 1$ (where o denotes the only vertex in the doubly-infinite geodesic g such that $\ell(o) = 0$);
- $\forall n \geqslant 1$, $\tilde{Q}_n$ is the Calderón–Zygmund set with root node x_n that is the father node of x_{n-1}, i.e. it is the only neighbour of x_{n-1} that lies above x_{n-1}, and height $h_n = h_{n-1} + 1$ (then we have $h_n = n + 1$).

We first show that $\tilde{Q}_n \subset \tilde{Q}_{n+1}$. Let $x \in \tilde{Q}_n$. By definition x lies below x_n, then by construction x also lies below x_{n+1}. Moreover, $\ell(x_{n+1}) - \ell(x) = \ell(x_n) + 1 - \ell(x)$ and we have that

$$\ell(x_{n+1}) - \ell(x) < 4h_n + 1 < 4h_{n+1} \,, \qquad \ell(x_{n+1}) - \ell(x) \geqslant \frac{h_n}{2} + 1 \geqslant \frac{h_{n+1}}{2} \,.$$

So we conclude that $x \in \tilde{Q}_{n+1}$.

To show that $\bigcup_n \tilde{Q}_n = \mathcal{V}$, consider $x \in \mathcal{V}$. Denote by k the smallest index such that x lies below x_k (and so x lies below x_j, $\forall j \geqslant k$) and set $\ell(x_k) - \ell(x) = d$. We seek for an index $j \geqslant k$ such that $x \in \tilde{Q}_j$, that is

$$\frac{j+1}{2} = \frac{h_j}{2} \leqslant \ell(x_j) - \ell(x) < 4h_j = 4(j+1) \,.$$

Observe that $\ell(x_j)-\ell(x)=\ell(x_j)-\ell(x_k)+\ell(x_k)-\ell(x)=j-k+d$, so that

$$\frac{j+1}{2}\leqslant j-k+d<4(j+1)\qquad\Longleftrightarrow\qquad\begin{cases} j>\frac{d-k-4}{3}\\ j\geqslant 1+2(k-d)\end{cases}$$

Thus it is sufficient to take $j\geqslant\max\left\{k,1+\frac{d-k-4}{3},1+2(k-d)\right\}$. □

For every $p\in(1,\infty)$ and for every $n\in\mathbb{N}$, let X_n^p be the space $L_0^p(\tilde{Q}_n)$ of all functions in $L^p(\mu)$ which are supported in the set $\tilde{Q}_n$ introduced in Proposition 7 and have vanishing integral. The space $(X_n^p,\|\cdot\|_{L^p})$ is a Banach space. We denote by X^p the space $L_{c,0}^p(\mu)$ of all functions in $L^p(\mu)$ with compact support and vanishing integral, interpreted as the strict inductive limit of the spaces X_n^p (see [2, II, p. 33] for the definition of the strict inductive limit topology). This space will be a key ingredient of next section. In particular, we shall use the following fact.

Proposition 8. *Let $p\in(1,\infty)$. For every function $F:\mathcal{V}\to\mathbb{C}$, the functional defined by*

$$\ell(g)=\int Fg\,\mathrm{d}\mu=\int(F+c)g\,\mathrm{d}\mu\qquad\forall g\in X^p,c\in\mathbb{C},\tag{10}$$

lies in the dual of X^p.On the other hand, for every functional ℓ in the dual of X^p there exists $F:\mathcal{V}\to\mathbb{C}$ such that (10) *holds.*

Proof. On one hand, let $F:\mathcal{V}\to\mathbb{C}$ and consider the linear functional $\ell:X^p\to\mathbb{C}$ defined by $\ell(g)=\int Fg\,\mathrm{d}\mu$. Then for every $n\in\mathbb{N}$ by Hölder's inequality we have

$$\left|\int Fg\,\mathrm{d}\mu\right|\leqslant\|F\|_{L^q(\tilde{Q}_n)}\,\|g\|_{L^p(\tilde{Q}_n)}\qquad\forall g\in X_n^p\,,$$

where $q=p'$. Hence, ℓ is continuous on every X_n^p and continuous on X^p.

Suppose now that ℓ is a continuous linear functional on X^p. Then, for every $n\in\mathbb{N}$, $\ell\in(X_n^p)^*$, hence it can be extended to a bounded linear functional on $L^p(\tilde{Q}_n)$ and there exists a function $F_n\in L^q(\tilde{Q}_n)$ such that

$$\ell(g)=\int(F_n+c)g\,\mathrm{d}\mu\qquad\forall g\in X_n^p,\quad c\in\mathbb{C}\,.$$

Notice that we used the fact that g has vanishing integral in the formula above. For every $n\in\mathbb{N}$ we choose the constant c_n such that $\int_{\tilde{Q}_1}(F_n+c_n)\,\mathrm{d}\mu=0$. This implies that the restriction of F_n+c_n on $\tilde{Q}_j$ coincide with F_j+c_j for every $j<n$. Hence for every $x\in\mathcal{V}$ we can define

$$F(x)=F_n(x)+c_n\,,$$

where n is any integer such that $x\in\tilde{Q}_n$. Then

$$\ell(g)=\int(F+c)\,g\,\mathrm{d}\mu\qquad\forall g\in X^p,\quad c\in\mathbb{C}\,,$$

as required. □

3. Hardy and *BMO* spaces

In this section we first recall the definition of atomic Hardy spaces given in [1].

Definition 9. *A function a is a* $(1,p)$-atom*, for $p\in(1,\infty]$, if it satisfies the following properties:*

(i) *a is supported in a Calderón–Zygmund set $\tilde{R}$;*

(ii) $\|a\|_{L^p} \leqslant \mu(\tilde{R})^{1/p-1}$;

(iii) $\int_{\mathcal{V}} a \,\mathrm{d}\mu = 0$.

Definition 10. *The Hardy space $H^{1,p}(\mu)$ is the space of all functions g in $L^1(\mu)$ such that $g = \sum_{j\in\mathbb{N}} \lambda_j\, a_j$, where a_j are $(1,p)$-atoms and λ_j are complex numbers such that $\sum_{j\in\mathbb{N}} |\lambda_j| < \infty$. We denote by $\|g\|_{H^{1,p}}$ the infimum of $\sum_{j\in\mathbb{N}} |\lambda_j|$ over all decompositions $g = \sum_{j\in\mathbb{N}} \lambda_j\, a_j$, where a_j are $(1,p)$-atoms.*

The space $H^{1,p}(\mu)$ endowed with the norm $\|\cdot\|_{H^{1,p}}$ is a Banach space.

For every $p \in (1,\infty]$ we also introduce the spaces

$$H^{1,p}_{\mathrm{fin}}(\mu) = \Big\{ g \in L^1(\mu) : \ g = \sum_{j=1}^{N} \lambda_j\, a_j,\ a_j\ (1,p) - atoms, \lambda_j \in \mathbb{C}, N \in \mathbb{N} \Big\}.$$

Proposition 11. *For any $p \in (1,\infty)$, the following hold:*

(i) *$H^{1,p}(\mu) = H^{1,\infty}(\mu)$ and there exists a constant C_p such that*

$$\|g\|_{H^{1,p}} \leqslant \|g\|_{H^{1,\infty}} \leqslant C_p \|g\|_{H^{1,p}};$$

(ii) *for every $L \in (H^1(\mu))^*$, $\|L\|_{(H^{1,p})^*} \leqslant C_p \|L\|_{(H^1)^*}$;*

(iii) *$H^{1,\infty}_{\mathrm{fin}}(\mu) \subset H^{1,p}_{\mathrm{fin}}(\mu)$;*

(iv) *for every Calderón–Zygmund set $\tilde{R}$, $L^p_0(\tilde{R}) \subset H^{1,\infty}_{\mathrm{fin}}(\mu)$.*

Proof. Property (i) follows from [1, Proposition 5]. Take now $L \in (H^1(\mu))^*$ and $g \in H^1(\mu)$. Then

$$|L(g)| \leqslant \|L\|_{(H^1)^*} \|g\|_{H^{1,\infty}} \leqslant C_p \|L\|_{(H^1)^*} \|g\|_{H^{1,p}},$$

so that (ii) follows.

Property (iii) holds since every $(1,\infty)$-atom is a $(1,p)$-atom.

To prove (iv) let $\tilde{R}$ be a Calderón–Zygmund set and $g \in L^p_0(\tilde{R})$ be a function in $L^p(\mu)$ supported in $\tilde{R}$, with vanishing integral. Then $\|g\|_{L^\infty} = \max_{x\in\tilde{R}} |g(x)| < \infty$. Hence, $\mu(\tilde{R})^{-1}\|g\|^{-1}_{L^\infty} g$ is a $(1,\infty)$-atom. This proves (iv). □

In the sequel we shall denote by $H^1(\mu)$ the space $H^{1,\infty}(\mu)$ endowed with the norm $\|\cdot\|_{H^1} = \|\cdot\|_{H^{1,\infty}}$.

We now introduce the space of functions of bounded mean oscillation. For every locally integrable function f and every Calderón–Zygmund set $\tilde{R}$ we denote by $f_{\tilde{R}}$ the average of f on $\tilde{R}$, i.e., $f_{\tilde{R}} = \frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} f \,\mathrm{d}\mu$.

Definition 12. *Let $q \in [1,\infty)$. The space $\mathcal{BMO}_q(\mu)$ is the space of all functions in $L^q_{\mathrm{loc}}(\mu)$ such that*

$$\sup_{\tilde{R}\in\mathcal{R}} \Big(\frac{1}{\mu(\tilde{R})} \int_{\tilde{R}} |f - f_{\tilde{R}}|^q \,\mathrm{d}\mu \Big)^{1/q} < \infty.$$

The space $BMO_q(\mu)$ is the quotient of $\mathcal{BMO}_q(\mu)$ by constant functions. It is a Banach space endowed with the norm

$$\|f\|_{BMO_q} = \sup \Big\{ \Big(\frac{1}{\mu(\tilde{R})} \int_{\tilde{R}} |f - f_{\tilde{R}}|^q \,\mathrm{d}\mu \Big)^{1/q} : \ \tilde{R} \in \mathcal{R} \Big\}.$$

We now prove the duality result between $BMO_1(\mu)$ and $H^1(\mu)$ and then show, as a consequence, that all $BMO_q(\mu)$ spaces coincide with $BMO_1(\mu)$, for $q \in (1,\infty)$.

Theorem 13. *The following hold:*

(i) *for every function f in $BMO_1(\mu)$ there exists a bounded linear functional L_f on $H^1(\mu)$ such that*
$$L_f(g) = \int_{\mathcal{V}} f\, g \,\mathrm{d}\mu \qquad \forall g \in H^{1,\infty}_{\mathrm{fin}}(\mu)\,, \tag{11}$$
and there exists $A > 0$ such that $\|L_f\|_{(H^1)^*} \leqslant A\, \|f\|_{BMO_1}$;

(ii) *for every bounded linear functional L on $H^1(\mu)$ there exists a unique function $f \in BMO_1(\mu)$ such that $L = L_f$ and $\|f\|_{BMO_1} \leqslant C_2\, \|L\|_{(H^1)^*}$, where C_2 is the constant which appears in Proposition 11(i).*

Proof. We first prove (i) in the case when $f \in L^\infty(\mu)$. Let g be a function in $H^1(\mu)$ and choose an atomic decomposition $g = \sum_j \lambda_j a_j$ such that a_j are $(1,\infty)$-atoms supported in Calderón–Zygmund sets $\tilde{R}_j$. Since $f \in L^\infty(\mu)$ we have
$$\int f\, g \,\mathrm{d}\mu = \sum_j \lambda_j \int f\, a_j \,\mathrm{d}\mu\,. \tag{12}$$
For every j
$$\begin{aligned}\Big|\int f\, a_j \,\mathrm{d}\mu\Big| &= \Big|\int_{\tilde{R}_j} (f - f_{\tilde{R}_j})\, a_j \,\mathrm{d}\mu\Big| \\ &\leqslant \int_{\tilde{R}_j} |f - f_{\tilde{R}_j}|\, |a_j| \,\mathrm{d}\mu \\ &\leqslant \mu(\tilde{R}_j)^{-1} \int_{\tilde{R}_j} |f - f_{\tilde{R}_j}| \,\mathrm{d}\mu \\ &\leqslant \|f\|_{BMO_1}\,.\end{aligned}$$
By (12) we deduce that
$$\Big|\int f\, g \,\mathrm{d}\mu\Big| \leqslant \sum_j |\lambda_j| \|f\|_{BMO_1}\,.$$
Taking the infimum over all atomic decompositions of g we get
$$\Big|\int f\, g \,\mathrm{d}\mu\Big| \leqslant \|g\|_{H^1} \|f\|_{BMO_1}\,. \tag{13}$$
Let now $f \in BMO_1(\mu)$ be real-valued and define for every $k \in \mathbb{N}$ the function $f_k : \mathcal{V} \to \mathbb{R}$ by
$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant k \\ k\,\frac{f(x)}{|f(x)|} & \text{if } |f(x)| > k. \end{cases}$$
Then $\|f_k\|_{L^\infty} \leqslant k$ and $\|f_k\|_{BMO_1} \leqslant C\|f\|_{BMO_1}$. Moreover $|f_k - f|$ tends monotonically to zero when k tends to ∞. Let $g \in H^{1,\infty}_{\mathrm{fin}}(\mu)$. By (13) we deduce that
$$\Big|\int f_k\, g \,\mathrm{d}\mu\Big| \leqslant \|g\|_{H^1} \|f_k\|_{BMO_1} \leqslant C\, \|g\|_{H^1} \|f\|_{BMO_1}\,.$$
Since $f_k g$ tends to fg everywhere, g is compactly supported and f is integrable on the support of g, by the dominated convergence theorem
$$\Big|\int f\, g \,\mathrm{d}\mu\Big| = \lim_{k\to\infty} \Big|\int f_k\, g \,\mathrm{d}\mu\Big| \leqslant C\, \|g\|_{H^1} \|f\|_{BMO_1} \qquad \forall g \in H^{1,\infty}_{\mathrm{fin}}(\mu)\,. \tag{14}$$

Since $H^{1,\infty}_{\rm fin}(\mu)$ is dense in $H^1(\mu)$, it follows that the functional L_f extends to a bounded functional on $H^1(\mu)$ and there exists a positive constant A such that $\|L_f\|_{(H^1)^*} \leqslant A\|f\|_{BMO_1}$.

If $f \in BMO_1(\mu)$ is complex-valued, then we apply (14) to $\operatorname{Re} f$ and $\operatorname{Im} f$ and prove (i).

We now prove (ii). Let $\{\tilde{Q}_n\}$ be the sequence of Calderón–Zygmund sets constructed in Proposition 7 and, for any $n \in \mathbb{N}$, let X^2_n and X^2 be the spaces introduced at the end of Section 2. Observe that $H^{1,2}_{\rm fin}(\mu)$ and X^2 agree as vector spaces. For any $g \in X^2_n$ the function $\mu(\tilde{Q}_n)^{-1/2}\|g\|^{-1}_{L^2}\, g$ is a (1,2)-atom, so that g is in $H^{1,2}(\mu)$ and $\|g\|_{H^{1,2}} \leqslant \mu(\tilde{Q}_n)^{1/2}\|g\|_{L^2}$. Hence $X^2 \subset H^{1,2}(\mu)$ and the inclusion is continuous.

Let L be in $(H^1(\mu))^* = (H^{1,2}(\mu))^*$. Hence L lies in the dual of X^2. Then by Proposition 8 there exists a function $f : \mathcal{V} \to \mathbb{C}$ such that

$$L(g) = \int f\, g\, {\rm d}\mu \qquad \forall g \in X^2\,.$$

We now show that $f \in BMO_2(\mu)$. Take a Calderón–Zygmund set $\tilde{R}$. For any $g \in X^2$ supported in $\tilde{R}$ the function $\mu(\tilde{R})^{-1/2}\|g\|^{-1}_{L^2} g$ is a (1,2)-atom. We then have

$$\Big|\int f g\, {\rm d}\mu\Big| = |L(g)| \leqslant \|L\|_{(H^{1,2})^*}\|g\|_{L^2}\mu(\tilde{R})^{1/2}\,.$$

This implies that $f - f_{\tilde{R}}$ is a function in $L^2_0(\tilde{R})$ which represents the restriction of the bounded linear functional L on $L^2_0(\tilde{R})$. Hence

$$\|f - f_{\tilde{R}}\|_{L^2(\tilde{R})} \leqslant \|L\|_{(L^2_0(\tilde{R}))^*} \leqslant \mu(\tilde{R})^{1/2}\|L\|_{(H^{1,2})^*}.$$

It follows that

$$\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - f_{\tilde{R}}|\, {\rm d}\mu \leqslant \|L\|_{(H^{1,2})^*} \leqslant C_2\|L\|_{(H^1)^*}\,,$$

so that $f \in BMO_1(\mu)$ and $\|f\|_{BMO_1} \leqslant C_2\|L\|_{(H^1)^*}$. □

Corollary 14. *For every q in $(1,\infty)$ the space $BMO_q(\mu)$ coincides with $BMO_1(\mu)$ and*

$$\|f\|_{BMO_1} \leqslant \|f\|_{BMO_q} \leqslant AC_p\|f\|_{BMO_1} \qquad \forall f \in BMO_q(\mu)\,,$$

where $p = q'$, C_p is the constant which appears in Proposition 11(i) and A is the constant which appears in Theorem 13(i).

Proof. It follows from Hölder's inequality that

$$\|f\|_{BMO_1} \leqslant \|f\|_{BMO_q} \qquad \forall f \in BMO_q(\mu)\,.$$

Take now $f \in BMO_1(\mu)$ and let L_f be the functional on $H^1(\mu)$ such that

$$L_f(g) = \int f\, g\, {\rm d}\mu \qquad \forall g \in H^{1,\infty}_{\rm fin}(\mu)\,.$$

Let $q \in (1,\infty)$ and $p = q'$.

Let $\{\tilde{Q}_n\}$ be the sequence of Calderón–Zygmund sets constructed in Proposition 7. For any $n \in \mathbb{N}$, let X^p_n and X^p be the spaces introduced in Section 2. Observe that $H^{1,p}_{\rm fin}(\mu)$ and X^p agree as vector spaces. For any $g \in X^p_n$ the function $\mu(\tilde{Q}_n)^{-1+1/p}\|g\|^{-1}_{L^p}\, g$ is a $(1,p)$-atom, so that g is in $H^{1,p}(\mu)$ and $\|g\|_{H^{1,p}} \leqslant \mu(\tilde{Q}_n)^{1-1/p}\|g\|_{L^p}$. Hence $X^p \subset H^{1,p}(\mu)$ and the inclusion is continuous.

Hence L_f lies in the dual of X^p. Then by Proposition 8 there exists a function $F:\mathcal{V}\to\mathbb{C}$ such that

$$L_f(g)=\int F\,g\,\mathrm{d}\mu \qquad \forall g\in X^p\,.$$

We now show that $F\in BMO_q(\mu)$. Take a Calderón–Zygmund set $\tilde{R}$. For any $g\in X^p$ supported in $\tilde{R}$ the function $\mu(\tilde{R})^{-1+1/p}\|g\|_{L^p}^{-1}g$ is a $(1,p)$-atom. We then have

$$\Big|\int F\,g\,\mathrm{d}\mu\Big| = |L(g)| \leqslant \|L_f\|_{(H^{1,p})^*}\|g\|_{L^p}\,\mu(\tilde{R})^{1-1/p}\,.$$

This implies that $F-F_{\tilde{R}}$ is a function in $L^q_0(\tilde{R})$ which represents the restriction of the bounded linear functional L_f on $L^p_0(\tilde{R})$. Hence

$$\|F-F_{\tilde{R}}\|_{L^q(\tilde{R})}\leqslant \|L_f\|_{(L^p_0(\tilde{R}))^*}\leqslant \mu(\tilde{R})^{1/q}\|L_f\|_{(H^{1,p})^*}.$$

It follows that

$$\Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}}|F-F_{\tilde{R}}|^q\,\mathrm{d}\mu\Big)^{1/q}\leqslant \|L_f\|_{(H^{1,p})^*}\leqslant C_p\|L_f\|_{(H^1)^*}\leqslant AC_p\|f\|_{BMO_1}\,,$$

so that $\|F\|_{BMO_q}\leqslant AC_p\|f\|_{BMO_1}$, where C_p is the constant which appears in Proposition 11(i).

Take any Calderón–Zygmund set $\tilde{R}$. Since by Proposition 11(iii) $L^p_0(\tilde{R})\subset H^{1,\infty}_{\mathrm{fin}}(\mu)$, we have that

$$L_f(g)=\int f g\,\mathrm{d}\mu=\int F\,g\,\mathrm{d}\mu \qquad \forall g\in L^p_0(\tilde{R})\,.$$

Hence there exists a positive constant $c_{\tilde{R}}$ such that $f=F-c_{\tilde{R}}$ on $\tilde{R}$, so that

$$\Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}}|f-f_{\tilde{R}}|^q\,\mathrm{d}\mu\Big)^{1/q}=\Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}}|F-F_{\tilde{R}}|^q\,\mathrm{d}\mu\Big)^{1/q}.$$

In conclusion, $f\in BMO_q(\mu)$ and $\|f\|_{BMO_q}=\|F\|_{BMO_q}\leqslant AC_p\|f\|_{BMO_1}$.

□

In the sequel we shall denote by $BMO(\mu)$ the space $BMO_1(\mu)$ endowed with the norm $\|\cdot\|_{BMO}=\|\cdot\|_{BMO_1}$.

As a consequence of the duality result, by [1, Theorem 2], arguing exactly as in [18, Section 5] we can deduce the following real interpolation results involving Hardy and BMO spaces.

Corollary 15. *Suppose that* $1\leqslant r_1<r<\infty$, $\frac{1}{r}=\frac{1-\theta}{r_1}$, $\theta\in(0,1)$. *Then*

$$[L^{r_1}(\mu),BMO(\mu)]_{\theta,q}=L^r(\mu)\,.$$

Moreover, if $\frac{1}{r}=1-\theta$, *with* $\theta\in(0,1)$, *then*

$$[H^1(\mu),BMO(\mu)]_{\theta,r}=L^r(\mu)\,.$$

As a consequence of the duality result and of [1, Theorem 3] we deduce that integral operators whose kernels satisfy a suitable integral Hörmander condition are bounded from $L^\infty(\mu)$ to $BMO(\mu)$.

Corollary 16. *Let T be a linear operator which is bounded on $L^2(\mu)$ and admits a locally integrable kernel K off the diagonal that satisfies the condition*

$$\sup_{\tilde{R}}\sup_{y,\,z\in\tilde{R}}\int_{(\tilde{R}^*)^c}|K(y,x)-K(z,x)|\,\mathrm{d}\mu(x)<\infty\,,$$

where the supremum is taken over all Calderón-Zygmund sets $\tilde{R}$ and $\tilde{R}^*$ is defined as in Definition 6. Then T extends to a bounded operator from $L^\infty(\mu)$ to $BMO(\mu)$.

4. The sharp maximal function

In this section we introduce the sharp maximal function associated with the family of Calderón–Zygmund sets and investigate some of its properties. This sharp maximal function is related with the definition of the BMO-space and might be useful to study interpolation properties of such space.

Definition 17. *Let $q \in [1,\infty)$. For every function f in $L^q_{\mathrm{loc}}(\mu)$ its sharp maximal function $f^{\sharp,q}$ is defined by*

$$f^{\sharp,q}(x) = \sup_{\tilde{R}\in\mathcal{R}, x\in\tilde{R}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - f_{\tilde{R}}|^q \,\mathrm{d}\mu\Big)^{1/q} \qquad \forall x \in \mathcal{V}\,.$$

Notice that $\|f^{\sharp,q}\|_{L^\infty} = \|f\|_{BMO_q}$ for every function $f \in BMO(\mu)$.

Proposition 18. *Let $q \in [1,\infty)$ and $f, g \in BMO(\mu)$. The following hold:*

(i)
$$\frac{1}{2} f^{\sharp,q}(x) \leqslant \sup_{\tilde{R}\in\mathcal{R}, x\in\tilde{R}} \inf_{c\in\mathbb{C}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - c|^q \,\mathrm{d}\mu\Big)^{1/q} \leqslant f^{\sharp,q}(x) \qquad \forall x \in \mathcal{V}\,;$$

(ii) *for every $x \in \mathcal{V}$, $|f|^{\sharp,q}(x) \leqslant 2 f^{\sharp,q}(x)$. Then $|f| \in BMO(\mu)$ and*
$$\|\,|f|\,\|_{BMO_q} \leqslant 2\|f\|_{BMO_q}\,;$$

(iii) *$(f+g)^{\sharp,q}(x) \leqslant f^{\sharp,q}(x) + g^{\sharp,q}(x)$ for every $x \in \mathcal{V}$;*

(iv) *there exists a positive constant C such that if f and g have real values, then for every $x \in \mathcal{V}$,*
$$[\max(f,g)]^{\sharp,q}(x) \leqslant C\left(|f|^{\sharp,q}(x) + |g|^{\sharp,q}(x)\right), \qquad [\min(f,g)]^{\sharp,q}(x) \leqslant C\left(|f|^{\sharp,q}(x) + |g|^{\sharp,q}(x)\right).$$

Proof. We first prove (i). Given $x \in \mathcal{V}$ and $\tilde{R} \in \mathcal{R}$ which contains x for every $\varepsilon > 0$ we choose a constant $c_{\tilde{R}}$ such that

$$\Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - c_{\tilde{R}}|^q \,\mathrm{d}\mu\Big)^{1/q} \leqslant \inf_{c\in\mathbb{C}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - c|^q \,\mathrm{d}\mu\Big)^{1/q} + \varepsilon$$

Then, by applying Hölder's inequality, we obtain

$$\begin{aligned}\Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - f_{\tilde{R}}|^q \,\mathrm{d}\mu\Big)^{1/q} &\leqslant \frac{1}{\mu(\tilde{R})^{1/q}}\|f - c_{\tilde{R}}\|_{L^q(\tilde{R})} + \frac{1}{\mu(\tilde{R})^{1/q}}\|f_{\tilde{R}} - c_{\tilde{R}}\|_{L^q(\tilde{R})}\\ &\leqslant 2\inf_{c\in\mathbb{C}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f - c|^q \,\mathrm{d}\mu\Big)^{1/q} + 2\varepsilon\,.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and taking the supremum over all sets $\tilde{R} \in \mathcal{R}$ we obtain the first inequality in (i); the second one is immediate.

Let us now prove (ii). For every $\tilde{R} \in \mathcal{R}$, we have

$$\begin{aligned}
|f|^{\sharp,q}(x) &\leqslant 2 \sup_{\tilde{R}\in\mathcal{R},x\in\tilde{R}} \inf_{c\in\mathbb{C}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} \big||f|-c\big|^q \,\mathrm{d}\mu\Big)^{1/q} \\
&\leqslant 2 \sup_{\tilde{R}\in\mathcal{R},x\in\tilde{R}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} \big||f|-|f_{\tilde{R}}|\big|^q \,\mathrm{d}\mu\Big)^{1/q} \\
&\leqslant 2 \sup_{\tilde{R}\in\mathcal{R},x\in\tilde{R}} \Big(\frac{1}{\mu(\tilde{R})}\int_{\tilde{R}} |f-f_{\tilde{R}}|^q \,\mathrm{d}\mu\Big)^{1/q} \\
&= 2f^{\sharp,q}(x)\,.
\end{aligned}$$

Property (iii) is an immediate consequence of the definition of the sharp maximal function.

Property (iv) follows from (ii) and the fact that

$$\max(f,g) = \frac{f+g+|f-g|}{2} \qquad \text{and} \qquad \min(f,g) = \frac{f+g-|f-g|}{2}\,.$$

□

Notice that, as a consequence of Proposition 18 (iv), the set of real valued functions in $BMO(\mu)$ is a lattice.

In the following result, we explain how the duality of $H^1(\mu)$ with $BMO(\mu)$ can be quantitatively expressed in terms of the sharp maximal function.

Proposition 19. *Let $q \in (1,\infty)$. There exists a positive constant C such that for every $f \in L^\infty(\mu)$ and $g \in H^1(\mu) \cap L^q(\mu)$*

$$\Big|\int f\, g \,\mathrm{d}\mu\Big| \leqslant C \int f^{\sharp,q'} (M(|g|^q))^{1/q}\,\mathrm{d}\mu\,,$$

where

$$M(|g|^q)(x) = \sup_{x\in R} \frac{1}{\mu(R)}\int_R |g|^q \,\mathrm{d}\mu \qquad \forall x \in \mathcal{V}\,,$$

where the supremum is taken over all admissible trapezoids that contain x.

Proof. Take $g \in H^1(\mu) \cap L^q(\mu)$. For every $j \in \mathbb{Z}$ arguing as in [1, Lemma 1] we can construct a family of disjoint trapezoids R_k^j, a function g^j and functions b_k^j such that

- $\bigcup_k R_k^j \subset \Omega_j = \{x \in \mathcal{V} : M(|g|^q) > 2^{jq}\} \subset \bigcup_k \tilde{R}_k^j$;
- $g = g^j + \sum_k b_k^j = g^j + b^j$;
- $|g^j| \leqslant C2^j$;
- b_k^j is supported in $\tilde{R}_k^j$, has vanishing integral and $\|b_k^j\|_{L^q} \leqslant C2^j\mu(\tilde{R}_k^j)^{1/q}$.

These facts implies that g^j tends to 0 uniformly when $j \to -\infty$ and $\|b^j\|_{H^1} \leqslant C2^{j(1-q)}\|g\|_{L^q}^q$, hence b^j tends to 0 in $H^1(\mu)$ when $j \to +\infty$. Thus

$$g = \lim_{N\to+\infty} \sum_{j=-\infty}^{N} (g^{j+1}-g^j) = \lim_{N\to+\infty} \sum_{j=-\infty}^{N} (b^j - b^{j+1})\,. \tag{15}$$

Notice that we can write $b_k^j = \lambda_k^j a_k^j$, where $\lambda_k^j = C2^j\mu(\tilde{R}_k^j)$ and a_k^j is a $(1,q)$-atom supported in $\tilde{R}_k^j$.

Take now $f \in L^\infty(\mu)$. We have

$$\Big|\int f\, a_k^j \,\mathrm{d}\mu\Big| \leqslant \Big|\int [f - f_{\tilde{R}_k^j}]\, a_k^j \,\mathrm{d}\mu\Big| \leqslant \|a_k^j\|_{L^q}\|f-f_{\tilde{R}_k^j}\|_{L^{q'}(\tilde{R}_k^j)} \leqslant \mu(\tilde{R}_k^j)^{-1+1/q}\|f-f_{\tilde{R}_k^j}\|_{L^{q'}(\tilde{R}_k^j)} \leqslant f^{\sharp,q'}(x)\,,$$

for every $x \in \tilde{R}_k^j$. Hence

$$\Big|\int f\, a_k^j \,\mathrm{d}\mu\Big| \leqslant \frac{1}{\mu(R_k^j)}\int_{R_k^j} f^{\sharp,q'}\,\mathrm{d}\mu\,.$$

Using (15) and Proposition 5, it follows that

$$\begin{aligned}
\Big|\int f\, g\,\mathrm{d}\mu\Big| &\leqslant \sum_{j,k}\lambda_k^j\Big|\int f\, a_k^j\,\mathrm{d}\mu\Big| + \sum_{j,\ell}\lambda_\ell^{j+1}\Big|\int f\, a_\ell^{j+1}\,\mathrm{d}\mu\Big| \\
&\leqslant C\sum_{j,k}2^j\mu(\tilde{R}_k^j)\frac{1}{\mu(R_k^j)}\int_{R_k^j} f^{\sharp,q'}\,\mathrm{d}\mu + C\sum_{j,\ell}2^{j+1}\mu(\tilde{R}_\ell^{j+1})\frac{1}{\mu(R_\ell^{j+1})}\int_{R_\ell^{j+1}} f^{\sharp,q'}\,\mathrm{d}\mu \\
&\leqslant C\sum_j\int_{\bigcup_k R_k^j} 2^j f^{\sharp,q'}\,\mathrm{d}\mu + C\sum_j\int_{\bigcup_\ell R_\ell^{j+1}} 2^{j+1} f^{\sharp,q'}\,\mathrm{d}\mu \\
&\leqslant C\sum_j\int_{\Omega_j} 2^j f^{\sharp,q'}\,\mathrm{d}\mu + C\sum_j\int_{\Omega_{j+1}} 2^{j+1} f^{\sharp,q'}\,\mathrm{d}\mu \\
&\leqslant C\int_{\mathcal{V}} f^{\sharp,q'}(x)\sum_{2^j<(M|g|^q)^{1/q}(x)} 2^j\,\mathrm{d}\mu(x) + C\int_{\mathcal{V}} f^{\sharp,q'}(x)\sum_{2^{j+1}<(M|g|^q)^{1/q}(x)} 2^{j+1}\,\mathrm{d}\mu(x) \\
&\leqslant C\int_{\mathcal{V}} f^{\sharp,q'}(x)(M|g|^q)^{1/q}(x)\,\mathrm{d}\mu(x)\,,
\end{aligned}$$

as required. □

The following theorem can be thought as a weak version of the classical L^p-inequality involving the sharp maximal function in the Euclidean setting (see [15, Theorem 2, §2, Ch. IV]).

Theorem 20. *Let $p \in (1,\infty)$ and $p_0 \in (1,p)$. There exists a positive constant C such that*

$$\|f\|_{L^p} \leqslant C\,\|f^{\sharp,p_0}\|_{L^p} \qquad \forall f \in L^{p_0}(\mu)\,. \tag{16}$$

Proof. Given $f \in L^{p_0}(\mu)$ real-valued we define for every $k \in \mathbb{N}$

$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqslant k \\ k\frac{f(x)}{|f(x)|} & \text{if } |f(x)| > k. \end{cases}$$

Then f_k converges to f in $L^{p_0}(\mu)$ when $k \to +\infty$, $f_k \in L^\infty(\mu)$ and by Proposition 18 (iv)

$$f_k^{\sharp,p_0} \leqslant C\, f^{\sharp,p_0}\,. \tag{17}$$

Denote by q and q_0 the conjugate exponents of p and p_0, respectively. Take a function $g \in L^q(\mu) \cap L^{q_0}(\mu)$ with $\|g\|_{L^q} \leqslant 1$. For every $k \in \mathbb{N}$ define

$$\tilde{g}_k = g\chi_{\tilde{Q}_k} \qquad \text{and} \qquad I_k = \int \tilde{g}_k\,\mathrm{d}\mu\,,$$

where $\tilde{Q}_k$ are the sets introduced in Proposition 7. We then define

$$g_k = \tilde{g}_k - I_k\,\mu(\tilde{Q}_{2^k})^{-1}\chi_{\tilde{Q}_{2^k}} = \tilde{g}_k - r_k\,.$$

The function g_k is in $L^q(\mu)$, it has vanishing integral and is supported in $\tilde{Q}_{2^k}$; hence g_k lies in $H^1(\mu)$. Moreover, g_k tends to g pointwise for $k \to +\infty$ and $|g_k| \leqslant |g| + r$, where $r = \sum_k |r_k|$. We have that

$$\|r_k\|_{L^q} \leqslant \mu(\tilde{Q}_k)^{1/q'} \mu(\tilde{Q}_{2^k})^{-1+1/q} = \Big(\frac{q^k(k+1)}{q^{2^k}(2^k+1)}\Big)^{1/q'},$$

and then $r \in L^q(\mu)$. Hence g_k tends to g also in $L^q(\mu)$ for $k \to +\infty$.

By applying Proposition 19 to f_k and g_k and Hölder's inequality we get

$$\Big|\int f_k\, g_k \,\mathrm{d}\mu\Big| \leqslant \int f_k^{\sharp,p_0} (M(|g_k|^{q_0}))^{1/q_0} \,\mathrm{d}\mu \leqslant \|f_k^{\sharp,p_0}\|_{L^p} \,\|(M(|g_k|^{q_0}))^{1/q_0}\|_{L^q}\,.$$

Taking the limit for $k \to +\infty$ on the left-hand side and by applying (17) and the boundedness of the Hardy–Littlewood maximal function on $L^{q/q_0}(\mu)$ (see [1]) we deduce that

$$\Big|\int f\, g \,\mathrm{d}\mu\Big| \leqslant C\,\|f^{\sharp,p_0}\|_{L^p}\,\|g_k\|_{L^q} \leqslant C\,\|f^{\sharp,p_0}\|_{L^p}\,\|g\|_{L^q} \leqslant C\,\|f^{\sharp,p_0}\|_{L^p}\,. \tag{18}$$

Since the previous inequality holds for every $g \in L^q(\mu) \cap L^{q_0}(\mu)$ with $\|g\|_{L^q} \leqslant 1$ with constants independent of g, we deduce that $\|f\|_{L^p} \leqslant C\,\|f^{\sharp,p_0}\|_{L^p}$.

The case when f is complex-valued follows by applying 18 to $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ and arguing as above. □

Remark 21. *Proposition 19 and Theorem 20 are inspired by similar results involving the sharp maximal function in the Euclidean setting (see* [15, §2, Ch. IV]*). More precisely, Proposition 19 differs from* [15, Formula (16) Ch. IV] *because we require an extra integrability condition on the function $g \in H^1(\mu)$ and the maximal function involved here is a variant of the Hardy–Littlewood maximal function. This is due to the fact that a maximal characterization of the Hardy space $H^1(\mu)$ is not available in our setting.*

The inequality (16) *is a weak version of the inequality*

$$\|f\|_{L^p} \leqslant C\,\|f^{\sharp,1}\|_{L^p}\,,$$

which is still unknown in the setting of the present paper. The proof of such inequality would probably require a distributional inequality involving both the Hardy–Littlewood and the sharp maximal functions (or a dyadic version of them) that is still work in progress.

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