Practical numbers among the binomial coefficients

PRACTICAL NUMBERS AMONG THE BINOMIAL COEFFICIENTS

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ABSTRACT. A *practical number* is a positive integer n such that every positive integer less than n can be written as a sum of distinct divisors of n. We prove that most of the binomial coefficients are practical numbers. Precisely, letting $f(n)$ denote the number of binomial coefficients $\binom{n}{k}$, with $0 \le k \le n$, that are not practical numbers, we show that

$$f(n) < n^{1-(\log 2 - \delta)/\log\log n}$$

for all integers $n \in [3, x]$, but at most $O_\gamma(x^{1-(\delta-\gamma)/\log\log x})$ exceptions, for all $x \ge 3$ and $0 < \gamma < \delta < \log 2$. Furthermore, we prove that the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \le x$ but at most $O(x^{0.88097})$ exceptions. We also pose some questions on this topic.

1. Introduction

A *practical number* is a positive integer n such that every positive integer less than n can be written as a sum of distinct divisors of n. This property has been introduced by Srinivasan [19]. Estimates for the counting function of practical numbers have been given by Hausman–Shapiro [5], Tenenbaum [20], Margenstern [9], Saias [15], and finally Weingartner [21], who proved that there are asymptotically $Cx/\log x$ practical numbers less than x, for some constant $C > 0$, as previously conjectured by Margenstern [9]. On another direction, Melfi [11] proved that every positive even integer is the sum of two practical numbers, and that there are infinitely many triples $(n, n+2, n+4)$ of practical numbers. Also, Melfi [10] proved that in every Lucas sequence, satisfying some mild conditions, there are infinitely many practical numbers, and Sanna [17] gave a lower bound for their counting function.

In this work, we study the binomial coefficients which are also practical numbers. Our first result, informally, states that for almost all positive integers n

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there is a negligible amount of binomial coefficients $\binom{n}{k}$, with $0 \le k \le n$, which are not practical. Precisely, for each positive integer n, define

$$f(n) := \#\left\{0 \le k \le n : \binom{n}{k} \text{ is not a practical number}\right\}.$$

Our first result is the following.

Theorem 1.1. *For all $x \ge 3$ and $0 < \gamma < \delta < \log 2$, we have*

$$f(n) < n^{1-(\log 2-\delta)/\log\log n}$$

for all integers $n \in [3, x]$, but at most $O_\gamma(x^{1-(\delta-\gamma)/\log\log x})$ exceptions.

As a consequence, we obtain that as $x \to +\infty$ almost all binomial coefficients $\binom{n}{k}$, with $0 \le k \le n \le x$, are practical numbers.

Corollary 1.1. *We have*

$$\sum_{n \le x} f(n) \ll_\varepsilon x^{2-(\frac{1}{2}\log 2-\varepsilon)/\log\log x},$$

for all $x \ge 3$ and $\varepsilon > 0$.

Among the binomial coefficients, the *central binomial coefficients* $\binom{2n}{n}$ are of great interest. In particular, several authors have studied their arithmetic and divisibility properties, see e.g. [1, 2, 14, 16, 18].

In this direction, our second result, again informally, states that almost all central binomial coefficients $\binom{2n}{n}$ are practical numbers.

Theorem 1.2. *For $x \ge 1$, the central binomial coefficient $\binom{2n}{n}$ is a practical number for all positive integers $n \le x$ but at most $O(x^{0.88097})$ exceptions.*

Probably, there are only finitely many positive integers n such that $\binom{2n}{n}$ is not a practical number. By a computer search, we found only three of them below 10^6, namely $n = 4, 10, 256$. However, proving the finiteness seems to be out of reach with actual techniques. Indeed, on the one hand, if n is a power of 2 whose base 3 representation contains only the digits 0 and 1, then it can be shown that $\binom{2n}{n}$ is not a practical number (see Proposition 2.1 below). On the other hand, it is an open problem to establish whether there are finitely or infinitely many powers of 2 of this type [4, 6, 8, 12].

We conclude by leaving two open questions. Note that since $\binom{n}{0} = \binom{n}{n} = 1$, we have $0 \le f(n) \le n - 1$ for all positive integers n. It is natural to ask when one of the equalities is satisfied.

Question 1.1. What are the positive integers n such that $f(n) = 0$?

Question 1.2. What are the positive integers n such that $f(n) = n - 1$?

Regarding Question 1.1, if $f(n) = 0$ then n must be a power of 2, otherwise there would exist (see Lemma 2.4 below) an odd binomial coefficient $\binom{n}{k}$, with $0 < k < n$, and since 1 is the only odd practical number, we would have $f(n) > 0$. However, this is not a sufficient condition, since $f(8) = 1$. Regarding Question 1.2, if $n = 2^k - 1$, for some positive integer k, then $f(n) = n - 1$, because all the binomial coefficients $\binom{n}{k}$, with $0 < k < n$, are odd (see Lemma 2.4 below) and greater than 1, and consequently they are not practical numbers. However, this is not a necessary condition, since $f(5) = 4$.

Notation. We employ the Landau–Bachmann "Big Oh" notation O and the associated Vinogradov symbol $\ll$. In particular, any dependence of the implied constants is indicated with subscripts. We write p_i for the ith prime number.

2. Preliminaries

This section is devoted to some preliminary results needed in the later proofs. We begin with some lemmas about practical numbers.

Lemma 2.1. *If m is a practical number and $n \leq 2m$ is a positive integer, then mn is a practical number.*

Proof. See [10, Lemma 4]. □

Lemma 2.2. *If d is a practical number and n is a positive integer divisible by d and having all prime factors not exceeding $2d$, then n is a practical number.*

Proof. By hypothesis, there exist positive integers $q_1, \dots, q_k \leq 2d$ such that $n = dq_1 \cdots q_k$. Then, using Lemma 2.1, it follows by induction that $dq_1 \cdots q_m$ is practical for all $m = 1, \dots, k$. In particular, n is practical. □

Lemma 2.3. *We have that $p_1^{a_1} \cdots p_s^{a_s}$ is a practical number, for all positive integers $a_1, \dots, a_s$.*

Proof. It follows easily by induction on s, using Lemma 2.1 and Bertrand's postulate $p_{i+1} < 2p_i$. □

For each prime number p and for each positive integer n, put

$$T_p(n) := \#\left\{0 \leq k \leq n : p \nmid \binom{n}{k}\right\}.$$

We have the following formula for $T_p(n)$.

Lemma 2.4. *Let p be a prime number and let*

$$n = \sum_{j=0}^{s} d_j p^j, \quad d_0, \dots, d_s \in \{0, \dots, p-1\}, \quad d_s \neq 0,$$

be the representation in base p of the positive integer n. Then we have

$$T_p(n) = \prod_{j=0}^{s} (d_j + 1).$$

Proof. See [3, Theorem 2]. □

For each prime number p, let us define

$$\omega_p := \frac{\log((p+1)/2)}{\log p}.$$

The quantity ω_p appears in the following upper bound for $T_p(n)$.

Lemma 2.5. *Let p be a prime number and fix $\varepsilon \in (0, 1/2)$. Then, for all $x \geq 1$, we have*

$$T_p(n) < n^{\omega_p + \varepsilon}$$

for all positive integers $n \leq x$ but at most $O(p^3 x^{1-\varepsilon})$ exceptions.

Proof. For $x \geq 1$, let k be the smallest integer such that $x < p^k$. Clearly, we have

$$\begin{aligned} E(x) &:= \#\{n \leq x : T_p(n) \geq n^{\omega_p+\varepsilon}\} \\ &\leq \sum_{j=1}^{k} \#\{p^{j-1} \leq n < p^j : T_p(n) \geq p^{(j-1)(\omega_p+\varepsilon)}\}. \end{aligned} \tag{1}$$

Moreover, thanks to Lemma 2.4, we have

$$\sum_{p^{j-1} < n \leq p^j} T_p(n) \leq \sum_{0 \leq d_0, \ldots, d_{j-1} < p} \prod_{i=0}^{j-1} (d_i + 1) = \left(\sum_{d=0}^{p-1} (d+1) \right)^j = \left(\frac{p(p+1)}{2} \right)^j,$$

and consequently

$$\begin{aligned} \#\{p^{j-1} \leq n < p^j : T_p(n) \geq p^{(j-1)(\omega_p+\varepsilon)}\} &\leq \frac{1}{p^{(j-1)(\omega_p+\varepsilon)}} \sum_{p^{j-1} < n \leq p^j} T_p(n) \\ &\leq \frac{1}{p^{(j-1)(\omega_p+\varepsilon)}} \left(\frac{p(p+1)}{2} \right)^j = \frac{p(p+1)}{2} \, p^{(1-\varepsilon)(j-1)} < p^{2+(1-\varepsilon)(j-1)}, \end{aligned} \tag{2}$$

for all positive integers j. Therefore, putting together (1) and (2), and using that $\varepsilon < 1/2$, we obtain

$$E(x) < \sum_{j=1}^{k} p^{2+(1-\varepsilon)(j-1)} \ll p^{2+(1-\varepsilon)k} \leq p^{2+(1-\varepsilon)(\log x/\log p + 1)} < p^3 x^{1-\varepsilon}, \tag{3}$$

as desired. □

Remark 2.1. The constant $1/2$ in the statement of Lemma 2.5 has no particular importance, it is only needed to justify the $\ll$ in (3). Any other real number less than 1 would be fine.

For all $x \geq 1$, let $\kappa(x)$ be the smallest integer $k \geq 1$ such that $p_1 \cdots p_k \geq x$.

Lemma 2.6. *We have*

$$\kappa(x) \sim \frac{\log x}{\log \log x} \quad \textit{and} \quad p_{\kappa(x)} \sim \log x,$$

as $x \to \infty$.

Proof. As a well-known consequence of the Prime Number Theorem, we have

$$\log(p_1 \cdots p_k) \sim p_k \sim k \log k, \tag{4}$$

as $k \to +\infty$. Since

$$\log(p_1 \cdots p_{\kappa(x)-1}) < \log x \leq \log(p_1 \cdots p_{\kappa(x)}),$$

and $\kappa(x) \to +\infty$ as $x \to +\infty$, by (4) we obtain

$$p_{\kappa(x)} \sim \kappa(x) \log \kappa(x) \sim \log x,$$

which in turn implies

$$\kappa(x) \sim \frac{\kappa(x) \log \kappa(x)}{\log \kappa(x) + \log \log \kappa(x)} \sim \frac{\log x}{\log \log x},$$

as desired. □

For every prime number p and every positive integer n, let $\beta_p(n)$ be the p-adic valuation of the central binomial coefficient $\binom{2n}{n}$.

Lemma 2.7. *For each prime p and all positive integers n, we have that $\beta_p(n)$ is equal to the number of digits of n in base p which are greater than $(p-1)/2$.*

Proof. The claim is a straightforward consequence of a theorem of Kummer [7] which says that, for positive integers m, n, the p-adic valuation of $\binom{m+n}{n}$ is equal to the number of carries in the addition $m + n$ done in base p. □

Proposition 2.1. *If n is a power of 2 and if all the digits of n in base 3 are equal to 0 or 1, then $\binom{2n}{n}$ is not a practical number.*

Proof. It follows by Lemma 2.7 that $\beta_2(n) = 1$ and $\beta_3(n) = 0$, that is, $\binom{2n}{n}$ is an integer of the form $12k \pm 2$. However, it is known that, other than 1 and 2, every practical number is divisible by 4 or 6, see [19]. □

We will make use of the following result of probability theory.

Lemma 2.8. *Let X be a random variable following a binomial distribution with j trials and probability of success α. Then*

$$\mathbb{P}[X \leq (\alpha - \varepsilon)j] \leq e^{-2\varepsilon^2 j}$$

for all $\varepsilon > 0$.

Proof. See [13, Theorem 1]. □

For each prime number p, let us define

$$\alpha_p := \frac{1}{p}\left\lceil \frac{p-1}{2} \right\rceil,$$

so that α_p is the probability that a random digit in base p is greater than $(p-1)/2$.

Lemma 2.9. *Let p be a prime number and fix $\varepsilon \in (0, 1/2)$. Then, for all $x \geq 1$, we have*

$$\beta_p(n) > (\alpha_p - \varepsilon)\frac{\log n}{\log p}$$

for all positive integers $n \leq x$ but at most $O(px^{1-2\varepsilon^2/\log p})$ exceptions.

Proof. For $x \geq 1$, let k be the smallest integer such that $x < p^k$. Clearly, we have

$$\begin{aligned} E(x) &:= \#\left\{n \leq x : \beta_p(n) \leq (\alpha_p - \varepsilon)\frac{\log n}{\log p}\right\} \\ &\leq \sum_{j=1}^{k} \#\big\{p^{j-1} \leq n < p^j : \beta_p(n) \leq (\alpha_p - \varepsilon)j\big\} \\ &\leq \sum_{j=1}^{k} \#\big\{0 \leq n < p^j : \beta_p(n) \leq (\alpha_p - \varepsilon)j\big\}. \end{aligned} \tag{5}$$

Given an integer $j \geq 1$, let us for a moment consider n as a random variable uniformly distributed in $\{0, \ldots, p^j - 1\}$. Then, the digits of n in base p are j independent random variables uniformly distributed in $\{0, \ldots, p-1\}$. Hence, as a consequence of Lemma 2.7, we obtain that $\beta_p(n)$ follows a binomial distribution with j trials and probability of success α_p. In turn, Lemma 2.8 yields

$$\#\{0 \leq n < p^j : \beta_p(n) \leq (\alpha_p - \varepsilon)j\} \leq p^j e^{-2\varepsilon^2 j}. \tag{6}$$

Therefore, putting together (5) and (6), and using that $\varepsilon < 1/2$, we get

$$E(x) \leq \sum_{j=1}^{k} p^j e^{-2\varepsilon^2 j} \ll (pe^{-2\varepsilon^2})^k \leq (pe^{-2\varepsilon^2})^{\log x/\log p + 1} < px^{1-2\varepsilon^2/\log p}, \tag{7}$$

as desired. □

Remark 2.2. The constant 1/2 in the statement of Lemma 2.9 has no particular importance, it is only needed to justify the $\ll$ in (7). Any other real number less than $(\frac{1}{2}\log 2)^{1/2}$ would be fine.

3. Proof of Theorem 1.1

Assume $x \geq 3$ sufficiently large, and put

$$\varepsilon := \frac{\delta - \gamma}{\log\log x} + \frac{4\log\log x}{\log x} \in (0, 1/2).$$

Let n be a positive integer. By Lemma 2.3 and by the definition of $\kappa(n)$, we know that $p_1 \cdots p_{\kappa(n)}$ is a practical number greater than or equal to n. Since all the prime factors of $\binom{n}{k}$ are not exceeding n, Lemma 2.2 tell us that if $p_1 \cdots p_{\kappa(n)}$ divides $\binom{n}{k}$ then $\binom{n}{k}$ is practical. Consequently, we have

$$f(n) \leq \#\left\{0 \leq k \leq n : p_1 \cdots p_{\kappa(n)} \nmid \binom{n}{k}\right\} \leq \sum_{j=1}^{\kappa(n)} T_{p_j}(n).$$

Therefore, it follows from Lemma 2.5 that

$$f(n) < \sum_{j=1}^{\kappa(n)} n^{\omega_{p_j} + \varepsilon}, \tag{8}$$

for all positive integers $n \leq x$ but at most

$$\ll \sum_{j=1}^{\kappa(x)} p_j^3 x^{1-\varepsilon} \ll p_{\kappa(x)}^4 x^{1-\varepsilon} \ll (\log x)^4 x^{1-\varepsilon} = x^{1-(\delta-\gamma)/\log\log x}$$

exceptions, where we also used Lemma 2.6.

Suppose that n satisfies (8). Since ω_p is a monotone increasing function of p, we get that

$$f(n) < \kappa(n) n^{\omega_{p_{\kappa(n)}} + \varepsilon} = n^{\omega_{p_{\kappa(n)}} + \log\kappa(n)/\log n + \varepsilon}. \tag{9}$$

Moreover, for $n \gg_\gamma 1$ we have

$$\omega_{p_{\kappa(n)}} < 1 - \frac{\log 2}{\log p_{\kappa(n)}} + \frac{1}{p_{\kappa(n)}\log p_{\kappa(n)}} < 1 - \frac{\log 2 - \gamma/4}{\log\log n}, \tag{10}$$

and

$$\frac{\log\kappa(n)}{\log n} < \frac{\gamma/4}{\log\log n}, \tag{11}$$

where we used Lemma 2.6. Furthermore, since $n \leq x$, we have

$$\varepsilon < \frac{\delta - \gamma/2}{\log\log n}. \tag{12}$$

Consequently, putting together (10), (11), and (12), we obtain

$$\omega_{p_{\kappa(n)}} + \frac{\log \kappa(n)}{\log n} + \varepsilon < 1 - \frac{\log 2 - \delta}{\log\log n},$$

which, inserted into (9), gives

$$f(n) < n^{1-(\log 2-\delta)/\log\log n}$$

as desired. The proof is complete.

4. Proof of Corollary 1.1

Obviously, we can assume $\varepsilon < \frac{1}{2}\log 2$. Put $\gamma := 2\varepsilon$ and $\delta := \frac{1}{2}\log 2 + \varepsilon$, so that $0 < \gamma < \delta < \log 2$. For all $x \geq 3$, let $\mathcal{E}(x)$ be the set of exceptional $n \leq x$ of Theorem 1.1. Then we have

$$\sum_{n \le x} f(n) = \sum_{n \notin \mathcal{E}(x)} f(n) + \sum_{n \in \mathcal{E}(x)} f(n) < \sum_{n \le x} n^{1-(\log 2-\delta)/\log\log n} + \#\mathcal{E}(x)\, x$$
$$\ll_\varepsilon x^{2-(\log 2-\delta)/\log\log x} + x^{2-(\delta-\gamma)/\log\log x} \ll x^{2-(\frac{1}{2}\log 2-\varepsilon)/\log\log x},$$

as claimed.

5. Proof of Theorem 1.2

For the sake of notation, put

$$s := 16, \quad \eta_s := \frac{\sum_{i=1}^s \alpha_{p_i} - 1}{\sum_{i=1}^s \sqrt{\log p_i}}, \quad \varepsilon_j := \eta_s \sqrt{\log p_j},$$

for $j = 1, \dots, s$. A computation shows that $\varepsilon_j \in (0, 1/2)$ for $j = 1, \dots, s$.

For $x \geq 1$, it follows from Lemma 2.9 that

$$\sum_{j=1}^{s} \beta_{p_j}(n) \log p_j > \sum_{j=1}^{s} (\alpha_{p_j} - \varepsilon_j) \log n = \log n, \tag{13}$$

for all positive integers $n \leq x$, but at most

$$\ll \sum_{j=1}^{s} p_j x^{1-2\varepsilon_j^2/\log p_j} \ll x^{1-2\eta_s^2} < x^{0.88097}$$

exceptions. Suppose that n is a positive integer satisfying (13). Then,

$$d := \prod_{j=1}^{s} p_j^{\beta_{p_j}(n)} > n.$$

Also, by Lemma 2.3 we have that d is a practical number, and by the definition of $\beta_{p_j}(n)$ we have that d divides $\binom{2n}{n}$. Moreover, since all the prime factors of

$\binom{2n}{n}$ are not exceeding $2d$, Lemma 2.2 yields that $\binom{2n}{n}$ is practical. The proof is complete.

Remark 5.1. A comment is in order to explain the choice of the parameters ε_j in the proof of Theorem 1.2. Given a positive integer s, one could fix some prime numbers $q_1 < \cdots < q_s$ and some real numbers $\varepsilon_1, \ldots, \varepsilon_s \in (0, 1/2)$ such that $q_1 \cdots q_s$ is a practical number and $\sum_{j=1}^{s}(\alpha_{q_j} - \varepsilon_j) \geq 1$. Everything would proceed similarly, with an estimate of the number of exceptions given by

$$O\Big(x^{\max\{1-2\varepsilon_1^2/\log q_1, \ldots, 1-2\varepsilon_s^2/\log q_s\}}\Big).$$

To minimize the exponent of x, the optimal choice for ε_j is

$$\varepsilon_j = \eta_s(q_1, \ldots, q_s)\sqrt{\log q_j}, \quad \eta_s(q_1, \ldots, q_s) := \frac{\sum_{i=1}^{s} \alpha_{q_i} - 1}{\sum_{i=1}^{s} \sqrt{\log q_j}},$$

for $j = 1, \ldots, s$, which gives the estimate

$$O\Big(x^{1-2\eta_s(q_1, \ldots, q_s)^2}\Big).$$

Since $\alpha_p = \frac{1}{2} + O(\frac{1}{p})$ for each prime number p, we get that $\eta_s(q_1, \ldots, q_s)$ is maximized when $q_j = p_j$, for $j = 1, \ldots, s$, and that $\eta_s(p_1, \ldots, p_s) \to 0$ as $s \to +\infty$. Lastly, some numeratical computations verify that the maximum of $\eta_s(p_1, \ldots, p_s)$ is reached for $s = 16$.

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