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Roberto Camporesi
Dipartimento di Scienze Matematiche, Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino Italy
e-mail: camporesi@polito.it *

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Abstract

Let $S = NA$ be a Damek-Ricci space, identified with the unit ball B in $\mathfrak{s}$ via the Cayley transform. Let $S^{p+q} = \partial B$ be the unit sphere in $\mathfrak{s}$, $p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z}$. The metric in the ball model was computed in [1] both in Euclidean (or geodesic) polar coordinates and in Cartesian coordinates on B. The induced metric on the Euclidean sphere $S(R)$ of radius R is the sum of a constant curvature term, plus a correction term proportional to h_1, where h_1 is a suitable differential expression which is smooth on $S(R)$ for $R < 1$, but becomes (possibly) singular on the unit sphere at the pole $(0, 0, 1)$. It has a simple geometric interpretation, namely $h_1 = |\Theta|^2$, where Θ is, up to a conformal factor, the pull-back of the canonical 1-form on the group N (defining the horizontal distribution on N) by the generalized stereographic projection. In the symmetric case h_1, as well as the transported distribution on $S^{p+q} \setminus \{(0, 0, 1)\}$, have a smooth extension to the whole sphere. This can be interpreted by the Hopf fibration of S^{p+q}. In the general case no such structure is allowed on the unit sphere, and the question was left open in [1] whether or not h_1 extends smoothly at the pole. In this paper we prove that h_1 does not extend, except in the symmetric case. More precisely, writing h_1 in the coordinates (V, Z) on S^{p+q} as $h_1 = \sum h_{ij}^{(\mathfrak{z})} \, dz_i \, dz_j + \sum h_{ij}^{(\mathfrak{v})} \, dv_i \, dv_j + \sum h_{ij}^{(\mathfrak{z}\mathfrak{v})} \, dz_i \, dv_j$, we prove that, in the non-symmetric case, the coefficients $h_{ij}^{(\mathfrak{z})}$ do not have a limit at the pole, but remain bounded there, whereas the coefficients $h_{ij}^{(\mathfrak{v})}$ and $h_{ij}^{(\mathfrak{z}\mathfrak{v})}$ extend smoothly at the pole. In order to do this, we obtain an explicit formula for the 1-form Θ valid for any Damek-Ricci space. From this formula we deduce that Θ does not extend to the pole, except for $q = 1$ (hermitian symmetric case). The square of Θ and the distribution $\ker \Theta$ do not extend, unless S is symmetric. Indeed, we prove that the singular part of h_1 vanishes identically iff the J^2-condition holds.

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1 Introduction

Let $S = NA$ be a Damek-Ricci space, i.e., the semidirect product of a (connected and simply connected) nilpotent Lie group N of Heisenberg type [7] and the one-dimensional Lie group $A \cong \mathbb{R}^+$ acting on N by anisotropic dilations. When S is equipped with a suitable left-invariant Riemannian metric γ_S, S becomes a (noncompact, simply connected) homogeneous harmonic Riemannian space [3, 4]. Conversely, every such space is a Damek-Ricci space if we exclude $\mathbb{R}^n$ and the "degenerate" case of real hyperbolic spaces (see [6], Corollary 1.2). We take the basic notation from [9], to which we refer for a nice introduction to the geometry and harmonic analysis on Damek-Ricci spaces.

We use the ball model B of S, namely we identify S with the unit ball B in the Lie algebra $\mathfrak{s}$ via the Cayley transform $C : NA \to B$ [2, 9],

$$S = NA \overset{C}{\cong} B = \{(V, Z, t) \in \mathfrak{s} : \ R^2 = |V|^2 + |Z|^2 + t^2 < 1\}.$$

Here $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, where $\mathfrak{a} \simeq \mathbb{R}$, $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement in $\mathfrak{n}$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathfrak{s}$, and by $|\cdot|$ the associated norm. For any $Z \in \mathfrak{z}$ we have the linear map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_Z V, V' \rangle = \langle Z, [V, V'] \rangle, \qquad \forall V, V' \in \mathfrak{v}.$$

The Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ of N is a two-step real nilpotent Lie algebra of Heisenberg type (or H-type), i.e., the map J_Z satisfies

$$J_Z^2 = -|Z|^2 I, \qquad \forall Z \in \mathfrak{z},$$

where I denotes the identity mapping. This implies that the map $Z \to J_Z$ extends to a representation of the real Clifford algebra $\mathrm{Cl}(\mathfrak{z})$ on $\mathfrak{v}$. This procedure can be reversed and yields a general method for constructing H-type Lie algebras [7].

We let $p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z}$, and let S^{p+q} be the unit sphere in $\mathfrak{s}$:

$$S^{p+q} = \partial B = \{(V, Z, t) \in \mathfrak{s} : \ |V|^2 + |Z|^2 + t^2 = 1\}.$$

Let γ_S be the left-invariant Riemannian metric on S given by [1], (1.1). The transported metric $\gamma_B = C^{-1*}(\gamma_S)$ was computed in [1], Theorem 3.1, in Euclidean polar coordinates $(R, (V, Z, t)) \in [0, 1) \times S^{p+q}$. It is given by

$$\gamma_B = \frac{4\, dR^2}{(1 - R^2)^2} + \gamma_{S(R)}, \tag{1.1}$$

where the induced metric on the Euclidean sphere $S(R)$ of radius $R < 1$ is

$$\gamma_{S(R)}|_{R(V,Z,t)} = \frac{4R^2}{1 - R^2} \gamma_{S^{p+q}}|_{(V,Z,t)} + \frac{4}{(1 - R^2)^2} h_1|_{R(V,Z,t)}.$$

Here $\gamma_{S^{p+q}}$ is the round metric on S^{p+q}, and h_1 is the following differential expression on $S^{p+q} \setminus \{(0, 0, 1)\}$:

$$\begin{aligned} h_1|_{(V,Z,t)} = {} & \big|[V, dV] + t dZ - Z dt\big|^2 + |Z|^2 |dZ|^2 - \langle Z, dZ \rangle^2 \\ & + 2\Big(\langle V, dV \rangle \langle Z, dZ \rangle - \big\langle J_{dZ} V, J_Z dV \big\rangle\Big) + k_1(V, Z, t), \end{aligned} \tag{1.2}$$

where

$$k_1(V,Z,t) = \frac{1}{((1-t)^2+|Z|^2)^2}\Big\{|V|^4\Big(\langle Z,dZ\rangle^2 - |Z|^2|dZ|^2\Big) + \big|[J_ZV, J_{dZ}V]\big|^2$$
$$+ 4(1-t)\Big(|V|^2\Big(\big\langle J_ZV, dV\big\rangle\langle Z,dZ\rangle - |Z|^2\big\langle J_{dZ}V, dV\big\rangle\Big) + \big\langle J_{[J_ZV,dV]}J_ZV, J_{dZ}V\big\rangle\Big)$$
$$- 4(1-t)\left(|Z|^2-(1-t)^2\right)\Big(\big\langle J_ZV,dV\big\rangle\langle V,dV\rangle + \big\langle J_{[V,dV]}J_ZV, dV\big\rangle\Big)$$
$$+ 4(1-t)^2\Big(\big\langle J_ZV,dV\big\rangle^2 - |Z|^2\big|[V,dV]\big|^2 + \big|[J_ZV,dV]\big|^2 - |Z|^2\langle V,dV\rangle^2\Big)$$
$$+ 2\big(|Z|^2-(1-t)^2\big)\Big(|V|^2\Big(\langle V,dV\rangle\langle Z,dZ\rangle - \big\langle J_{dZ}V, J_ZdV\big\rangle\Big) - \big\langle J_{[V,dV]}J_ZV, J_{dZ}V\big\rangle\Big)\Big\}. \tag{1.3}$$

(See [1], (3.3), and observe that $R^4h_R|_{(V,Z,t)} = h_1|_{R(V,Z,t)}$, and $k_1(R(V,Z,t))$ is just R^4 times the term with the curly bracket in [1], (3.3). In the notations of [1], $h_1 = \lim_{R\to 1} h_R$.)

For $R<1$, the expression $h_1|_{R(V,Z,t)}$ is smooth $\forall (V,Z,t)\in S^{p+q}$. For $R=1$, $h_1|_{(V,Z,t)}$ in (1.2)-(1.3) is smooth for $(V,Z,t)\neq(0,0,1)$, but it could be singular at the pole $(0,0,1)$. The question whether or not h_1 extends smoothly at the pole was left open in [1], p. 330.

Note that we can rewrite the ball metric (1.1) in Cartesian coordinates $(V',Z',t') = R(V,Z,t)\in B$ as

$$\gamma_B|_{(V',Z',t')} = 4\,\frac{|dV'|^2+|dZ'|^2+dt'^2}{1-R^2} + \frac{4}{(1-R^2)^2}\Big\{R^2dR^2 + h_1|_{(V',Z',t')}\Big\}, \tag{1.4}$$

where $RdR = \langle V',dV'\rangle + \langle Z',dZ'\rangle + t'dt'$ (cf. [1], (3.1), (3.11), (3.12)). The question is then whether or not the curly bracket in (1.4) admits a continuous extension to the boundary $R=1$ (namely at the pole $(0,0,1)$).

In this paper we address these questions. We prove that h_1 does not extend to the pole, except when the curly bracket in k_1 vanishes identically. Using the coordinates (V,Z) on S^{p+q} to write $h_1 = \sum h^{(\mathfrak{z})}_{ij}\,dz_i\,dz_j + \sum h^{(\mathfrak{v})}_{ij}\,dv_i\,dv_j + \sum h^{(\mathfrak{zv})}_{ij}\,dz_i\,dv_j$, we will see that, in the non-symmetric case, the coefficients $h^{(\mathfrak{z})}_{ij}$ do not have a limit at the pole, but remain bounded there, whereas the coefficients $h^{(\mathfrak{v})}_{ij}$ and $h^{(\mathfrak{zv})}_{ij}$ extend smoothly to zero at the pole.

In section 2 we prove that the first two terms of the bracket in (1.3) vanish if and only if the J^2-condition holds, i.e., if and only if S is symmetric [2]. Combined with Proposition 3.1, this implies that $k_1 = 0$ iff the J^2-condition holds. We then briefly discuss the example of the 7-dimensional non-symmetric Damek-Ricci space with N the complex Heisenberg group. In this case it is easily proved that the $h^{(\mathfrak{z})}_{ij}$ do not extend.

In section 3 we approach the problem using $\mathfrak{z}$-valued 1-forms on N and S^{p+q}. Let $\mathcal{C}: N\to S^{p+q}\setminus\{(0,0,1)\}$ be the generalized stereographic projection, given by $\mathcal{C}(n) = \lim_{t\to-\infty} C(ne^t)$, and let $\Omega'|_{(V,Z)} = dZ - \frac{1}{2}[V,dV]$ be the canonical 1-form on the group N, whose kernel is the horizontal distribution on N. Then $h_1 = \Theta^2 \equiv |\Theta|^2$, where $\Theta = \lambda^{-1}(\mathcal{C}^{-1*}\Omega')$, with $\lambda(V,Z,t) = -2/[(1-t)^2+|Z|^2]$. We obtain an explicit formula for the 1-form Θ. We discuss the symmetric case and the 7-dim example in detail. Then we conclude with the general result valid for any Damek-Ricci space (Theorem 3.7).

2 The vanishing of k_1 and the J^2-condition

Consider the limit of $k_1(V,Z,t)$ as $(V,Z,t) \to (0,0,1)$. The term $((1-t)^2+|Z|^2)^{-2}$ blows up, while the curly bracket tends to zero. Obviously, the limit is either zero or does not exist. We will see that this limit does not exist, and h_1 does not extend to the pole, except when the curly bracket in k_1 vanishes identically.

In order to prove this, we have to work with the coordinates (V,Z) on the sphere S^{p+q}. Fix orthonormal bases $\{U_i\}_{i=1}^q$ of $\mathfrak{z}$ and $\{V_j\}_{j=1}^p$ of $\mathfrak{v}$, and set $Z=\sum z_i U_i$, $V=\sum v_j V_j$. Then we write k_1 in (1.3) as

$$k_1 = \sum_{i,j=1}^{q} k_{ij}^{(\mathfrak{z})}\, dz_i\, dz_j + \sum_{i,j=1}^{p} k_{ij}^{(\mathfrak{v})}\, dv_i\, dv_j + \sum_{i=1}^{q}\sum_{j=1}^{p} k_{ij}^{(\mathfrak{z}\mathfrak{v})}\, dz_i\, dv_j, \tag{2.1}$$

and take the limit of the components k_{ij} as $(V,Z) \to (0,0)$. We can assume $t>0$ and $t=\sqrt{1-|V|^2-|Z|^2}$. The mixed components, as well as the V-components, are complicated, in general. However the Z-components only involve the quantity

$$|V|^4\Big(\langle Z, dZ\rangle^2 - |Z|^2|dZ|^2\Big) + \big|[J_Z V, J_{dZ} V]\big|^2, \tag{2.2}$$

i.e., the first two terms of (1.3). For later use we note that, in coordinates, we have

$$|Z|^2|dZ|^2 - \langle Z, dZ\rangle^2 = \sum_{i<j} \left(z_i dz_j - z_j dz_i\right)^2, \tag{2.3}$$

$$[J_Z V, J_{dZ} V] = \sum_{i<j} \left(z_i dz_j - z_j dz_i\right)[J_i V, J_j V], \tag{2.4}$$

where $J_i \equiv J_{U_i}$, with $J_i^2 = -I$, $J_i J_j = -J_j J_i$ $(i \neq j)$ (see [1], p. 325).

The quantity (2.2), divided by $((1-t)^2+|Z|^2)^2$, is just $\sum k_{ij}^{(\mathfrak{z})}\, dz_i\, dz_j$ (cf. (1.3) and (2.1)). It will turn out that this is precisely the singular part of k_1, namely the coefficients $k_{ij}^{(\mathfrak{z})}$ do not have a limit at the pole unless $k_{ij}^{(\mathfrak{z})} = 0$, whereas the $k_{ij}^{(\mathfrak{v})}$ and $k_{ij}^{(\mathfrak{z}\mathfrak{v})}$ tend to zero at the pole. We shall prove this later in a simple example, and in the next section in the general case.

First, let us prove that the expression (2.2) vanishes if and only if the J^2-condition holds. We recall here the definition of the J^2-condition (see [2], Definition 2.10).

Definition 2.1. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an H-type Lie algebra. We say that $\mathfrak{n}$ satisfies the J^2-condition if for all V in $\mathfrak{v}$ and all Z, Z' in $\mathfrak{z}$ such that $\langle Z, Z'\rangle = 0$, there exists Z'' in $\mathfrak{z}$ (possibly depending on V, Z and Z') such that*

$$J_Z J_{Z'} V = J_{Z''} V.$$

In [2], Proposition 4.1 and Theorem 4.5, it is proved that the J^2-condition is equivalent to S being symmetric.

Theorem 2.2. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be an H-type Lie algebra. The following are equivalent:*

(i) $\big|[J_Z V, J_{Z'} V]\big|^2 = |V|^4\Big(|Z|^2|Z'|^2 - \langle Z, Z'\rangle^2\Big), \quad \forall V \in \mathfrak{v},\ \forall Z, Z' \in \mathfrak{z}.$

(ii) $\mathfrak{n}$ satisfies the J^2-condition.

Thus the expression (2.2) vanishes, i.e., $k_{ij}^{(\mathfrak{z})} = 0$, $\forall i, j$, iff the J^2-condition holds.

Proof. (i) ⇒ (ii). Let Z' be orthogonal to Z. Using $[J_Z V, J_{Z'} V] = [V, J_Z J_{Z'} V]$ and (i), we have

$$\big|[V, J_Z J_{Z'} V]\big|^2 = |Z|^2 |Z'|^2 |V|^4. \tag{2.5}$$

Now suppose $V \neq 0$, and recall the orthogonal direct sum decomposition

$$\mathfrak{v} = \mathbb{R}V \oplus J_{\mathfrak{z}}(V) \oplus \mathfrak{k}(V),$$

where

$$J_{\mathfrak{z}}(V) = \{J_Z V : \ Z \in \mathfrak{z}\},$$

and $\mathfrak{k}(V)$ is the orthogonal complement of $\mathbb{R}V$ in $\ker \operatorname{ad}(V)$. We decompose $J_Z J_{Z'} V$ accordingly:

$$J_Z J_{Z'} V = cV + J_{Z''} V + W, \tag{2.6}$$

for some $c \in \mathbb{R}$, $Z'' \in \mathfrak{z}$, and $W \in \mathfrak{k}(V)$. It follows that

$$\big|J_Z J_{Z'} V\big|^2 = |Z|^2 |Z'|^2 |V|^2 = c^2 |V|^2 + |Z''|^2 |V|^2 + |W|^2.$$

On the other hand $[V, J_Z J_{Z'} V] = [V, J_{Z''} V] = |V|^2 Z''$, and (2.5) implies

$$|Z''|^2 |V|^4 = |Z|^2 |Z'|^2 |V|^4.$$

Comparing the latter two equalities, we obtain $c = 0$ and $W = 0$, whence $J_Z J_{Z'} V = J_{Z''} V$. This proves (ii).

(ii) ⇒ (i). Assuming (ii), for any $V \in \mathfrak{v}$ and any orthogonal vectors $Z, Z' \in \mathfrak{z}$, there exists $Z'' \in \mathfrak{z}$ such that $J_Z J_{Z'} V = J_{Z''} V$.

On the one hand $[V, J_Z J_{Z'} V] = [V, J_{Z''} V] = |V|^2 Z''$, therefore

$$\big|[J_Z V, J_{Z'} V]\big|^2 = \big|[V, J_Z J_{Z'} V]\big|^2 = |Z''|^2 |V|^4.$$

On the other hand $\big|J_Z J_{Z'} V\big|^2 = \big|J_{Z''} V\big|^2$, so $|Z|^2 |Z'|^2 |V|^2 = |Z''|^2 |V|^2$, and (i) follows for Z' orthogonal to Z.

Finally, decomposing an arbitrary $Z' \in \mathfrak{z}$ as $Z' = \lambda Z + Z^{\perp}$, with $Z^{\perp} \in \mathfrak{z}$ orthogonal to Z, we have

$$\begin{aligned} \big|[J_Z V, J_{Z'} V]\big|^2 &= \big|[J_Z V, J_{Z^{\perp}} V]\big|^2 = |Z|^2 |Z^{\perp}|^2 |V|^4 \\ &= \Big(|Z|^2 |Z'|^2 - \langle Z, Z' \rangle^2\Big) |V|^4. \end{aligned}$$

□

Remark 2.3. This theorem implies that the expression k_1 in (1.3) vanishes iff the J^2-condition holds. Indeed if k_1 vanishes then (2.2) vanishes and the J^2-condition holds. Conversely, if this condition holds, one can prove the vanishing of the remaining part of the curly bracket in k_1. For instance, using the identity (3.17) below (which is equivalent to the J^2-condition, see Remark 3.2), one can easily prove the vanishing of the mixed components $k_{ij}^{(\mathfrak{z}\mathfrak{v})}$ in k_1. In a similar way one proves the vanishing of the V-components $k_{ij}^{(\mathfrak{v})}$. We omit the details because this will follow somewhat more transparently from the approach below using $\mathfrak{z}$-valued 1-forms (see Proposition 3.1 and Remark 3.6).

Now let us prove that the quantity (2.2) does not have a limit at the pole in the non-symmetric case. We consider a simple example here, and treat the general case in the next section. Consider the lowest (=7) dimensional non-symmetric Damek-Ricci space, namely $S = NA$, where N is the complexified Heisenberg group. Here $q = 2$, $p = 4$, $\mathfrak{z} = \mathbb{R}^2$ and $\mathfrak{v} = \mathbb{R}^4$, with commutations (see [9], p. 67)

$$\begin{aligned}[V, V'] &= [(a, b, c, d), (a', b', c', d')] \\ &= (ab' - ba' + dc' - cd', ac' - ca' + bd' - db').\end{aligned}$$

One computes $[J_Z V, J_{dZ} V] = 0$, so that by (2.3) and (2.2) we get

$$\sum_{i,j=1}^{q} k_{ij}^{(\mathfrak{z})}\, dz_i\, dz_j = -\frac{|V|^4 (z_1 dz_2 - z_2 dz_1)^2}{((1-t)^2 + |Z|^2)^2} = -\beta^2, \tag{2.7}$$

where β is the 1-form

$$\beta|_{(V,Z,t)} = \frac{|V|^2 (z_1 dz_2 - z_2 dz_1)}{(1-t)^2 + |Z|^2} = -\frac{z_2 |V|^2}{(1-t)^2 + |Z|^2} dz_1 + \frac{z_1 |V|^2}{(1-t)^2 + |Z|^2} dz_2.$$

Consider the first component of β. Using $(1-t)^2 + |Z|^2 = 2 - |V|^2 - 2t$, we get

$$\begin{aligned}\lim_{(V,Z,t)\to(0,0,1)} \frac{z_2|V|^2}{(1-t)^2 + |Z|^2} &= \lim_{(V,Z)\to(0,0)} \frac{z_2|V|^2}{2 - |V|^2 - 2\sqrt{1 - |V|^2 - |Z|^2}} \\ = \lim_{(V,Z)\to(0,0)} &\frac{z_2|V|^2 \left(2 - |V|^2 + 2\sqrt{1 - |V|^2 - |Z|^2}\right)}{(2 - |V|^2)^2 - 4(1 - |V|^2 - |Z|^2)} \\ = 4 \lim_{(V,Z)\to(0,0)} &\frac{z_2|V|^2}{|V|^4 + 4|Z|^2}.\end{aligned}$$

This is either zero or does not exist. Taking $z_1 = 0$, $z_2 = |V|^2$, or $z_1 = z_2 = |V|^2$, we get a nonzero value, thus the limit does not exist. Alternatively, set $1 - t = \rho\cos\phi$, $|Z| = \rho\sin\phi$, and $z_2 = |Z|\sin\alpha$, then $|V|^2 = \rho(2\cos\phi - \rho)$ and

$$\frac{z_2|V|^2}{(1-t)^2 + |Z|^2} = \sin\alpha \sin\phi(2\cos\phi - \rho).$$

This does not have a limit at the pole, where $\rho \to 0$ but ϕ and α are undefined. The same conclusion holds for the second component of β and for the coefficients $k_{ij}^{(\mathfrak{z})}$. We shall see later that the components $k_{ij}^{(\mathfrak{v})}$ and $k_{ij}^{(\mathfrak{zv})}$ of k_1 tend to zero at the pole. These results will then be generalized to any Damek-Ricci space.

Remark 2.4. In order to compute the limit $\lim_{(V,Z,t)\to(0,0,1)} k_1(V, Z, t)$, we could follow the suggestion in [1], p. 330, to use bispherical coordinates $(\rho, \phi, \omega_1, \omega_2)$, or equivalently, $(|V|, |Z|, \omega_1, \omega_2)$, on S^{p+q} ([1], p. 332). The expression of h_1 in these coordinates is given by [1], (4.14), with $R = 1$, k_1 being the term with the curly bracket there. Recall that

$$V = |V|\omega_1, \quad Z = |Z|\omega_2, \quad |Z| = \rho\sin\phi, \quad t = \rho\cos\phi, \quad |V|^2 = 1 - \rho^2,$$

and $\omega_1 \in S^{p-1}$, $\omega_2 \in S^{q-1}$ (the unit spheres in $\mathfrak{v}$ and $\mathfrak{z}$, respectively). Let $\varepsilon = 1-t$. Using [1], (4.14), we can rewrite $k_1(V,Z,1-\varepsilon)$ in (1.3) in terms of $(|V|,|Z|,\varepsilon,\omega_1,\omega_2)$ as

$$k_1(V,Z,1-\varepsilon) = \frac{|Z||V|^4}{(\varepsilon^2+|Z|^2)^2}\Big\{A|Z|^3+B|Z|^2\varepsilon+C\varepsilon(|Z|^2-\varepsilon^2)+D\varepsilon^2|Z|+E|Z|(|Z|^2-\varepsilon^2)\Big\}, \tag{2.8}$$

where A,B,C,D,E are the following differential expressions on $S^{p-1}\times S^{q-1}$:

$$\begin{aligned}
A &= -\gamma_{S^{q-1}} + \big|\,[J_{\omega_2}\omega_1, J_{d\omega_2}\omega_1]\,\big|^2,\\
B &= 4\Big(-\big\langle J_{d\omega_2}\omega_1, d\omega_1\big\rangle + \big\langle J_{[J_{\omega_2}\omega_1,d\omega_1]}J_{\omega_2}\omega_1, J_{d\omega_2}\omega_1\big\rangle\Big),\\
C &= -4\big\langle J_{[\omega_1,d\omega_1]}J_{\omega_2}\omega_1, d\omega_1\big\rangle,\\
D &= 4\Big(\big\langle J_{\omega_2}\omega_1, d\omega_1\big\rangle^2 - \big|\,[\omega_1,d\omega_1]\,\big|^2 + \big|\,[J_{\omega_2}\omega_1,d\omega_1]\,\big|^2\Big),\\
E &= 2\Big(\big\langle J_{\omega_2}J_{d\omega_2}\omega_1, d\omega_1\big\rangle - \big\langle J_{[\omega_1,d\omega_1]}J_{\omega_2}\omega_1, J_{d\omega_2}\omega_1\big\rangle\Big),
\end{aligned}$$

$\gamma_{S^{q-1}} = |d\omega_2|^2$ being the round metric on S^{q-1}. Now the functions

$$\frac{|Z|^4}{(\varepsilon^2+|Z|^2)^2},\quad \frac{\varepsilon|Z|^3}{(\varepsilon^2+|Z|^2)^2},\quad \frac{\varepsilon|Z|(|Z|^2-\varepsilon^2)}{(\varepsilon^2+|Z|^2)^2},\quad \frac{\varepsilon^2|Z|^2}{(\varepsilon^2+|Z|^2)^2},\quad \frac{|Z|^2(|Z|^2-\varepsilon^2)}{(\varepsilon^2+|Z|^2)^2} \tag{2.9}$$

are bounded in a neighborhood of $(Z,\varepsilon)=(0,0)$, and it would seem from (2.8) that

$$\lim_{(V,Z,\varepsilon)\to(0,0,0)} k_1(V,Z,1-\varepsilon) = 0, \tag{2.10}$$

so that (1.2) would imply

$$\lim_{(V,Z,t)\to(0,0,1)} h_1|_{(V,Z,t)} = |dZ|^2. \tag{2.11}$$

The curly bracket in (1.4) would then extend continuously to the boundary $R=1$, with the value $dt'^2+|dZ'|^2$ at the pole $(0,0,1)$.

Unfortunately, this result is wrong, as seen above in the 7-dim example. The point is that we cannot use bispherical coordinates to compute the limit in (2.10), because these coordinates are singular (undefined) precisely at the pole. In (2.8) we have products of biradial quantities (namely the functions in (2.9) multiplied by $|V|^4$), that tend to zero at the pole, times the "angular" expressions A,B,C,D,E, that are undefined and do not have a limit at the pole. Note that A,B,C,D,E are not scalar-valued but tensor-valued (they are quadratic in the differentials of the angular coordinates ω_1,ω_2). By no means can they be regarded as bounded quantities. Thus we cannot conclude that these products tend to zero and extend smoothly. For instance in the 7-dim example, let $\omega_2 = (\cos\alpha,\sin\alpha)\in S^1$, then $z_1dz_2 - z_2dz_1 = |Z|^2d\alpha$, $A=-d\alpha^2$, and (2.7) becomes

$$\sum_{i,j} k_{ij}^{(\mathfrak{z})}\, dz_i\, dz_j = -\frac{|V|^4|Z|^4}{((1-t)^2+|Z|^2)^2}\, d\alpha^2.$$

The scalar quantity multiplying $d\alpha^2$ tends to zero as $(V,Z,t)\to(0,0,1)$. However, the 1-form $d\alpha$ is unbounded around the pole with respect to the Euclidean norm, being $\|d\alpha\| = 1/|Z| \to \infty$ as $Z\to 0$. In fact, this expression does not extend smoothly at the pole, since the coefficients $k_{ij}^{(\mathfrak{z})}$ in Cartesian coordinates (z_1,z_2) do not have a limit there.

3 The approach by $\mathfrak{z}$-valued 1-forms

Let us recall the following geometric interpretation of the differential expression h_1. Consider the stereographic projection $\mathcal{C} : N \to S^{p+q} \setminus \{(0,0,1)\}$. This is the diffeomorphism defined by $\mathcal{C}(n) = \lim_{t\to-\infty} C(ne^t) \in \partial B$ (see [9], section 4.6), and given explicitly by $(V, Z) \to (V', Z', t')$, where

$$\begin{cases} V' = \frac{\left(1+\frac{1}{4}|V|^2\right)V - J_Z V}{\left(1+\frac{1}{4}|V|^2\right)^2 + |Z|^2}, \\ Z' = \frac{2Z}{\left(1+\frac{1}{4}|V|^2\right)^2 + |Z|^2}, \\ t' = \frac{-1+\left(\frac{1}{4}|V|^2\right)^2 + |Z|^2}{\left(1+\frac{1}{4}|V|^2\right)^2 + |Z|^2}, \end{cases}$$

with inverse

$$\begin{cases} V = 2\frac{(1-t')V' + J_{Z'}V'}{(1-t')^2 + |Z'|^2}, \\ Z = \frac{2Z'}{(1-t')^2+|Z'|^2}. \end{cases}$$

Recall the generalized contact structure on the H-type group N. The horizontal subbundle $HN \subset TN$ is spanned by the left-invariant vector fields X such that $X_e \in \mathfrak{v}$. The bundle HN can be represented as the kernel of the following $\mathfrak{z}$-valued 1-form on N:

$$\Omega'|_{(V,Z)} = dZ - \tfrac{1}{2}[V, dV] \tag{3.1}$$

(see [1], p. 329). We define the horizontal distribution $HS^\star$ on the punctured sphere $S^\star = S^{p+q} \setminus \{(0,0,1)\}$ to be

$$HS^\star = \mathcal{C}_* HN = \mathcal{C}_* \ker \Omega' = \ker(\mathcal{C}^{-1*}\Omega'). \tag{3.2}$$

The pull-back $\mathcal{C}^{-1*}\Omega'$ can be computed by calculating dV, dZ in terms of dV', dZ', dt' and then substituting in (3.1). The result is (dropping primes):

$$\begin{aligned} \left(\mathcal{C}^{-1*}\Omega'\right)|_{(V,Z,t)} = {}& \frac{2}{((1-t)^2+|Z|^2)^2} \Big\{ dZ\Big((1-t)^2 + |Z|^2 - (1-t)|V|^2\Big) \\ & + Z\Big((2-|V|^2)dt + 2t\langle V, dV\rangle \ - 2\langle J_Z V, dV\rangle\Big) \\ & + \left(|Z|^2 - (1-t)^2\right)[V, dV] - 2(1-t)[J_Z V, dV] - [J_Z V, J_{dZ} V]\Big\}, \end{aligned} \tag{3.3}$$

for any $(V, Z, t) \neq (0, 0, 1)$ (cf. [1], (3.18)). Here, of course, $\langle V, dV\rangle + \langle Z, dZ\rangle + t\,dt = 0$. The norm squared of this $\mathfrak{z}$-valued 1-form on $S^\star$ is related to h_1. Indeed, by [1], Proposition 3.4 (with $h_1 = \lim_{R\to1} h_R$), we have

$$\left|\mathcal{C}^{-1*}\Omega'\right|^2|_{(V,Z,t)} = \big(\lambda(V,Z,t)\big)^2\, h_1|_{(V,Z,t)} \tag{3.4}$$

for $(V, Z, t) \neq (0, 0, 1)$, where

$$\lambda(V, Z, t) = -\frac{2}{(1-t)^2 + |Z|^2}. \tag{3.5}$$

If we could prove that

$$\lim_{(V,Z,t)\to(0,0,1)} \left(\big(\lambda(V,Z,t)\big)^{-1} (\mathcal{C}^{-1\,*}\Omega')|_{(V,Z,t)} \right) = dZ|_{(0,0,1)}, \tag{3.6}$$

then (2.11) would follow, being $h_1 = |\lambda^{-1}(\mathcal{C}^{-1\,*}\Omega')|^2$ by (3.4). However, (3.6) does not hold, in general, i.e., the $\mathfrak{z}$-valued 1-form

$$\Theta = \lambda^{-1}(\mathcal{C}^{-1\,*}\Omega'), \tag{3.7}$$

such that $h_1 = \Theta^2 \equiv |\Theta|^2$, does not extend continuously to the pole, in general. We will actually see that the limit in (3.6) does not exist, except when $q = 1$.

3.1 The symmetric case

Let $S = NA$ be a symmetric Damek-Ricci space. Then S can be identified with a noncompact Riemannian symmetric space of rank one $X = G/K$, by viewing NA as the solvable component in the Iwasawa decomposition $G = NAK$ of a noncompact simple Lie group G of real rank one. By suitably scaling the metric, S is isometric to one of the following hyperbolic spaces: $\mathbb{C}H^n$ (complex hyperbolic spaces, $q = 1$, $p = 2(n-1)$, $n \geq 2$); $\mathbb{H}H^n$ (quaternionic hyperbolic spaces, $q = 3$, $p = 4(n-1)$, $n \geq 2$); $\mathbb{O}H^2$ (octonionic hyperbolic plane, $q = 7$, $p = 8$). (See [9], Proposition 27, p. 97.)

The unit sphere S^{p+q} is a fibre bundle with fibre S^q over a suitable projective space (the generalized Hopf fibration, see [5]):

$$S^q \hookrightarrow S^{p+q} \to \mathbb{K}P^{p/(q+1)}, \tag{3.8}$$

where $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ for $q = 1, 3, 7$, respectively, and $\mathbb{O}P^1 \equiv S^8$. Explicitly, we have the fibrations [5]

$$\begin{aligned} S^1 \hookrightarrow S^{2n-1} \to \mathbb{C}P^{n-1} &= \text{complex projective } n-1 \text{ space}, \\ S^3 \hookrightarrow S^{4n-1} \to \mathbb{H}P^{n-1} &= \text{quaternionic projective } n-1 \text{ space}, \\ S^7 \hookrightarrow S^{15} \to S^8. & \end{aligned}$$

Let Ω be the $\mathfrak{z}$-valued 1-form on S^{p+q} given by (cf. [1], (3.14)):

$$\Omega|_{(V,Z,t)} = [V, dV] + t dZ - Z dt + \tfrac{1}{|V|^2}[J_Z V, J_{dZ} V]. \tag{3.9}$$

For $q = 1, 3$ the bundle (3.8) is principal, and Ω is a connection 1-form. Thus the kernel of Ω defines a p-dimensional distribution (the horizontal subbundle $HT(S^{p+q})$), with supplementary (vertical) subspace $VT(S^{p+q})$ provided by the fibers of the Hopf fibration. For $q = 7$ the bundle (3.8) is not principal, but we have a similar interpretation of Ω as a connection 1-form. However, in this case, Ω is undefined and has no limit at the points $V = 0$, $Z \neq 0$ of S^{15}, due to the last term in (3.9). This term explicitly depends on V (unlike $q = 1, 3$, see below for details), and has no limit at

$$\{(0, Z, t) : \ |Z|^2 + t^2 = 1, \ Z \neq 0\} \equiv S^7 \setminus \{(0, 0, \pm 1)\}.$$

Here S^7 (the unit sphere in $\mathfrak{z} \oplus \mathfrak{a}$) is just the Hopf fiber through the poles $(0, 0, \pm 1)$ of S^{15}. (We identify $\mathfrak{z} \oplus \mathfrak{a} \simeq \mathbb{O} \simeq \mathbb{R}^8$ with the Cayley line $L_\infty = \{(0, u) : u \in \mathbb{O}\}$ in $\mathbb{O}^2 \simeq \mathbb{R}^{16}$, corresponding to the south pole of the base space $S^8 \simeq \{L_m,\ m \in \mathbb{O}\} \cup \{L_\infty\}$, see [5].) At the poles the last term in (3.9) tends to zero, and Ω tends to $\pm dZ|_{(0,0,\pm 1)}$. Nevertheless, we prove in Proposition 3.1 that the norm squared of Ω is well defined and smooth on the whole sphere and equals $h_1 - k_1$ in (1.2) (cf. (3.18)). The kernel of Ω^2 is then smooth on S^{15}, and defines the horizontal distribution $HT(S^{15})$ (of dimension 8). Of course, at the points $x = (0, Z, t)$, we have $HT_x(S^{15}) = \mathfrak{v}$ and $VT_x(S^{15}) \oplus \mathbb{R}x = \mathfrak{z} \oplus \mathfrak{a}$.

To see that $\ker \Omega^2$ agrees with the distribution (3.2) on $S^\star$, we need the relationship between Ω and $\mathcal{C}^{-1*}\Omega'$. In the hermitian case of $q = 1$, we have $\mathfrak{v} = \mathbb{C}^{n-1} = \mathbb{R}^{2(n-1)}$, $\mathfrak{z} = \operatorname{Im}\mathbb{C} = \mathbb{R}U_1$, $J_{U_1}V = iV = (iV_1, \ldots, iV_{n-1})$, and we compute $[J_Z V, J_{dZ} V] = 0$, so $\Omega|_{(V,Z,t)} = [V, dV] + t dZ - Z dt$ is smooth on $S^{p+q} = S^{2n-1}$, and (cf. [1], p. 330)

$$\mathcal{C}^{-1*}\Omega' = \lambda\,\Omega \qquad (q = 1) \tag{3.10}$$

(λ given by (3.5)), i.e., $\Theta = \Omega$ on $S^\star$. Thus Θ and $\ker \Theta$ extend continuously to the whole sphere S^{2n-1}, and (3.6) holds. For $q = 3, 7$, the relationship between $\mathcal{C}^{-1*}\Omega'$ and Ω is more complicated. If $\Omega' = (\alpha_1, \ldots, \alpha_q)$ and $\Omega = (\omega_1, \ldots, \omega_q)$, one gets (see Remark 3.6)

$$\mathcal{C}^{-1*}\alpha_i = \lambda \sum_j r_{ij}\omega_j \qquad (q = 3, 7) \tag{3.11}$$

(λ given by (3.5)), i.e., $\Theta = \mathcal{R}(\Omega)$, where $\mathcal{R} = (r_{ij})$ is a smooth function on $S^\star$ (for $q = 3$) or $S^{15} \setminus S^7$ (for $q = 7$) with values in $SO(q)$. This function does not have a limit as $(V, Z, t) \to (0, 0, 1)$. We discuss this separately for $q = 3$ and $q = 7$.

$\boldsymbol{q = 3}$. In the quaternionic case, we have $\mathfrak{v} = \mathbb{H}^{n-1} = \mathbb{R}^{4(n-1)}$, $\mathfrak{z} = \operatorname{Im}\mathbb{H} = \mathbb{R}^3$, and we take $U_1 \cdot U_2 = U_3$ plus permutations (where $\{U_1, U_2, U_3\}$ is an orthonormal basis of $\mathfrak{z}$), and $J_Z V = Z \cdot V = (Z \cdot V_1, \ldots, Z \cdot V_{n-1})$. Then $J_1 J_2 = J_3$ plus permutations (where $J_i = J_{U_i}$), and we compute from (2.4)

$$\begin{aligned}[J_Z V, J_{dZ} V] = {} & |V|^2(z_2 dz_3 - z_3 dz_2)U_1 + |V|^2(z_3 dz_1 - z_1 dz_3)U_2 \\ & + |V|^2(z_1 dz_2 - z_2 dz_1)U_3.\end{aligned}$$

Formula (3.9) yields

$$\begin{cases} \omega_1 = [V, dV]_1 + t dz_1 - z_1 dt + z_2 dz_3 - z_3 dz_2, \\ \omega_2 = [V, dV]_2 + t dz_2 - z_2 dt + z_3 dz_1 - z_1 dz_3, \\ \omega_3 = [V, dV]_3 + t dz_3 - z_3 dt + z_1 dz_2 - z_2 dz_1, \end{cases}$$

and Ω is smooth on $S^{p+q} = S^{4n-1}$. Using (3.3), one obtains the general formula (3.24)-(3.25) for the 1-form Θ. Specializing this formula to our case, we get (3.28), i.e., $\Theta = \mathcal{R}(\Omega)$, with

$$\mathcal{R}(V, Z, t) = \begin{pmatrix} \frac{(1-t)^2 + z_1^2 - z_2^2 - z_3^2}{(1-t)^2 + |Z|^2} & \frac{2(z_1 z_2 - z_3(1-t))}{(1-t)^2 + |Z|^2} & \frac{2(z_1 z_3 + z_2(1-t))}{(1-t)^2 + |Z|^2} \\ \frac{2(z_1 z_2 + z_3(1-t))}{(1-t)^2 + |Z|^2} & \frac{(1-t)^2 + z_2^2 - z_1^2 - z_3^2}{(1-t)^2 + |Z|^2} & \frac{2(z_2 z_3 - z_1(1-t))}{(1-t)^2 + |Z|^2} \\ \frac{2(z_1 z_3 - z_2(1-t))}{(1-t)^2 + |Z|^2} & \frac{2(z_2 z_3 + z_1(1-t))}{(1-t)^2 + |Z|^2} & \frac{(1-t)^2 + z_3^2 - z_1^2 - z_2^2}{(1-t)^2 + |Z|^2} \end{pmatrix}. \tag{3.12}$$

It is easy to check that $\mathcal{R}(V,Z,t) \in SO(3)$. Using bispherical coordinates $(|V|,|Z|,\omega_1,\omega_2)$ and then polar coordinates $1-t=\rho\cos\phi$, $|Z|=\rho\sin\phi$, we see that the entries r_{ij} are bounded around the pole $(0,0,1)$ but do not have a limit there. Thus the 1-form Θ in (3.7) does not extend to the pole, and (3.6) does not hold.

Nevertheless, from (3.11) or (3.28) we get $|\mathcal{C}^{-1*}\Omega'|^2 = \lambda^2|\Omega|^2$, i.e., $\Theta^2=\Omega^2$, so the square of Θ does extend to the pole. Moreover, it follows from (3.11) that $\ker\Theta = \ker\Omega$ on $S^\star$, i.e., the horizontal distribution (3.2) coincides with the horizontal subbundle of the Hopf bundle on $S^\star$, and thereby extends continuously to the whole sphere S^{4n-1}.

$\boldsymbol{q=7}$. A similar analysis can be repeated in the octonionic case. The details are more complicated, due to the non-associativity of the product in $\mathfrak{v}\simeq\mathbb{O}$ and $\mathfrak{z}\simeq \mathrm{Im}(\mathbb{O})$. In particular, the claim made in [1], Example 3, p. 326, about the "multiplication table" for the products J_iJ_j with $i\neq j$ (where $J_i=J_{U_i}$, $\{U_i\}_{i=1}^7$ an orthonormal basis of $\mathfrak{z}$), is incorrect. (See also Remark 3.3.) Recall that the operators J_Z are defined as left (or right) multiplication in $\mathbb{O}$, say $J_ZV = Z\cdot V$. Then $J_iJ_jV = U_i\cdot(U_j\cdot V)$ is different from $(U_i\cdot U_j)\cdot V = J_{U_i\cdot U_j}V$, in general, and the products J_iJ_j do not follow the multiplication table of octonions. We only have the J^2-condition:

$$J_iJ_jV = J_{Z_{ij}(V)}V \qquad (i\neq j), \tag{3.13}$$

where $\mathfrak{v}\setminus\{0\}\ni V\to Z_{ij}(V)\in\mathfrak{z}$ are nontrivial functions. (For $q=3$, the Z_{ij} are independent of V, being $Z_{ij}(V)=U_i\cdot U_j$, $\forall V\neq 0$.) Using $[V,J_ZV]=|V|^2Z$, we get from (3.13)

$$Z_{ij}(V) = \tfrac{1}{|V|^2}[V, J_iJ_jV].$$

Defining the components $Z_{ij}(V)=\sum_k Z_{ij}^k(V)U_k$, we have

$$Z_{ij}^k(V) = \tfrac{1}{|V|^2}\langle J_iJ_jV, J_kV\rangle. \tag{3.14}$$

Note that $Z_{ij}^i(V)=0=Z_{ij}^j(V)$, and $Z_{ij}^k(V)=-Z_{ik}^j(V)$, so that $Z_{ijk}\equiv Z_{ij}^k$ is totally antisymmetric. By (2.4) we compute

$$\tfrac{1}{|V|^2}[J_ZV, J_{dZ}V] = \sum_{i<j}(z_idz_j - z_jdz_i)Z_{ij}(V) = \sum_k\sum_{i<j}(z_idz_j-z_jdz_i)Z_{ij}^k(V)U_k,$$

and the connection 1-form $\Omega=\sum\omega_kU_k$ in (3.9) has components

$$\omega_k|_{(V,Z,t)} = [V,dV]_k + tdz_k - z_kdt + \sum_{i<j}(z_idz_j - z_jdz_i)Z_{ij}^k(V).$$

Note that the Z_{ij}, Z_{ij}^k are actually functions of $\omega_1 = V/|V|\in S^{p-1}=S^7$ (the unit sphere in $\mathfrak{v}$), and they are bounded since by (3.13) we get

$$|Z_{ij}(V)|^2 = \sum_k |Z_{ij}^k(V)|^2 = 1, \;\; \forall i\neq j,\; \forall V\neq 0.$$

It follows that $\lim_{(V,Z,t)\to(0,0,1)}\omega_k|_{(V,Z,t)} = dz_k$, but Ω has no limit at the points $(0,Z,t)\in S^{15}$ with $Z\neq 0$, although it is obviously bounded there. We shall see in Proposition 3.1 that Ω^2 is well defined and smooth on S^{15}, with $\Omega^2 = h_1 - k_1$ in (1.2) (cf. (3.18)).

Here is a simple algorithm to compute the functions Z_{ij}. Fix a multiplication table in $\mathfrak{v} \simeq \mathbb{O}$, and identify $\mathfrak{z} \simeq \mathrm{Im}(\mathbb{O})$. Each $V \in \mathfrak{v}$ is written as $v_0 U_0 + \sum_1^7 v_j U_j$, where U_0 is the neutral element and U_j $(1 \leq j \leq 7)$ are the imaginary units, with $U_j^2 = -U_0$. Given $i < j$, determine k from the table such that $U_i \cdot U_j = \pm U_k$, and let $T = \pm J_k J_i J_j$ (same sign). Then $T^* = T$, $T^2 = I$, and $\mathfrak{v} = \mathfrak{v}_+ \oplus \mathfrak{v}_-$, where $\mathfrak{v}_\pm$ are the eigenspaces of T with eigenvalues ± 1, respectively. Let $\{a, b, c, d\} = \{1, \dots, 7\} \setminus \{i, j, k\}$. Each $V \in \mathfrak{v}$ can be written as $V = V_+ + V_-$, where $V_+ = v_a U_a + v_b U_b + v_c U_c + v_d U_d \in \mathfrak{v}_+$, and $V_- = v_0 U_0 + v_i U_i + v_j U_j + v_k U_k \in \mathfrak{v}_-$. Then $Z_{ij}(V_\pm) = \mp U_i \cdot U_j$, but in general

$$Z_{ij}(V) = Z_{ij}^k(V) U_k + Z_{ij}^a(V) U_a + Z_{ij}^b(V) U_b + Z_{ij}^c(V) U_c + Z_{ij}^d(V) U_d,$$

where by (3.14) we compute

$$\begin{aligned} Z_{ij}^k(V) &= \pm \frac{|V_-|^2 - |V_+|^2}{|V|^2}, \\ Z_{ij}^l(V) &= \pm \frac{2}{|V|^2} \langle J_k V_-, J_l V_+ \rangle \qquad (l = a, b, c, d) \end{aligned}$$

(same sign as in $U_i \cdot U_j = \pm U_k$). It is easy to see from these formulas that the functions $\omega_1 \to Z_{ij}^m(\omega_1)$ are spherical harmonics of degree 2 on S^{p-1}.

Again the general formula (3.24)-(3.25) for the 1-form Θ yields (3.28), i.e., $\Theta = \mathcal{R}(\Omega)$, where $\mathcal{R} = (r_{ij})$ is given as follows:

$$r_{ii}(V, Z, t) = \frac{(1-t)^2 + z_i^2 - \sum_{j \neq i} z_j^2}{(1-t)^2 + |Z|^2} \qquad (1 \leq i \leq 7), \tag{3.15}$$

$$\begin{aligned} r_{ij}(V, Z, t) &= \frac{2}{(1-t)^2 + |Z|^2} \Big\{ z_i z_j + (1-t) \sum_{k \neq i,j} z_k Z_{ik}^j(V) \Big\} \\ &= \frac{2}{(1-t)^2 + |Z|^2} \Big\{ z_i z_j - (1-t) \big\langle Z, Z_{ij}(V) \big\rangle \Big\} \qquad (i \neq j), \end{aligned} \tag{3.16}$$

where we used the identities $Z_{ik}^j(V) = -Z_{ij}^k(V)$ to write

$$\sum_k z_k Z_{ik}^j(V) = -\sum_k z_k Z_{ij}^k(V) = -\langle Z, Z_{ij}(V) \rangle.$$

The non-diagonal entries r_{ij} and r_{ji} are related by a sign change in the second term of the curly bracket in (3.16) (as in (3.12)). It then follows that $\mathcal{R}(V, Z, t) \in SO(7)$, and the entries r_{ij} are bounded around the pole $(0, 0, 1)$ but do not have a limit there. Thus Θ does not extend to the pole but its square does, being $\Theta^2 = \Omega^2$ with Ω^2 smooth on S^{15} (cf. (3.18)). Again $\ker \Theta = \ker \Omega^2$ on $S^\star = S^{15} \setminus \{(0, 0, 1)\}$, and the horizontal distribution (3.2) extends continuously to the whole sphere S^{15}. Note that Θ is smooth on $S^\star$ (by (3.25)), so the singularities of $\mathcal{R}$ and Ω at the points $(0, Z, t)$, $Z \neq 0$ (due to the functions Z_{ij}) cancel out in $\Theta = \mathcal{R}(\Omega)$.

We can now easily prove that $k_1 = 0$ in (1.3).

Proposition 3.1. *Let $\mathfrak{n}$ satisfy the J^2-condition. Then in (1.2) we have $h_1 - k_1 = \Omega^2$, so since $h_1 = \Theta^2 = \Omega^2$, we get $k_1 = 0$.*

Proof. From (3.9) we have

$$\Omega^2|_{(V,Z,t)} = \big|[V, dV] + tdZ - Zdt\big|^2 + \big||V|^{-2}[J_Z V, J_{dZ} V]\big|^2 \\ + 2\big\langle [V, dV], \tfrac{1}{|V|^2}[J_Z V, J_{dZ} V]\big\rangle + 2\big\langle tdZ - Zdt, \tfrac{1}{|V|^2}[J_Z V, J_{dZ} V]\big\rangle.$$

The last term in this expression vanishes, as easily seen. The second term equals $\big(|Z|^2|dZ|^2 - \langle Z, dZ\rangle^2\big)$ (by Theorem 2.2). Let us prove that

$$\big\langle [V, V'], \tfrac{1}{|V|^2}[J_Z V, J_{Z'} V]\big\rangle = \langle V, V'\rangle\langle Z, Z'\rangle - \langle J_{Z'} V, J_Z V'\rangle, \tag{3.17}$$

for all $V, V' \in \mathfrak{v}$, $V \neq 0$, and $Z, Z' \in \mathfrak{z}$. This will establish that $\Omega^2 = h_1 - k_1$ in (1.2), i.e.,

$$\Omega^2|_{(V,Z,t)} = \big|[V, dV] + tdZ - Zdt\big|^2 + |Z|^2|dZ|^2 - \langle Z, dZ\rangle^2 \\ + 2\Big(\langle V, dV\rangle\langle Z, dZ\rangle - \big\langle J_{dZ} V, J_Z dV\big\rangle\Big), \qquad \forall (V, Z, t) \in S^{p+q}. \tag{3.18}$$

First let $Z' \in \mathfrak{z}$ with $\langle Z, Z'\rangle = 0$. Then, by the J^2-condition, there is $Z'' \in \mathfrak{z}$ such that $J_Z J_{Z'} V = J_{Z''} V$. Therefore,

$$[J_Z V, J_{Z'} V] = [V, J_Z J_{Z'} V] = [V, J_{Z''} V] = |V|^2 Z'', \tag{3.19}$$

and

$$-\big\langle J_{Z'} V, J_Z V'\big\rangle = \big\langle V', J_Z J_{Z'} V\big\rangle = \big\langle V', J_{Z''} V\big\rangle \\ = \big\langle Z'', [V, V']\big\rangle.$$

Thus (3.17) follows for Z' orthogonal to Z.

For an arbitrary $Z' \in \mathfrak{z}$, we decompose $Z' = \lambda Z + Z^\perp$, with $Z^\perp \in \mathfrak{z}$ orthogonal to Z. Let Z'' be determined by $J_Z J_{Z^\perp} V = J_{Z''} V$. Then we have

$$[J_Z V, J_{Z'} V] = [J_Z V, J_{Z^\perp} V] = |V|^2 Z'',$$

where we used (3.19) with $Z^\perp$ in place of Z'. The left-hand side of (3.17) is then equal to $\big\langle Z'', [V, V']\big\rangle$. The right-hand side of (3.17) equals

$$\langle V, V'\rangle\lambda|Z|^2 - \big\langle J_{\lambda Z} V, J_Z V'\big\rangle - \big\langle J_{Z^\perp} V, J_Z V'\big\rangle \\ = \big\langle J_Z J_{Z^\perp} V, V'\big\rangle = \big\langle J_{Z''} V, V'\big\rangle = \big\langle Z'', [V, V']\big\rangle.$$

This proves the proposition. □

Remark 3.2. If $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is an H-type Lie algebra such that (3.17) holds, then the J^2-condition holds. Indeed let $\langle Z, Z'\rangle = 0$, and decompose $J_Z J_{Z'} V$ as in (2.6). Then the left-hand side of (3.17) equals $\langle [V, V'], Z''\rangle$, and we get $\langle J_{Z''} V, V'\rangle = \langle J_Z J_{Z'} V, V'\rangle$, for all $V, V' \in \mathfrak{v}$. Thus $J_Z J_{Z'} V = J_{Z''} V$, and the J^2-condition holds. This condition is then equivalent to (3.17).

Remark 3.3. Consider the $\mathfrak{z}$-valued 1-form

$$\Phi|_{(V,Z,t)} = [V, dV] + tdZ - Zdt + \sum_{i<j}(z_i dz_j - z_j dz_i) U_i \cdot U_j.$$

This is smooth on S^{p+q} and it reduces to Ω for $q = 1, 3$, but for $q = 7$, $\Phi \neq \Omega$. In this case, the component 1-forms ϕ_k of $\Phi = \sum \phi_k U_k$ are given by

$$\phi_k|_{(V,Z,t)} = [V, dV]_k + tdz_k - z_k dt + \sum_{i<j}(z_i dz_j - z_j dz_i) C_{ijk},$$

where C_{ijk} are the structure constants defined by $U_i \cdot U_j = \sum_{k=1}^{7} C_{ijk}\, U_k$ from any given multiplication table in $\mathbb{O}$. Note that C_{ijk} is totally antisymmetric. It is natural to ask whether $\Phi^2 = \Omega^2$ for $q = 7$. Formula (3.15) in [1] would then apply, with the component 1-forms $\omega_1, \ldots, \omega_7$ being just the ϕ_k. Unfortunately this is not true, i.e., Φ^2 does not agree with Ω^2 on S^{15}, as easily seen. Thus there is no simple formula for the vertical part of the metric $h_1 = \Theta^2 = \Omega^2$ in terms of the 1-forms ϕ_k.

To summarize, in the symmetric case the 1-form Θ does not extend to the pole $(0, 0, 1)$ of S^{p+q} for $q = 3, 7$, but its square does, together with the distribution. This can be explained by the generalized Hopf fibration.

The horizontal-vertical distributions on the unit sphere S^{p+q} can also be described as follows. Recall that S^{p+q} is a homogeneous space K/M, where K is the subgroup of the isometry group of $S = NA$ that fixes the origin, and M is the group of orthogonal automorphisms of NA, namely the automorphisms of S that preserve the inner product on the Lie algebra $\mathfrak{s}$. The origin eM in K/M corresponds to the north pole $(0, 0, 1)$, and the tangent space $T_{eM}S^{p+q}$ decomposes as $\mathfrak{v} \oplus \mathfrak{z}$. The horizontal subbundle is then the assignment $kM \to k_*\mathfrak{v}$, the vertical one is $kM \to k_*\mathfrak{z}$. If we use Euclidean polar coordinates on the unit ball B in $\mathfrak{s}$ to write $b = R\omega \in B$, with $R > 0$ and $\omega = kM \in S^{p+q}$, then we have the orthogonal splitting of the tangent space $T_bB = T_b^{(1)} \oplus T_b^{(2)} \oplus \mathbb{R}b$, where $T_b^{(1)} = k_*\mathfrak{v}$ and $T_b^{(2)} = k_*\mathfrak{z}$. See [2], Theorem 7.10, for an explicit description of $T_b^{(1)}$ and $T_b^{(2)} \oplus \mathbb{R}b$ in the coordinates $\omega = (V, Z, t) \in S^{p+q}$.

3.2 The general case

In the non-symmetric case, the situation is as follows. The unit sphere S^{p+q} is no longer a fibration with fiber S^q, and the horizontal distribution (3.2) does not extend smoothly, in general. For instance for q even, so that $p+q$ is even, there are no smooth distributions on S^{p+q} of dimension k (smooth fields of k-planes) for $1 \leq k \leq p + q - 1$ ([10], Theorem 27.18). It was proved recently that the horizontal distribution extends iff the J^2-condition holds, iff S is symmetric ([8], Proposition 4.3).

The distribution (3.2) on $S^\star = S^{p+q} \setminus \{(0, 0, 1)\}$ is just the kernel of the $\mathfrak{z}$-valued 1-form Θ in (3.7) (λ in (3.5) being smooth and non-vanishing on $S^\star$). This distribution can also be described as the kernel of the rank-2 tensor $h_1 = \Theta^2$. Indeed, recall that

$$\ker h_1 = \{X \in TS^\star : \; h_1(X, Y) = 0, \; \forall Y \in TS^\star\}.$$

If $\Theta = (\theta_1, \ldots, \theta_q)$, then $\Theta^2 = \theta_1^2 + \cdots + \theta_q^2$, and $\ker \Theta = \bigcap_j \ker \theta_j \subseteq \ker \Theta^2$. Conversely, if $X \in \ker h_1 = \ker \Theta^2$, then taking $Y = X$ in the definition above gives

$$0 = h_1(X, X) = \Theta(X)^2 = \theta_1(X)^2 + \cdots + \theta_q(X)^2,$$

whence $\theta_j(X) = 0, \forall j$, i.e., $X \in \ker \Theta$. As mentioned before, h_1 does not extend to the pole $(0, 0, 1)$ in the non-symmetric case, in agreement with the non-extendability of the distribution. We saw this in the 7-dimensional example, to which we now return.

3.2.1 The 7-dimensional example

We want to write down the 1-form Θ more explicitly, and show it does not extend. Let U_1, U_2 be an orthonormal basis of $\mathfrak{z}$, and $J_i = J_{U_i}$, $i = 1, 2$, as usual. If $Z = (z_1, z_2) \in \mathfrak{z}$ and $V = (v_1, v_2, v_3, v_4) \in \mathfrak{v}$, we have

$$\begin{cases} [V, dV]_1 = \langle J_1 V, dV \rangle = v_1 dv_2 - v_2 dv_1 + v_4 dv_3 - v_3 dv_4, \\ [V, dV]_2 = \langle J_2 V, dV \rangle = v_1 dv_3 - v_3 dv_1 + v_2 dv_4 - v_4 dv_2, \\ \langle J_1 J_2 V, dV \rangle = v_4 dv_1 - v_1 dv_4 + v_2 dv_3 - v_3 dv_2, \end{cases}$$

and we compute $[J_Z V, J_{dZ} V] = 0$,

$$[J_Z V, dV] = -\langle V, dV \rangle Z + \langle J_1 J_2 V, dV \rangle (z_2 U_1 - z_1 U_2).$$

By (3.3), we obtain the following formula for the 1-form Θ in (3.7):

$$\begin{aligned} \Theta|_{(V,Z,t)} &= [V, dV] + t dZ - Z dt + \frac{2(z_2 U_1 - z_1 U_2)}{(1-t)^2 + |Z|^2} \times \\ &\times \Big\{ z_1 dz_2 - z_2 dz_1 + z_1 [V, dV]_2 - z_2 [V, dV]_1 + (1-t) \langle J_1 J_2 V, dV \rangle \Big\}. \end{aligned} \tag{3.20}$$

Letting $\Theta = (\theta_1, \theta_2)$, we get the following 1-forms on $S^6 \setminus \{(0, 0, 1)\}$:

$$\begin{cases} \theta_1|_{(V,Z,t)} = [V, dV]_1 + t dz_1 - z_1 dt + \frac{2z_2}{(1-t)^2 + |Z|^2} \times \\ \times \Big\{ z_1 dz_2 - z_2 dz_1 + z_1 [V, dV]_2 - z_2 [V, dV]_1 + (1-t) \langle J_1 J_2 V, dV \rangle \Big\}, \\ \theta_2|_{(V,Z,t)} = [V, dV]_2 + t\, dz_2 - z_2 dt - \frac{2z_1}{(1-t)^2 + |Z|^2} \times \\ \Big\{ z_1 dz_2 - z_2 dz_1 + z_1 [V, dV]_2 - z_2 [V, dV]_1 + (1-t) \langle J_1 J_2 V, dV \rangle \Big\}. \end{cases}$$

We can now prove that these 1-forms do not extend to the pole, the problem being due to the coefficients of the dz_j.

Theorem 3.4. *Let $S = NA$ be the non-symmetric Damek-Ricci space of dimension 7 with $p = 4$, $q = 2$, and N the complexified Heisenberg group. Then the 1-form Θ, given by (3.20), does not extend to the pole $(0, 0, 1)$. More precisely, the coefficients of the dz_j do not have a limit at the pole, but remain bounded there, whereas the coefficients of the dv_j tend to zero at the pole.*

Proof. Consider the coefficients of the dz_j in the term with the curly bracket in (3.20), namely the functions

$$\frac{z_1 z_2}{(1-t)^2+|Z|^2}, \qquad \frac{z_1^2}{(1-t)^2+|Z|^2}, \qquad \frac{z_2^2}{(1-t)^2+|Z|^2}. \tag{3.21}$$

These functions do not have a limit at the pole. For instance, we get

$$\begin{aligned}\lim_{(V,Z)\to(0,0)} \frac{z_1 z_2}{(1-t)^2+|Z|^2} &= \lim_{(V,Z)\to(0,0)} \frac{z_1 z_2}{2-|V|^2-2\sqrt{1-|V|^2-|Z|^2}} \\ &= \lim_{(V,Z)\to(0,0)} \frac{z_1 z_2\Big(2-|V|^2+2\sqrt{1-|V|^2-|Z|^2}\Big)}{(2-|V|^2)^2-4(1-|V|^2-|Z|^2)} \\ &= 4 \lim_{(V,Z)\to(0,0)} \frac{z_1 z_2}{|V|^4+4|Z|^2}.\end{aligned}$$

This is either zero or does not exist. Taking $z_1 = z_2 = |V|^2$ we get a nonzero value, thus the limit does not exist. Alternatively, use bispherical coordinates $(|V|, |Z|, \omega_1, \omega_2)$ on S^6, defined by

$$V = |V|\omega_1, \;\; \omega_1 = (a_1, a_2, a_3, a_4) \in S^3, \quad Z = |Z|\omega_2, \quad \omega_2 = (\cos\alpha, \sin\alpha) \in S^1.$$

Then

$$\frac{z_1 z_2}{(1-t)^2+|Z|^2} = \frac{|Z|^2}{(1-t)^2+|Z|^2} \sin\alpha\cos\alpha.$$

Now take polar coordinates $1-t = \rho\cos\phi$, $|Z| = \rho\sin\phi$, then $\rho \to 0$ when $(V, Z, t) \to (0, 0, 1)$, while ϕ, α are undefined at the pole. The first function in (3.21) reduces to $(\sin\phi)^2 \sin\alpha\cos\alpha$, so the limit does not exist. However, this function remains bounded. A similar analysis can be repeated for the other functions in (3.21): they do not have a limit but remain bounded around the pole.

Now look at the coefficients of the dv_j. Consider, for instance, the first component θ_1 of Θ. The coefficient of dv_1 in the term with the curly bracket is

$$f(V, Z, t) = \frac{2z_2}{(1-t)^2+|Z|^2}\Big\{ - z_1 v_3 + z_2 v_2 + (1-t)v_4\Big\}.$$

It is easy to see that this tends to zero at the pole. Indeed, in bispherical coordinates, it reads

$$f(V, Z, t) = \frac{2|Z|\sin\alpha}{(1-t)^2+|Z|^2}\Big\{|Z||V|(-a_3\cos\alpha + a_2\sin\alpha) + (1-t)|V|a_4\Big\}.$$

Now take polar coordinates $1-t = \rho\cos\phi$, $|Z| = \rho\sin\phi$, then

$$\begin{aligned}|V|^2 &= 1-t^2-|Z|^2 = (1-t)(1+t)-|Z|^2 \\ &= \rho(2\cos\phi - \rho).\end{aligned}$$

The range of (ρ,ϕ) is $0\le\rho\le 2\cos\phi$, $0\le\phi\le\pi/2$, and corresponds to the semicircle $t^2+|Z|^2\le 1$, $0\le|Z|\le 1$ in the $(t,|Z|)$ plane. In this region we have $0\le 2\cos\phi-\rho\le 2$. Now when $(V,Z,t)\to(0,0,1)$, $\rho\to 0$, while ϕ,α,a_j are undefined in the limit. We get

$$f(V,Z,t)=\frac{2\rho\sin\phi\sin\alpha}{\rho^2}\rho^{3/2}\sqrt{2\cos\phi-\rho}\Big\{\sin\phi(-a_3\cos\alpha+a_2\sin\alpha)+a_4\cos\phi\Big\}.$$

This tends to zero as $\rho^{1/2}$ when $\rho\to 0$, the remaining expression being bounded. A similar analysis shows that the coefficients of the dv_j in the curly bracket in (3.20) extend to zero at the pole. The remaining coefficients of the dv_j in (3.20) obviously tend to zero at the pole. □

Using (3.20), we can compute the square of Θ. The result can be written as follows:

$$\Theta^2|_{(V,Z,t)}=\big|[V,dV]+tdZ-Zdt\big|^2+\big(z_1dz_2-z_2dz_1+\langle J_1J_2V,dV\rangle\big)^2-\theta^2|_{(V,Z,t)}, \quad (3.22)$$

where θ is the 1-form

$$\begin{aligned}\theta|_{(V,Z,t)}=&\frac{1}{(1-t)^2+|Z|^2}\Big(|V|^2(z_1dz_2-z_2dz_1)\\&+2(1-t)\big(z_1[V,dV]_2-z_2[V,dV]_1\big)-\big(|Z|^2-(1-t)^2\big)\langle J_1J_2V,dV\rangle\Big)\\&\equiv(f_1\,dz_1+f_2\,dz_2)+(g_1\,dv_1+g_2\,dv_2+g_3\,dv_3+g_4\,dv_4)\equiv\beta+\gamma.\end{aligned}$$

Formula (3.22) for Θ^2 agrees with the known formula for h_1 ($=\lim_{R\to 1}h_R$, see [1], (3.16)), in agreement with (3.4)-(3.7). The precise identification of k_1 in (1.2)-(1.3) is:

$$k_1(V,Z,t)=\langle J_1J_2V,dV\rangle^2-\theta^2|_{(V,Z,t)}. \quad (3.23)$$

The 1-forms θ,β do not extend to the pole, whereas γ extends to zero there. Indeed, the functions f_1,f_2 do not have a limit (but remain bounded), whereas the functions g_j, $1\le j\le 4$, tend to zero as $(V,Z,t)\to(0,0,1)$. The proof is the same as above: use bispherical coordinates followed by polar coordinates for $1-t$, $|Z|$. We can then complete the analysis of the differential expression k_1 as $(V,Z,t)\to(0,0,1)$. Since

$$\begin{aligned}\theta^2&=\sum_{i,j=1}^{2}f_if_j\,dz_i\,dz_j+\sum_{i,j=1}^{4}g_ig_j\,dv_i\,dv_j+2\sum_{i=1}^{2}\sum_{j=1}^{4}f_ig_j\,dz_i\,dv_j\\&=\beta^2+\gamma^2+2\beta\gamma,\end{aligned}$$

we get, comparing (3.23) with (2.1), $k_{ij}^{(\mathfrak{z})}=-f_if_j$, $k_{ij}^{(\mathfrak{zv})}=-2f_ig_j$, and

$$\sum k_{ij}^{(\mathfrak{v})}dv_idv_j=-\sum g_ig_jdv_idv_j+\langle J_1J_2V,dV\rangle^2.$$

Therefore, as $(V,Z,t)\to(0,0,1)$, the coefficients $k_{ij}^{(\mathfrak{z})}$ do not have a limit (as we already know), whereas the $k_{ij}^{(\mathfrak{v})}$ tend to zero, as well as the $k_{ij}^{(\mathfrak{zv})}$ (since $g_j\to 0$ and the f_i are bounded around the pole).

As regards the horizontal distribution $\ker h_1$, this is smooth on $S^\star = S^6 \setminus \{(0,0,1)\}$, with dimension $p = 4$. In principle, it could extend smoothly on S^6, but it would have to change dimension at the pole. Indeed on S^6 there are no continuous k-dimensional distributions (continuous fields of k-planes) for $1 \le k \le 5$ ([10], Theorem 27.18). For $k = 1$, this is the well known result that even spheres do not admit continuous nowhere vanishing vector fields, or 1-forms by duality. However, $\ker h_1$ can not extend smoothly on S^6, since $h_1 = \Theta^2$ is not smooth at the pole

Again note that, in bispherical coordinates, $z_1 dz_2 - z_2 dz_1 = |Z|^2 d\alpha$, and if we take the limit of Θ in these coordinates we seem to get the result (3.6), i.e., that Θ extends to the pole. This proof would be wrong for the same reasons discussed before (Remark 2.4).

3.2.2 The 1-form Θ in the general case

In order to generalize the above calculations, we examine in more detail the 1-form Θ in (3.7). We would like to write it in a more explicit form, analogous to (3.20).

Theorem 3.5. *Let S be any Damek-Ricci space. Fix an orthonormal basis $\{U_i\}_{i=1}^q$ of $\mathfrak{z}$, and set $Z = \sum z_i U_i$, $J_i \equiv J_{U_i}$. Then the 1-form Θ in (3.7) can be written as*

$$\begin{aligned}\Theta|_{(V,Z,t)} = [V, dV] + t dZ - Z dt + \frac{1}{(1-t)^2 + |Z|^2} \sum_{i<j} \Big\{ 2\big(z_j U_i - z_i U_j\big) \times \\ \times \Big(z_i dz_j - z_j dz_i + z_i [V, dV]_j - z_j [V, dV]_i + (1-t)\langle J_i J_j V, dV \rangle \Big) \\ + (z_i dz_j - z_j dz_i)[J_i V, J_j V] \Big\}. \end{aligned} \tag{3.24}$$

Proof. We insert the quantity $0 = -2(\langle V, dV\rangle + \langle Z, dZ\rangle + t dt)$ in the curly bracket in (3.3), and use the identity

$$(1-t)^2 + |Z|^2 - (1-t)|V|^2 = 2(t^2 + |Z|^2) - t(2 - |V|^2),$$

to get, after some algebra,

$$\begin{aligned}\Theta|_{(V,Z,t)} = [V, dV] + t dZ - Z dt + \frac{1}{(1-t)^2 + |Z|^2} \Big\{ 2\Big(\langle Z, dZ \rangle Z - |Z|^2 dZ \Big) \\ + 2\Big(Z \big\langle J_Z V, dV \big\rangle - |Z|^2 [V, dV] \Big) + 2(1-t)\Big(Z\langle V, dV\rangle + [J_Z V, dV] \Big) \\ + [J_Z V, J_{dZ} V] \Big\}. \end{aligned} \tag{3.25}$$

The following identities are easily proved:

$$\langle Z, dZ \rangle Z - |Z|^2 dZ = \sum_{i<j} \big(z_j U_i - z_i U_j\big)\big(z_i dz_j - z_j dz_i\big),$$

$$Z \big\langle J_Z V, dV \big\rangle - |Z|^2 [V, dV] = \sum_{i<j} \big(z_j U_i - z_i U_j\big)\big(z_i [V, dV]_j - z_j [V, dV]_i\big),$$

$$Z\langle V, dV\rangle + [J_Z V, dV] = \sum_{i<j} \big(z_j U_i - z_i U_j\big)\langle J_i J_j V, dV\rangle.$$

Using these and (2.4) in (3.25), gives (3.24). □

Remark 3.6. For $q = 1$, the curly bracket in (3.25) vanishes, and we get back (3.10). In the non-symmetric example of $q = 2$, the last term in the curly bracket of (3.25) vanishes, and we obtain formula (3.20). In the symmetric case of $q = 3, 7$, we use the J^2-condition (3.13) to write Θ in (3.24) as

$$\Theta = \Omega + \Lambda, \tag{3.26}$$

where $\Omega = (\omega_1, \ldots, \omega_q)$ is the connection 1-form (3.9), and Λ is the 1-form

$$\begin{aligned} \Lambda|_{(V,Z,t)} = \frac{2}{(1-t)^2 + |Z|^2} \sum_{i<j} \Big\{ &\big(z_j U_i - z_i U_j\big) \times \\ &\times \Big(z_i dz_j - z_j dz_i + z_i [V, dV]_j - z_j [V, dV]_i + (1-t)\langle J_{Z_{ij}(V)} V, dV\rangle\Big) \\ &+ \big(|V|^2 - (1-t)\big)(z_i dz_j - z_j dz_i) Z_{ij}(V)\Big\}. \end{aligned} \tag{3.27}$$

Using (3.26)-(3.27), we can easily rewrite $\Theta = (\theta_1, \ldots, \theta_q)$ in the form

$$\Theta = \mathcal{R}(\Omega), \quad \text{i.e.,} \quad \theta_i = \sum_j r_{ij}\omega_j, \tag{3.28}$$

where $\mathcal{R} = (r_{ij})$ is given respectively by (3.12) and (3.15)-(3.16). Thus $\Theta^2 = \Omega^2$, and $\Lambda = \mathcal{R}(\Omega) - \Omega$ satisfies $\langle \Lambda + 2\Omega, \Lambda\rangle = 0$. We conclude that although Θ and Λ do not extend to the pole $(0, 0, 1)$, Θ^2 does, as already discussed.

We can now repeat a similar analysis as was done before in the 7-dimensional example. We obtain the following result.

Theorem 3.7. *Let $S = NA$ be any Damek-Ricci space.*

1) The $\mathfrak{z}$-valued 1-form Θ, defined in (3.7) and given by (3.24), does not extend to the pole $(0, 0, 1)$, except for $q = 1$ (hermitian symmetric case). More precisely, the coefficients of the dz_j in Θ do not have a limit at the pole for $q > 1$, but remain bounded there, whereas the coefficients of the dv_j tend to zero at the pole.

2) The differential expression $h_1 = \Theta^2$ in (1.2)-(1.3) does not extend to the pole, unless S is symmetric. More precisely, the differential expression k_1 in (1.3) vanishes identically in the symmetric case, but it is nonzero for S non-symmetric. In this case, writing k_1 in the form (2.1), the metric coefficients $k_{ij}^{(\mathfrak{z})}$ do not have a limit as $(V, Z, t) \to (0, 0, 1)$, but remain bounded, whereas the coefficients $k_{ij}^{(\mathfrak{v})}$ and $k_{ij}^{(\mathfrak{zv})}$ tend to zero at the pole.

3) The horizontal distribution $\ker\Theta = \ker h_1$ on $S^{p+q} \setminus \{(0, 0, 1)\}$ does not extend smoothly to the pole unless S is symmetric.

Proof. Let $\Theta = (\theta_1, \ldots, \theta_q) = \sum \theta_i U_i$. From (3.24) we get, for $1 \le i \le q$,

$$\theta_i|_{(V,Z,t)} = [V, dV]_i + t dz_i - z_i dt + \frac{1}{(1-t)^2 + |Z|^2} \times$$
$$\times \left\{ 2 \sum_{j \neq i} z_j \Big(z_i dz_j - z_j dz_i + z_i [V, dV]_j - z_j [V, dV]_i + (1-t) \langle J_i J_j V, dV \rangle \Big) \right.$$
$$\left. + \sum_{j<k} (z_j dz_k - z_k dz_j) \langle J_i J_j V, J_k V \rangle \right\}. \tag{3.29}$$

Consider the coefficients of the differentials dz_l for $l = i$ and $l = j \neq i$ in the term with the curly bracket in (3.29). They involve the functions

$$\frac{z_j^2}{(1-t)^2 + |Z|^2}, \qquad \frac{z_i z_j}{(1-t)^2 + |Z|^2}, \qquad \frac{z_k \langle J_i J_j V, J_k V \rangle}{(1-t)^2 + |Z|^2}. \tag{3.30}$$

These functions do not have a limit at the pole. Indeed in bispherical coordinates $V = |V|\omega_1$, $Z = |Z|\omega_2$, where $\omega_1 \in S^{p-1}$, $\omega_2 \in S^{q-1}$, we have, e.g.,

$$\frac{z_i z_j}{(1-t)^2 + |Z|^2} = \frac{|Z|^2}{(1-t)^2 + |Z|^2} (\omega_2)_i (\omega_2)_j.$$

Letting $1 - t = \rho \cos \phi$, $|Z| = \rho \sin \phi$, then $\rho \to 0$ when $(V, Z, t) \to (0, 0, 1)$, and the functions above do not have a limit. In a similar way we get, being $|V|^2 = \rho(2\cos\phi - \rho)$,

$$\begin{aligned} \frac{z_k \langle J_i J_j V, J_k V \rangle}{(1-t)^2 + |Z|^2} &= \frac{|Z||V|^2}{(1-t)^2 + |Z|^2} (\omega_2)_k \langle J_i J_j \omega_1, J_k \omega_1 \rangle \\ &= \sin\phi (2\cos\phi - \rho)(\omega_2)_k \langle J_i J_j \omega_1, J_k \omega_1 \rangle, \end{aligned}$$

which does not have a limit when $\rho \to 0$. Notice, however, that these functions remain bounded around the pole.

As regards the coefficients of the dv_l in the curly bracket in (3.29), they are generated by the 1-forms

$$\frac{z_i z_j [V, dV]_j}{(1-t)^2 + |Z|^2}, \qquad \frac{z_j^2 [V, dV]_i}{(1-t)^2 + |Z|^2}, \qquad \frac{z_j (1-t) \langle J_i J_j V, dV \rangle}{(1-t)^2 + |Z|^2}, \tag{3.31}$$

and tend to zero at the pole. Indeed, the terms $[V, dV]_j$ and $\langle J_i J_j V, dV \rangle$ are linear expressions in the coordinates v_k and the differentials dv_l, for instance, we have

$$[V, dV]_j = \sum_{k,l=1}^{p} a_{kl}^{(j)} v_k dv_l \qquad (1 \le j \le q),$$

for suitable constants $a_{kl}^{(j)}$. Using again bispherical coordinates and then polar coordinates for $(1 - t, |Z|)$, we get, e.g.,

$$\begin{aligned} \frac{z_i z_j [V, dV]_j}{(1-t)^2 + |Z|^2} &= \frac{|Z|^2 |V|}{(1-t)^2 + |Z|^2} (\omega_2)_i (\omega_2)_j \sum_{k,l} a_{kl}^{(j)} (\omega_1)_k dv_l \\ &= \sqrt{\rho} \sqrt{2\cos\phi - \rho}\, (\sin\phi)^2 (\omega_2)_i (\omega_2)_j \sum_{k,l} a_{kl}^{(j)} (\omega_1)_k dv_l. \end{aligned}$$

The coefficients of the dv_l in this formula go to zero as $\sqrt{\rho}$ when $\rho \to 0$, the remaining expressions being bounded. The same result holds for the other terms in (3.31).

Thus Θ does not extend smoothly on S^{p+q} unless $q = 1$ (in which case the curly bracket in (3.29) vanishes). Its square does not extend either, except in the symmetric case, and the same holds for the differential expression h_1 in (1.2), being

$$h_1 = \Theta^2 = \theta_1^2 + \cdots + \theta_q^2.$$

Indeed, in the symmetric case the quantity k_1 in (1.3) vanishes identically (Proposition 3.1), and $h_1 = \Theta^2 = \Omega^2$ extends to the pole.

In the non-symmetric case k_1 is nonzero, indeed the expression (2.2) vanishes iff the J^2-condition holds (Theorem 2.2). By writing k_1 in the form (2.1), and computing the square of Θ from (3.29), we see that the coefficients $k_{ij}^{(\mathfrak{z})}$ involve products of the functions in (3.30) (or products of those functions and t), the $k_{ij}^{(\mathfrak{v})}$ are generated by products of the 1-forms in (3.31) (or products of those 1-forms and $[V, dV]_i$), and the $k_{ij}^{(\mathfrak{zv})}$ arise from products of the functions in (3.30) and the 1-forms in (3.31) (or those in (3.30) and $[V, dV]_i$, or those in (3.31) and t). Therefore, the coefficients $k_{ij}^{(\mathfrak{z})}$ do not have a limit as $(V, Z, t) \to (0, 0, 1)$, but remain bounded there, whereas the coefficients $k_{ij}^{(\mathfrak{v})}$ and $k_{ij}^{(\mathfrak{zv})}$ tend to zero at the pole. Finally, the horizontal distribution $\ker \Theta = \ker h_1$ on $S^\star$ does not extend smoothly to S^{p+q} in the non-symmetric case, since $\Theta^2 = h_1$ is not smooth at the pole. □

As a final remark, we record the following formula for h_1, that generalizes (3.22) to any Damek-Ricci space.

Proposition 3.8. *The differential expression h_1 in (1.2)-(1.3) can be written (in the notations of Theorem (3.5)) as*

$$\begin{aligned}
h_1|_{(V,Z,t)} &= \big|[V, dV] + t dZ - Z dt\big|^2 + \sum_{i<j} \big(z_i dz_j - z_j dz_i + \langle J_i J_j V, dV\rangle\big)^2 \\
&+ \frac{1}{((1-t)^2 + |Z|^2)^2} \Big\{ \big|[J_Z V, J_{dZ} V]\big|^2 + 4(1-t)\,\big\langle [J_Z V, J_{dZ} V], [J_Z V, dV]\big\rangle \\
&- 2\big(|Z|^2 - (1-t)^2\big)\,\big\langle [J_Z V, J_{dZ} V], [V, dV]\big\rangle \\
&- \sum_{i<j} \Big(|V|^2\,(z_i dz_j - z_j dz_i) + 2(1-t)\big(z_i [V, dV]_j - z_j [V, dV]_i\big) \\
&- \big(|Z|^2 - (1-t)^2\big)\langle J_i J_j V, dV\rangle \Big)^2 \\
&- 4(1-t)^2 \sum_{i<j<k} \Big(z_i \langle J_j J_k V, dV\rangle + z_j \langle J_k J_i V, dV\rangle + z_k \langle J_i J_j V, dV\rangle \Big)^2 \Big\}.
\end{aligned} \tag{3.32}$$

The quantity $k_1(V, Z, t)$ in (1.3) is then $\sum_{i<j} \langle J_i J_j V, dV\rangle^2$ plus the term with the curly bracket in (3.32) (this generalizes (3.23)).

Proof. We use the identities in [1], p. 325 and top p. 326, in (1.2)-(1.3). □

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