Constructive Estimates of the Pull-In Range for Synchronization Circuit Described by Integro-Differential Equations

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Anton V. Proskurnikov[1], Vera B. Smirnova[2]

***Abstract*— The pull-in range, known also as the acquisition or capture range, is an important characteristics of synchronization circuits such as e.g. phase-, frequency- and delay-locked loops (PLL/FLL/DLL). For PLLs, the pull-in range characterizes the maximal frequency detuning under which the system provides phase locking (mathematically, every solution of the system converges to one of the equilibria). The presence of periodic nonlinearities (characteristics of phase detectors) and infinite sequences of equilibria makes rigorous analysis of PLLs very difficult in spite of their seeming simplicity. The models of PLLs can be featured by multi-stability, hidden attractors and even chaotic trajectories. For this reason, the pull-in range is typically estimated numerically by e.g. using harmonic balance or Galerkin approximations. Analytic results presented in the literature are not numerous and primarily deal with ordinary differential equations. In this paper, we propose an analytic method for pull-in range estimation, applicable to synchronization systems with infinite-dimensional linear part, in particular, for PLLs with delays. The results are illustrated by analysis of a PLL described by second-order delay equations.**

***Index Terms*— PLL, pull-in range, nonlinear system, stability**

I. Introduction

Mathematical models of phase-locked loops (PLLs) and other nonlinear synchronization circuits has been long studied in the literature [1]–[4]. In spite of their seeming simplicity, PLLs can have non-trivial dynamics featured by chaotic behaviors [5] and hidden attractors [6]–[8]. These effects are essentially nonlinear and cannot be understood by linearization-based analysis and are caused by presence of periodic nonlinearities, describing the phase detector. Mathematical methods able to cope with general periodic nonlinear systems (or, equivalently, dynamics on cylindrical manifolds) have been developed quite recently and originate from dynamical systems and control theories [9]–[12].

In this paper, we study one of the classical problems related to PLL circuits stability, namely, estimation of PLL's *pull-in* range. The pull-in range characterizes capturing capabilities of the PLL and, following [1], is defined mathematically [6] as the interval of frequency detuning (deviation between the reference and controlled oscillators' frequencies), for which phase locking is guaranteed. Mathematically, phase locking can be characterized as the convergence of each solution to one of the equilibria of the system.

Starting from the pioneer works on the pull-in range estimation [13], [14], the engineering literature has been primarily dealing with approximate numerical methods. On one hand, the pull-in range cannot be broader than the *hold-in* range [6], that is, the set of initial detuning for which the system may have a (locally) stable equilibrium. Local stability analysis of equilibria thus gives a rough upper estimate of the pull-in interval. On the other hand, global stability obviously excludes the existence of periodic solutions (cycles). The boundary of the pull-in interval is thus often estimated as a point of bifurcation, at which a cycle (possibly, degenerating to a saddle-point separatrix loop) emerges [15]–[17]. To find cycles, harmonic balance [14], [18]–[20] or Galerkin approximation [16], [21] techniques can be used; some special methods also exist for low-order systems [22]. It should be noticed, however, that mathematically the absence of cycles and global stability are *different* properties. Using the Poincare-Bendixsson theory [23], their equivalence can be proved for second-order models of PLLs, whereas three- and higher dimensional systems may have non-periodic attractors [7]. Mathematically rigorous estimates for pull-in ranges are scarce and surveyed in recent works [6], [8], [24].

The aforementioned techniques for the pull-in range estimation are mainly confined to nonlinear circuits described by ordinary differential equations (ODE). In practice, the actual dynamics of PLL may appear to be infinite-dimensional for two reasons. First, many PLL circuits contain a non-negligible *delay* in the feedback loop [20], [25]–[28] or in the reference signal [29]. Second, a PLL may contain a loop-filter whose transfer function is non-rational, e.g. fractional-order low pass filter [30]. Analytic estimates for the pull-in frequency detuning range in the infinite-dimensional case are quite limited. For delayed second-order PLLs, some numerical results are available in [28], [31]. The aforementioned method of harmonic balance to find cycles in delayed PLLs was employed in [20]. Bifurcation analysis of a simplistic first-order PLL model has been performed in [32], [33].

In this paper, we propose an estimate applicable to a more general class of synchronization systems that arise as feedback superpositions of linear blocks and periodic nonlinearities. Unlike many results in engineering literature, we do not simply exclude periodic oscillations, but prove global stability of the system. We illustrate our results by analyzing some models of PLL circuits with delay [20], [28].

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II. The general model of PLL. Problem setup.

The minimal structure of a PLL circuit is shown in Fig. 1 and comprises the *phase detector* (comparator), the low-pass *loop filter* and the *voltage control oscillator* (VCO), which has to be synchronized with the *reference oscillator* (RO) signal. In practice, synchronization circuits often include frequency dividers, charge pumps and other elements.

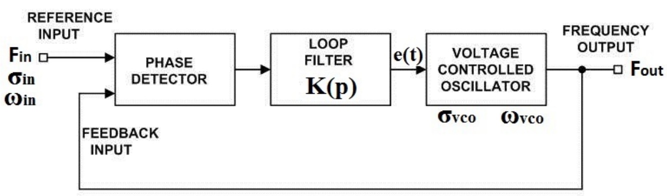

Fig. 1. The minimal structure of a PLL circuit

Assume that the input to PLL is a harmonic signal $F_{in}(t) = \sin\sigma_{in}(t) = \sin(\omega_{in}t + \sigma_{in}^0)$ of constant frequency $\omega_{in} > 0$. The VCO has a *free-run* (natural) frequency ω_{VCO}^0, whereas its instantaneous frequency is controlled as

$$\omega_{VCO}(t) = \dot\sigma_{VCO}(t) = \omega_{VCO}^0 + e(t),$$

where $e(t)$ is the filtered phase error, computed by the detector. The VCO output $F_{out}(t) = \sin\omega_{VCO}(t)$ has to be *synchronized* with the reference input, that is,

$$\begin{aligned} \sigma(t) \overset{\Delta}{=} \sigma_{in}(t) - \sigma_{VCO}(t) \xrightarrow[t\to\infty]{} \sigma_* = const, \\ \dot\sigma(t) = \omega_{in} - \dot\sigma_{VCO}(t) \xrightarrow[t\to\infty]{} 0. \end{aligned} \tag{1}$$

The phase detector (comparator) receives the input and output signals $F_{in}(t), F_{out}(t)$, and returns a sum of a "slowly" changing function, represented as $f(\sigma(t))$, and a "fast" oscillatory signal, e.g., by computing the product

$$F_{in}(t)F_{out}(t) = \underbrace{\frac{1}{2}\cos(\sigma(t))}_{f(\sigma(t))} - \underbrace{\frac{1}{2}\cos(\sigma_{VCO}(t) + \sigma_{in}(t))}_{\text{fast signal}}.$$

To simplify modeling, it is typically assumed that the filter perfectly rejects high-frequency components of the detector's output, and only the "slow" part of this output $f(\sigma(t))$ influences the VCO; note that $f(\sigma)$ is a periodic function. Typically, the loop filter is described by a stable linear time-invariant system, described by the convolution equation

$$e(t) = \rho f(\sigma(t)) + \int_0^t \chi(t-s) f(\sigma(s))\, ds, \tag{2}$$

where $\rho = const$ and $\chi(\cdot) \in L_1[0,\infty)$ are the filter's characteristics. The filter's transfer function is defined as

$$K(p) = \rho + \int_0^\infty \chi(t)e^{-pt}dt.$$

Combining (1) and (2), one arrives at the equation

$$\dot\sigma(t) = \underbrace{(\omega_{in} - \omega_{VCO}^0)}_{\Delta\omega} - \rho f(\sigma(t)) - \int_0^t \chi(t-\tau) f(\sigma(\tau))\, d\tau, \tag{3}$$

The constant $\Delta\omega$ is said to be the *frequency detuning* of the PLL. Our main concern is to find the set of possible values $\Delta\omega$, under which the PLL provides phase locking (1) for every initial phase error $\sigma(0)$.

For technical reasons, it appears to be convenient to get rid of the constant $\Delta\omega$ by shifting the detector characteristics. Notice that in the case of ideal phase locking $\sigma_{VCO}(t) - \sigma_{in}(t) \equiv \sigma_* = const$, the steady phase error $\beta \overset{\Delta}{=} f(\sigma_*)$ can be found from (3) as follows

$$\beta = -\frac{\Delta\omega}{\rho + \int_0^\infty \chi(t)dt} = -\frac{\Delta\omega}{K(0)}. \tag{4}$$

It is convenient to introduce a shifted detector's characteristics $\varphi_\beta(\sigma) \overset{\Delta}{=} f(\sigma) - \beta$, vanishing at the equilibrium points σ_*. The equation (3) shapes into

$$\dot\sigma(t) = \sigma_0(t) - \rho\varphi_\beta(\sigma(t)) - \int_0^t \chi(t-\tau)\varphi_\beta(\sigma(\tau))ds, \tag{5}$$

where $\sigma_0(t) = \beta\int_t^\infty \chi(s)ds \xrightarrow[t\to\infty]{} 0$. Typically, the filter is an exponentially stable system, so that $|\chi(t)| \le ae^{-bt}$ for some $a, b > 0$. In this case, σ_0 also decays exponentially

$$|\sigma_0(t)| \le a|\beta| \int_t^\infty e^{-bs}ds = \frac{a|\beta|}{b}e^{-bt}.$$

In this paper, we consider a more general model of a PLL, allowing the presence of delays in the chain Detector-Filter-VCO. Assuming, for simplicity, that the transport delay $h \ge 0$ is constant and lumped, (5) is replaced by

$$\dot\sigma(t) = \sigma_0(t) - \rho\varphi_\beta(\sigma(t-h)) - \int_0^t \chi(t-\tau)\varphi_\beta(\sigma(\tau))ds. \tag{6}$$

The kernel $\chi(t)$ remains exponentially decaying, and the function $\sigma_0(t)$ depends not only on β and filter characteristics, but also on the initial condition $\sigma(t)$, $t \in [-h, 0]$. Furthermore, we also allow *noises* in the PLL circuit [34], [35] that are supposed to decay sufficiently fast (yet not exponentially) and can also be included into the term $\sigma_0(t)$. For this reason, we do not require exponential convergence for σ_0, but only suppose that $\sigma_0 \in L_1[0,\infty) \cap L_2[0,\infty)$. Considering the delay as an element of the loop filter, its transfer function becomes as follows

$$K(p) = \left(\rho + \int_0^\infty \chi(t)e^{-pt}dt\right)e^{-ph}, \quad p \in \mathbb{C}. \tag{7}$$

We are now ready to formulate the problem in question.

Problem. *Given the phase detector characteristics* $f(\sigma)$ *and the loop filter's transfer function* $K(p)$*, find the set of* $\beta \in \mathbb{R}$ *such that the solution of* (6) *with* $\varphi_\beta(\sigma) = f(\sigma) - \beta$ *satisfies* (1) *for every* $\sigma_0(\cdot) \in L_1[0,\infty) \cap L_2[0,\infty)$.

In view of (4), the frequency detuning $\Delta\omega = -K(0)\beta$ belongs to the pull-in range of the synchronization circuit. We thus give a *sufficient* condition for belonging of $\Delta\omega$ to the pull-in range (our criteria are, however, not necessary).

III. Stability criterion

Henceforth the following assumptions are adopted:

1) the phase detector nonlinearity is continuously differentiable and periodic with known period $\Delta > 0$, that is, $f(\sigma + \Delta) = f(\sigma)$;
2) for every $\beta \in \mathbb{R}$, the equation

$$\varphi_\beta(\sigma) = f(\sigma) - \beta = 0, \quad \sigma \in [0, \Delta], \tag{8}$$

has a finite non-zero number of solutions;
3) the filter is an exponentially stable linear block, that is, $|\chi(t)| \le ae^{-bt}$ for all $t \ge 0$ and some $a, b > 0$;

We also introduce some notation. Let

$$\alpha_1 \stackrel{\Delta}{=} \inf_{\zeta \in [0,\Delta)} f'(\zeta); \quad \alpha_2 \stackrel{\Delta}{=} \sup_{\zeta \in [0,\Delta)} f'(\zeta) \tag{9}$$

$$\Phi(\zeta) \stackrel{\Delta}{=} \sqrt{\left(1 - \alpha_1^{-1} f'(\zeta)\right)\left(1 - \alpha_2^{-1} f'(\zeta)\right)}, \tag{10}$$

$$\nu(\beta) \stackrel{\Delta}{=} \frac{\int_0^\Delta \varphi_\beta(\zeta)\, d\zeta}{\int_0^\Delta |\varphi_\beta(\zeta)| d\zeta}, \quad \nu_0(\beta) \stackrel{\Delta}{=} \frac{\int_0^\Delta \varphi_\beta(\zeta)\, d\zeta}{\int_0^\Delta \Phi(\zeta)\, |\varphi_\beta(\zeta)|\, d\zeta}. \tag{11}$$

In view of periodicity of $f(\cdot)$, we have $\alpha_1 < 0 < \alpha_2$. In general, the explicit computation of functions $\nu(\beta), \nu_0(\beta)$ is not very simple, however, for special phase detectors (e.g. one with sinusoidal characteristics $f(\sigma) = \sin\sigma$) their closed-form expressions are available (see Example 1).

To estimate the pull-in range, the will use the following stability criterion established in [36].

Theorem 1: *Suppose there exist the numbers $a \in [0, 1]$, $\varepsilon > 0, \delta > 0, \tau > 0$ and $\varkappa \in \mathbb{R}$, such that the following requirements are true:*
1) for all $\omega \in \mathbb{R}$ the inequality

$$\Pi(\omega) \stackrel{\Delta}{=} \mathrm{Re}\{\varkappa K(\imath\omega) - \tau(K(\imath\omega) + \imath\alpha_1^{-1}\omega)^*(K(\imath\omega) + \\ + \imath\alpha_2^{-1}\omega)\} - \varepsilon|K(\imath\omega)|^2 \ge \delta \tag{12}$$

(where $*$ stands for complex conjugation) is valid.
2) the following quadratic form is positive definite:

$$Q(x, y, z) \stackrel{\Delta}{=} \varepsilon x^2 + \delta y^2 + \tau z^2 + \\ + a\varkappa\nu(\beta)xy + (1 - a)\varkappa\nu_0(\beta)yz > 0 \\ \forall x, y, z : |x| + |y| + |z| \neq 0. \tag{13}$$

Then every solution of (6) *(corresponding to some function $\sigma_0 \in L_1 \cap L_2$) converges to an equilibrium*

$$\dot\sigma(t) \xrightarrow[t\to\infty]{} 0, \quad \sigma(t) \xrightarrow[t\to\infty]{} \sigma_{eq} \tag{14}$$

where $f(\sigma_{eq}) = \beta$. In other words, the frequency detuning $\Delta\omega = -K(0)\beta$ is within the PLL's pull-in range.

Remarks. It should be noticed that Theorem 1 is applicable to every PLL representable in the form (6), not only low-order systems. Unlike many results, available in engineering literature, it ensures stability of (1) rather than the absence of periodic cycles. Notice that Theorem 1 does not say anything about (local) stability of a specific equilibrium point. Considering the simple model of a viscously damped pendulum [37], one may notice that typically the system has both stable and unstable equilibria. The frequency-domain condition (12) does not involve β and depends only on the properties of linear filter and the slopes α_i of the phase detector characteristics. However, conditions (12) and (13) are entangled, involving the same scalar parameters $\delta, \varepsilon, \varkappa, \tau$.

A. Simplification of the conditions from Theorem 1

The condition (13) can be rewritten in a simpler way

$$4\varepsilon\delta\tau > \varkappa^2(a^2\nu(\beta)^2\tau + (1 - a)^2\nu_0(\beta)^2\varepsilon). \tag{15}$$

One way to prove this is to use the standard Sylvester criterion for positive definiteness; alternatively, one may minimize $Q(x, y, z)$ with respect to z and check that (13) boils down to the positive definiteness of the quadratic form

$$Q_{min}(x, y) \stackrel{\Delta}{=} \min_z Q(x, y, z) = \\ = \varepsilon x^2 + \left[\delta - \frac{(1 - a)^2\varkappa^2\nu_0(\beta)^2}{4\tau}\right] y^2 + a\varkappa\nu(\beta)xy,$$

which is equivalent to the negativity of the discriminant

$$a^2\varkappa^2\nu(\beta)^2 - 4\varepsilon\delta + \frac{\varepsilon(1 - a)^2\varkappa^2\nu_0(\beta)^2}{\tau} < 0.$$

The latter inequality is equivalent to (15).

Notice now that the parameter a appears only in (15), and the right-hand side of (15) attains its minimum for $a = (\varepsilon\nu_0^2)/(\varepsilon\nu_0^2 + \tau\nu^2) \in [0, 1]$. Substituting this value into (15), one shows that (15) holds *for some* $a \in [0, 1]$ if and only if

$$4\varepsilon\delta\tau > \frac{\varkappa^2\tau\varepsilon\nu(\beta)^2\nu_0(\beta)^2}{\varepsilon\nu_0(\beta)^2 + \tau\nu(\beta)^2} \Leftrightarrow 4\delta > \frac{\varkappa^2\nu(\beta)^2\nu_0(\beta)^2}{\varepsilon\nu_0(\beta)^2 + \tau\nu(\beta)^2}.$$

The latter inequality, obviously, holds when $\varkappa = 0$. This inequality, as well as (12), retain their validity if one scales all parameters $\varkappa, \varepsilon, \tau, \delta$ by a positive constant. For this reason, one may always assume that either $\varkappa = 0$ or $\varkappa = \pm 1$. Theorem 1 can be now restated in a simpler form.

Theorem 2: Suppose that three real numbers $\varepsilon, \tau, \delta > 0$ and an integer $\varkappa \in \{-1, 0, 1\}$ exist such (12) holds and

$$\delta > \varkappa^2 \frac{\nu(\beta)^2\nu_0(\beta)^2}{4(\varepsilon\nu_0(\beta)^2 + \tau\nu(\beta)^2)}. \tag{16}$$

Then every solution of (6) converges to an equilibrium (14).

Notice that essentially one can get rid of the parameter δ, rewriting (12) and (16) as a single inequality

$$\inf_{\omega\in\mathbb{R}} \Pi(\omega) > \varkappa^2 \frac{\nu(\beta)^2\nu_0(\beta)^2}{4(\varepsilon\nu_0(\beta)^2 + \tau\nu(\beta)^2)}, \tag{17}$$

which involves only two real parameters $\varepsilon, \tau > 0$ and one discrete parameter $\varkappa \in \{0, \pm 1\}$.

B. A special case of system (6): delayed differential equation

A typical example of the infinite-dimensional system (6) is the PLL described by delay equations

$$\frac{dz(t)}{dt} = Az(t) - b\varphi_\beta(\sigma(t - h)) \in \mathbb{R}^m \\ \frac{d\sigma(t)}{dt} = c^\top z(t) - \rho\varphi_\beta(\sigma(t - h)) \in \mathbb{R}, \tag{18}$$

where A is a Hurwitz (stable) matrix, b,c are vectors, $\rho \in \mathbb{R}$ and $z(t)$ is the internal state variable of the filter. The solution is uniquely defined by initial conditions $z(0)$ and $\sigma : [-h, 0] \to \mathbb{R}$, it is supposed that σ is continuous at $t = 0$ so that $\sigma(0) = \lim_{t\to -0} \sigma(t)$. The filter's transfer function is

$$K(p) = \left(\rho + c^\top (pI - A)^{-1} b\right) e^{-ph},$$

which corresponds to the following kernel of convolution

$$\chi(t) = \begin{cases} 0, & \text{if } t < h, \\ -c^\top e^{A(t-h)} b, & \text{if } t > h. \end{cases}$$

In the case of filter (18), phase locking (1) also implies that $z(t) \xrightarrow[t\to 0]{} 0$, since A is Hurwitz and $\varphi_\beta(\sigma(t)) \xrightarrow[t\to\infty]{} 0$.

IV. Numerical examples

In this section, we illustrate Theorem 2 by examining a delayed PLL with a proportional-integrating filter and a sine-shaped detector (with the period is $\Delta = 2\pi$) [20]

$$K(p) = T\frac{sTp+1}{Tp+1}e^{-ph}, \quad f(\sigma) = \sin\sigma,\ \varphi_\beta(\sigma) = \sin\sigma - \beta.$$

Notice that the equation (8) has solutions only for $\beta \in [-1, 1]$. Due to space limitations, we consider only non-negative values of $\beta \geq 0$ (estimating thus a "half-plane" pull-in range [16]). Here $T > 0$ and $s \in (0, 1)$ are some constants, $h \geq 0$ is the delay. The slopes (9) of $f(\cdot)$ are $\alpha_2 = 1 = -\alpha_1$, thus $\Phi(\sigma) = |\sin\sigma|$ and one can show that

$$|\nu(\beta)| = \frac{\pi\beta}{2(\beta \arcsin\beta + \sqrt{1-\beta^2})} \tag{19}$$

$$|\nu_0(\beta)| = \frac{2\pi\beta}{4\beta + \pi - 2\arcsin\beta - 2\beta\sqrt{1-\beta^2}}. \tag{20}$$

Example 1. Consider first the undelayed PLL case $h = 0$. Notice first that for $K(0) = T > 0$ and $\mathrm{Re}\{(K(\imath\omega) + \imath\alpha_1^{-1}\omega)^*(K(\imath\omega) + \imath\alpha_2^{-1}\omega)\} = |K(\imath\omega)|^2 - \omega^2$, the inequality (12) may hold only with $\varkappa > 0$, so we choose $\varkappa = 1$. Also, it is convenient to redesignate the parameters: $\tau' = \tau/\mu, \varepsilon' = \varepsilon/\mu, \delta' = \delta\mu$, where $\mu \triangleq \frac{1}{T}$. Obviously, this change of the parameter does not change the relation (16).

Denoting $y \triangleq \omega^2$, (12) is written as

$$Ay^2 + By + C \geq 0,\ \forall y \geq 0 \tag{21}$$

$$\begin{gathered} A \triangleq \tau'\mu^3, B \triangleq \tau'\mu^5 - (\tau' + \varepsilon')s^2\mu + s\mu - \delta'\mu, \\ C \triangleq \mu^3(1 - \tau' - \varepsilon' - \delta'). \end{gathered} \tag{22}$$

It is clear that (21) holds if and only if $C \geq 0$ (the value at $y = 0$) and either $B \geq 0$ or the discriminant of the quadratic function $B^2 - 4AC \leq 0$. In other words, $\tau' + \varepsilon' + \delta' \leq 1$ and at least one of the following conditions should hold:

$$\tau'\mu^4 - (\tau' + \varepsilon')s^2 + s - \delta' \geq 0, \tag{23}$$

$$(\tau'\mu^4 - (\tau' + \varepsilon')s^2 + s - \delta')^2 \leq 4\tau'\mu^4(1 - \varepsilon' - \delta' - \tau') \tag{24}$$

It can be shown that the right-hand side of (16) is an increasing function of β. Changing, with a sufficiently small step, the values of $\delta' \in (0, 1)$, $\varepsilon' \in (0, 1 - \delta')$ and $\tau' \in (0, 1 - \delta' - \varepsilon')$ and taking such triples that one of the relations (23) or (24) holds, one can estimate the maximal β satisfying (16) (with $\varkappa = 1$). Taking the maximum over all admissible triples $(\delta, \varepsilon, \tau)$, one estimates the pull-in range for the fixed values of μ, s. The dependence between β and μ for different $s \in (0, 1)$ is shown in Fig.1.

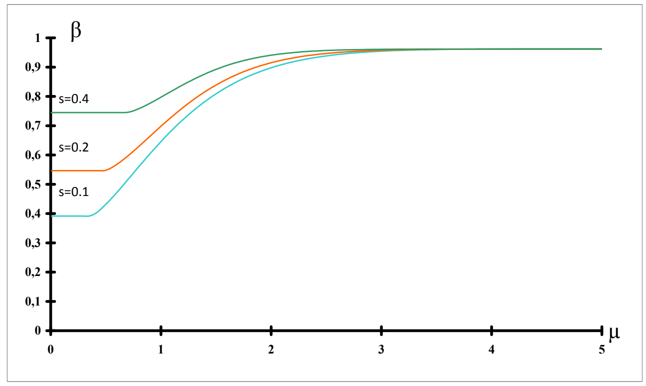

Fig. 2. Stability domains for PLL's with the proportional–integrating low-pass filter and the sine–shaped characteristic of phase detector

Notice that the genuine value for the pull-in range boundary reported in [38] for $s = 0.4$ is $\beta_0 = 0.84$ for $\mu < 0.2$ and $\beta_0 = 1$ for $\mu \geq 1$. Theorem 2 gives the value $\beta \approx 0.74$ for $\mu < 0.2$, $\beta \approx 0.8$ for $\mu = 1$ and $\beta \approx 0.96$ for $\mu > 2.2$.

Example 2. Consider now the case where $h > 0$. The frequency–domain inequality (12) with $\varkappa = 1$ takes the form

$$\begin{gathered} \tau\mu^2\omega^4 + \omega^2(\tau\mu^4 - \delta\mu^2 + \mu s\cos(\omega h) - \\ -(\varepsilon + \tau)s^2) - \mu^2(1 - s)\omega\sin(\omega h) + \mu^3\cos(\omega h) - \\ -(\varepsilon + \tau)\mu^2 - \delta\mu^4 \geq 0, \quad \forall\omega \geq 0. \end{gathered} \tag{25}$$

Substituting $\omega = 0$, one shows that

$$\mu^3 - (\varepsilon + \tau)\mu^2 - \delta\mu^4 \geq 0,$$

and therefore $\mu^2\delta + \varepsilon + \tau \leq \mu$. For every pair (μ, β) the inequality (25) has been checked numerically. It is obvious that it holds for $\omega > \Omega$, where Ω is sufficiently large. To check it on the interval $[0, \Omega]$, we scan this interval with sufficiently small step $h_\omega > 0$.

Scanning with small steps $h_\delta, h_\varepsilon, h_\tau$ the intervals

$$\delta \in \left(0, \frac{1}{\mu}\right), \varepsilon \in (0, \mu - \mu^2\delta),\ \tau \in (0, \mu - \mu^2\delta - \varepsilon),$$

we find the triples $(\delta, \varepsilon, \tau)$ that satisfy (25) and estimate the maximal value of β satisfying (16) for every such triple. Taking the maximum over all feasible triples $(\delta, \varepsilon, \tau)$, we estimate the pull-in range. For $s = 0.2, h = 0.01$ and $\mu = T = 1$ the estimated value is $\beta = 0.7$, whereas the genuine pull-in range reported in [31] is $\beta_0 = 0.93$.

V. Future works

The pull-in range estimates obtained in this paper are confined to analog PLL circuits with smooth characteristics of phase detector. Their extensions to continuous-time models with continuous yet non-smooth (e.g. Lipschitz) and discontinuous nonlinearities and, more important, to models of *digital PLLs* are subjects of ongoing research.

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