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Trees in Connected Graphs

Ashish Gupta
Montclair State University

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Trees in Connected Graphs

by

Ashish Gupta

A Masters Thesis Submitted to the Faculty of

Montclair State University

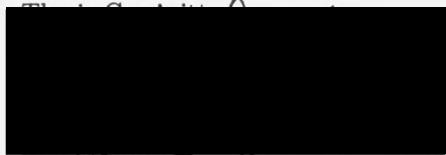In Partial Fulfillment of the Requirements

For the Degree of

Master of Science

August 2010

College/School: College of Science and Mathematics
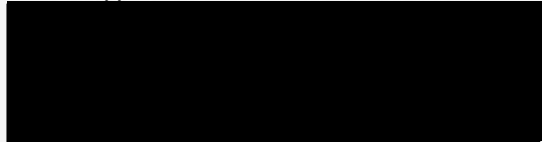Department: Mathematical Sciences

Certified by:

Dean of College of Science and Mathematics
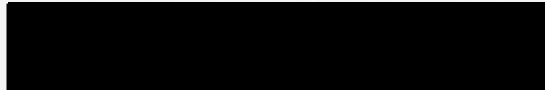Dr. Robert Prezant

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(date)

Thesis Sponsor: Dr. Jonathan Cutler

Committee Member: Dr. Evan Fuller

Committee Member: Dr. Aihua Li

Department Chair: Dr. Helen Roberts

Abstract

The focus of the Master's Thesis will be the investigation of current research involving trees that cover subsets of the vertex set of a connected graph. The primary goal is the extension of some recent results and a conjecture of Horak and McAvaney. Given certain conditions, we will reformulate their conjecture that states that if a graph can be spanned by a number of edge-disjoint trees, we can provide a bound on the maximum degree of this collection of edge-disjoint trees. We are able to show that this conjecture is true if the number of trees used to span the graph is one. We will then look at a specific class of graphs, namely series-parallel graphs, and present several new extremal examples to show that these "tree-like" graphs are difficult to analyze. A comprehensive survey of related literature is also included.

TREES IN CONNECTED GRAPHS

A MASTERS THESIS

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by

ASHISH GUPTA

Montclair State University

Montclair, NJ

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Chapter 1

Introduction

A well-studied problem in the field of graph theory is the determination of whether the vertex set, $V(G)$, of a connected graph, G, can be covered by a single subgraph of G that is a tree called a spanning tree. Note n equals the number of vertices in a graph G or $n = |V(G)|$. This problem involves determining the number of subgraphs needed to cover the vertices of G as well as a upper bound on the degree of the set of subgraphs.

The focus of the thesis will be to expand upon the results found by Horak and McAvaney [12]. In their paper, they tackled the problem of finding the minimum number of subtrees of a maximum degree, k, whose union covers all vertices of a connected graph, G. In this case, we will denote the minimum number of subtrees as s and $T_1, \ldots, T_s$ is the collection of subtrees with $\Delta(T_i) \leq k$ for $i = 1, \ldots, s$. We will only be considering the case where T_i's are edge-disjoint trees. Horak and McAvaney's conjecture is as follows.

Conjecture 1. *Let G be a connected graph on n vertices, $\delta = \delta(G)$, and $k \geq 2$. Then the vertices of G can be covered by $s \leq \left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil$ edge-disjoint trees of maximum degree at most k.*

If we consider when $s = 1$, then $V(G)$ can be spanned by a single tree of maximum degree at most k. Note that throughout this paper, if G contains a spanning tree (i.e., if G is connected), we will denote some spanning tree of G as T_*. There have been several theorems whose given conditions on G determine not only whether $V(G)$ can be covered by a single spanning tree but the maximum degree of a spanning tree. The first group of these theorems are based on whether a graph is Hamiltonian. A *Hamiltonian cycle* in a graph G is a spanning cycle C where $V(C) = V(G)$. We say that a graph is *Hamiltonian* if it contains

a Hamiltonian cycle. If a graph, G, is Hamiltonian, then the removal of exactly one edge of the Hamiltonian cycle results in a *Hamiltonian path* which is a type of spanning tree. Thus, if a graph G is Hamiltonian, then there exists a single spanning tree (Hamiltonian path) with maximum degree two. If G is not Hamiltonian or does not contain a Hamiltonian path, there may still exist a T_* that covers $V(G)$ with $\Delta(T_*) > 2$. We will look at theorems that provide conditions for a graph to be spanned by a single tree, T_*, of maximum degree, k. We will then consider theorems for a set, $A \subseteq V(G)$, to be covered by a tree. This lets us determine the maximum number of edge-disjoint trees necessary to cover $V(G)$. Note that if $A = V(G)$ then G can be covered by a single spanning tree.

Our results will extend upon Horak and McAvaney's conjecture such that given G contains a spanning tree T_*, we can determine a bound of the maximum degree of T_* that is $\Delta(T_*) \leq k$. We will also consider when G is spanned by a collection of edge-disjoint trees, the maximum degree of the set of trees. We will also investigate a specific group of graphs called series-parallel graphs and utilize them to exemplify Horak and McAvaney's conjecture. We will consider another group of graphs called random graphs and show how Horak and McAvaney's conjecture is trivial for these types of graphs.

Chapter 2

Background

We will look at previous theorems that investigate conditions of G that help calculate the number of trees necessary to cover $V(G)$ and the maximum degree of the collection of trees.

Note that for a spanning tree with maximum degree one, the case is trivial because the only possible graphs are either the single vertex graph, K_1, or the graph, K_2, with two vertices connected by a single edge. In both cases the graph itself is its spanning tree with maximum degree one. If we consider the number of trees with maximum degree one necessary to cover a graph G where $n \geq 3$, the number of spanning trees necessary to cover the vertices of G is the matching number of G plus the number of vertices not matched. The *matching number*, $v(G)$, of a graph G is the size of the largest possible set of pair-wise non-adjacent edges in G. In our paper, we will only consider connected graphs with $n \geq 3$ and $\Delta(G) \geq 2$.

The best possible non-trivial case for Conjecture 1 is when G contains a spanning tree, T_*, with maximum degree two (Hamiltonian path). We first consider theorems that provide conditions for G to be Hamiltonian or contain a Hamiltonian path. The next group of theorems to consider is when there exists a T_* in G but $\Delta(T_*) > 2$. There are theroems stating that for a set $A \subseteq V(G)$, A can be covered by a tree. If $A \neq V(G)$, these theorems provide a bound on the maximum number of trees necessary to cover G. Finally, we will further investigate the findings of Horak and McAvaney.

2.1 Hamiltonian graphs and Hamiltonian paths

The simplest spanning tree for a graph G for $n \geq 3$ is a T_* with $\Delta(T_*) = 2$ (Hamiltonian path). Conditions of a graph, G, to be Hamiltonian or contain a Hamiltonian path lead to the same important results: $V(G)$ can be covered by a single tree, T_* with $\Delta(T_*) = 2$. We begin by looking at the degrees of vertices of G in determining whether a graph is Hamiltonian. The study of spanning trees in graphs with maximum degree two can be said to have started with the classical theorem of Dirac [8] which provides a sufficient condition for the existence of a Hamiltonian cycle in a graph. We include the proof for completeness.

Dirac's theorem is as follows.

Theorem 2. *If G is a connected graph on $n \geq 3$ vertices and $\delta(G) \geq \frac{n}{2}$, then G is Hamiltonian.*

Proof. Suppose the contrary. Let G be an edge-maximal counterexample, that is G is not Hamiltonian, $n \geq 3$, $\delta(G) \geq \frac{n}{2}$, and the addition of any edge joining two non-adjacent vertices in G results in a Hamiltonian graph. Since K_n is obviously Hamilonian, $G \neq K_n$. So there exist x and y in G such that $xy \notin E(G)$. Since $G + xy$ contains a Hamiltonian cycle, there exists a Hamiltonian path in G with x and y as the endpoints. Let $x = v_1, v_2, \ldots, v_{n-1}, v_n = y$ be such a path.

Suppose $v_1 v_i \in E(V)$ such that $2 \leq i \leq n - 1$. Then $v_{i-1} v_n \notin E(G)$ because then there exists a Hamiltonian cycle, $v_1, v_2, \ldots, v_{i-1}, v_n, v_{n-1}, \ldots, v_i, v_1$ as in Figure 2.2.
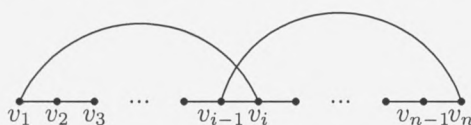

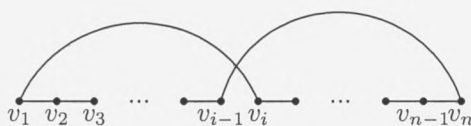Figure 2.1: Graph of example for proof of Dirac's theorem.

Figure 2.2: Hamiltonian graph of example for proof of Dirac's theorem.

Let $A := \{v_i : v_1 v_i \in E(G)\}$ and $B := \{v_i : v_{i-1} v_n \in E(G)\}$. We must have $A \cap B = \emptyset$ for otherwise there exists a vertex, v_i, such that there exists a Hamiltonian cycle as in

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Figure 2.2. Note, $v_i \notin A$ by the definition of A and $v_1 \notin B$ for there is no v_0 in the graph. As a result, since $v_1 \notin A \cup B$, $n - |\{v_1\}| \geq |A \cup B|$ and the following inequalities result.

$$n - |\{v_1\}| \geq |A \cup B|$$
$$n - 1 \geq |A \cup B|$$
$$= |A| + |B| - |A \cap B|$$
$$= d_G(v_1) + d_G(v_n) - 0$$
$$\geq \frac{n}{2} + \frac{n}{2} \tag{2.1}$$
$$= n,$$

a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Thus, by Dirac if $\delta(G) \geq \frac{n}{2}$ for a connected graph G, there exists a T_* with a maximum degree $k = 2$. In order to help us understand the types of graphs when the conditions of Dirac's theorem hold, consider the graph G where G is a complete bipartite graph, $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. This is a complete bipartite graph where $\Delta(G) = \lceil \frac{n}{2} \rceil$ and $\delta(G) = \lfloor \frac{n}{2} \rfloor$. This is an extremal example of Dirac's Theorem when n is even for then $\Delta(G) = \delta(G) = \frac{n}{2}$ and by Figure 2.3, we can see that G is Hamiltonian and therefore also contains a Hamiltonian path.
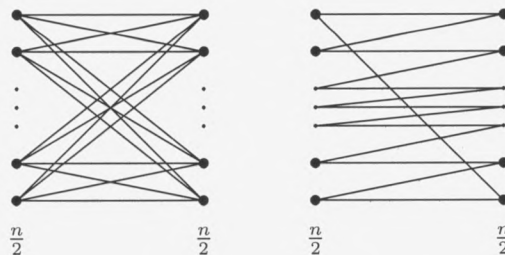
$$\frac{n}{2} \qquad\qquad \frac{n}{2} \qquad\qquad \frac{n}{2} \qquad\qquad \frac{n}{2}$$

Figure 2.3: $K_{\frac{n}{2}, \frac{n}{2}}$ and a Hamiltonian cycle of $K_{\frac{n}{2}, \frac{n}{2}}$ for n is even.

Note that when n is not even in a complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, $\delta(G) = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ and the condition for Dirac's theorem is not satisfied. Let us consider an example of this graph, $K_{3,4}$, and investigate if the graph is still Hamiltonian. By Figure 2.4, we note that $\delta(G) = 3 < \lceil \frac{n}{2} \rceil = 4$, so Dirac's condition does not hold. If we choose any vertex, x, of the independent set 4 and construct a maximal path from x, the path will end at another vertex, y, in the independent set of size 4. Since x and y belong to an independent set,
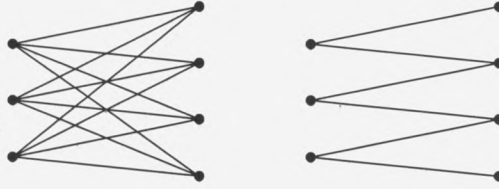
Figure 2.4: $K_{3,4}$ and a Hamiltonian path of $K_{3,4}$.

there is no edge in G to complete the cycle. Thus, there does not exist a Hamiltonian cycle and G is not Hamiltonian. This is not implied by Dirac, but the result shows that while a graph may not be Hamiltonian, the graph may contain a Hamiltonian path.

From the example of $K_{\frac{n}{2},\frac{n}{2}}$ where n is even and the example of $K_{3,4}$, we see that complete bipartite graphs are not necessarily Hamiltonian but may have a Hamiltonian path. When a complete bipartite graph, G, is in the form $K_{k,k}$ where $n = 2k$, the graph is Hamiltonian. In order to have a Hamiltonian path, a complete bipartite graph must be in the form $K_{k,k+i}$ where $i \in \{0, 1\}$. Otherwise, if $i > 1$, there is an independent set of at least $k + 2$ vertices and only k vertices available to be between each vertex of the independent set. Let X be the independent set of size k where $X = \{x_1, x_2, \ldots, x_{k-1}, x_k\}$ and Y be the independent set of size $k + 2$ where $Y = \{y_1, y_2, \ldots, y_k, y_{k+1}, y_{k+2}\}$. Then a longest path, P, in G is of length $k + 1$ where $P = \{y_1, x_1, y_2, x_2, \ldots, y_k, x_k, y_{k+1}\}$ which leads to vertex y_{k+2} not being covered. Thus, in order to have a Hamiltonian path, a complete bipartite graph must be in the form of $K_{k,k+i}$ where $i \in \{0, 1\}$.

Now, let us consider any graph that is a cycle, C_n, where $n \geq 3$. The graph $G = C_n$ is Hamiltonian for the graph itself is the Hamiltonian cycle. But, $\Delta(G) = \delta(G) = 2$ and for $n \geq 5$, $\frac{n}{2}$ becomes significantly greater than the actual $\delta(G)$. This illustrates that while Dirac provided a lower bound on $\delta(G)$ that guarantees G being Hamiltonian, the converse is false. Determining a necessary and sufficient condition for Hamiltonicity has proved to be a very difficult problem.

Following Dirac, we find that if $\delta \geq \frac{n}{2}$ then there exists a T_* of maximum degree two; however, this leaves a large group of graphs that may or may not be Hamiltonian that need to be investigated. Ore [15] proved that a lower bound on the degree sum of any pair of nonadjacent vertices of G leads to G being Hamiltonian.

Theorem 3. *If G is a graph of order $n \geq 3$ such that for each pair of nonadjacent vertices*

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x and y, $d_G(x) + d_G(y) \geq n$, then G is Hamiltonian.

Proof. The same outline of the proof for Theorem 2 can be used to prove Theorem 3 where the conditions of G are changed from $\delta(G) \geq \frac{n}{2}$ to the condition that for each pair of nonadjacent vertices x and y in G, $d_G(x) + d_G(y) \geq n$. Given the same construction of a maximal counterexample, G is not Hamiltonian and the addition of any edge between nonadjacent vertices of G makes a Hamiltonian cycle. Let x and y be defined as before, since x and y are not adjacent, $d_G(x) + d_G(y) \geq n$. From the same inequality, the same contradiction results with the omission of step 2.1 in the proof of Theorem 2. $\square$

While Dirac's theorem is significant, it leaves room for improvement. Ore's theorem generalizes Dirc's theorem slightly. Let us consider a Hamiltonian graph, G, that satisfies the conditions of Ore's theorem but do not satisfy the conditions of Dirac.
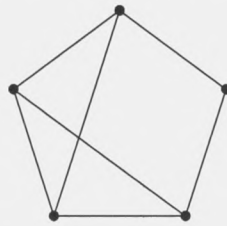
Figure 2.5: Example of graph that satisfies the conditions of Ore's theorem.

By Figure 2.5, $\delta(G) = 2 < \lceil \frac{n}{2} \rceil = \lceil \frac{5}{2} \rceil = 3$ and for each pair of nonadjacent vertices x and y, $d_G(x) + d_G(y) \geq 5$. Thus, while Ore's condition for Hamiltonicity is satisfied, Dirac's it not. This illustrates that Ore's Theorem is an extension of Dirac's and also shows that while Ore's condition is less strict than that of Dirac, it provides a sufficient condition for Hamiltonicity.

In our investigation of theorems beyond Dirac, we try and find the minimum conditions for a graph to be Hamiltonian or contain a Hamiltonian path. Pósa [16] proved an extension of Ore's theorem which goes beyond just the Hamiltonicity of a graph but rather provides a minimum number of vertices that can be covered by a path which is a type of tree with maximum degree two. Pósa's theorem is as follows.

Theorem 4. *Let G be a connected graph of order $n \geq 3$ such that for any two non-adjacent vertices x and y we have*

$$d_G(x) + d_G(y) \geq k.$$

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If $k = n$ then G is Hamiltonian, and if $k < n$ then G contains a path of length k and a cycle of length at least $\lceil \frac{k+2}{2} \rceil$.

For $k = n$, the theorem is the same as Ore's. However, let us consider a connected graph G when $k < n$ that satisfies Theorem 4 and results in a Hamiltonian path but fails the conditions of Theorem 3. Consider the graph, G, on 4 vertices, (see Figure 2.6) that
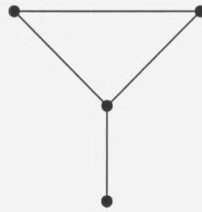
Figure 2.6: Example of graph that satisfies the conditions of Pósa's theorem.

contains a C_3 with an additional edge from one vertex of the C_3 to the fourth vertex. For any nonadjacent vertices x and y in $V(G)$, $d_G(x) + d_G(y) \geq 3$. While Ore's condition for Hamiltonicity fails (so G is not Hamiltonian), by Theorem 4, for $k = 3 < 4 = n$, G contains a path of length $k = 3$ and a cycle of length at least $\lceil \frac{3+2}{2} \rceil = 3$. Thus, while G does not contain a Hamiltonian cycle (need cycle to be length equal 4), there exists a path(Hamiltonian path) the covers all 4 vertices in G.

Thus, a corollary of Pósa's theorem is that if for any nonadjacent vertices x and y in $V(G)$, $d_G(x) + d_G(y) \geq n - 1$, G contains a Hamiltonian path. The previous example shows that G containing a Hamiltonian path does not necessarily result in G being Hamiltonian. We have only considered the Hamiltonicity of G because it guarantees the existence of a Hamiltonian path in G which is a spanning tree T_* with maximum degree two. Thus this corollary suffices for our line of research.

In the previous theorems, the Hamiltonicity of G or the existence of a Hamiltonian path in G were based on the degree of vertices. Bondy and Chvátal [3] provided a necessary and sufficient condition for graphs being Hamiltonian based upon the closure of the graph. Given a graph, G, on n vertices, the *closure* of G, $cl(G)$, is the graph constructed by successively adding new edges uv for all nonadjacent pairs of vertices u and v of G with $d_G(v) + d_G(u) \geq n$. Bondy and Chvátal's theorem is as follows.

Theorem 5. *A graph, G, is Hamiltonian if and only if its closure is Hamiltonian.*

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Determining $cl(G)$ is Hamiltonian requires investigating the degree sum of all nonadjacent pairs of vertices and thus constructing a new graph if the degree sums are greater than n. While this provides a definitive method in determining the Hamiltonicity of a graph, it can prove difficult and tedious as this requires examining each vertex, v, and $d_G(v)$ in comparison to each nonadjacent vertices of v which is not efficient.

We now look at a theorem whose condition is the connectivity of G. Chvátal and Erdős [5] proved that if given specific connectivity conditions on connected graph G then G had a Hamiltonian path. Note a connected graph G is *k-connected* if deleting any set of vertices of size $k - 1$ of G results in G still being connected. Chvátal and Erdős's theorem is as follows.

Theorem 6. *Let G be a k-connected graph with no independent set of k+2 vertices. Then G has a Hamiltonian path.*

In order to exemplify this theorem, let G be the complete bipartite graph $K_{k,k+1}$. Then G is k-connected for deleting any $k - 1$ vertices leaves G still connected and the largest independent set is of size $k + 2$. As seen in the earlier example of $K_{3,4}$, while G is not Hamiltonian, G does contain a Hamiltonian path. Chvátal and Erdős used the condition of connectivity in determining the existence of a Hamiltonian path while Pósa used the degree sum of nonadjancent vertices.

So far we have only considered theorems where given certain conditions of the graph G were satisfied, G is Hamiltonian or contains a Hamiltonian path. That is, if the respective conditions of these theorems were satisfied, there exists spanning tree $T_* \in G$ where $\Delta(T_*) = 2$ (Hamiltonian path). But when G does not necessarily contain a Hamiltonian path, we will try to determine if G contains some spanning tree, T_*, with a maximum degree of the tree, $\Delta(T_*)$.

2.2 Graphs containing a spanning tree but not Hamiltonian path

We will now consider theorems whose results are that the graph G contains a spanning tree that is not necessarily a Hamiltonian path. By Conjecture 1, this is the case where $s = 1$ with $k \geq 2$. We now consider k to be a general upper bound as the maximum degree of the

spanning tree, T_*. Win [17] proved that given certain degree conditions of a specific group of vertices, G contains a spanning tree of maximum degree k. A *k-tree* in a graph G is a spanning tree, T, with $\Delta(T) \leq k$. (Note that a 2-tree is simply a Hamiltonian path.) Win's Theorem is as follows.

Theorem 7. *Let $k \geq 2$ be an integer and let G be a connected graph on n vertices satisfying:*

$$\sum_{x \in I} d_G(x) \geq n - 1$$

for every k-element independent set $I \subset V(G)$. Then G contains a spanning tree T_* with $\Delta(T_*) \leq k$.

The assumption of this theorem generalizes that of Pósa's Theorem in that if $k = 2$, then $d_G(x) + d_G(y) \geq n - 1$ for every independent set $\{x, y\} \in V(G)$. This guarantees the existence of a spanning tree of maximum degree two, i.e., a Hamiltonian path. Note that the condition of Theorem 7 is satisfied whenever $\delta(G) \geq \frac{n-1}{k}$.

An extremal example of this is any *star*, S_k, which is defined as a bipartite graph $K_{1,k}$ where $k \geq 2$. Given $G = S_k$ with $k \geq 2$, let y be the lone vertex with degree greater than 1. The only independent set, I, where $|I| = k \geq 2$, is $I = V(G) \setminus \{y\}$. For all $x \in I$, $d_G(x) = 1$ and $\sum_{x \in I} d_G(x) = 1|I| = n - 1$. Thus, S_k with $k \geq 2$ is an extremal example of Theorem 7 since G is itself a tree, and so is its own unique spanning subtree.

Czygrinov et al. [7] proved an extension of Theorem 7. By Theorem 7 given the conditions are satisfied, there exists a spanning tree in G. But Czygrinov et al. expanded this theorem by describing the spanning tree as having three possible properties.

First, let n and k be positive integers and consider a sequence $\delta_1, \delta_2, \ldots, \delta_k$ of positive integers with $\sum_{i=1}^{k} \delta_i = n - 1$. Then let us define the graph, $G(\delta_1, \delta_2, \ldots, \delta_k)$ as being formed by taking k disjoint complete graphs, each of size δ_i for each $i = 1, 2, \ldots, k$ and then attaching a new vertex, x, that is adjacent to all the other vertices in G. (See Figure 2.7). A *caterpillar* is a type of tree, T, when there exists a path P in T so that every vertex of T which is not on the path P is adjacent to a point of P. The theorem by Czygrinov et al. is as follows.

Theorem 8. *Let $k \geq 2$ be an integer and let G be a connected graph on n vertices satisfying:*

$$\sum_{x \in I} d_G(x) \geq n - 1$$

for every k-element independent set $I \subset V(G)$. Then either:

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1. *G has a spanning tree with maximum degree less than k;*

2. *$G = G(\delta_1, \delta_2, \ldots, \delta_k)$ for some sequence $\delta_1, \delta_2, \ldots, \delta_k$ of positive integers with at least three $\delta_i s$ larger than 1; or*

3. *for every maximum length path P in G, there is a spanning tree T of G such that:*

   (a) T is a caterpillar,

   (b) $\Delta(T) = k$,

   (c) the spine of T is the path P, and

   (d) the set $\{v \in V(G) | d_T(v) \geq 3\}$ is independent in T.

In Theorem 8, (1) is the same result as that of Win in Theorem 7. (2) is shown by the graph G in Figure 2.7.
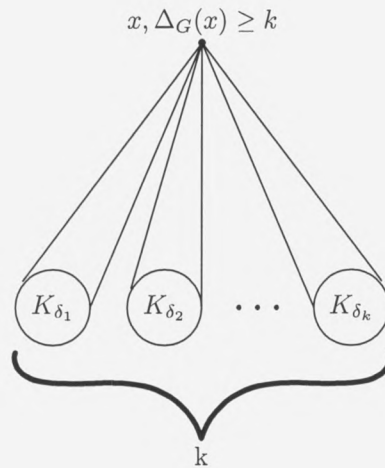
Figure 2.7: Example of Czygrinov et al.'s theorem(2).

If we consider K_{δ_i} for $i = 1, \ldots, k$ separately, each K_{δ_i} is Hamiltonian for each contains the Hamiltonian cycle C_{δ_i}). Therefore, each K_δ for $i = 1, \ldots, k$ also contains a Hamiltonian path. Since x is adjacent to all other vertices in G, there exists a Hamiltonian path with x as the endpoint in each $K_{\delta_i} \cup x$ for $i = 1, \ldots, k$. Thus, if we take the union of each of these Hamiltonian paths, we have a spanning tree, T_* in G, with maximum degree $\Delta_G(T_*) = k$.

Finally, the third possible structure of the spanning tree T_* is described in (3). This is another specific type of spanning tree T_* called a caterpillar with a maximum path P and each vertex, $v \notin P$ being adjacent to a vertex in P. and the maximum degree of

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the spanning tree, T_*, is k. See Figure 2.8 for an example of a caterpillar described in Theorem 8(3).

Figure 2.8: Example of Czygrinov et al.'s theorem(3).

Another generalization of the Ore condition that is related to our work is that of Kyaw [14]. A few definitions needed are as follows. A nonempty set S of independent vertices of G is called *frame* of G if $G - S'$ is connected for any $S' \subseteq S$. If $|S| = k$ then S is called a *k-frame*. For a set $S \subseteq V(G)$, we denote the set of vertices with exactly i neighbors in S by $N_i(S)$, so that $N_i(S) = \{v \in V(G) : |N_G(v) \cap S| = i\}$. The theorem of Kyaw is as follows:

Theorem 9. *Let G be a connected graph and $k \geq 2$ be an integer. If*

$$\sum_{s \in S} d_G(s) + \sum_{i=2}^{k+1} (k - i)|N_i(S)| \geq n - 1$$

for every $(k + 1)$-frame S in G, then G has a k-tree.

The results of this theorem implies Theorem 7. The condition is technical and requires significant investigation in the properties of G in order to find G. Note that if the condition holds for every 3-frame S in G, then G contains a 2-tree or a Hamiltonian path.

So far we have considered theorems where if the given conditions were satisfied, G contained a spanning tree, T_*. In the case of G being Hamiltonian or containing a Hamiltonian path, this results in $\Delta(T_*) = 2$. In the more recently provided theorems, G contains a $\Delta(T_*)$ with a maximum degree of $k \geq 2$. Now we will consider when the set $A \subseteq V(G)$ can be covered by a tree and note that if $A \neq V(G)$, how many trees are necessary to cover $V(G)$.

2.3 Subtrees that cover the vertices of a set A where $A \subseteq V(G)$

We now proceed to theorems where G may not contain a spanning tree T_*. In this case, we will try to determine the maximum number of vertices, $A \subseteq V(G)$, that can be covered

by a single subtree. Note that for $A \neq V(G)$, this is not directly related to our problem but does provide some bounds on the number of trees necessary to cover G for if A can be covered by a tree of maximum degree k, then the maximum number of trees necessary to cover $V(G)$ is less than or equal to $n - |A| + 1$.

Caro, Krasikov, and Roditty [4] provided an extension of Win's theorem that gives a minimum number of vertices covered by a tree in a graph G. The theorem is as follows.

Theorem 10. *Let $k \geq 2$. Then every connected graph G contains a tree T of maximum degree at most k that either spans G or has order at least $k\delta(G) + 1$.*

While Win's theorem has an extra condition, Theorem 10 follows from Win's theorem for given a subtree T in G where $k = \Delta(T)$, k is also the the minimum size of an independent set I in T. Therefore, the minimum number of vertices able to be covered in G is $k\delta(G) + 1$ and there exists a spanning tree in G when $\delta(G) \geq \frac{n-1}{\Delta(T)}$. Although the subtree, T, does not necessarily cover all the vertices of G, the theorem provides a bound on the maximum number of trees necessary to cover G which is $n - (k\delta(G) + 1) + 1 = n - k\delta(G)$.

Egawa and Morimoto [9] provided an extension to the theorems of Ore and Pósa in regards to the degree of non-adjacent vertices in a graph G but a minimum number of vertices that could be covered by a cycle. The theorem is as follows.

Theorem 11. *If G is a graph such that $d_G(x) + d_G(y) \geq \left\lceil \frac{2n}{p} \right\rceil$ for some integer p and any nonadjacent distinct vertices x and y of $V(G)$, then G has a cycle of length at least $\left\lceil \frac{n}{p-1} \right\rceil$.*

Note that when $p = 2$, Theorem 11 is the same as that of Ore. The extension does not explicitly state that the cycle in G is a Hamiltonian but at least of length $\left\lceil \frac{n}{p-1} \right\rceil$. Pósa's theorem states that if the degree sum of nonadjacent vertices is greater than k, then there exists a cycle of at least length $\frac{k+2}{2}$. If we compare this to the conditions of Theorem 11, and let $k = \frac{2n}{p}$, by Pósa, there exists a cycle of length at least $\frac{n}{p} + 1$ compared to a cycle of length at least $\left\lceil \frac{n}{p-1} \right\rceil$. The result of Theorem 11 is that if the conditions are satisfied then the maximum number of spanning trees necessary to cover $V(G)$ is $s \leq n - \left\lceil \frac{n}{p-1} \right\rceil + 1$ with maximum degree is 2.

Bollobás and Brightwell [2] proved an extension of Theorem 11. For a graph G, such that $W \subseteq V(G)$ and t is an integer, a (t, W)-*cycle* is a cycle in G containing at least t vertices of W. Bollobás and Brightwell's theorem is as follows.

Theorem 12. *Let G be a graph on n vertices where $W \subseteq V(G)$, $w = |W|$, $d = \delta_G(W)$. Suppose that $s \geq 2$ and for some integer $l \geq 2$, $w \geq s(l-1) + 1$ and $n \leq dl$. Then G contains an $(s+1, W)$-cycle.*

Let us consider two examples that show this result is best possible given n, w, and d. Let $k = \left\lceil \frac{n}{d} \right\rceil$ and $s = \lceil w/(k-1) \rceil - 1$.

For $s \leq d - 1$, we will show in Figure 2.9 a sharp example of Theorem 12.
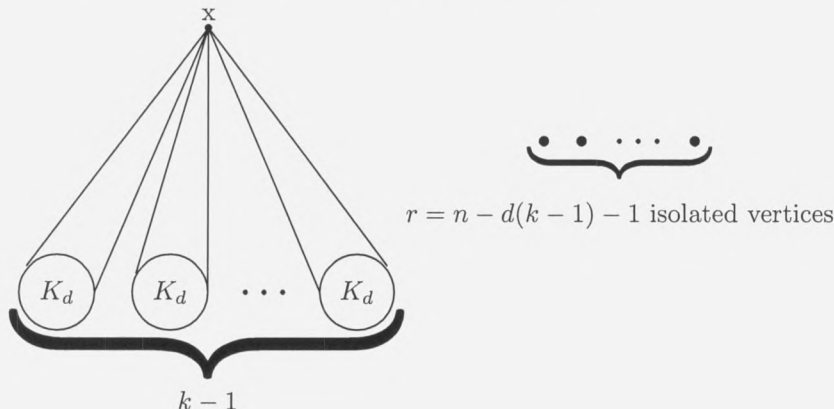

Figure 2.9: Bollobás and Brightwell case for $s \leq d - 1$.

In Figure 2.9, we define G by taking $k-1$ disjoint copies of K_d, all attached to a separate vertex, x, and a set of $r = n - d(k-1) - 1$ isolated vertices where $r \neq 0$. For $s = \left\lceil \frac{w}{(k-1)} \right\rceil - 1$, W is chosen so that it contains at most $s + 1$ vertices from each K_d. Then for $s = d - 1$, $d = s + 1$ and W is the union of $s + 1$ disjoint complete graphs K_d and G contains an $(s + 1, W)$-cycle in one of the K_d's which makes this example sharp.

Let us now consider when $s \geq d$.

In Figure 2.10, we see that every vertex of W has degree at least $s \geq d$. Then for $s = d$, the largest cycle in W is the cycle contained in the complete graph K_{s+1} by $K_s \cup q$ where $q \in Q$. Thus, there is no $(s + 2, W)$-cycle and the largest cycle is an $(s + 1, W)$-cycle which makes the example sharp.

Given this theorem, we note that at least $s + 1$ vertices can be covered by a cycle and thus a path. Therefore the maximum number of trees necessary to cover G is $n - (s+1) + 1 = (n - s)$.

A theorem of Cutler [6] gives a degree condition for existence of a tree through specified vertices. To state the theorem, we need a bit of notation. Namely, for a graph G and a set of
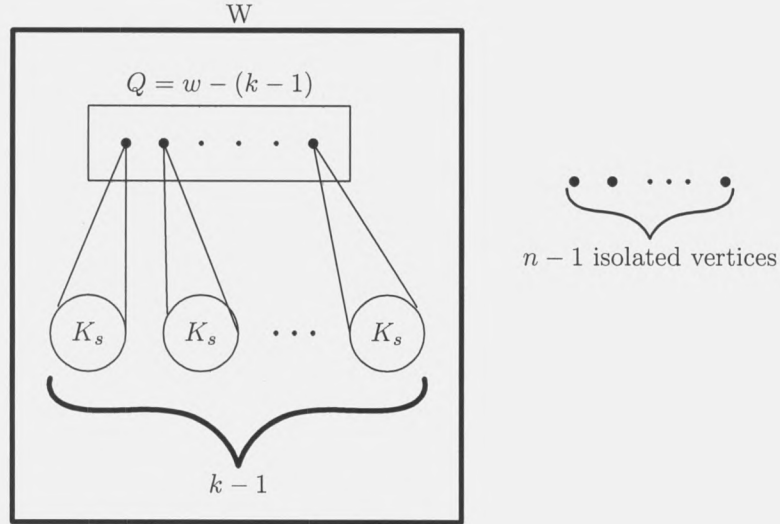
Figure 2.10: Bollobás and Brightwell case for $s \geq d$.

vertices $A \subset V(G)$, we let $\delta_G(A) = \min\{d_G(v) : v \in A\}$ and likewise $\Delta_G(A) = \max\{d_G(v) : v \in A\}$. Cutler's theorem is as follows.

Theorem 13. *Let k be such that $1 \leq k \leq n - 2$. Then, given a connected graph G on n vertices and $A \subseteq V(G)$ such that $\delta_G(A) \geq k$, there exists a subtree T of G such that $A \subseteq V(T)$ and $\Delta_T(A) \leq \lceil \frac{n-1}{k} \rceil$.*

Let us look two different sharp examples of Cutler's theorem the first where $A = V(G)$ and the second where $A \subset V(G)$.

Example. For $1 \leq k \leq n - 2$, let $G = K_{k,n-k}$ and let $A = V(G)$ as in Figure 2.11. Without loss of generality let $k \leq \frac{n}{2}$ ($k \leq n - k$), then $\delta = \delta(G) = \delta_G(A) = k$. Since $A = V(G)$, in order to prevent confusion we will denote everything in terms of G. Let $X \subset V(G)$ be the independent set of vertices of size k and let $Y \subset V(G)$ be the independent set of vertices of size $n - k$ where $X \cup Y = V(G)$. Let each vertex in X be denoted x_i for $i = 1, \ldots, k$ such that $X = \{x_i | i = 1, \ldots, k\}$. Construct subtrees by first distributing the $n - k$ vertices of Y among the vertices of X as evenly as possible. Now we have k disjoint trees, $T_1, \ldots, T_k$ with the maximum degree of the collection of trees being $\lceil \frac{n-k}{k} \rceil = \lceil \frac{n}{k} \rceil - 1$ and $\Delta(T_i) = \Delta_{T_i}(x_i)$ for $i = 1, \ldots, k$. In order to connect the disjoint trees, add an edge between one y vertex in each T_i to $x_{i+1} \in V(T_{i+1})$ for $i = 1, \ldots, k-1$ since the T_k tree will be connected by the edge added from $y \in Tk - 1$. Now we have a spanning tree, T_* with

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$\Delta(T_*) = \left\lceil \frac{n}{k} \right\rceil - 1 + 1 = \left\lceil \frac{n}{k} \right\rceil$. However, note that $d_{T_*}(x_1) = \left\lceil \frac{n}{k} \right\rceil - 1$ since its degree did not increase when connecting the disjoint trees. But Cutler's theorem requires $\Delta(T) \leq \left\lceil \frac{n-1}{k} \right\rceil$. The only time $\Delta(T) \leq \left\lceil \frac{n-1}{k} \right\rceil \neq \left\lceil \frac{n}{k} \right\rceil$ is for $n \equiv 1 \bmod k$. But, if this is the case, since $d_{T_*}(x_1) = \left\lceil \frac{n}{k} \right\rceil - 1$ when constructing T_* let the extra vertex in y be connected to x_i which does not increase $\Delta(T_*)$ and results in $\Delta(T_*) = \left\lceil \frac{n-1}{k} \right\rceil$. So, for the graph G being $K_{k,n-k}$ with $2 \leq k \leq n-2$ and $A = V(G)$, we have shown that there exists a subtree T of G such that $A = V(T)$ and $\Delta_T(A) = \left\lceil \frac{n-1}{k} \right\rceil$ thus making G a sharp example.
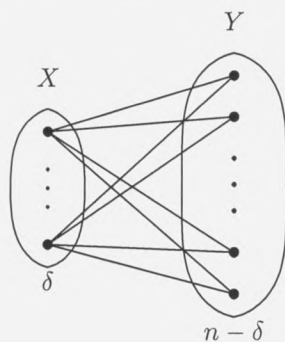

Figure 2.11: Sharp example of Cutler's theorem with $A = V(G)$.

The previous example was introduced not only to illustrate the theorem is sharp and best possible for $A = V(G)$ but also to introduce techniques in the construction of spanning trees that will be utilized later.

Let us look at another sharp example of Cutler's theorem where $A \subsetneq V(G)$. First, per Cutler's theorem, let us consider the following inequality:

$$1 \leq k \leq \delta_G(A)\Delta_T(A) \leq \left\lceil \frac{n-1}{k} \right\rceil \leq n-2.$$

Since all values represent integers we can restate the inequality without the ceiling as

$$1 \leq k \leq \delta_G(A)\Delta_T(A) \leq \frac{n-1}{k} \leq n-2.$$

Then $k \leq \frac{n-1}{k}$ leads to $k \leq \sqrt{n-1}$. Thus, for $k = \sqrt{n-1}$, $k = \delta_G(A) = \Delta_T(A) = \frac{n-1}{k}$. This signifies for a graph G with $A \subset V(G)$ and $\delta_G(A) \geq k$ when $k \leq \sqrt{n-1}$, G is an extremal example of Cutler's theorem. Let us now consider the sharp extremal example where $A \subsetneq G$.

Example. Let there exist a vertex, x, such that $d_G(x) = \Delta(G)$ and $N_G(x)$ is an independent set. Let $A \subset V(G)$ where $A = \{x, N_G(x)\}$ and for each vertex, $y \in N_G(x)$, let $d_G(y) = k$

as in Figure 2.12. We will denote $\hat{N}_G(x)$ or the *neighborhood of the neighborhood of x in G*, as the set of vertices $\hat{N}_G(x) := N_G(N(x)) \setminus [N(x) \cup \{x\}]$. Let $\Delta = \Delta(G) = \Delta(A)$ then $d_G(x) = d_A(x) = \Delta$.
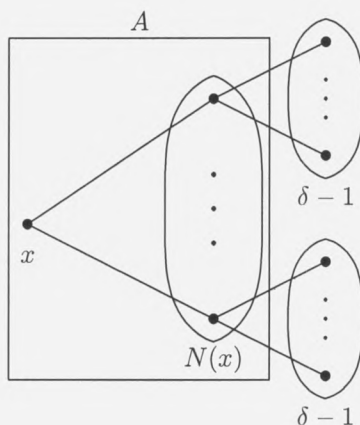
Figure 2.12: Sharp example of Cutler's theorem with $A \neq V(G)$.

If we calculate n, we get the following equalities:

$$n = |x| + |N_G(x)| + |\hat{N}_G(x)|$$
$$= |x| + |N_G(x)| + |N_G(x)|(k-1)$$
$$= 1 + \Delta + \Delta(k-1)$$
$$= 1 + \Delta k.$$

This implies that

$$n - 1 = \Delta k,$$

so that

$$\Delta = \frac{n-1}{k} \geq k,$$

where the last inequality follows from the assumption that $k \leq \sqrt{n-1}$ as showed earlier. Then there exists a subtree T of G such that for $A \subset V(T)$ and in this case $A \neq V(T)$, where $\Delta_T(A) = \frac{n-1}{k}$ when $\Delta = \frac{n-1}{k} = k$ making G a sharp example.

A corollary to this theorem is as follows. The proof of Theorem 13 is built on the same ideas. We include the proof because the proof of Horak and McAvaney's theorem may

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follow the same ingredients. We consider only the case when $k \leq \sqrt{n-1}$ since the proof is much simpler in this case. Let us consider a corollary by Cutler to Cutler's theorem.

Corollary 14. *Let G be a connected graph with $\delta(G) \geq k$ and $1 \leq k \leq \sqrt{n-1}$. Then there exists a spanning tree, T, such that $\Delta(T) \leq \left\lceil \frac{n-1}{k} \right\rceil$.*

Proof. We will be proving Corollary 14 by contradiction. We will suppose that the conditions are satisfied and further that a counterexample exists. If G be a counterexample, then every spanning tree of G has maximum degree at least $\left\lceil \frac{n-1}{k} \right\rceil + 1$. Let $\mathcal{T}$ be the set of spanning trees of G. Then let the subset $\mathcal{T}_\Delta \subseteq \mathcal{T}$ be the set of spanning trees with minimal maximum degree, say Δ, so that, by assumption, $\Delta \geq \left\lceil \frac{n-1}{k} \right\rceil + 1$. We define the subset $\mathcal{T}_\Delta^{\min} \subseteq \mathcal{T}_\Delta$ as the set of spanning trees with the minimum number of vertices of degree Δ. Finally, let $T \in \mathcal{T}_\Delta^{\min}$ and suppose that $x \in V(T)$ with $d_T(x) = \Delta$.

We begin by showing that if u and v are T-neighbors of x and that $uv \in E(G)$, then both u and v must have large T-degree.

Claim. *If $u, v \in N_T(x)$ with $uv \in E(G)$, then $d_T(u), d_T(v) \geq \Delta - 1$.*

Proof of claim. Suppose the contrary. Then there exists $u, v \in N_T(x)$ with $uv \in E(G)$ such that without loss of generality $d_T(u) \leq \Delta - 2$. Since $u, v \in N_T(x)$, $ux, vx \in E(T)$. (Note $uv \notin E(T)$ because then T has cycle.) Then we can remove the edge vx from T and add graph edge uv to T to form a new spanning tree T'. However, this decreases $d_T(x)$ and we have a spanning tree T' where $d_{T'}(x) = \Delta - 1$ and $d_{T'}(u) = d_T(u) + 1 \leq \Delta - 2 + 1 \leq \Delta - 1$ and $d_{T'}(v) = d_T(v)$. This contradicts our assumption that $T \in \mathcal{T}_\Delta^{\min}$ with the minimum number of vertices with minimum maximum degree Δ. Also, note that if x was the only vertex in T with degree Δ, then $\Delta(T') = \Delta - 1$ and $T \notin \mathcal{T}_\Delta$. $\qquad\square$

We use the above claim to show that each vertex in $N_T(x)$ has at least $k - 1$ neighbors outside of $\{x\} \cup N_T(x)$. To do this, we need to show that if $k \leq \sqrt{n-1}$ then $\Delta - 1 \geq k$. First if $k \leq \sqrt{n-1}$ then $k^2 \leq n-1$ and so $k \leq \frac{n-1}{k}$. By definition, $\Delta \geq \left\lceil \frac{n-1}{k} \right\rceil + 1$ and we

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have the following string of inequalities.

$$\Delta - 1 \geq \left\lceil \frac{n-1}{k} \right\rceil + 1 - 1$$

$$= \left\lceil \frac{n-1}{k} \right\rceil$$

$$\geq \frac{n-1}{k}$$

$$\geq k.$$

In order to get to a contradiction, we would like to estimate the size of $\hat{N}_T(x) = N_G(N_T(x)) \setminus (\{x\} \cup N_T(x))$. We know, by the claim above, that any vertex in $N_T(x)$ is adjacent to at least $k-1$ vertices in $\hat{N}_T(x)$. If we could map the edges between $N := N_T(x)$ and $\hat{N} = \hat{N}_T(x)$ by an injection to vertices in $V(G) \setminus (\{x\} \cup N)$, then the following would be true.

$$n \geq |\{x\}| + |N| + |V(G) \setminus (\{x\} \cup N)|$$

$$\geq 1 + \Delta + \Delta(k-1)$$

$$\geq 1 + \Delta k$$

$$\geq 1 + \left(\frac{n-1}{k} + 1 \right) k$$

$$\geq 1 + n - 1 + k$$

$$\geq n + k,$$

a contradiction since $k \geq 1$. Unfortunately, we cannot guarantee that this map will be an injection, but we can keep track of how much we lose for each vertex in $\hat{N}$.

Partition $\hat{N} = S \cup U$, where S represents vertices with two or more G-neighbors in N and U has only one G-neighbor in N. (Note this is in respect to G not T for otherwise T would have a cycle and no longer be a tree.) Our next claim shows that vertices in S must have large T-degree.

Claim. *For each $s \in S$, $d_T(s) \geq \Delta - 1$.*

Proof of claim. Suppose not, so that $d_T(s) < \Delta - 1$. Then since $s \in S$, we know there exists a vertex, $u \in N$, such that $su \in E(G)$ and u does not lie on the unique (s, x)-path in T. Then we can add the edge su to T and delete the edge xu. This results in a contradiction for $d_T(x)$ has decreased which contradicts our choice of T. $\square$

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We would like to show that there are no edges of T inside of S, but cannot assume this is necessarily true of T. However, we can find another tree in $\mathcal{T}_\Delta^{\min}$. Let $B \subseteq N$ be the set of vertices $x \in N$ with $d_T(x) \geq \Delta - 1$. This allows us to note, by our earlier claim, that $G[N \setminus B]$ is empty. We modify G by deleting all edges of $G - T$ between N and $\hat{N}$ incident with vertices of B. Note that all the claims above still hold.

Claim. *There is some tree $T' \in \mathcal{T}_\Delta^{\min}$ such that $T'[S]$ contains no edges.*

Proof of claim. Suppose not and let $T' \notin \mathcal{T}_\Delta^{\min}$ with minimum number of edges in $T'[S]$. Let $s, t \in S$ and $st \in E(T)$. Then, without loss of generality, s has no T-neighbors in N. (Both s and t cannot have T-neighbors, or T would contain a cycle.) Further, s has a G-neighbor in N which is not on the unique (s, x)-path in T, say u. We can then add su to T and delete st. This does not create a new vertex of degree Δ since $u \notin B$ by our above work. This contradicts our assumption that T' had the minimum number of vertices in $T'[S]$. $\square$

Claim. *For each $s \in S$, there exists $\Delta - 1$ unique vertices $G - (\{x\} \cup N \cup U)$*

Proof of claim. Given that $d(s) \geq \Delta - 1$ for all $s \in S$, we can algorithmically go down paths from x and group $\Delta - 1$ vertices in each closed neighborhood (neighborhood that includes the vertex) at each step. $\square$

In order to conclude the proof of the Corollary 14 we must show that $|S| \leq k - 1$, since it is for these vertices that we lose one in our map from edges to vertices. If we try and calculate the lower bound of $n = |V(T)|$ we have the following inequality.

$$n \geq |\{x\}| + |N| + |S|(\Delta - 1)$$
$$\geq 1 + \Delta + |S|(\Delta - 1)$$
$$\geq 1 + \left\lceil \frac{n-1}{k} \right\rceil + |S| \left(\left\lceil \frac{n-1}{k} \right\rceil \right),$$

which implies that

$$|S| \leq \frac{n - \left\lceil \frac{n-1}{k} \right\rceil - 2}{\left\lceil \frac{n-1}{k} \right\rceil}$$

$$= \frac{n-2}{\left\lceil \frac{n-1}{k} \right\rceil} - 1$$

$$\leq \left(\frac{n-2}{n-1} \right) k - 1$$

$$\leq k - 1.$$

Now given we have shown $|S| \leq k - 1$, we can provide a lower bound for n which leads to a contradiction. Note that for each vertex of S, while it may be adjacent to all of N in G, we can find only $\Delta - 1$ vertices to associate with them. Thus, we lose one in our correspondence for each. Thus, we have

$$n \geq |\{x\}| + |N| + |N|(k-1) - |S|$$

$$= 1 + \Delta + \Delta(k-1) - |S|$$

$$\geq 1 + \Delta k - (k-1)$$

$$\geq n + k - (k-1)$$

$$= n + 1,$$

a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

While we have not been able to modify this proof to help prove Conjecture 1, we do believe that this may be possible. Looking at simplified cases such as $k \leq \sqrt{n-1}$ may lead to partial results in this direction.

These results illustrate the different approaches and conditions that have been been proven for if $A = V(G)$ a connected graph G has a spanning tree, T_* with maximum degree k and if $A \leq V(G)$, the maximum number of spanning trees to cover G are $n - |A| + 1$.

Chapter 3

Our research

3.1 Horak and McAvaney

Let us begin by restating Horak and McAvaney's original conjecture.

Conjecture 1. *Let G be a connected graph, and $k \geq 2$. Then the vertices of G can be covered by $s \leq \left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil$ edge-disjoint trees of maximum degree at most k.*

So far we have considered three different groups of theorems. The first were where given conditions of G, G contained a Hamiltonian path that results in G containing a spanning tree, T_*, of maximum degree two. The second group of theorems we considered stated that given conditions on G, G contains a spanning tree, T_*, of maximum degree k. Finally, we considered theorems where given conditions, a set $A \subseteq V(G)$, there exists a tree in G that covers A which leads to the maximum number of trees necessary to cover $V(G)$ as $n - |A| + 1$ of maximum degree k. Note that for the case where we considered G contains a single spanning tree, edge-disjoint is irrelevant. However, when we will be considering more than one tree that spans $V(G)$ we will be considering edge-disjoint trees since this is the conjecture of Horak and McAvaney we have considered. A different set of questions comes up if the trees considered have either no condition on them or are vertex-disjoint.

We will now look at specific cases where given a graph G, a positive integer k, and $\delta = \delta(G)$, $V(G)$ can be covered by $\left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil$ edge-disjoint trees of maximum degree at most k. Horak and McAvaney proved that for $\delta = 1$ and $k \geq 2$, Conjecture 1 was true. If $\delta = 1$ and $k \geq 2$, Horak and McAvaney's conjecture states that for a graph G, $V(G)$ can be covered by $\left\lceil \frac{n-1}{1(k-1)+1} \right\rceil = \left\lceil \frac{n-1}{k} \right\rceil$ edge-disjoint trees of maximum degree k. Note that this

follows directly from Cutler's theorem which showed that G then contains a spanning tree, T_* (T_* is the one edge-disjoint tree). The proof was provided earlier.

Let us consider the graph $K_{\delta,n-\delta}$ and show that $V(G)$ can be covered by $\left\lceil \frac{n-1}{\delta(k-1)+1} \right\rceil$ edge-disjoint trees with maximum degree $k = 2$. For $k = 2$ and $\delta(G) = 1$, by Conjecture 1, $V(G)$ can be spanned by at most $\left\lceil \frac{n-1}{2} \right\rceil$ edge-disjoint trees of maximum degree two. By
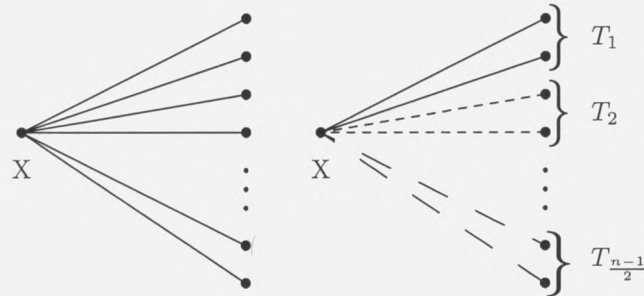

Figure 3.1: $K_{1,n-1}$ and edge-disjoint trees of $K_{1,n-1}$ of maximum degree two.

Figure 3.1, if n is odd, G can be spanned by $\frac{n-1}{2}$ edge-disjoint trees of maximum degree two. If n is even, then there will be an isolated vertex and therefore, G, will be spanned by $\frac{n}{2}$ edge-disjoint trees of maximum degree two with one tree being an isolated vertex.

Let us now consider when $k = 2$ and $\delta \geq 1$.

For $k = 2$ and $\delta \geq 1$, we can find a connected graph, G that can be covered by $s \leq \left\lceil \frac{n-\delta}{\delta(2-1)+1} \right\rceil = \left\lceil \frac{n-\delta}{\delta+1} \right\rceil$ edge-disjoint trees of maximum degree $k = 2$. This follows in the same fashion as the previous example where in this case we will find the maximum length paths and each of these paths will the edge-disjoint trees that cover $V(G)$. Let G be the graph $K_{\delta,n-\delta}$ and $\delta \geq 1$ as in Figure 3.2. Without loss of generality let $\delta < n - \delta$. For $k = 2$ by Figure 3.2 we see the maximum number of edge-disjoint trees necessary to cover $V(G)$ is $\left\lceil \frac{n-\delta}{\delta+1} \right\rceil$.

Theorem 15. *Let k be a positive integer. Every connected graph on n vertices can be covered by $\left\lceil \frac{n-1}{k} \right\rceil$ edge-disjoint trees of maximum degree at most k.*

Also, Horak and McAvaney [12] were able to show the case when $k = 2$ and all $\delta \geq 2$.

Theorem 16. *Every connected graph G can be covered by $\left\lceil \frac{n-\delta}{\delta+1} \right\rceil$ edge-disjoint paths.*
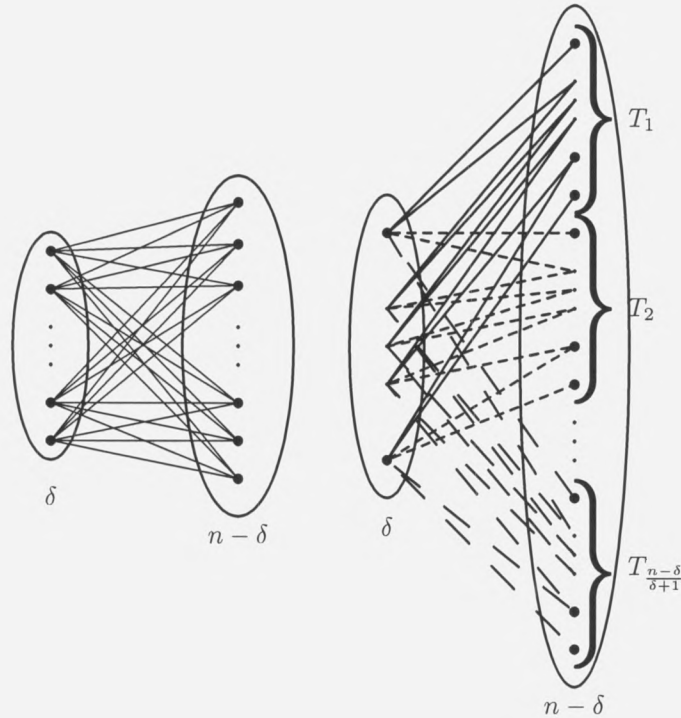
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Figure 3.2: $K_{\delta, n-\delta}$ and edge-disjoint trees of $K_{\delta, n-\delta}$ of maximum degree two.

3.2 Restatement of the conjecture

We slightly shift the scope of investigation by restating Conjecture 1. This allows us to prove some other cases of the conjecture. We begin by noting that Dirac's theorem implies Conjecture 1 for graphs G on $n \geq 3$ with $\delta(G) \geq n/2$, which can also be read out of Theorem 16. To this end, we let $\tau(n, \delta, k) = \left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil$. Thus, Conjecture 1 can be restated: If G is a connected graph, then G can be covered by at most $\tau(n, \delta(G), k)$ edge-disjoint trees of maximum degree at most k. We have the following theorem.

Theorem 17. *Let G be a connected graph on $n \geq 3$ vertices with $\delta(G) \geq n/2$. Then we can cover the vertices with at most $\tau(n, \delta(G), k)$ edge-disjoint trees of maximum degree $2 \leq k$. In particular, G has a Hamiltonian path, i.e., a spanning tree of maximum degree two.*

Proof. We begin by noting that $\tau(n, \delta(G), k) \geq 1$ and so the result follows from Dirac's theorem by deleting an edge from the Hamiltonian cycle in G. $\qquad \square$

The idea behind our restatement of Conjecture 1 is that we split not according to values of δ or k, but rather values of s, the number of trees used to span the vertices of the graph.

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Our restatement follows from a bit of algebra on the relationship between $\tau(n, \delta(G), k)$ and s.

Conjecture 18. *If G is a connected graph on n vertices with $\delta(G) = \delta$, then the vertices of G can be spanned by at most q edge-disjoint trees, each of maximum degree $\left\lceil \frac{n+\delta(q-1)-q}{\delta q} \right\rceil$.*

We now prove the following which shows that our restatement is at least as strong as Horak and McAvaney's conjecture.

Theorem 19. *Conjecture 18 implies Conjecture 1.*

Proof. Assuming Conjecture 18, we know that G can be covered by at most q edge-disjoint trees of maximum degree $\left\lceil \frac{n+\delta(q-1)-q}{\delta q} \right\rceil$. To show that this implies Conjecture 1, we would like to show that if $q = \tau(n, \delta, k)$, then $k \geq \left\lceil \frac{n+\delta(q-1)-q}{\delta q} \right\rceil$, since then the trees that Conjecture 18 would guarantee would also work for Conjecture 1. To this end, let

$$q = \tau(n, \delta, k) = \left\lceil \frac{n-\delta}{\delta(k-1)+1} \right\rceil,$$

and so,

$$q \geq \frac{n-\delta}{\delta(k-1)+1}.$$

Solving for k, we see that

$$k \geq \frac{n + \delta(q-1) - q}{\delta q}.$$

Since k is an integer, we have that

$$k \geq \left\lceil \frac{n + \delta(q-1) - q}{\delta q} \right\rceil.$$

$\square$

This version of the conjecture allows us to derive the case $\tau(n, \delta, k) = 1$ from Theorem 13 which results in the following theorem. This corresponds to the case $s = 1$ in Conjecture 1 and is one advantage of the restatement.

Theorem 20. *Conjecture 18 is true in the case when $q = 1$; i.e., if G is a connected graph on n vertices with $\delta(G) = \delta$, then G has a spanning tree of maximum degree $\left\lceil \frac{n-1}{\delta} \right\rceil$.*

Proof. By Theorem 13, with $A = V(G)$, we know that there is a spanning tree with maximum degree $\left\lceil \frac{n-1}{\delta} \right\rceil$. This is equivalent to Conjecture 18 in the case $q = 1$ since then

$$\left\lceil \frac{n + \delta(q-1) - q}{\delta q} \right\rceil = \left\lceil \frac{n-1}{\delta} \right\rceil.$$

$\square$

3.3 Series-parallel graphs

As of this point, we have had difficulty making progress a proof of Conjecture 18. We have focused our attention on a very specific case of the conjecture, namely when $q = 2$ (the first unsolved case). Further, we have focused on proving the conjecture for the class of graphs known as series-parallel graphs. One reason for doing this is that they have a "tree-like" structure, and so if the conjecture is able to be proved easily, it should be easy in this case. However, we believe that the conjecture is difficult even in this case, which leads us to believe that both Conjecture 1 and Conjecture 18 are difficult.

Series-parallel graphs (SP-graphs) are a type of two-terminal graphs. A *two-terminal graph* is a graph with two distinguished vertices, s and t called source and sink, respectively. The motivation for this terminology comes from having a circuit with a positive side, s, and negative side, t, of a battery. An SP-graph is formed by either identifying the source of one two-terminal graph to the sink of another two-terminal graph (a series connection) or identifying the both sources of two two-terminal graphs and their sinks (a parallel connection). The simplest two-terminal graph we will consider is K_2 with one endpoint the source and the other the sink. It is well-known that if G is an SP-graph and $v \in V(G) \setminus \{s, t\}$, then $\delta_G(v) \geq 2$. In our research, we will only consider simple SP-graphs with no loops and multiple edges.

Since Conjecture 1 has already been proved for graphs G with $\delta(G) = 1$, we need only consider those SP-graphs G with $\delta(G) \geq 2$. Since vertices of degree one in SP-graphs can only occur at the source or the sink, we do not eliminate many SP-graphs by doing this. Since we are interested in Conjecture 18 with $q = 2$, we want to show that if G is an SP-graph, then $V(G)$ can be spanned by 2 edge-disjoint trees, T_1 and T_2, where $\Delta(T_i) \leq \lceil \frac{n}{4} \rceil$ for $i = 1, 2$. Let us state this simplified conjecture precisely.

Conjecture 21. *If G is a series-parallel graph with $\delta(G) = 2$, then there exist two edge-disjoint subtrees T_1 and T_2 such that*

$$V(T_1) \cup V(T_2) = V(G) \quad and \quad \Delta(T_i) \leq \left\lceil \frac{n}{4} \right\rceil \quad for\ i = 1, 2.$$

Let us consider an SP-graph of the form $E_2 \vee E_{n-2}$, which is isomorphic to $K_{2,n-2}$, and was already shown to be extremal in [12]. Note that E_n is a graph on n vertices where the set of edges is empty. That is E_n is a graph of n isolated vertices. (The *join of graphs G*

and H, denoted $G \vee H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}$.) We will introduce notation for a group of SP-graphs that generalize $E_2 \vee E_{n-2}$ as in Figure 3.3. We denote this type of graph as $\mathcal{S}(n; k_1, \ldots, k_\ell)$, which is formed by taking SP-graphs $E_2 \vee E_{k_i}$ for $i = 1, \ldots, \ell$, where one of the vertices in the E_2 is the source and the other the sink, and connecting them in series. Thus, in order for $\mathcal{S}(n; k_1, \ldots, k_\ell)$ to be defined, we need $n = 1 + \sum_{i=1}^{\ell}(k_i + 1)$. So, for graph $E_2 \vee E_{n-2}$, we would denote this graph as $\mathcal{S}(n; n-2)$ since there is only one independent set, k_1. Later we will look at the case where $\ell = 2$.
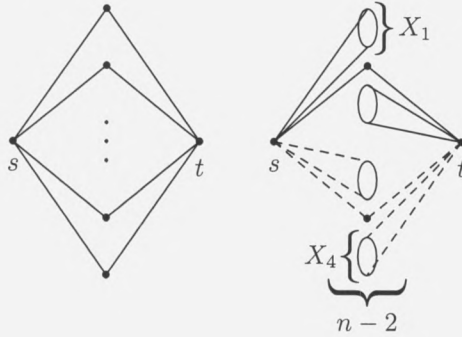

Figure 3.3: $\mathcal{S}(n; n-2)$ and two edge-disjoint trees of $\mathcal{S}(n; n-2)$.

Figure 3.3 shows that G can be spanned by two edge-disjoint trees, T_1 and T_2, where T_1 has solid edges and T_2 dashed edges. For the $n-2$ middle vertices of $\mathcal{S}(n; n-2)$, we reserve two vertices as "connecting" vertices and divide the remaining $n-4$ vertices into four sets as equally as possible. To be precise, divide these $n-4$ vertices into four sets X_1, X_2, X_3, X_4 where $\lfloor \frac{n-4}{4} \rfloor \leq |X_j| \leq \lceil \frac{n-4}{4} \rceil$ for $j = 1, \ldots, 4$. Thus, for either $i = 1$ or $i = 2$, we have that $\Delta(T_i) = \lceil \frac{n-4}{4} \rceil + 1 = \lceil \frac{n}{4} \rceil$ and so this is an extremal example for Conjecture 21.

In fact, since $\mathcal{S}(n; n-2) = K_{2,n-2}$, the fact that this example is extremal was already noted by Horak and McAvaney. Further, if we add an edge between the two terminals, this provides another extremal example which is noted in [12]. We attempted to find an inductive proof of Conjecture 21, but were unable to get it to go through. Another possible method of attack would be to try to make any series-parallel graph "more extremal" via some operation, but we realized that this technique would be difficult as we found that the above two examples are not the only extremal cases. In what follows, we will present further extremal examples.

Let us now consider the series union of two of these types of SP-graphs, which we have

denoted $\mathcal{S}(n; k, n-k-3)$. It is easy to see in Figure 3.4 that as long as $k \geq 2$ and $n \geq 7$, then
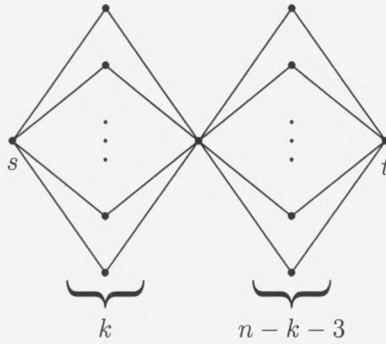

Figure 3.4: $\mathcal{S}(n; k, n - k - 3)$.

$\delta(\mathcal{S}(n; k, n - k - 3)) = 2$. We begin by showing that in a particular case, we can do better than $\left\lceil \frac{n}{4} \right\rceil$, and so this is not an extremal example in general. Suppose that $n - 3$ is even, and consider the graph $\mathcal{S}(n; \frac{n-3}{2}, \frac{n-3}{2})$, so that the two independent sets have the same size. In each of the independent sets of size $\frac{n-3}{2}$, we again reserve two vertices as "connectors" and divide each of the remaining sets of vertices into three parts, as equally as possible. Connect each of the terminal vertices to one of the three sets in each tree and connect the middle vertex to the remaining set on the left in one of the trees and the remaining set to the right in the other tree (see Figure 3.5). In this case, $\Delta(T_i) = \left\lceil \frac{1}{3} \left(\frac{n-3}{2} - 2 \right) \right\rceil + 1 \sim \frac{n}{6}$ when n is large. Thus, this is not an extremal example.
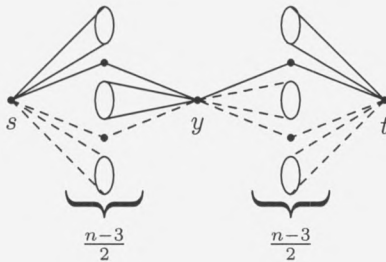

Figure 3.5: T_1 and T_2 of $\mathcal{S}(n; \frac{n-3}{2}, \frac{n-3}{2})$.

However, there are an infinite number of graphs of the form $\mathcal{S}(n; k, n - k - 3)$ that are extremal for Conjecture 21. We found that if $n \equiv 3 \pmod 4$, then $\mathcal{S}(n; 2, n - 5)$ is an extremal example for the conjecture. See Figure 3.6 for a depiction of $\mathcal{S}(n; 2, n - 5)$. Since $n \equiv 3 \pmod 4$, then $n - 7 \equiv 0 \pmod 4$. Again, we reserve two "connector" vertices in
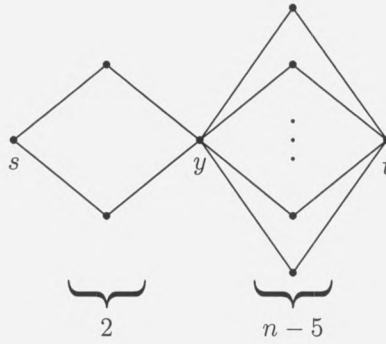
Figure 3.6: $\mathcal{S}(n; 2, n - 5)$.

the independent set of size $n - 5$, and divide the remaining $n - 7$ vertices into four equal parts (which is possible since $n - 7$ is divisible by 4). Then connecting the middle vertex, which we will call y, and t as we did above (see Figure 3.7), and then connecting y to one of vertices in T_1, we get that

$$d_{T_1}(y) = \frac{n - 7}{4} + 2 = \frac{n + 1}{4} = \left\lceil \frac{n}{4} \right\rceil.$$

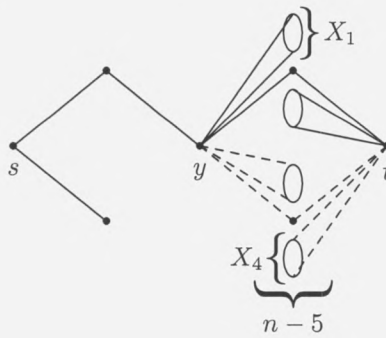The same argument works for $n \equiv 0 \pmod 4$, and so $\mathcal{S}(n; 2, n-5)$ is also always an extremal

Figure 3.7: T_1 and T_2 of $\mathcal{S}(n; 2, n - 5)$.

example for Conjecture 21. The graph of the trees is similar to Figure 3.7. However, the additional vertex in the independent set of size $n - 5$ can be connected to t in either tree.

Here we presented different extremal examples for Conjecture 18 within two different types of SP-graphs both of the form $\mathcal{S}(n; k, n - k - 5)$. This does seem to indicate that the conjecture may be difficult to prove. In our investigations, we found that extending the cycle on the left of $\mathcal{S}(n; 2, n - 5)$ led to further extremal examples, but this only worked

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for very small cycles. Also, an investigation of whether $\mathcal{S}(n; k, \ell, n - k - l - 4)$ is extremal would be an interesting area of future research. This leads us to ask the following question.

Question. Are there extremal examples for Conjecture 21 with arbitrarily large diameter?

3.4   Random graphs

A natural question is whether Conjecture 1 holds for random graphs, as introduced by Erdős and Rényi [10]. Given n and M, we define the *random graph*, denoted $G(n, M)$, as a graph on n vertices and M edges chosen uniformly at random from the set of all graphs on n vertices and M edges. Thus, if H is any graph on n vertices and M edges, we have

$$\mathbb{P}(G(n, M) = H) = \left(\binom{\binom{n}{2}}{M} \right)^{-1}.$$

Of course, if $M = 0$, then $G(n, 0) = E_n$, the empty graph on n vertices, and if $M = \binom{n}{2}$, then $G(n, \binom{n}{2}) = K_n$, the complete graph on n vertices. In order for Conjecture 1 to hold, we need that the random graph is connected. We say that an event, $A(n)$, depending on n occurs *asymptotically almost surely*, or a.a.s., if $\mathbb{P}(A(n)) \to 1$ as $n \to \infty$.

Erdős and Rényi [11] were able to show the following.

Theorem 22. *Let $\omega(n)$ be any function approaching ∞ as $n \to \infty$. Then*

1. If $M(n) = \frac{n}{2}(\log n - \omega(n))$, then $G(n, M(n))$ is a.a.s. disconnected.

2. If $M(n) = \frac{n}{2}(\log n + \omega(n))$, then $G(n, M(n))$ is a.a.s. connected.

In this case, we say that $(n \log n)/2$ is the *threshold* for connectivity in the random graph $G(n, M)$. If the random graph is Hamiltonian, then, as noted above, Conjecture 1 is certainly true, since we can simply delete an edge from the Hamiltonian cycle and are left with a spanning tree of maximum degree two. While Pósa [16] gave a rougher result much earlier, Komlós and Szemerédi [13] and Bollobás [1] were able to show that the threshold for Hamiltonicity occurs only slightly later, which is also when the random graph becomes 2-connected.

Theorem 23. *Let $\omega(n)$ be any function approaching ∞ as $n \to \infty$. Then*

1. If $M(n) = \frac{n}{2}(\log n + \log \log n - \omega(n))$, then $G(n, M(n))$ is a.a.s. not Hamiltonian.

2. If $M(n) = \frac{n}{2}(\log n + \log \log n + \omega(n))$, then $G(n, M(n))$ is a.a.s. Hamiltonian.

Thus, Conjecture 1 becomes trivial as soon as the threshold for Hamiltonicity is reached, which occurs essentially when connectivity is reached. The question of whether the conjecture is true for M such that $(n \log n)/2 \leq M \leq n(\log n + \log \log n)/2$ is perhaps an interesting one, but outside the scope of this thesis.

Chapter 4

Conclusion

In this thesis, we have shown that the idea that one or more trees of a maximum degree span a connected graph has been explored extensively. Here we have expanded upon more recent concepts by Horak and McAvaney. They conjectured a bound for the maximum number of edge-disjoint trees of a maximum degree necessary to cover the vertices of a connected graph. As provided in the background section, we have illustrated sharp examples of Horak and McAvaney's conjecture. We have also formulated our own conjecture that if Horak and McAvaney's conjecture always holds true, we have a bound for the maximum degree of the set of trees. Within this bound, we found extremal examples among different types of series-parallel graphs. In the future, we look to try and prove the validity of our conjecture for all series-parallel graph specifically those with arbitrarily large diameters.

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