A Fractional-Order Predator-Prey Model with Age Structure on Predator and Nonlinear Harvesting on Prey

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ABSTRACT

In this manuscript, the dynamics of a fractional-order predator-prey model with age structure on predator and nonlinear harvesting on prey are studied. The Caputo fractional-order derivative is used as the operator of the model by considering its capability to explain the present state as the impact of all of the previous conditions. Three biological equilibrium points are successfully identified including their existing properties. The local dynamical behaviors around each equilibrium point are investigated by utilizing the Matignon condition along with the linearization process. The numerical simulations are demonstrated not only to show the local stability which confirms all of the previous analytical results but also to show the existence of periodic signal as the impact of the occurrence of Hopf bifurcation.

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1. Introduction

Two species with prey and predator relationships remain a priority consideration of most mathematical scholars in developing ecological modeling. The main basis of this state of affairs lies in the importance of maintaining the availability of biological resources. Since the existence of prey depends on the way they can protect themselves from the presence of a predator and the growth rate of predator stand on the availability of prey as their foods, studying the predator-prey interaction with mathematical modeling growing more and more. Some modifications based on the biological behaviors are integrated to construct a better model. For example, the predator-prey model involving the effect of fear [1–4], the impact of Allee to the existence of prey and predator [5–8], and the exploitation of biological resources by harvesting [9–11].

In this paper, we study the dynamical behaviors of the prey and predator relationship which assume that (i) the predator is divided into two compartments namely the juvenile and adult predators as the impact of the inability of the juvenile predator in hunting prey, and (ii) the harvesting on prey by humans for foods. The first assumption known as age structure which can be seen in [12–14], the second assumption is given by nonlinear (or Michaelis-Menten) harvesting which states the harvesting has a saturation point as in [15–18], and references therein. We also include the memory effect which expresses the impact of all previous biological behaviors on the present condition of the population by applying the fractional-order derivative to replace the conventional first-order derivative as the operator [19–22]. The memory effect state that the population dynamics of the present states depend on all of the previous condition that are saved on their system memory such as their previous experience in finding foods, the best place to get protection, the right time to migrate to another places, and so forth. There are several famous fractional-order derivative used for the operator such as the Riemann-Liouville, Caputo, Caputo-Fabrizio, and Atangana-Baleanu [23–27]. Taking into account the availability of the analytical tools, the Caputo fractional-order the derivative is chosen for the operator of the given model in this paper.

To make this manuscript more structured, we organize the content as follows: the mathematical methods including model formulation and replacing operator are given in Section 2. The Section 3 is provided to explore the dynamical behaviors of the model by considering the biological equilibria, local dynamics, and numerical simulations. We end the discussion of the paper by giving a conclusion in Section 4.

2. Methods

2.1. Model Formulation

The interaction between prey and its predator is modeled by adopting a classical Lotka-Volterra model proposed by Alfred J. Lotka [28] and Vito Volterra [29]. We symbolize the density of predator as $x(t)$ and the density of prey as $P(t)$. Let prey grow logistically following Verhulst model [30] with r is its intrinsic growth rate of prey and K is its environmental carrying capacity. The prey is then hunted by a predator for foods bilinearly with m as the predation rate (or called Holling type-I predator functional response, see [31] and references therein). The birth rate of a predator depends on how much the predation process could supply the foods for breeding with n as the parameter which shows the predation conversion rate to the birth rate of the predator. The density of the predator decreases as the impact of the natural death rate denotes by δ_1. The given assumptions are formulated as

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - mxy, \quad \frac{dP}{dt} = nxP - \delta_1 P.$$

Suppose that the predator is divided into two compartments namely juvenile predator $(y(t))$ and adult predator $(z(t))$. Only adult predator has capability for hunting. Therefore, the following model is obtained.

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - mxz, \quad \frac{dy}{dt} = nxz - \beta y - \delta_1 y, \quad \frac{dz}{dt} = \beta y - \delta_2 z,$$

where β is the transition rate from juvenile to adult predators and δ_2 is the natural death rate of adult predator. Since the adult predator has responsibility for hunting and

providing food for the juvenile predator, we assume that the foods for them are sufficient to satisfy their needs so that competition among adult predators does not exist. Different circumstance occurs among juvenile predators. They compete with each other to get food from the adult predator because of their inability to find their foods and only hope on food brought by adult predators. Obeying the intraspecific competition concept given by Bazykin [32], the model becomes

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - mxz, \\
\frac{dy}{dt} &= nxz - \beta y - \delta_1 y - \omega y^2 \\
\frac{dz}{dt} &= \beta y - \delta_2 z
\end{aligned}
\tag{1}
$$

where ω the death rate of juvenile predator due to competition. The interspecific competition population is succesfully applied in several model as given in [8, 33–35]. In several cases, human intervention in the ecosystem also affects the existence of a population. Here, we assume that prey is harvested following nonlinear type harvesting or known as Michaelis-Menten harvesting. See [17, 36–38] for another example of this type of harvesting. Therefore, we acquire

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - mxz - \frac{hx}{c + x}, \\
\frac{dy}{dt} &= nxz - \beta y - \delta_1 y - \omega y^2 \\
\frac{dz}{dt} &= \beta y - \delta_2 z,
\end{aligned}
\tag{2}
$$

where h is the harvesting rate and c is the half saturation constant of harvesting.

2.2. Model with Caputo Operator

A Caputo operator is a fractional-order derivative which defined by

Definition 1. [23, 24] Let $0 < \alpha \leq 1$. The Caputo fractional derivative with order-α is defined as

$$
{}^C\mathcal{D}_t^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} u'(\tau) d\tau,
\tag{3}
$$

where $t \geq 0$, $u \in C^n([0, +\infty), \mathbb{R})$, and $\Gamma(\cdot)$ is a Gamma Euler function.

Now, we adopt the similar manner using by Panigoro, et al. [5, 19, 20]. The first-first order derivative at the left-hand side of model (2) is replaced by the Caputo fractional-order derivative given in Definition 1. We obtain

$$
\begin{aligned}
{}^C\mathcal{D}_t^\alpha x(t) &= rx\left(1 - \frac{x}{K}\right) - mxz - \frac{hx}{c + x} \\
{}^C\mathcal{D}_t^\alpha y(t) &= nxz - \beta y - \delta_1 y - \omega y^2 \\
{}^C\mathcal{D}_t^\alpha z(t) &= \beta y - \delta_2 z.
\end{aligned}
\tag{4}
$$

When the operator is replaced, the time's dimension at the left-hand side is scaled from t to t^α. The model becomes inconsistent because some parameter such as $r, m, h, n, \beta, \delta_1,$

δ_2, and ω stil have dimension of time t^1 which are different from the other side of model (4). This conditions can be adjusted by rescale all the propitious parameters. Thus the appropriate model is given by

$$
\begin{aligned}
{}^C\mathcal{D}_t^\alpha x(t) &= r^\alpha x \left(1 - \frac{x}{K}\right) - m^\alpha xz - \frac{h^\alpha x}{c+x} \\
{}^C\mathcal{D}_t^\alpha y(t) &= n^\alpha xz - \beta^\alpha y - \delta_1^\alpha y - \omega^\alpha y^2 \\
{}^C\mathcal{D}_t^\alpha z(t) &= \beta^\alpha y - \delta_2^\alpha z.
\end{aligned}
\tag{5}
$$

Let $\bar{r} = r^\alpha$, $\bar{m} = m^\alpha$, $\bar{h} = h^\alpha$, $\bar{n} = n^\alpha$, $\bar{\beta} = \beta^\alpha$, $\bar{\delta}_1 = \delta_1^\alpha$, $\bar{\delta}_2 = \delta_2^\alpha$, and $\bar{\omega} = \omega^\alpha$. Model (5) becomes

$$
\begin{aligned}
{}^C\mathcal{D}_t^\alpha x(t) &= \bar{r}x \left(1 - \frac{x}{K}\right) - \bar{m}xz - \frac{\bar{h}x}{c+x} \\
{}^C\mathcal{D}_t^\alpha y(t) &= \bar{n}xz - \bar{\beta}y - \bar{\delta}_1 y - \bar{\omega}y^2 \\
{}^C\mathcal{D}_t^\alpha z(t) &= \bar{\beta}y - \bar{\delta}_2 z.
\end{aligned}
\tag{6}
$$

For simplicity, the parameters are resymbolized by dropping the bar $^-$ on each parameter. Thus we get the final model as follows.

$$
\begin{aligned}
{}^C\mathcal{D}_t^\alpha x(t) &= rx \left(1 - \frac{x}{K}\right) - mxz - \frac{hx}{c+x} \\
{}^C\mathcal{D}_t^\alpha y(t) &= nxz - \beta y - \delta_1 y - \omega y^2 \\
{}^C\mathcal{D}_t^\alpha z(t) &= \beta y - \delta_2 z.
\end{aligned}
\tag{7}
$$

3. Results and Discussions

In this section, we present the analytical and numerical results including their biological interpretations. We first identify the existence of biological equilibria, investigate the local dynamics, and ended by computing the numerical solutions to show the dynamical behaviors numerically.

3.1. Biological Equilibria

Biological equilibria is basically the equilibrium point of model (7) which exists in $\mathbb{R}_+^3 := \{(x,y,z) : x \geq 0, y \geq 0, z \geq 0, (x,y,z) \in \mathbb{R}^3\}$. Therefore, the following equations are needed to solve.

$$
\begin{aligned}
\left[r\left(1 - \frac{x}{K}\right) - mz - \frac{h}{c+x}\right]x &= 0, \\
nxz - \beta y - \delta_1 y - \omega y^2 &= 0, \\
\beta y - \delta_2 z &= 0.
\end{aligned}
$$

The first equilibrium point is given by $E_0 = (0,0,0)$. This equilibrium point always exists which represents the extinction of both prey, juvenile predator, and adult predator. The second equilibrium point is presented by $E_1 = (\hat{x}, 0, 0)$ which represents the predator-free condition where $\hat{x}$ is the positive root of a quadratic equation as follows.

$$
x^2 + (c - K)x + \left(\frac{h}{r} - c\right)K = 0
\tag{8}
$$

The existence of E_1 is given by the following theorem.

Theorem 1. *Suppose that*

$$h^* = \frac{(c+K)^2 r}{4K},$$

$$\hat{x}_1 = \frac{K-c}{2} + \sqrt{(h^* - h)\frac{K}{r}},$$

$$\hat{x}_2 = \frac{K-c}{2} - \sqrt{(h^* - h)\frac{K}{r}}.$$

(i) *If $h > h^*$ then E_1 does not exist.*

(ii) *If $h = h^*$ and*

    (ii.a) *$K > c$ then a unique E_1 exist given by $E_1 = \left(\frac{K-c}{2}, 0, 0\right)$, or;*

    (ii.b) *$K < c$ then E_1 does not exist.*

(iii) *If $h < h^*$ and*

    (iii.a) *$c < \min\left\{\frac{h}{r}, k\right\}$ then there exists two equilibrium points $E_{11} = (\hat{x}_1, 0, 0)$ and $E_{12} = (\hat{x}_2, 0, 0)$, or;*

    (iii.b) *$c > \frac{h}{r}$ then there exist a single $E_1 = (\hat{x}_1, 0, 0)$, or;*

    (iii.c) *$K < c < \frac{h}{r}$ then E_1 does not exist.*

Proof. Note that $E_1 \in \mathbb{R}^3_+$ if $\hat{x} > 0$ or the root of eq. (8) respect to x is positive. By simple computation gives $\hat{x}_1$ and $\hat{x}_2$ are the roots of eq. (8). Before the positive root is recognized, the existence of real solution is then confirmed. It is clear that this situation depends on the value of h. When $h > h^*$, both roots are complex conjugate numbers and hence E_1 does not exist. For $h = h^*$, $\hat{x}_1 = \hat{x}_2 = \frac{K-c}{2}$ which is positive when $K > c$. Thus, only $K > c$ gives a unique E_1. When $h < h^*$, $\hat{x}_1$ and $\hat{x}_2$ are real numbers. If $c < \min\left\{\frac{h}{r}, k\right\}$ then both roots are positive numbers, if $c > \frac{h}{r}$ then only $\hat{x}_1$ is positive, and if $K < c < \frac{h}{r}$ then $\hat{x}_i < 0$, $i = 1, 2$. The existence of E_1 are completely explored. $\qquad\square$

At the last of this subsection, the third equilibrium point lie on the interior is observed. This equilibrium is given by $E^* = (x^*, y^*, z^*)$ which define the existence of all populations given by $x^* = \frac{(\beta + \delta_1 + \omega y^*)\delta_2}{\beta n}$, $z^* = \frac{\beta y^*}{\delta_2}$, and y^* is positive root respect to y of a quadratic equation $a_1 y^2 + a_2 y + a_3 = 0$ where

$$a_1 = \left(\frac{\omega \delta_2 r}{\beta n}\right)^2 + \frac{\omega m K}{n}$$

$$a_2 = (\beta + \delta_1)\frac{mK}{n} + \frac{\beta c m K}{\delta_2} + (c - K)\frac{\omega \delta_2 r}{\beta n}$$

$$a_3 = hK + \left(1 + \frac{\delta_1}{\beta}\right)(c - K)\frac{\delta_2}{n} + (\beta^2 + \delta_1^2)\frac{r\delta_2^2}{(\beta n)^2} - crK$$

$$x^* = \frac{(\beta + \delta_1 + \omega y*)\delta_2}{\beta n}$$

$$y^* = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}$$

$$z^* = \frac{\beta y*}{\delta_2}$$

3.2. Local Dynamics

In this subsection, the dynamical behavior around each equilibrium point is studied. The Matignon condition [39] is utilized after the Jacobian matrix is evaluated. As results, Theorems 2 to 4 are successfully constructed to describe the local stability of all biological equilibria. Two possible local dynamics are provided here namely locally asymptotically stable (LASEP) equilibrium point and saddle equilibrium point (SEP).

Theorem 2. $E_0 = (0, 0, 0)$ *LASEP if* $r < \dfrac{h}{c}$, *otherwise a SEP.*

Proof. The following Jacobian matrix are achieved as a proceeding of computes it at $E_0 = (0, 0, 0)$.

$$J(x, y, z)|_{E_0} = \begin{bmatrix} r - \dfrac{h}{c} & 0 & 0 \\ 0 & -(\beta + \delta_1) & 0 \\ 0 & \beta & -\delta_2 \end{bmatrix}.$$

Thus, we have three eigenvalues $\lambda_1 = r - \dfrac{h}{c}$, $\lambda_2 = -(\beta + \delta_1)$, and $\lambda_3 = -\delta_2$. Since $\lambda_i < 0$, $i = 2, 3$, we obtain $|\arg(\lambda_i)| > \dfrac{\alpha \pi}{2}$, $i = 2, 3$. Accordingly, the stability of E_0 determined by the value of λ_1 which gives $|\arg(\lambda_1)| > \dfrac{\alpha \pi}{2}$ if $r < \dfrac{h}{c}$ and $|\arg(\lambda_1)| < \dfrac{\alpha \pi}{2}$ if $r > \dfrac{h}{c}$. obeying Matignon condition [39], the LASEP and SEP requirements given by Theorem 2 are completely verificated. $\square$

Theorem 3. *If* $h < \dfrac{(c + \hat{x})^2 r}{K}$ *and* $\hat{x} < \dfrac{(\beta + \delta_1)\delta_2}{\beta n}$ *then* E_1 *is LASEP. The SEP condition of* E_1 *occurs when* h *or* $\hat{x}$ *has the opposite sign.*

Proof. The Jacobian matrix at $E_1 = (\hat{x}, 0, 0)$ is

$$J(E_1) = \begin{bmatrix} \left(\dfrac{h}{(c + \hat{x})^2} - \dfrac{r}{K}\right)\hat{x} & 0 & -m\hat{x} \\ 0 & -(\beta + \delta_1) & n\hat{x} \\ 0 & \beta & -\delta_2 \end{bmatrix},$$

which gives an eigen values $\lambda_1 = \left(\dfrac{h}{(c + \hat{x})^2} - \dfrac{r}{K}\right)\hat{x}$ and a pair of eigen values $\lambda_{2,3}$ obtained from quadratic polynomial characteristic as follows.

$$\lambda^2 + (\beta + \delta_1 + \delta_2)\lambda + (\beta + \delta_1)\delta_2 - \beta n \hat{x} = 0. \tag{9}$$

Equation (9) gives a pair eigenvalues as follows.

$$\lambda_2 = -\frac{1}{2}\left[\beta + \delta_1 + \delta_2 - \sqrt{(\beta + \delta_1 - \delta_2)^2 + 4\beta n\hat{x}}\right],$$

$$\lambda_3 = -\frac{1}{2}\left[\beta + \delta_1 + \delta_2 + \sqrt{(\beta + \delta_1 - \delta_2)^2 + 4\beta n\hat{x}}\right].$$

It could be clarified that $\lambda_i \in \mathbb{R}$, $i = 2,3$. Moreover, $\lambda_3 < 0$ and hence $|\arg(\lambda_3)| > \frac{\alpha\pi}{2}$. Thus, the stability is confirmed from the argument of λ_1 and λ_2. If $h < \frac{(c+\hat{x})^2 r}{K}$, then we have negative sign of λ_1 which impact $|\arg(\lambda_1)| > \frac{\alpha\pi}{2}$. The opposite condition occurs for the diferent sign of h. If $\hat{x} < \frac{(\beta + \delta_1)\delta_2}{\beta n}$ then $\lambda_2 < 0$ and hence $|\arg(\lambda_2)| > \frac{\alpha\pi}{2}$. The different values of λ_2 given by $\hat{x} > \frac{(\beta + \delta_1)\delta_2}{\beta n}$. All possible dynamics are shown when the Matignon condition is applied [39]. This ends proof. $\square$

Theorem 4. *Suppose that*

$$\xi_1 = \delta_1 + \delta_2 + \beta + 2\omega y^* + \left(\frac{c}{c + x^*} - 1\right)\frac{h}{c + x^*}$$

$$\xi_2 = \delta_1\delta_2 + \beta\delta_2 + 2\omega\delta_2 y^* + (\delta_1 + \beta + 2\omega y^* + \delta_2)\frac{ch}{(c+x)^2}$$

$$- (\delta_1 + \beta + 2\omega\delta_2 y^* + \delta_2)\frac{h}{c + x^*} - \beta nx^*$$

$$\xi_3 = \beta mnx^*z^* + (\beta nx^* - \delta_1\delta_2 - \delta_2 x^* - 2\omega\delta_2 y^*)\frac{h}{c + x^*}$$

$$+ (\delta_1\delta_2 + \delta_2 x^* + 2\delta_2\omega y - \beta nx^*)\frac{ch}{(c + x^*)^2}$$

$$\Delta = 18\xi_1\xi_2\xi_3 + (\xi_1\xi_2)^2 - 4\xi_3\xi_1^3 - 4\xi_2^3 - 27\xi_3^2.$$

LASEP condition is satisfied by $E^* = (x^*, y^*, z^*)$ *if one of the following statements hold.*

(i) $\Delta > 0$, $\xi_1 > 0$, $\xi_3 > 0$, and $\xi_1\xi_2 > \xi_3$ or;
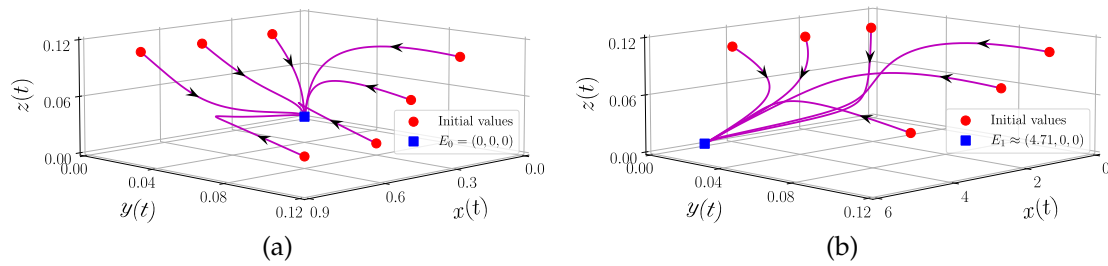
(ii) $\Delta < 0$, $\xi_1 \geq 0$, $\xi_2 \geq 0$, $\xi_3 > 0$ and $\alpha < \frac{2}{3}$ or;

(iii) $\Delta < 0$, $\xi_1 < 0$, $\xi_2 < 0$ and $\alpha > \frac{2}{3}$ or;

(iv) $\Delta < 0$, $\xi_1 > 0$, $\xi_2 > 0$, $\xi_1\xi_2 = \xi_3$ for all $0 < \alpha < 1$.

Proof. Computing the Jacobian matrix evaluated at E^*, we get

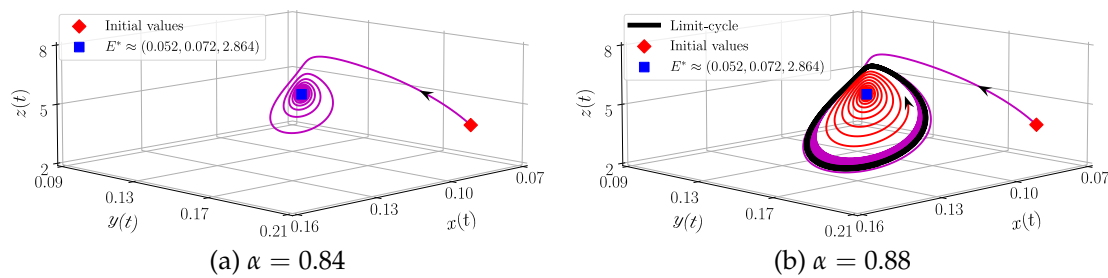$$J(E^*) = \begin{bmatrix} \left(1 - \frac{c}{c + x^*}\right)\frac{h}{c + x^*} & 0 & -mx^* \\ nz^* & -(-\delta_1 + \beta + 2\omega y^*) & nx^* \\ 0 & \beta & -\delta_2 \end{bmatrix},$$

which gives a polynomical characteristic equation a follows.

$$\lambda^3 + \xi_1\lambda^2 + \xi_2\lambda + \xi_3 = 0. \tag{10}$$

Figure 1. Phase portrait of model (7) around E_0 and E_1 with parameter values are given by (a) eq. (11) (b) eq. (12)

Figure 2. Phase portrait of model (7) around E_0 with parameter values are given by eq. (13)

Applying Proposition 1 in Ahmed, et al. [40], all statements given in Theorem 4 can be validly confirmed. □

3.3. Numerical Simulations

In this section, we apply a generalized predictor-corrector numerical scheme for fractional-order differential equation given by Diethelm, et al. [41] to investigate the local dynamics around equilibrium points. The numerical schemes are given by the following equations.

$$x_h(t_{n+1}) = x(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} F_1\left(x_h^P(t_{n+1}), y_h^P(t_{n+1}), z_h^P(t_{n+1})\right)$$
$$+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} F_1\left(x_h(t_j), y_h(t_j), z_h(t_j)\right),$$

$$y_h(t_{n+1}) = y(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} F_2\left(x_h^P(t_{n+1}), y_h^P(t_{n+1}), z_h^P(t_{n+1})\right)$$
$$+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} F_2\left(x_h(t_j), y_h(t_j), z_h(t_j)\right),$$

$$z_h(t_{n+1}) = z(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} F_3\left(x_h^P(t_{n+1}), y_h^P(t_{n+1}), z_h^P(t_{n+1})\right)$$
$$+ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} F_3\left(x_h(t_j), y_h(t_j), z_h(t_j)\right),$$

$$x_h^P(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_1\left(x_h(t_j), y_h(t_j), z_h(t_j)\right),$$

$$y_h^P(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_2\left(x_h(t_j), y_h(t_j), z_h(t_j)\right),$$

$$z_h^P(t_{n+1}) = z(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} F_3\left(x_h(t_j), y_h(t_j), z_h(t_j)\right).$$

where x_h, y_h, and z_h are the corrector schemes for prey, juvenile predator, and adult predator, respectively; x_h^P, y_h^P, and z_h^P are respectively the predictor schemes for prey, juvenile predator, and adult predator; h is the step-size; and

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{jika } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{jika } 1 \leq j \leq n, \\ 1, & \text{jika } j = n+1, \end{cases}$$

$$b_{j,n+1} = \frac{h}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right).$$

Now, we start our first simulation by setting the parameter values as follows.

$$r = 0.1,\ k = 5,\ m = 0.25,\ h = 0.1,\ c = 0.5,\ n = 0.01,\ \beta = 0.06,$$
$$\delta_1 = 0.05,\ \omega = 0.1,\ \delta_2 = 0.05,\ \text{and}\ \alpha = 0.95. \quad (11)$$

It is clear from analytical results that Theorem 2 is satisfied and E_0 is LASEP. By numerical simulation, we portray local dynamics as a phase portrait in Figure 1(a). All nearby solutions converge to E_0 which states for low population densities of both prey and predator effect cause their extinction as time goes on.

For the second simulation, the parameters are setted as follows.

$$r = 0.1,\ k = 5,\ m = 0.25,\ h = 0.03,\ c = 0.5,\ n = 0.01,\ \beta = 0.06,$$
$$\delta_1 = 0.05,\ \omega = 0.1,\ \delta_2 = 0.05,\ \text{and}\ \alpha = 0.95. \quad (12)$$

If we choose the parameter as in eq. (12), E_1 becomes LASEP and this confirm Theorem 3. As result, we have a phase portrait given by Figure 2(b). All nearby solutions converge to E_1 as $t \to \infty$. The population of both juvenile and adult predators will become extinct and on the other hand, the prey success to maintain its existence.

we give the following parameter values as our last numerical simulations.

$$r = 0.8,\ k = 5,\ m = 0.25,\ h = 0.01,\ c = 0.08,\ n = 0.2,\ \beta = 0.4,$$
$$\delta_1 = 0.01,\ \omega = 0.1,\ \text{and}\ \delta_2 = 0.01. \quad (13)$$

According to Theorem 4(iii), the stability of E_* depends on the values of α. We set $\alpha = 0.84$ and 0.88 and plot in into Figure 2(a) and 2(b). When $\alpha = 0.84$, we have a LASEP E^* which confirm Theorem 4(iii). The stability of E^* vanishes and is replaced by unstable E^* without changing its value when $\alpha = 0.88$. The interesting dynamics show around E^* where all nearby solutions converge to a periodic signal namely Hopf bifurcation. See [19, 20] and references therein for further information about this bifurcation. From a biological point of view, this phenomenon shows the existence of both populations in another way. Both populations' density changes periodically around their interior point. This means, that by changing the order of the derivative, the prey and predator change

their behavior to maintain their existence.
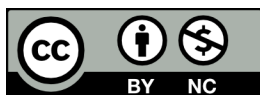
4. Conclusion

A fractional-order predator-prey model with age structure on predator and nonlinear harvesting on prey has been studied. The dynamical behaviors including the existence of biological equilibria and local asymptotic stability have been investigated. Three possible equilibrium points have been found namely the origin, axial point, and the interior point are identified in both their existence conditions and local asymptotic stability. Some numerical simulations have been provided to support analytical findings. The existence of Hopf bifurcation has been shown numerically which indicates that there exists a condition where the interior point is unstable but the existence of three populations is still maintained due to the occurrence of a periodic signal called limit-cycle. From all analytical and numerical results, we also conclude that it is impossible for prey to extinct when the predator exists. This makes sense when we take note of the model assumes that the prey is the only food resource for the predator.

Reference

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