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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL RELAXATION INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. The aim of this paper is to study the existence and uniqueness of solutions for nonlinear fractional relaxation integro-differential equations with boundary conditions. Some results about the existence and uniqueness of solutions have been established by using the Banach contraction mapping principle and the Schauder fixed point theorem. An example is provided which illustrates the theoretical results.
Key words: Fractional relaxation integro-differential equations, Riemann-Liouville fractional derivative, Liouville-Caputo fractional derivative, existence, uniqueness, fixed point.

1. Introduction

Fractional differential equations have many applications in different problems and phenomenons in science and engineering, see [1]-[16], [18]-[20].

In [10], Chidouh, Guezane-Lakoud and Bebbouchi studied the existence and uniqueness of positive solutions of the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{LC}D^{\alpha}x\left(t\right)+\lambda x\left(t\right)=f\left(t,x\left(t\right)\right), \ 0<t\leq 1, \\ x\left(0\right)=x_0>0, \end{cases}$$

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where ${}^{LC}D^{\alpha}$ is the Liouville-Caputo fractional derivative, $0 < \alpha \leq 1$. By using the method of the upper and lower solutions and the Schauder and Banach fixed point theorems, the existence and uniqueness of solutions have been established.

In [11], Guezane Lakoud, Khaldi and Kilicman discussed the existence of solutions for the following nonlinear differential equation with boundary conditions

$$\begin{cases} {}^{LC}D_{1-}^{\alpha} D_{0+}^{\beta} x(t) = f(t, x(t)), \ t \in (0,1), \\ x(0) = x'(0) = x(1) = 0, \end{cases}$$

where ${}^{LC}D_{1-}^{\alpha}$ and D_{0+}^{β} are the right Liouville-Caputo and the left Riemann-Liouville fractional derivatives respectively, $0 < \alpha \leq 1$, $1 < \beta \leq 2$. By employing the Krasnoselskii fixed point theorem, the authors obtained existence results.

In [2], Abdo, Wahash and Panchat investigated the existence and uniqueness of positive solutions of the following nonlinear fractional differential equation with integral boundary conditions

$$\begin{cases} {}^{LC}D^{\alpha} x(t) = f(t, x(t)), \ 0 < t \leq T, \\ x(0) = a \int_0^T x(s)\, ds + b, \end{cases}$$

where $1 < \alpha < 1$. By applying the method of the upper and lower solutions and the Schauder and Banach fixed point theorems, the existence and uniqueness of solutions have been provided.

Inspired and motivated by the works mentioned above, by using the Banach and Schauder fixed point theorems, we study the existence and uniqueness of solutions for the following nonlinear fractional relaxation integro-differential equation

$$\begin{cases} D^{\beta}\, {}^{LC}D^{\alpha} x(t) + \lambda x(t) = f(t, x(t), I^{\gamma} x(t)), \ t \in (0,T), \ \lambda \in \mathbb{R}, \\ {}^{LC}D^{\alpha} x(0) = {}^{LC}D^{\alpha}\, x(T) = 0, \ x(0) = a \int_0^T x(s)\, ds + b, \ a, b \in \mathbb{R}, \end{cases} \tag{1.1}$$

where D^{β} and ${}^{LC}D^{\alpha}$ are the Riemann-Liouville fractional derivative and Liouville-Caputo fractional derivative of orders β and α respectively, $1 < \beta < 2$, $0 < \alpha < 1$, I^{γ} is the Riemann-Liouville fractional integral of order $\gamma \in (0,1)$, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function.

The remaining part of the paper is organized in four sections. In Section 2, some notations, definitions of fractional calculus and fixed point theorems are presented. In Section 3, some useful results about the existence and uniqueness of nonlinear fractional relaxation integro-differential equations are obtained. In Section 4, an example is provided to illustrate the theoretical results.

2. Preliminaries

Some definitions, notations and results of the fractional calculus are introduced throughout this section which will be utilized in this paper.

Let $J = [0, T]$. Denote by $\mathcal{C} = C(J)$ the Banach space of all continuous functions defined on J endowed with the norm

$$\|x\| = \sup \{|x(t)| : t \in J\}.$$

And $AC(J)$ is the space of absolutely continuous valued functions from J into $\mathbb{R}$, and set

$$AC^{m}(J)=\left\{x:J\rightarrow\mathbb{R}:x,x',x'',\quad ,x^{m-1}\in\mathcal{C}\text{ and }x^{m-1}\in AC(J)\right\}.$$

Now we're giving out some fractional calculus results and properties.

Definition 2.1. [14] The fractional integral of order $\alpha>0$ of a function $h:J\rightarrow\mathbb{R}$ is defined by

$$I^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}h(s)\,ds,$$

provided the integral exists.

Definition 2.2. [14] The Liouville-Caputo fractional derivative of order $\alpha>0$ of function $h:J\rightarrow\mathbb{R}$ is defined by

$${}^{LC}D^{\alpha}h(t)=D^{\alpha}\left[h(t)-\sum_{k=0}^{m-1}\frac{h^{(k)}(0)}{k!}t^{k}\right],$$

where

(2.1) $$m=[\alpha]+1\text{ for }\alpha\notin\mathbb{N}_0,\ m=\alpha\text{ for }\alpha\in\mathbb{N}_0,$$

and D^{α}_{0+} is a fractional derivative in Riemann-Liouville sense of order α given by

$$D^{\alpha}h(t)=D^{m}I^{m-\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{m}}{dt^{m}}\int_{0}^{t}(t-s)^{m-\alpha-1}h(s)\,ds.$$

The Liouville-Caputo fractional derivative ${}^{LC}D^{\alpha}_{0+}$ exists for x belonging to $AC^{m}(J)$. In this case, it is defined by

$${}^{LC}D^{\alpha}h(t)=I^{m-\alpha}x^{(m)}(t)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{m-\alpha-1}h^{(m)}(s)\,ds.$$

Remark that when $\alpha=m$, we get ${}^{LC}D^{\alpha}h(t)=h^{(m)}(t)$.

Lemma 2.1. [14] *Let $\alpha>0$ and m be given by* (2.1)*. If $h\in AC^{m}(J,\mathbb{R})$, then*

$$\left(I^{\alpha\,LC}D^{\alpha}h\right)(t)=h(t)-\sum_{k=0}^{m-1}\frac{h^{(k)}(0)}{k!}t^{k},$$

where $h^{(k)}$ is the usual derivative of h of order k.

Lemma 2.2. [14] *For $\alpha > 0$ and m be given by (2.1), then the Liouville-Caputo fractional differential equation ${}^{LC}D^{\alpha}h(t) = 0$ has a general solution*

$$h(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{m-1} t^{m-1},$$

where $a_i \in \mathbb{R}$, $i = 0, 1, 2, ..., m-1$. Further, the Riemann-Liouville fractional differential equation

$$D^{\alpha} h(t) = 0,$$

has a general solution

$$h(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + a_3 t^{\alpha-3} + ... + a_m t^{\alpha-m},\ a_i \in \mathbb{R},\ i = 1, 2, ..., m.$$

Lemma 2.3. [14] *For any $\alpha, \beta \in [0, \infty)$ and $\mu > -1$, then*

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\beta-1} s^{\alpha-1} ds = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}.$$

Lemma 2.4. (Banach fixed point theorem [17]) *Let Ω be a nonempty closed convex subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Lemma 2.5. (Schauder fixed point theorem [17]) *Let Ω be a nonempty bounded closed convex subset of a Banach space S and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then has a fixed point in Ω.*

To obtain our results, we need the following lemma.

Lemma 2.6. *For any $h \in C(J)$, then the problem*

$$\left\{ \begin{array}{l} D^{\beta}\ {}^{LC}D^{\alpha} x(t) + \lambda x(t) = h(t),\ t \in (0, T),\ \lambda \in \mathbb{R}, \\ {}^{LC}D^{\alpha} x(0) = {}^{LC}D^{\alpha}\, x(T) = 0,\ x(0) = a \int_0^T x(s)\, ds + b,\ a, b \in \mathbb{R}, \end{array} \right. \tag{2.2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= I^{\alpha+\beta} h(t) - \lambda I^{\alpha+\beta} x(t) - \frac{t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(I^{\beta} h(T) - \lambda I^{\beta} x(T) \right) \\ &\quad + a \int_0^T x(s)\, ds + b \\ &= \frac{1}{\Gamma(\alpha+\beta)} \left(\int_0^t (t-s)^{\alpha+\beta-1} h(s)\, ds - \lambda \int_0^t (t-s)^{\alpha+\beta-1} x(s)\, ds \right) \\ &\quad - \frac{t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\int_0^T (T-s)^{\beta-1} h(s)\, ds - \lambda \int_0^T (T-s)^{\beta-1} x(s)\, ds \right) \\ &\quad + a \int_0^T x(s)\, ds + b. \end{aligned} \tag{2.3}$$

Proof. Taking the integrator operator I^{β} to the first equation of (2.2), and from Lemma 2.2, we get

$$^{LC}D^{\alpha}x(t) = I^{\beta}h(t) - \lambda I^{\beta}x(t) + a_1 t^{\beta-1} + a_2 t^{\beta-2}. \tag{2.4}$$

According to conditions $^{LC}D^{\alpha}x(0) = {}^{LC}D^{\alpha}\, x(T) = 0$, it yields

$$a_1 = \frac{1}{T^{\beta-1}}\left(\lambda I^{\beta}x(T) - I^{\beta}h(T)\right),\ a_2 = 0.$$

Replacing a_1 and a_2 by their values in (2.4), we get

$$^{LC}D^{\alpha}x(t) = I^{\beta}h(t) - \lambda I^{\beta}x(t) + \frac{t^{\beta-1}}{T^{\beta-1}}\left(\lambda I^{\beta}x(T) - I^{\beta}h(T)\right).$$

Taking the integrator operator I^{α} again to the above equation and using Lemmas 2.2 and 2.3 we obtain

$$x(t) = I^{\alpha+\beta}h(t) - \lambda I^{\alpha+\beta}x(t) - \frac{\Gamma(\beta)\, t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(I^{\beta}h(T) - \lambda I^{\beta}x(T)\right) + a_3. \tag{2.5}$$

Using the integral condition, we find

$$a_3 = a\int_0^T x(s)\, ds + b.$$

Substituting the value of a_3 into (2.5), we obtain the integral equation (2.3). The reverse is followed by a direct calculation which finishes the proof. □

3. Main results

In the following we employ fixed point theorems to prove existence and uniqueness results for the problem (1.1).

For obtaining our results, we need the following hypotheses

(H1) There exist constants $l_1, l_2 > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le l_1|x_1 - x_2| + l_2|y_1 - y_2|,$$

for any $t \in J$ and each $x_i, y_i \in \mathbb{R}$, $i = 1, 2$.

(H2) There exists a function $\Psi \in L^1(J, \mathbb{R}^+)$ such that

$$|f(t, x, y)| \le \Psi(t),\ \forall (t, x, y) \in J \times \mathbb{R} \times \mathbb{R}.$$

3.1. Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.1. *Let* (H1) *holds. If*

$$\theta = \left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{2\beta+\alpha-1}}{\beta T^{\beta-1}\Gamma(\beta+\alpha)}\right)\left(l_1 + l_2\frac{T^{\eta}}{\Gamma(\eta+1)} + |\lambda|\right) + |a|\, T < 1, \tag{3.1}$$

then (1.1) *has at least one solution.*

Proof. We convert the problem (1.1) into a fixed point problem by defining the operator $\Phi : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}(\Phi x)(t) &= \frac{1}{\Gamma(\alpha+\beta)}\left(\int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s), I^\gamma x(s))\, ds\right. \\ &\left. -\lambda \int_0^t (t-s)^{\alpha+\beta-1} x(s)\, ds\right) \\ &-\frac{t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(\int_0^T (T-s)^{\beta-1} f(s, x(s), I^\gamma x(s))\, ds\right. \\ &\left. -\lambda \int_0^T (T-s)^{\beta-1} x(s)\, ds\right) + a\int_0^T x(s)\, ds + b. \end{aligned} \tag{3.2}$$

Obviously, the fixed points of operator Φ are solutions of problem (1.1). By (H1), for each $x, y \in \mathcal{C}$ and $t \in J$, we get

$$\begin{aligned} & |(\Phi x)(t) - (\Phi y)(t)| \\ \leq\; & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s, x(s), I^\eta x(s)) - f(s, y(s), I^\eta y(s))|\, ds \\ & + \frac{|\lambda|}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |x(s) - y(s)|\, ds \\ & + \frac{t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\int_0^T (T-s)^{\beta-1} |f(s, x(s), I^\eta x(s))\right. \\ & \left. - f(s, y(s), I^\eta y(s))|\, ds + |\lambda| \int_0^T (T-s)^{\beta-1} |x(s) - y(s)|\, ds\right) \\ & + |a| \int_0^T |x(s) - y(s)|\, ds \\ \leq\; & \left(\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{2\beta+\alpha-1}}{\beta T^{\beta-1}\Gamma(\beta+\alpha)}\right)\right. \\ & \left. \times \left(l_1 + l_2 \frac{T^\eta}{\Gamma(\eta+1)} + |\lambda|\right) + |a|\, T\right) \|x - y\|. \end{aligned}$$

Thus

$$\|\Phi x - \Phi y\| \leq \theta \|x - y\|.$$

From (3.1), Φ is a contraction. As a result of Banach's fixed point theorem, Φ has a unique fixed point which is the unique solution of the problem (1.1) on J. This finishes the proof. □

3.2. Existence results via Schauder's fixed point theorem

For the sake convenience, we put

$$\Lambda_1 = \frac{\Psi^* T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Psi^* T^{\alpha+2\beta-1}}{\beta T^{\beta-1}\Gamma(\alpha+\beta)} + |b|,$$

where $\Psi^* = \sup\{\Psi(t) : t \in J\}$.

Theorem 3.2. *Assume that the hypotheses* (H1) *and* (H2) *are satisfied. If*

$$\omega = |\lambda| \left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{2\beta+\alpha-1}}{\beta T^{\beta-1}\Gamma(\alpha+\beta)} \right) + |a| T < 1,$$

then (1.1) *has at least one solution on* J.

Proof. We consider the nonempty closed bounded convex subset

$$\Omega = \{x \in \mathcal{C} : \|x\| \leq M\}$$

of $\mathcal{C}$, where M is chosen such

$$M \geq \frac{\Lambda_1}{1-\omega}.$$

Notice that, the continuity of the operator Φ follows from the continuity of the function f. Now, we need to show that the operator Φ is compact by using the Arzela-Ascoli theorem. So, we will prove that $\Phi(\Omega) \subset \Omega$ and $\Phi(\Omega)$ is uniformly bounded and equicontinuous set. For $x \in \Omega$, we have

$$\begin{aligned}
& |(\Phi x)(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s, x(s), I^\eta x(s))| \, ds \\
& + \frac{|\lambda|}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |x(s)| \, ds \\
& + \frac{t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\int_0^T (T-s)^{\beta-1} |f(s, x(s), I^\eta x(s))| \, ds \right. \\
& \left. + |\lambda| \int_0^T (T-s)^{\beta-1} |x(s)| \, ds \right) + |a| \int_0^T |x(s)| \, ds + |b| \\
\leq\ & \frac{\Psi^* T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda| M \left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{2\beta+\alpha-1}}{\beta T^{\beta-1}\Gamma(\alpha+\beta)} \right) \\
& + \frac{\Psi^* T^{\alpha+2\beta-1}}{\beta T^{\beta-1}\Gamma(\alpha+\beta)} + |a| TM + |b| \\
\leq\ & M.
\end{aligned}$$

Then
$$\|\Phi x\| \leq M,$$
which means that $\Phi(\Omega) \subset \Omega$ and the set $\Phi(\Omega)$ is uniformly bounded. Next, we will prove that $\Phi(\Omega)$ is equicontinuous set. For $t_1, t_2 \in J$ such that $t_1 < t_2$ and for $x \in \Omega$, we get

$$\begin{aligned}
& |(\Phi x)(t_2) - (\Phi x)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \left((t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1} \right) |f(s, x(s), I^\eta x(s))| \, ds \\
& + \frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} |f(s, x(s), I^\eta x(s))| \, ds \\
& + \frac{|\lambda|}{\Gamma(\alpha+\beta)} \left(\int_0^{t_1} \left((t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1} \right) |x(s)| \, ds \right. \\
& \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} |x(s)| \, ds \right) \\
& + \frac{t_2^{\beta+\alpha-1} - t_1^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\int_0^T (T - s)^{\beta-1} |f(s, x(s), I^\eta x(s))| \, ds \right. \\
& \left. + |\lambda| \int_0^T (T - s)^{\beta-1} |x(s)| \, ds \right) \\
\leq\ & \frac{\Psi^*}{\Gamma(\alpha+\beta)} \left(\int_0^{t_1} \left((t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1} \right) ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} \, ds \right) \\
& + \frac{|\lambda| M}{\Gamma(\alpha+\beta)} \left(\int_0^{t_1} \left((t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1} \right) ds \right. \\
& \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} \, ds \right) + \frac{t_2^{\beta+\alpha-1} - t_1^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\frac{\Psi^* T^\beta}{\beta} + \frac{|\lambda| T^\beta M}{\beta} \right) \\
\leq\ & \frac{\Psi^*}{\Gamma(\alpha+\beta+1)} \left(t_2^{\alpha+\beta} - t_1^{\alpha+\beta} \right) + \frac{\left(t_2^{\beta+\alpha-1} - t_1^{\beta+\alpha-1} \right)}{T^{\beta-1}\Gamma(\beta+\alpha)} \left(\frac{\Psi^* T^\beta}{\beta} + \frac{|\lambda| T^\beta M}{\beta} \right).
\end{aligned}$$

As $t_1 \to t_2$, we see that the right hand side of the above inequality tends to zero and the convergence is independent of x in Ω, which means $\Phi(\Omega)$ is equicontinuous. The Arzela-Ascoli theorem implies that Φ is compact. Thus, by the Schauder fixed point theorem, we prove that Φ has at least one fixed point $x \in \Omega$ which is a solution of the problem (1.1) on J. □

4. Example

We take the following fractional relaxation integro-differential equation

$$\begin{cases} D^{\frac{3}{2}}\ {}^{LC}D^{\frac{1}{2}} x(t) + \frac{1}{4} x(t) = f\left(t, x(t), I^{\frac{1}{3}} x(t)\right), \ t \in (0,1), \\ {}^{LC}D^{\frac{1}{2}} x(0) = {}^{LC}D^{\frac{1}{2}} x(1) = 0, \ x(0) = \frac{1}{10} \int_0^1 x(s) \, ds + 2. \end{cases} \tag{4.1}$$

Here $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\eta = \frac{1}{3}$, $\lambda = \frac{1}{4}$, $a = \frac{1}{10}$ and $b = 2$. Set

$$f\left(t, x\left(t\right), I^{\frac{1}{3}}x\left(t\right)\right) = \frac{\sin\left(t\right)}{\exp\left(t^2\right)+7}\left(\frac{\left|x\left(t\right)\right|}{\left|x\left(t\right)\right|+1} + \frac{\left|I^{\frac{1}{3}}x\left(t\right)\right|}{1+\left|I^{\frac{1}{3}}x\left(t\right)\right|}\right).$$

For $x_i, y_i \in \mathbb{R}$, $i = 1, 2$, we have

$$\begin{aligned}
& \left|f(t, x_1, x_2) - f(t, y_1, y_2)\right| \\
= & \left|\frac{\sin\left(t\right)}{\exp\left(t^2\right)+7}\left(\left(\frac{|x_1|}{|x_1|+1} - \frac{|y_1|}{|y_1|+1}\right) + \left(\frac{|x_2|}{|x_2|+1} - \frac{|y_2|}{|y_2|+1}\right)\right)\right| \\
\leq & \frac{1}{\exp\left(t^2\right)+7}\left(\frac{|x_1 - y_1|}{\left(1+|x|\right)\left(1+|y|\right)} + \frac{|x_2 - y_2|}{\left(1+|x_2|\right)\left(1+|y_2|\right)}\right) \\
\leq & \frac{1}{8}\left(|x_1 - y_1| + |x_2 - y_2|\right),
\end{aligned}$$

thus, the assumption (H1) is satisfied with $l_1 = l_2 = \frac{1}{8}$. We will check that condition (3.1) is satisfied. Indeed

$$\begin{aligned}
\theta = & \left(\frac{T^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)} + \frac{T^{2\beta+\alpha-1}}{\beta T^{\beta-1}\Gamma\left(\beta+\alpha\right)}\right)\left(l_1 + l_2\frac{T^{\eta}}{\Gamma\left(\eta+1\right)} + |\lambda|\right) + |a|\,T \\
= & \left(\frac{1}{\Gamma\left(3\right)} + \frac{2}{3\Gamma\left(2\right)}\right)\left(\frac{1}{8} + \frac{1}{8}\frac{1}{\Gamma\left(\frac{1}{3}+1\right)} + \frac{1}{4}\right) + \frac{1}{10} \\
\simeq & \ 0.7 < 1.
\end{aligned}$$

Then by Theorem 3.1, the problem (4.1) has a unique solution on $[0, 1]$. Also we have

$$f\left(t, x, y\right) \leq \frac{2}{\exp(t^2)+7},\ \forall (t, x, y) \in J \times \mathbb{R} \times \mathbb{R}.$$

Hence condition (H2) holds with $\Psi(t) = \frac{2}{\exp(t^2)+7}$, it follows from Theorem 3.2 that the problem (4.1) has at least one solution on $[0, 1]$.

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