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SOLUTIONS FOR THE FRACTIONAL MATHEMATICAL MODELS OF DIFFUSION PROCESS

Suleyman Cetinkaya[1], Mine Aylin Bayrak[1], Ali Demir[1] and Dumitru Baleanu[2]

[1] **Faculty of Arts and Sciences, Department of Mathematics**
41380 Izmit, Kocaeli
[2] **Department of Mathematics, 06530 Balgat, Ankara**

Abstract. In this research we present two new approaches with Laplace transformation to form the truncated solution of space-time fractional differential equations (STFDE) with mixed boundary conditions. Since order of the fractional derivative of time derivative is taken between zero and one we have a sub-diffusive differential equation. First, we reduce STFDE into either a time or a space fractional differential equation which are easier to deal with. At the second step the Laplace transformation is applied to the reduced problem to obtain truncated solution. At the final step using the inverse transformations, we get the truncated solution of the problem we consider it. Presented examples illustrate the applicability and power of the approaches, used in this study.
Key words: Laplace transform, Liouville-Caputo derivative, time-space fractional differential equations.

1. Introduction

Mathematical models with fractional differential equations for many physical phenomena play important roles in various applied sciences such as mathematics physics, biology, dynamical systems, control systems, engineering and soon [13, 15, 16, 17]. As a result, there are many introductory overview about fractional calculus such as [18, 19, 20] in order to develop the knowledge about it. Fractional mathematical

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Corresponding Author: Suleyman Cetinkaya, Faculty of Arts and Sciences, Department of Mathematics, 41380 Izmit, Kocaeli | E-mail: suleyman.cetinkaya@kocaeli.edu.tr
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models plays a crucial role to model various physical and scientific processes since they have more degrees of freedom and memory to reflect the behavior of processes much more better. Therefore, many different processes such as the finite difference method [22], Kontrovich-Lebedev transform method[21], the homotopy perturbation method (HPM) [23, 12, 10] are introduced and applied so as to get the solution [14, 24, 25, 1, 4]. Moreover, various fractional derivatives are defined to make the mathematical modeling of physical phenomena much more better [2, 5, 6, 7, 8, 9]. Laplace transform appear frequently to solve complicated problems in applied sciences and engineering mathematics due to the special and useful properties of it [13, 15]. The purpose of this study to solve the following initial boundary value problem with STFDE:

$$D_t^{\alpha} u\left(x,t\right) = D_x^{\beta} u\left(x,t\right) + F, x \in \mathbb{R}, t \in \mathbb{R} \tag{1.1}$$

$$u\left(x,0\right) = \varphi(x), x \in \mathbb{R} \tag{1.2}$$

$$u\left(0,t\right) = \mu_1(t), t \in \mathbb{R} \tag{1.3}$$

$$u_x\left(0,t\right) = \mu_2(t), t \in \mathbb{R} \tag{1.4}$$

where $0 < \alpha \leqslant 1$, $1 < \beta \leqslant 2$. The restriction $0 < \alpha \leqslant 1$ implies that we have a sub-diffusive case. Since diffusion coefficient is 1 in our problem, the direction of the diffusion from higher density to lower density.

2. Preliminaries and Notations

In this section, preliminaries, notations and properties of the fractional calculus are given [15, 17]. Riemann-Liouville time-fractional integral of a real valued function $u(x,t)$ is defined as

$$I_t^{\alpha} u\left(x,t\right) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(t-s\right)^{\alpha-1} u(x,s) ds \tag{2.1}$$

where $\alpha > 0$ denotes the order of the integral.

α^{th} order the Liouville-Caputo time-fractional derivative operator of $u(x,t)$ is defined as

$$\begin{aligned} \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} &= I_t^{m-\alpha}\left[\frac{\partial^m u(x,t)}{\partial t^m}\right] \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \left(t-y\right)^{m-\alpha-1} \frac{\partial^m u(x,y)}{\partial y^m} dy, & m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, \alpha = m \end{cases} \end{aligned} \tag{2.2}$$

Two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}\left(z\right) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, Re\left(\alpha\right) > 0,\ z, \beta \in \mathbb{C}. \tag{2.3}$$

where α and β are parameters.

The following set of functions has Laplace transformation
$\left\{ f(t) \mid \exists M,\ \tau_1, \tau_2 > 0, \quad such\ that\ |f(x,t)| < M e^{|t|/\tau_j} \quad if\ t \in (-1)^j \times [0,\infty) \right\}$
and it is defined as

$$\mathcal{L}[f(t)] = F(\omega) = \int_0^\infty e^{-\omega t} f(t)\, dt, \tag{2.4}$$

which has the following property

$$\mathcal{L}\left[\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right] = \frac{1}{\omega^{n\alpha+1}}, \alpha > 0 \tag{2.5}$$

inverse Laplace inverse transform of $\frac{1}{\omega^{n\alpha+1}}$ is defined as

$$\mathcal{L}^{-1}\left[\frac{1}{\omega^{n\alpha+1}}\right] = \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, Re(\alpha) > 0 \tag{2.6}$$

where $n > 0$ [13, 15, 17].
For α^{th} order the Liouville-Caputo time-fractional derivative of $f(x,t)$, the Laplace transformation has the following form [13, 15, 17]:

$$\begin{aligned} \mathcal{L}\left[\frac{\partial^\alpha f(x,t)}{\partial t^\alpha}\right] &= \omega^\alpha \mathcal{L}[f(x,t)] \\ &- \sum_{k=0}^{n-1}\left[\omega^{\alpha-k-1}\frac{\partial^k f(x,0)}{\partial t^k}\right], n-1 < \alpha \leqslant n, n \in \mathbb{N}. \end{aligned} \tag{2.7}$$

Similarly, for β^{th} order the Liouville-Caputo space-fractional derivative of $f(x,t)$, the Laplace transformation has the following form

$$\begin{aligned} \mathcal{L}\left[\frac{\partial^\beta f(x,t)}{\partial x^\beta}\right] &= \omega^\beta \mathcal{L}[f(x,t)] \\ &- \sum_{k=0}^{m-1}\left[\omega^{\beta-k-1}\frac{\partial^k f(0,t)}{\partial x^k}\right], m-1 < \beta \leqslant m, m \in \mathbb{N}. \end{aligned} \tag{2.8}$$

3. Solving FPDE via Laplace Transformation with New Approaches

3.1. First approach with the transformation $u(x,t) = D_t^{1-\alpha} v(x,t)$

In this section, we consider the problem (1.1)-(1.4) below and apply new approach to construct truncated solution of it. First the order of time fractional derivative is made integer by the transformation $u(x,t) = D_t^{1-\alpha} v(x,t)$ and then we apply the Laplace transformation with respect to space variable x to obtain the truncated solution at the final step.

We make use of the transformation $u(x,t) = D_t^{1-\alpha} v(x,t)$ in Eq. (1.1) to make the order of fractional derivative with respect to time variable t, integer,. Hence we obtain the following problem:

$$D_t v(x,t) = D_x^{\beta}\left(D_t^{1-\alpha} v(x,t)\right) + F(t), \tag{3.1}$$

along with the initial and boundary conditions

$$v(x,0) = D_t^{\alpha-1}\varphi(x), \tag{3.2}$$
$$v(0,t) = D_t^{\alpha-1}\mu_1(t), \tag{3.3}$$
$$v_x(0,t) = D_t^{\alpha-1}\mu_2(t). \tag{3.4}$$

Now let us apply the transformation to Eq. (3.1)

$$\begin{aligned}\mathcal{L}\left[D_t^{1-\alpha} v(x,t)\right] &= w^{-\beta}\mathcal{L}\left[D_t v(x,t) - F(t)\right] \\ &\quad + \frac{1}{w} D_t^{1-\alpha} v(0,t) + \frac{1}{w^2} D_t^{1-\alpha} v_x(0,t).\end{aligned} \tag{3.5}$$

Hence the inverse transformation of Eq. (3.5) becomes

$$D_t^{1-\alpha} v(x,t) = \mathcal{L}^{-1}\left[w^{-\beta}\mathcal{L}\left[D_t v(x,t) - F(t)\right]\right] + \mu_1(t) + x\mu_2(t). \tag{3.6}$$

Taking the transformation $D_t^{\alpha-1} v$ of equation (3.6) leads to

$$v(x,t) = D_t^{\alpha-1}\left[\mathcal{L}^{-1}\left[w^{-\beta}\mathcal{L}\left[D_t v(x,t) - F(t)\right]\right]\right] + D_t^{\alpha-1}\mu_1(t) + xD_t^{\alpha-1}\mu_2(t). \tag{3.7}$$

After using the equation (3.7) successively, the following recurrence relation is obtained:

$$\left.\begin{array}{c} v_0(x,t) = D_t^{\alpha-1}\mu_1(t) + xD_t^{\alpha-1}\mu_2(t) \\ v_1(x,t) = \frac{x^{\beta}}{\Gamma(\beta+1)} D_t^{2\alpha-1}\left[\mu_1(t) - F(t)\right] + \frac{x^{\beta+1}}{\Gamma(\beta+2)} D_t^{2\alpha-1}\mu_2(t) \\ v_2(x,t) = \frac{x^{2\beta}}{\Gamma(2\beta+1)} D_t^{3\alpha-1}\left[\mu_1(t) - F(t)\right] + \frac{x^{2\beta+1}}{\Gamma(2\beta+2)} D_t^{3\alpha-1}\mu_2(t) \\ v_3(x,t) = \frac{x^{3\beta}}{\Gamma(3\beta+1)} D_t^{4\alpha-1}\left[\mu_1(t) - F(t)\right] + \frac{x^{3\beta+1}}{\Gamma(3\beta+2)} D_t^{4\alpha-1}\mu_2(t) \\ \vdots \\ v_n(x,t) = \frac{x^{n\beta}}{\Gamma(n\beta+1)} D_t^{(n+1)\alpha-1}\left[\mu_1(t) - F(t)\right] + \frac{x^{n\beta+1}}{\Gamma(n\beta+2)} D_t^{(n+1)\alpha-1}\mu_2(t) \end{array}\right\} \tag{3.8}$$

By taking the first n-term, the truncated solution for the problem (3.1)-(3.4) is constructed as follows:

$$v = \sum_{k=0}^{n} v_k(x,t) \tag{3.9}$$

For the convergence of the approximate solution see the paper [11],[3]. Hence by means of transformation $u(x,t) = D_t^{1-\alpha} v(x,t)$, the solution of the problem (1.1)-

(1.4) is obtained as follows.

$$\left.\begin{array}{c} u_0(x,t) = \mu_1(t) + x\mu_2(t) \\ u_1(x,t) = \frac{x^{\beta}}{\Gamma(\beta+1)} D_t^{\alpha}\left[\mu_1(t) - F(t)\right] + \frac{x^{\beta+1}}{\Gamma(\beta+2)} D_t^{\alpha}\mu_2(t) \\ u_2(x,t) = \frac{x^{2\beta}}{\Gamma(2\beta+1)} D_t^{2\alpha}\left[\mu_1(t) - F(t)\right] + \frac{x^{2\beta+1}}{\Gamma(2\beta+2)} D_t^{2\alpha}\mu_2(t) \\ u_3(x,t) = \frac{x^{3\beta}}{\Gamma(3\beta+1)} D_t^{3\alpha}\left[\mu_1(t) - F(t)\right] + \frac{x^{3\beta+1}}{\Gamma(3\beta+2)} D_t^{3\alpha}\mu_2(t) \\ \vdots \\ u_n(x,t) = \frac{x^{n\beta}}{\Gamma(n\beta+1)} D_t^{n\alpha}\left[\mu_1(t) - F(t)\right] + \frac{x^{n\beta+1}}{\Gamma(n\beta+2)} D_t^{n\alpha}\mu_2(t) \end{array}\right\} \tag{3.10}$$

The n-term truncated solution of the problem (1.1)-(1.4) is formed by

$$\begin{aligned} u(x,t) &= \mu_1(t) + x\mu_2(t) \\ &+ \sum_{k=1}^{n}\left[\frac{x^{k\beta}}{\Gamma(k\beta+1)} D_t^{k\alpha}\left[\mu_1(t) - F(t)\right] + \frac{x^{k\beta+1}}{\Gamma(k\beta+2)} D_t^{k\alpha}\mu_2(t)\right] \end{aligned} \tag{3.11}$$

Various initial boundary value problems with STFDE are illustrated with the approach above.

Example 1. Consider the following initial boundary value problems with STFDE

$$D_t^{\alpha}u(x,t) = D_x^{\beta}u(x,t), 0 < \alpha \leqslant 1, 1 < \beta \leqslant 2, \tag{3.12}$$

$$u(x,0) = e^x, \tag{3.13}$$

$$u(0,t) = e^t, \tag{3.14}$$

$$u_x(0,t) = e^t. \tag{3.15}$$

Via the equation (3.11), the solution of the problem (3.11)-(3.15) is obtained as follows:

$$u(x,t) = \sum_{n=0}^{\infty}\left[\frac{x^{n\beta}}{\Gamma(n\beta+1)} D_t^{n\alpha}e^t + \frac{x^{n\beta+1}}{\Gamma(n\beta+2)} D_t^{n\alpha}e^t\right] \tag{3.16}$$

Notice that for $\alpha = 1$ and $\beta = 2$, the obtained solution of problem (3.12)-(3.15) coincides with the exact solution $u(x,t) = e^{t+x}$.

As it can be seen from Table 3.1 that for the fixed values of x and for various values of t, the truncated solution $u(x,t)$ generally decreases as the order of the fractional derivatives α and β increases to 1 and 2 respectively. Moreover, for the fixed values of α, β and t and for various values of x the truncated solution $u(x,t)$ generally increases. Figure 3.1 and Figure 3.2 verify these analysis.

Example 2. Consider the following initial boundary value problems with STFDE

$$D_t^{\alpha}u(x,t) = D_x^{\beta}u(x,t), \tag{3.17}$$

Table 3.1: The values of the truncated solution for various values of α and β for Example 1.

x	t	$\alpha = 0.5$ $\beta = 1.5$	$\alpha = 0.75$ $\beta = 1.75$	$\alpha = 1$ $\beta = 2$	Exact
0.2	0.3	1.675391751023197	1.661611900072517	1.648721270700128	1,64872127070013
	0.6	2.283916841552046	2.250319144729066	2.225540928492467	2,22554092849247
	0.9	3.100190675952160	3.042543402023502	3.004166023946433	3,00416602394643
0.4	0.3	2.061251560254603	2.041884927738453	2.013752707470477	2,01375270747048
	0.6	2.853064443823610	2.783562915911935	2.718281828459045	2,71828182845905
	0.9	3.906280530683131	3.775744672641578	3.669296667619244	3,66929666761924
0.6	0.3	2.503100562482106	2.494912862533716	2.459603111156950	2,45960311115695
	0.6	3.524424504544300	3.429372923376531	3.320116922736547	3,32011692273655
	0.9	4.872515541590597	4.670760057580716	4.481689070338066	4,48168907033806
0.8	0.3	3.005168989384650	3.031418130123285	3.004166023946433	3,00416602394643
	0.6	4.308486262927710	4.205754361438392	4.055199966844675	4,05519996684467
	0.9	6.018093997314748	5.754562697087009	5.473947391727201	5,47394739172720

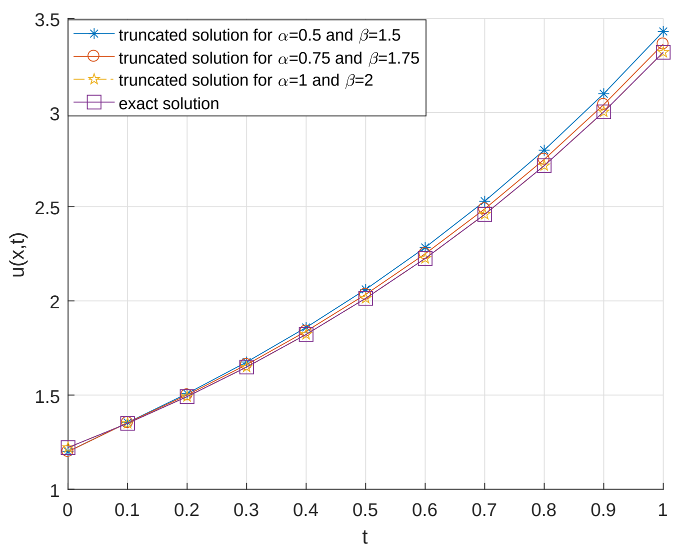

FIG. 3.1: The 10th-order truncated and exact solutions of the problem (3.12)-(3.15) for different α and β at $x = 0.2$.

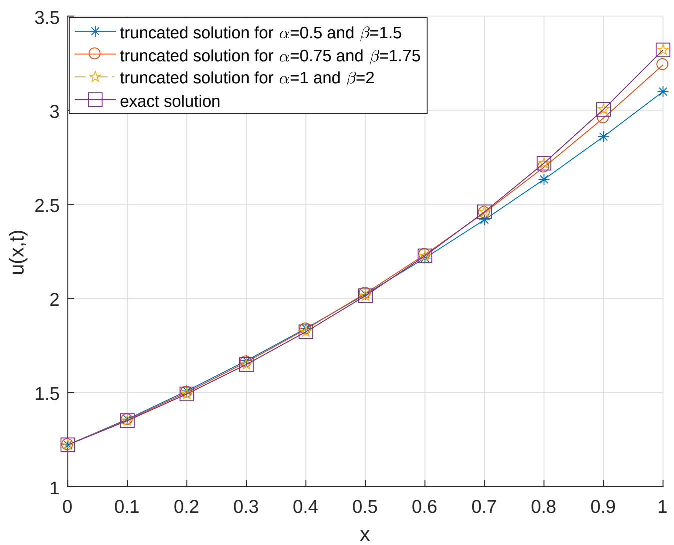

FIG. 3.2: The 10th-order truncated and exact solutions of the problem (3.12)-(3.15) for different α and β at $t = 0.2$.

$$u(x, 0) = \cos(x) \ , \tag{3.18}$$

$$u(0, t) = e^{-t}, \tag{3.19}$$

$$u_x(0, t) = 0. \tag{3.20}$$

Via the equation (3.11), the solution of the problem (3.17)-(3.20) is obtained as follows:

$$u(x, t) = \sum_{n=0}^{\infty} \frac{x^{n\beta}}{\Gamma(n\beta + 1)} D_t^{n\alpha} e^t \tag{3.21}$$

Notice that for $\alpha = 1$ and $\beta = 2$, the obtained solution of problem (3.17)-(3.20) coincides with the exact solution $u(x, t) = e^{-t}\cos(x)$.

Example 3. Consider the following initial boundary value problems with STFDE

$$D_t^{\alpha} u(x, t) = D_x^{\beta} u(x, t) + e^t, \tag{3.22}$$

$$u(x, 0) = e^x + 1, \tag{3.23}$$

Table 3.2: The values of the truncated solution for various values of α and β for Example 2.

x	t	$\alpha = 0.5$ $\beta = 1.5$	$\alpha = 0.75$ $\beta = 1.75$	$\alpha = 1$ $\beta = 2$	Exact
0.2	0.3	0,740472648604241	0,757764200108701	0,735084515424080	0,726051178345969
	0.6	0,548210064836476	0,564830235716550	0,539573137005435	0,537871942066125
	0.9	0,405778446537417	0,420771990105327	0,394990889736626	0,398465335076047
0.4	0.3	0,738053847242957	0,797806797805539	0,717923192590723	0,682338766716552
	0.6	0,543999806762696	0,602676230708350	0,511983040348768	0,505488991061114
	0.9	0,400241480080791	0,454322518697496	0,360491216360044	0,374475454932091
0.6	0.3	0,731490321530827	0,856606643012024	0,689453469660600	0,611423661702639
	0.6	0,532578497121307	0,658215581297093	0,466416576416867	0,452953789145250
	0.9	0,385226030471766	0,503529541784707	0,303777849742335	0,335556420125626
0.8	0.3	0,718716235427178	0,932090444852964	0,649873504996465	0,516133024755582
	0.6	0,510364398169066	0,729393151506932	0,403495894729401	0,382360749034503
	0.9	0,356041852908981	0,566492343465877	0,226020491016988	0,283259809758270

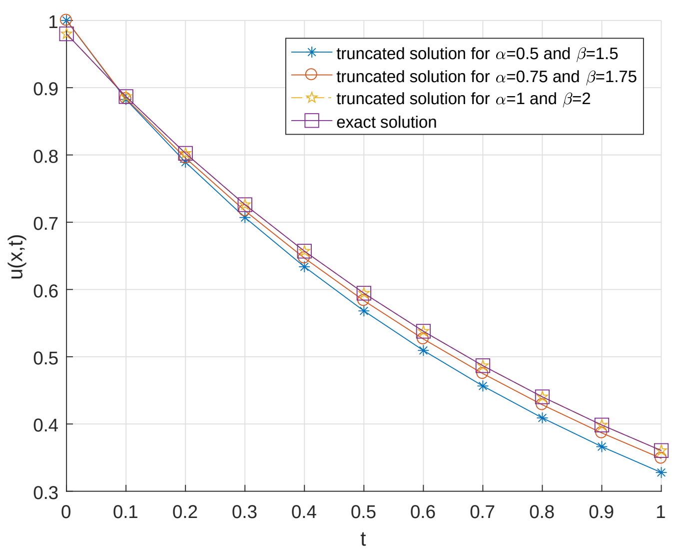

Fig. 3.3: The 10th-order truncated and exact solutions of the problem (3.17)-(3.20) for different α and β at $x = 0.2$.

$$u\,(0,t) = 2e^{t}, \tag{3.24}$$

$$u_x\,(0,t) = e^{t}. \tag{3.25}$$

Via the equation (3.11), the solution of the problem (3.22)-(3.25) is obtained as follows:

$$u\,(x,t) = 2e^{t} + xe^{t} + \sum_{k=1}^{n}\left[\frac{x^{k\beta}}{\Gamma\,(k\beta+1)}D_t^{k\alpha}e^{t} + \frac{x^{k\beta+1}}{\Gamma\,(k\beta+2)}D_t^{k\alpha}e^{t}\right] \tag{3.26}$$

Notice that for $\alpha = 1$ and $\beta = 2$, the obtained solution of problem (3.22)-(3.25) coincides with the exact solution $u\,(x,t) = e^{t}\,(e^{x}+1)$. Analysis of many examples lead us to the conclusion that the transformation $u(x,t) = D_t^{1-\alpha}v(x,t)$ decrease the effect of α and increase the effect of β.

3.2. Second approach with the transformation $u(x,t) = D_x^{2-\beta}v(x,t)$

In this section, we take the same problem (1.1)-(1.4) but apply a different approach to construct approximate solution of it. First the order of space fractional derivative is made integer by the transformation $u\,(x,t) = D_x^{2-\beta}v(x,t)$ and then we make the initial condition homogenous via the $w\,(x,t) = v\,(x,t) - v(x,0)$. Unlike the first approach, we apply the Laplace transformation to obtain the approximate solution at the final step. Now let us illustrate this approach step by step: First taking the transformation $u\,(x,t) = D_x^{2-\beta}v(x,t)$ into account, the problem (1.1)-(1.4) is turned into the problem below:

$$D_t^{\alpha}\left(D_x^{2-\beta}v(x,t)\right) = D_x^{2}v\,(x,t) + F(x), \tag{3.27}$$

along with the initial and boundary conditions

$$v\,(x,0) = D_x^{\beta-2}\varphi(x), \tag{3.28}$$

$$v\,(0,t) = 0, \tag{3.29}$$

$$v_x\,(0,t) = 0. \tag{3.30}$$

By $w\,(x,t) = v\,(x,t) - v(x,0)$, the problem (3.27)-(3.30) is transformed into initial boundary value problems with STFDE:

$$D_t^{\alpha}\left(D_x^{2-\beta}w(x,t)\right) = D_x^{2}w\,(x,t) + D_x^{\beta}\varphi\,(x) + F(x), \tag{3.31}$$

$$w\,(x,0) = 0, \tag{3.32}$$

$$w\,(0,t) = -D_x^{\beta-2}\varphi(0), \tag{3.33}$$

$$w_x\,(0,t) = -D_x^{\beta-1}\varphi(0). \tag{3.34}$$

Taking the Laplace transform of Eq. (3.31) with respect to time variable t

$$\mathcal{L}\left[D_x^{2-\beta}w\left(x,t\right)\right]=\frac{1}{z}D_x^{2-\beta}w\left(x,0\right)+\frac{1}{z^{\alpha}}\mathcal{L}\left[D_x^{2}w\left(x,t\right)+D_x^{\beta}\varphi\left(x\right)+F(x)\right]. \tag{3.35}$$

Applying the inverse Laplace transform of Eq. (3.35)

$$\begin{aligned} D_x^{2-\beta}w\left(x,t\right)&=\mathcal{L}^{-1}\left[\frac{1}{z}D_x^{2-\beta}w\left(x,0\right)\right]\\ &\quad+\mathcal{L}^{-1}\left[\frac{1}{z^{\alpha}}\mathcal{L}\left[D_x^{2}w\left(x,t\right)+D_x^{\beta}\varphi\left(x\right)+F(x)\right]\right]. \end{aligned} \tag{3.36}$$

Taking the integral of order $\beta-2$ of equation (3.36), we get

$$\begin{aligned} w\left(x,t\right)&=D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[\frac{1}{z}D_x^{2-\beta}w\left(x,0\right)\right]\right]\\ &\quad+D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[\frac{1}{z^{\alpha}}\mathcal{L}\left[D_x^{2}w\left(x,t\right)+D_x^{\beta}\varphi\left(x\right)+F(x)\right]\right]\right]. \end{aligned} \tag{3.37}$$

After using the equation (3.37) successively, the following recurrence relation is obtained:

$$\left.\begin{array}{c} w_0\left(x,t\right)=D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[D_x^{2-\beta}w\left(x,0\right)\right]\right]=0\\ w_1\left(x,t\right)=D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[z^{\alpha}\mathcal{L}\left[D_x^{\beta}\varphi\left(x\right)+F(x)\right]\right]\right]\\ w_2\left(x,t\right)=D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[z^{\alpha}\mathcal{L}\left[D_x^{2}w_1(x,t)\right]\right]\right]\\ w_3\left(x,t\right)=D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[z^{\alpha}\mathcal{L}\left[D_x^{2}w_2(x,t)\right]\right]\right]\\ \vdots\\ w_n\left(x,t\right)=D_x^{\beta-2}\left[\mathcal{L}^{-1}\left[z^{\alpha}\mathcal{L}\left[D_x^{2}w_{n-1}(x,t)\right]\right]\right] \end{array}\right\} \tag{3.38}$$

For the convergence of the approximate solution see the paper [11],[3].

The n-term truncated solution of the problem (3.31)-(3.34) is formed by

$$w\left(x,t\right)=\sum_{k=0}^{n}w_k(x,t) \tag{3.39}$$

From transformation $v\left(x,t\right)=w\left(x,t\right)+D_x^{\beta-2}\varphi(x)$, the n-term approximate solution of the problem (3.27)-(3.30) is obtained as follows.

$$v\left(x,t\right)=\sum_{k=0}^{n}w_k(x,t)+D_x^{\beta-2}\varphi(x) \tag{3.40}$$

From transformation $u\left(x,t\right)=D_x^{2-\beta}v(x,t)$, the n-term truncated solution of the problem (1.1)-(1.4) is obtained as follows:

$$u\left(x,t\right)=D_x^{2-\beta}\left[\sum_{k=0}^{n}w_k\left(x,t\right)\right]+\varphi(x) \tag{3.41}$$

Table 3.3: The values of the truncated solution for various values of α and β for Example 4.

x	t	$\alpha=0.5$ $\beta=1.5$	$\alpha=0.75$ $\beta=1.75$	$\alpha=1$ $\beta=2$	Exact
0.2	0.3	2,08837337953747	2,15027880106184	2,07603978323998	1,64872127070013
	0.6	2,55660975766192	3,02092619056800	3,22967909859111	2,22554092849247
	0.9	2,97068648944469	3,99928929179716	4,78692926898143	3,00416602394643
0.4	0.3	3,01154466102428	2,83650667907428	2,53568071729955	2,01375270747048
	0.6	3,91889994200771	4,16468209833847	3,94473895899143	2,71828182845905
	0.9	4,76788251532150	5,72187909306572	5,84676861225157	3,66929666761924
0.6	0.3	4,08672974637649	3,61399152945137	3,09708742192322	2,45960311115695
	0.6	5,53837499045954	5,43980865709289	4,81811504473401	3,32011692273655
	0.9	6,95383597076827	7,63616665641431	7,14125930932837	4,48168907033806
0.8	0.3	5,37868288669723	4,53486432006456	3,78279111940020	3,00416602394643
	0.6	7,51490086104965	6,93852313008294	5,88485900477113	4,05519996684467
	0.9	9,66790946048631	9,88215532465688	8,72235381715067	5,47394739172720

Example 4. Consider the following initial boundary value problems with STFDE

$$D_t^{\alpha}u(x,t)=D_x^{\beta}u(x,t), 0<\alpha\leqslant 1, 1<\beta\leqslant 2, \tag{3.42}$$

$$u\left(x,0\right)=e^{x}, \tag{3.43}$$

$$u\left(0,t\right)=e^{t}, \tag{3.44}$$

$$u_x\left(0,t\right)=e^{t}. \tag{3.45}$$

Via the equation (3.41), the solution of the problem (3.42)-(3.45) is obtained as follows:

$$u\left(x,t\right)=\sum_{n=1}^{\infty}\left[\frac{t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}D_x^{n\beta}e^{x}\right]+e^{x}=\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma\left(n\alpha+1\right)}D_x^{n\beta}e^{x} \tag{3.46}$$

Notice that for $\alpha=1$ and $\beta=2$, the obtained solution of problem (3.42)-(3.45) coincides with the exact solution $u\left(x,t\right)=e^{t+x}$

Example 5. Consider the following initial boundary value problems with STFDE

$$D_t^{\alpha}u(x,t)=D_x^{\beta}u(x,t), \tag{3.47}$$

$$u\left(x,0\right)=\cos(x)\ , \tag{3.48}$$

$$u\left(0,t\right)=e^{-t}, \tag{3.49}$$

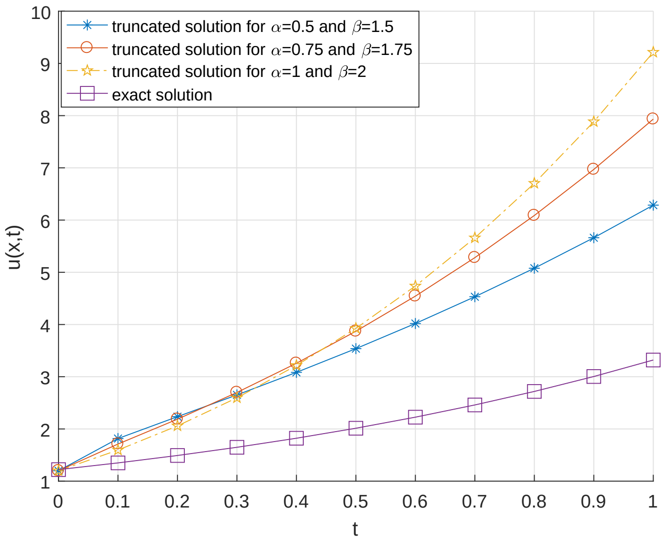

FIG. 3.4: The 10th-order truncated and exact solutions of the problem (3.47)-(3.50) for different α and β at $x = 0.2$.

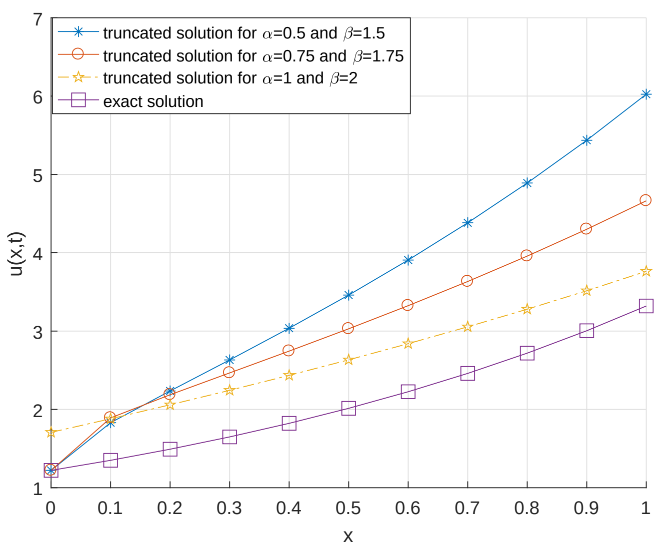

FIG. 3.5: The 10th-order truncated and exact solutions of the problem (3.47)-(3.50) for different α and β at $t = 0.2$.

Table 3.4: The values of the truncated solution for various values of α and β for Example 5.

x	t	$\alpha=0.5$ $\beta=1.5$	$\alpha=0.75$ $\beta=1.75$	$\alpha=1$ $\beta=2$	Exact
0.2	0.3	0.903862244232298	0.824980984781871	0.726051178346011	0,726051178345969
	0.6	0.841968005543826	0.707685126960147	0.537871942150945	0,537871942066125
	0.9	0.791500457499182	0.618430352394595	0.398465342240667	0,398465335076047
0.4	0.3	0.821148214301377	0.749209750922389	0.682338766716592	0,682338766716552
	0.6	0.746841346197896	0.626270832430415	0.505488991140827	0,505488991061114
	0.9	0.691152851212666	0.537571656548363	0.374475461665361	0,374475454932091
0.6	0.3	0.713607265803153	0.652547810042192	0.611423661702675	0,611423661702639
	0.6	0.636427951728122	0.534166558506156	0.452953789216679	0,452953789145250
	0.9	0.582550153991443	0.452260178714689	0.335556426159110	0,335556420125626
0.8	0.3	0.581636525788664	0.533892938724123	0.516133024755612	0,516133024755582
	0.6	0.507723734195925	0.426978492751225	0.382360749094800	0,382360749034503
	0.9	0.459721663506987	0.356091115676833	0.283259814851433	0,283259809758270

$$u_x\,(0,t) = 0. \tag{3.50}$$

Via the equation (3.41), the solution of the problem (3.47)-(3.50) is obtained as follows:

$$u\,(x,t) = \sum_{n=1}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} D_x^{n\beta}\cos(x) \;+\cos(x) \;= \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} D_x^{n\beta}\cos(x) \tag{3.51}$$

Particularly for $\alpha = 1$ and $\beta = 2$, eqs (3.47)-(3.50) coincides with the exact solution $u\,(x,t) = e^{-t}\cos(x)$.

As it can be seen from Table 3.4 that for the fixed values of x and for various values of t, the truncated solution $u\,(x,t)$ decreases as the order of the fractional derivatives α and β increases to 1 and 2 respectively. Moreover, for the fixed values of α, β and t and for various values of x the truncated solution $u\,(x,t)$ generally decreases. Figure 3.6 and Figure 3.7 verify these analysis.

Example 6. Consider the following initial boundary value problems with STFDE

$$D_t^{\alpha}u(x,t) = D_x^{\beta}u(x,t) + 3\cos(x)\ , \tag{3.52}$$

$$u\,(x,0) = 4\cos(x)\ , \tag{3.53}$$

$$u\,(0,t) = 3 + e^{-t}, \tag{3.54}$$

$$u_x\,(0,t) = 0. \tag{3.55}$$

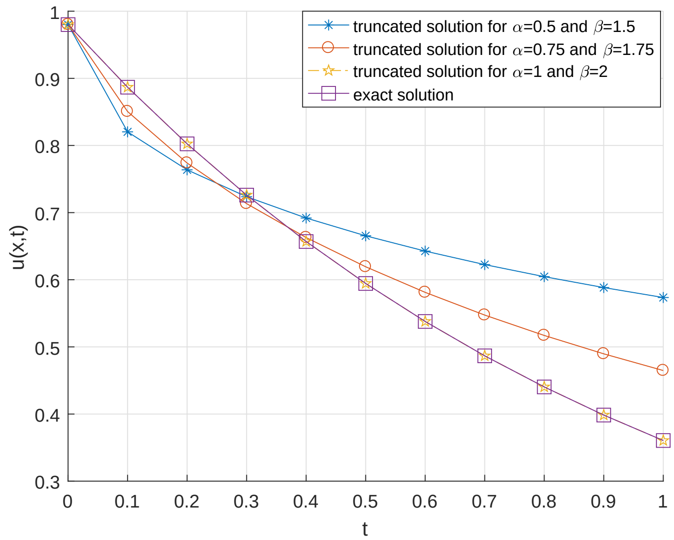

FIG. 3.6: The 10th-order truncated and exact solutions of the problem (3.47)-(3.50) for different α and β at $x = 0.2$.

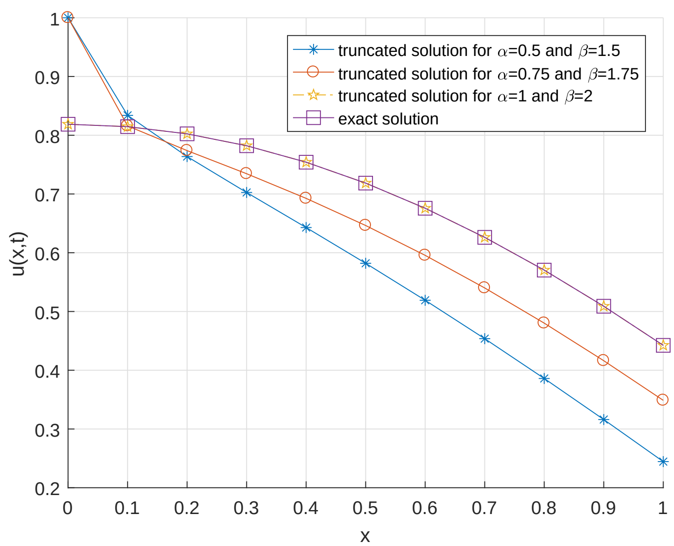

FIG. 3.7: The 10th-order truncated and exact solutions of the problem (3.47)-(3.50) for different α and β at $t = 0.2$.

Table 3.5: The values of the truncated solution for various values of α and β for Example 6.

x	t	$\alpha = 0.5$ $\beta = 1.5$	$\alpha = 0.75$ $\beta = 1.75$	$\alpha = 1$ $\beta = 2$	Exact
0.2	0.3	4.048265009071808	3.918430488600173	3.670349334015450	3,66625091186969
	0.6	4.112985147153213	3.897810154133708	3.508638348671645	3,47807167558985
	0.9	4.114426725609185	3.858405307965574	3.435133355333552	3,33866506859977
0.4	0.3	3.666999265151771	3.562520423402033	3.449373422540805	3,44552174872521
	0.6	3.605397852354764	3.466172423035573	3.297398358530328	3,26867197306977
	0.9	3.499439737620516	3.395199974912163	3.228318783980112	3,13765843694075
0.6	0.3	3.178909379366299	3.106622182184542	3.090881877836746	3,08743050643167
	0.6	3.045899567443888	2.974701495671091	2.954701501376648	2,92896063387429
	0.9	2.902313023871482	2.905580400098362	2.892801330258422	2,81156326485466
0.8	0.3	2.584462494159759	2.545682509109287	2.609166626505134	2,60625315279708
	0.6	2.412498120529409	2.398501460159810	2.494210019462853	2,47248087707600
	0.9	2.270933716497613	2.345283690540227	2.441957016261815	2,37337993779977

Via the equation (3.41), the solution of the problem (3.52)-(3.55) is obtained as follows:
(3.56)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \left[D_x^{n\beta}\left(4\cos(x)\right) + D_x^{(n-1)\beta}\left(3\cos(x)\right) \right] + \left(4\cos(x)\right)$$

Particularly for $\alpha = 1$ and $\beta = 2$, eqs (3.52)-(3.55) coincides with the exact solution $u(x,t) = \cos(x)\ (e^{-t} + 3)$.

As it is clear from Table 3.5 that for the fixed values of x and for various values of t, the truncated solution $u(x,t)$ generally decreases as the order of the fractional derivatives α and β increases to 1 and 2 respectively. Moreover, for the fixed values of α, β and t and for various values of x the truncated solution $u(x,t)$ decreases. Figure 3.8 and Figure 3.9 verify these analysis.

Analysis of many examples lead us to the conclusion that the transformation $u(x,t) = D_x^{2-\beta} v(x,t)$ increase the effects of α and β.

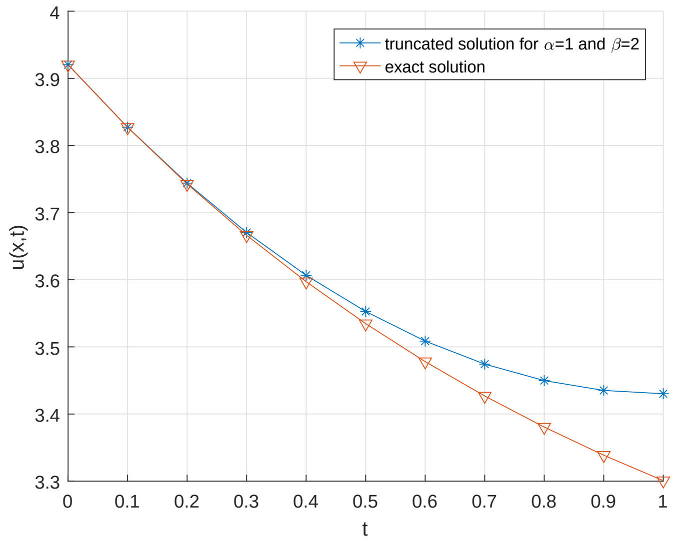

FIG. 3.8: The 10th-order truncated and exact solutions of the problem (3.52)-(3.55) for $\alpha = 1$ and $\beta = 2$ at $x = 0.2$.

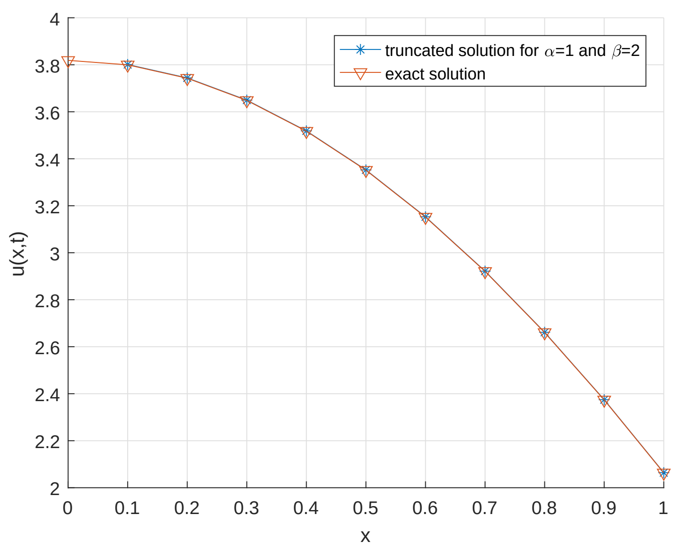

FIG. 3.9: The 10th-order truncated and exact solutions of the problem (3.52)-(3.55) for $\alpha = 1$ and $\beta = 2$ at $t = 0.2$.

4. Conclusion

In this study, Laplace transformation with two new additional transformations are applied to initial boundary value problems with STFDE including Liouville-Caputo fractional derivative to construct the series solution which is called semi analytic solution. By making use of these transformations, the original problem is reduced into either time or space fractional differential equation for which it is easier to implement the Laplace transformation to construct truncated solution. We reach the conclusion that the truncated solution obtained by first approach with the transformation $u(x,t) = D_t^{1-\alpha} v(x,t)$ is more accurate than the one obtained by second approach with the transformation $u(x,t) = D_x^{2-\beta} v(x,t)$. The reason for this result that the second approach include fractional derivative with respect to spatial variable x which affect the solution more than the time variable t when you construct the solution in series form. The numerical examples illustrated that the convergence and accuracy of the solution is very high.

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