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STOCHASTIC PERTURBATION OF A CUBIC ANHARMONIC OSCILLATOR

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Abstract. We perturb with an additive noise the Hamiltonian system associated to a cubic anharmonic oscillator. This gives rise to a system of stochastic differential equations with quadratic drift and degenerate diffusion matrix. Firstly, we show that such systems possess explosive solutions for certain initial conditions. Then, we carry a small noise expansion's analysis of the stochastic system which is assumed to start from initial conditions that guarantee the existence of a periodic solution for the unperturbed equation. We then investigate the probabilistic properties of the sequence of coefficients which turn out to be the unique strong solutions of stochastic perturbations of the well-known Lamé's equation. We also obtain explicit expressions of these in terms of Jacobi elliptic functions. Furthermore, we prove, in the case of Brownian noise, a lower bound for the probability that the truncated expansion stays close to the solution of the deterministic problem. Lastly, when the noise is bounded, we provide conditions for the almost sure convergence of the global expansion.

1. Introduction.

We investigate the second order stochastic differential equation

$$\ddot{x}(t) = x(t)^2 - \mathcal{B} + \sigma \dot{Z}(t), \quad x(0) = y \quad \dot{x}(0) = \eta \tag{1}$$

where σ is a positive constant, $\mathcal{B} \in \mathbb{R}$ and $\{Z(t)\}_{t\geq 0}$ is a continuous square integrable martingale starting at zero and defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is assumed to fulfil the usual completeness requirement. The rigorous formulation of equation (1) is achieved by considering the Itô-type stochastic Hamiltonian system

$$\begin{cases} dx(t) = \xi(t)dt, & x(0) = y \\ d\xi(t) = \left(x^2(t) - \mathcal{B}\right) dt + \sigma dZ(t), & \xi(0) = \eta. \end{cases} \tag{2}$$

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This very simple model comes essentially from the Hamiltonian

$$H(x,\xi) = \xi^2/2 - x^3/3 + \mathcal{B}x, \tag{3}$$

of a *cubic anharmonic oscillator* perturbed by noise. We may assume the initial data (y,η) to be deterministic.

The function (3) belongs to a class of Hamiltonians very thoroughly studied, see e.g. Delabaere and Trinh [8], Ferreira and Sesma [9] and the references therein, providing examples of quantum systems of interest in quantum field theory and in general theoretical physics. Perhaps slightly less known is the fact that (3) also arises as the major ingredient in a special canonical form in the theory of hyperbolic operators with double characteristics of non-effectively hyperbolic type according to Hörmander's classification [12], generating at the same time a number of problems on the regularity of the corresponding solutions and on the behaviour of the solutions of the associated Hamilton equations. In fact, a specific question the result of this paper aims to address is what can be said, studying the hyperbolic operator in $\mathbb{R}^{n+1} \ni x = (x_0, x_1, \dots, x_n)$, about its wellposedness properties:

$$P(x,D) = D_0^2 - 2x_1 D_0 D_n + D_1^2 + x_1^3 D_n^2 + \text{white noise} .$$

For the deterministic part it is known (see Bernardi and Nishitani [6], [7]) that the Cauchy problem is, surprisingly, not well posed in the C^∞ category, a fact which has been proved to be related to the behavior of the solutions of the Hamilton equations, the bicharacteristic strips in the cotagent bundle. The motivation was here to analyze how the effect of a suitable noise was going to reinforce or dampen this deterministic setup, through its equivalent formulation as a second order stochastic ODE. All this of course is meant to be a preliminary base to the full study of the (lack of) well posedness of the Cauchy problem for this class of hyperbolic stochastic partial differential equations with multiple characteristics.

System (2) is characterized by a drift vector which is linear in the first component and quadratic in the second one and a degenerate diffusion matrix as the first equation is not perturbed by the noise. The local lipschitzianity of the drift entails existence of a path-wise unique strong solution up to a possible almost surely finite stopping time at which the solution explodes. System (2) presents some distinguishing features that prevent the use of standard techniques in the analysis of existence and uniqueness of weak/strong solutions. First of all, due to the super-linear growth of the drift coefficient we are not allowed to employ (when we consider Brownian perturbations) the Girsanov theorem to construct weak solutions; this is one of the key tools for the investigation of almost sure properties of the solution (see Markus and Weerasinghe [16], [17]). Moreover, the Hamiltonian (3) is not bounded below and therefore classical methods based on the positivity of the energy have to be excluded (see Albeverio et al. [1], [2]).

We mention that in the paper Appleby et al. [3] (see also the references quoted there) the authors address the general problem of stabilization and destabilization by a Brownian noise perturbation that preserves the equilibrium of the corresponding noise free ordinary differential equation. However, the additive noise case considered in the current work is not covered by that investigation.

Our approach is as follows: firstly, we show that systems of the type (2) possess in general explosive solutions. This is accomplished via a careful analysis of the behaviour of a concrete example where the solution of the stochastic equation is compared with an explosive solution of a related deterministic problem. The particular polynomial growth in (2) prevents one from using the well known Lyapunov auxiliary functions techniques usually employed in determing stability or explosiveness of the solutions, see e.g. Theorem 3.6 in [15], page 77. Motivated by this fact, we then focus on initial conditions for the system (2) that guarantee existence of a periodic (hence global) solution for the associated deterministic problem (i.e. when $\sigma = 0$). The analysis of the system (2) corresponding to those initial conditions is carried out via a small noise expansion's technique in the spirit of Gardiner's book [10]. See also [22] for a general presentation of aymptotics analysis of SDEs via Fokker-Planck equation. More precisely, we set

$$X^\sigma(t) := \sum_{n\geq 0} \sigma^n x_n(t), \quad t \geq 0. \tag{4}$$

A formal substitution of this expression into equation (1) results in an equality between two power series. If we impose the coefficients of the corresponding powers of σ to be equal, we end up with a sequence of nested random/stochastic Cauchy problems for the sequence of functions $\{x_n(t)\}_{t\geq 0, n\geq 0}$. In fact, via a direct verification one gets that $\{x_0(t)\}_{t\geq 0}$ is associated with the deterministic equation

$$\ddot{x}_0(t) = x_0(t)^2 - \mathcal{B}, \quad x_0(0) = y \quad \dot{x}_0(0) = \eta. \tag{5}$$

This equation is the deterministic version of (2); its solution is the periodic function mentioned above. The function $\{x_1(t)\}_{t\geq 0}$ is linked to the linear stochastic differential equation

$$\ddot{x}_1(t) = 2x_0(t)x_1(t) + \dot{Z}(t), \quad x_1(0) = 0 \quad \dot{x}_1(0) = 0 \tag{6}$$

where $\{x_0(t)\}_{t\geq 0}$ solves (5). Equation (6) can be interpreted as a *stochastic Lamé's equation*, see e.g. Arscott [4], Arscott and Khabaza [5], Volker [21] and the website [25] for the deterministic case. We will see later in fact that, once $x_0(t)$ is solved in (5), (6) can be written as

$$\ddot{x}_1(t) + (h - \nu(\nu+1)k^2 \mathrm{sn}^2(t,k))x_1(t) = \dot{Z}(t) \ , \tag{7}$$

where $\mathrm{sn}(t,k)$ is the Jacobi elliptic function with modulus k and h, ν are suitable constants depending on $\mathcal{B}$ essentially.
For $n \geq 2$ the function $\{x_n(t)\}_{t\geq 0}$ solves the random differential equation

$$\ddot{x}_n(t) = 2x_0(t)x_n(t) + \sum_{j=1}^{n-1} x_j(t)x_{n-j}(t), \quad x_n(0) = 0 \quad \dot{x}_n(0) = 0. \tag{8}$$

We note that also in this case the equation to be solved is linear, the function $\{x_0(t)\}_{t\geq 0}$ solves (5) and the functions involved in the sum are the coefficients of lower order terms (with respect to the unknown) from the expansion (4).
Once the coefficients of expansion (4) are analysed, our investigation focuses on studying the truncated and global expansions. We first consider the case where $\{Z(t)\}_{t\geq 0}$ is a standard one dimensional Brownian motion: here, we prove lower bounds for the probability that the truncated expansion stays in a certain neighbourhood of x_0 (the solution of the deterministic problem). Then, under an assumption of boundedness for the noise $\{Z(t)\}_{t\geq 0}$ we obtain the almost sure convergence for the global expansion (4).

Remark 1. The techniques employed in this paper carry over the case of a multiplicative noise term as well, that means we are able to treat with only minor and straightforward modifications also the second order stochastic differential equation

$$\ddot{x}(t) = x(t)^2 - \mathcal{B} + \sigma x(t)\dot{Z}(t), \quad x(0) = y \quad \dot{x}(0) = \eta \tag{9}$$

where now the white noise is multiplied by the unknown $x(t)$. In fact, proceeding as explained above with the formal substitution of the power series (4) in the equation (9) one immediately finds that $\{x_0(t)\}_{t\geq 0}$ is again associated with the deterministic equation (5) while $\{x_1(t)\}_{t\geq 0}$ is now linked to the linear stochastic differential equation

$$\ddot{x}_1(t) = 2x_0(t)x_1(t) + x_0(t)\dot{Z}(t), \quad x_1(0) = 0 \quad \dot{x}_1(0) = 0. \tag{10}$$

We observe that the noise term in (10) is additive and corresponds to a minor modification of $\{Z_t\}_{t\geq 0}$. Therefore, equation (10) can be treated as equation (6). Moreover, for $n \geq 2$ the function $\{x_n(t)\}_{t\geq 0}$ now solves the random differential equation

$$\ddot{x}_n(t) = 2x_0(t)x_n(t) + \sum_{j=1}^{n-1} x_j(t)x_{n-j}(t) + x_{n-1}(t)\dot{Z}(t), \quad x_n(0) = 0 \quad \dot{x}_n(0) = 0. \tag{11}$$

The term $x_{n-1}(t)\dot{Z}(t)$ in (11), which is not present in (8) can be defined as a pathwise integral since the function $t \mapsto x_{n-1}(t)$ is almost surely continuously differentiable; as for the solution of (11), the new term can be absorbed by the sum on the right hand side and the analysis follows from the one employed for (8).
We also remark that the linearity of equations with index $n \geq 1$ derives from the asymptotic expansion approach. In fact, this feature is preserved even in the presence of multiplicative noises involving non linear functions of $x(t)$. See [10] for more details.

The paper is organized as follows: In Section 2 we prove through a detailed study of an example that systems of the type (2) possess in general explosive solutions; Section 3 is devoted to the small noise expansion corresponding to the stochastic system related to the deterministic problem with a periodic solution: more precisely, we analyse the coefficients of the power series (4) as solutions to certain stochastic/random differential equations, we provide explicit solutions and describe their fundamental probabilistic properties. In Section 4 we take the noise to be a Brownian motion and present several probabilistic lower bounds in terms of the (explicit) two independent solutions of the Lamé's equation for both the coefficients of series (4) and truncated expansion. Finally, in Section 5 under an assumption of boundedness of the noise we obtain the almost sure uniform convergence of the global expansion on a compact time interval whose length depend on the diffusion coefficient σ.

2. Explosive solutions.

In this section we prove that in general systems of the type (2) possess explosive solutions. To this aim we consider the system

$$\begin{cases} dx(t) = \xi(t)dt, & x(0) = \sqrt{3} + \delta \\ d\xi(t) = (x^2(t) - 1)dt + \sigma dB(t), & \xi(0) = 0, \end{cases} \tag{12}$$

which corresponds to (2) for $\mathcal{B} = 1$, $y = \sqrt{3} + \delta$, $\eta = 0$ and $\{Z_t\}_{t\geq 0}$ being a one dimensional Brownian motion $\{B_t\}_{t\geq 0}$; here δ is a fixed positive constant. We will show that the solution of (12) explodes in finite time with positive probability. Our analysis relies on the behaviour of the auxiliary deterministic system

$$\begin{cases} du(t) = v(t)dt, & u(0) = \sqrt{3} \\ dv(t) = (u^2(t) - 1)dt, & v(0) = 0. \end{cases} \tag{13}$$

Lemma 2.1. *The unique solution of system (13) is given by*

$$u(t) = \frac{\sqrt{3}}{\mathrm{cn}^2\left(\frac{t}{3^{1/4}}, q\right)}, \quad t \in [0, \tau[$$

where $\mathrm{cn}(\cdot, q)$ *denotes the elliptic Jacobi cosine function and* $\tau = \sqrt[4]{3}t_0$ *with* $t_0 = 1.85407$ *being the first positive zero of* $\mathrm{cn}(x, q)$ *with* $q^2 = 1/2$.

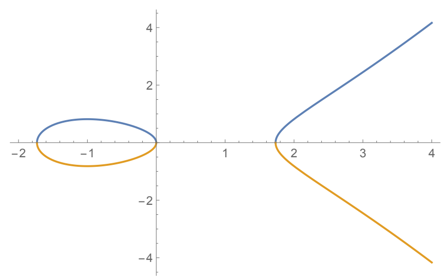

FIGURE 1. Energy surface $\xi^2/2 - (x^3/3 - x) = 0$ of system (13)

Proof. System 13 is equivalent to

$$\begin{cases} \ddot{u}(t) = u^2(t) - 1, \\ u(0) = \sqrt{3}, \quad \dot{u}(0) = 0. \end{cases}$$

Integrating once more we end up with

$$(\dot{u}(t))^2 = 2u(t)\left(\frac{u^2(t)}{3} - 1\right)$$

and, choosing the positive square root (i.e. working with $t \geq 0$), we get

$$\int_{\sqrt{3}}^{u} \frac{ds}{\sqrt{(s-3)s(s+3)}} = \sqrt{2/3}t.$$

On the other hand, from formula (3.131.7) in [11] we have that

$$\int_{\sqrt{3}}^{u} \frac{ds}{\sqrt{(s-3)s(s+3)}} = \frac{\sqrt{2}}{3^{1/4}}F(\mu, q)$$

where $q = \sqrt{1/2}$, $\mu = \arcsin\sqrt{1 - \frac{\sqrt{3}}{u}}$ and $F(\mu, q) = \int_0^q \frac{d\alpha}{\sqrt{1-q^2\sin^2\alpha}}$ is the elliptic integral of the first kind of parameters μ and q. Therefore, combining the two previous identities we get

$$\frac{t}{3^{1/4}} = \int_0^{\arcsin\sqrt{1-\frac{\sqrt{3}}{u}}} \frac{d\alpha}{\sqrt{1 - q^2\sin^2\alpha}}.$$

To conclude the proof, we recall that the Jacobi sine function $\mathrm{sn}(u, q)$ is defined via the equation $\mathrm{sn}(x, q) = \sin\phi$ where

$$x = \int_0^\phi \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}}.$$

Hence, we get $\mathrm{sn}\left(\frac{t}{3^{1/4}}, q\right) = \sqrt{1 - \frac{\sqrt{3}}{u}}$ and simple algebraic simplification yields the exact solution of system (13)

$$u(t) = \frac{\sqrt{3}}{\mathrm{cn}^2\left(\frac{t}{3^{1/4}}, q\right)}, \quad t \in [0, \tau[.$$

□

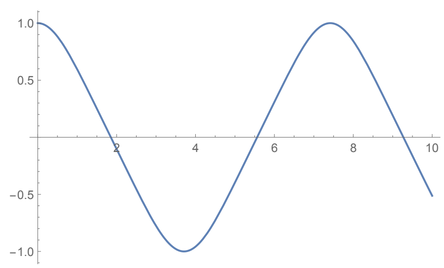

FIGURE 2. graph of $\mathrm{cn}(x, q)$

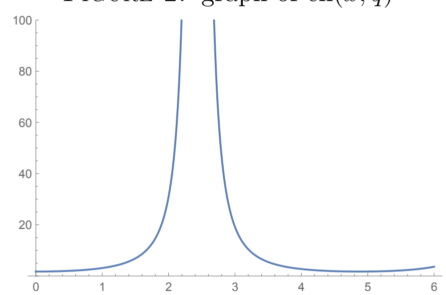

FIGURE 3. graph of the solution of system (13)

We now compare the stochastic solution of system (12) with the deterministic u just found.

Proposition 1. *There exists an event $E \subseteq \Omega$ such that $\mathbb{P}(E) > 0$ and for all $\omega \in E$ we have*

$$x(t) \geq u(t) \quad \text{for all } t < \tau.$$

In particular, if we set

$$\tau_\infty := \lim_{R \to +\infty} \tau_R := \lim_{R \to +\infty} \inf\{t \geq 0 : |x(t)| \geq R\},$$

we obtain $\mathbb{P}(\tau_\infty < +\infty) > 0$.

Proof. We start rewriting (12) as

$$\dot{x}(t) = \int_0^t (x^2(s) - 1)ds + \sigma B(t), \quad x(0) = \sqrt{3} + \delta$$

and (13) as

$$\dot{u}(t) = \int_0^t (u^2(s) - 1)ds, \quad u(0) = \sqrt{3}.$$

Then, denoting $z(t) := x(t) - u(t)$ for $t < \tau$ (remember that $\tau > 0$ is the first pole of the function u), we see that $z(0) = \delta$ and

$$\begin{aligned}\dot{z}(t) &= \int_0^t (x^2(s) - u^2(s))ds + \sigma B(t)\\ &= \int_0^t z(s)(2u(s) + z(s))ds + \sigma B(t).\end{aligned}$$

Integrating once more and using the identity

$$\int_0^t \int_0^s h(r)drds = \int_0^t (t-r)h(r)dr,$$

we get

$$\begin{aligned}z(t) &= \delta + 2\int_0^t (t-r)u(r)z(r)dr + \int_0^t (t-r)z^2(r)dr + \sigma \int_0^t B(r)dr\\ &\geq \delta + 2\int_0^t (t-r)u(r)z(r)dr + \sigma \int_0^t B(r)dr. \qquad (14)\end{aligned}$$

Let us now focus on the term $\int_0^t B(r)dr$ above. We recall that for each $t \geq 0$ the random variable $m_t := \inf_{0\leq s\leq t} B_s$ is continuous with probability density function $x \mapsto \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \mathbf{1}_{]-\infty,0]}(x)$. Therefore, for any $\varepsilon > 0$ the event $E_\varepsilon := \{m_\tau \geq -\varepsilon\}$ has positive probability; moreover, for any $t \in [0, \tau]$ and $\omega \in E_\varepsilon$ we have

$$\int_0^t B(r)dr \geq tm_t \geq tm_\tau \geq -t\varepsilon \geq -\varepsilon\tau$$

and hence

$$\inf_{t\in[0,\tau]} \int_0^t B(r)dr \geq -\varepsilon\tau.$$

In particular,

$$E_\varepsilon \subseteq \left\{\omega \in \Omega : \inf_{0\leq t\leq\tau} \int_0^t B(r)dr \geq -\epsilon\tau\right\} =: F_\varepsilon$$

which implies $\mathbb{P}(F_\varepsilon) > 0$. Now, for $\omega \in F_\varepsilon$ we get from (14) that

$$z(t,\omega) \geq \delta + 2\int_0^t (t-r)u(r)z(r,\omega)dr - \sigma\varepsilon\tau$$

which gives for ε in $]0, \frac{\delta}{2\sigma\tau}[$,

$$z(t,\omega) \geq \delta/2 + 2\int_0^t (t-r)u(r)z(r,\omega)dr. \qquad (15)$$

To conclude, we argue that the continuous function $[0,\tau) \ni t \to z(t,\omega)$ is non-negative for $\omega \in F_\varepsilon$. In fact, if we suppose that there exists $t_1 < \tau$ such that

$z(t_1,\omega) < 0$, then calling t_2 the first zero of $z(t,\omega)$ in the interval $[0,t_1]$ we would get from (15)

$$0 \geq \delta/2 + 2\sqrt{3}\int_0^{t_2}(t_2-r)u(r)z(r,\omega)dr$$

which is impossible since in that interval $(t_2-r)u(r)z(r,\omega)$ is always non-negative. Therefore, the stochastic process $x(t)$ solution of (12) is given by $x(t) = u(t) + z(t)$ for all $t \in [0,\tau[$; furthermore, the non negativity of $z(t)$ on that interval implies $x(t,\omega) \geq u(t)$ for $\omega \in F_\epsilon$. Since, $\lim_{t\to\tau} u(t) = +\infty$, we get $\tau_\infty \leq \tau$ on F_ϵ and the proof is complete. □

Remark 2. Following the details of the proof above, one sees that the explosive behaviour of $x(t)$ is not determined by some specific features of the driving noise. Therefore, a similar argument can be utilized to extend Proposition 1 to equations perturbed by different types of noise.

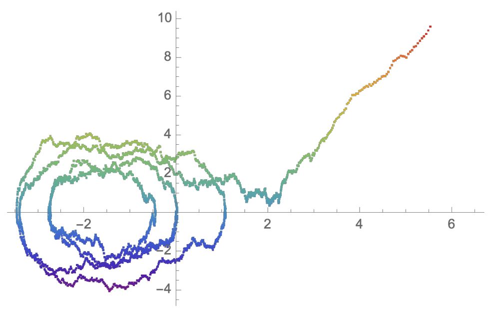

FIGURE 4. Explosive solution for the system (12) with $x(0) = 0$

3. **Small noise expansion: analysis of the coefficients.** In this section we investigate the properties of the coefficients in the expansion (4) described in the Introduction. For a general survey of asymptotic expansions in powers of σ see e.g. Gardiner [10] page 182.

3.1. **The equation for x_0: deterministic case.** We begin with the study of the deterministic system

$$\ddot{x}_0(t) = x_0(t)^2 - \mathcal{B}, \quad x_0(0) = y \quad \dot{x}_0(0) = \eta$$

which is equivalent to

$$\begin{cases} \dot{x}_0(t) = \xi_0(t), & x_0(0) = y \\ \dot{\xi}_0(t) = x_0^2(t) - \mathcal{B}, & \xi_0(0) = \eta. \end{cases} \tag{16}$$

The constant $\mathcal{B}$ and the initial data (y,η) will be chosen in such a way that the third order polynomial $x^3/3 - \mathcal{B}x + H(y,\eta)$ has three real roots. This implies the existence for (16) of a *periodic solution*, whose behavior under the stochastic perturbation is our concern here.

Then, slightly changing our notations, we start directly with the three real roots of the polynomial and we denote them by c, $-a-c$, a with $c < 0 < a$; imposing

$$x^3/3 - \mathcal{B}x + H(y,\eta) = (x-a)(x+a+c)(x-c)/3$$

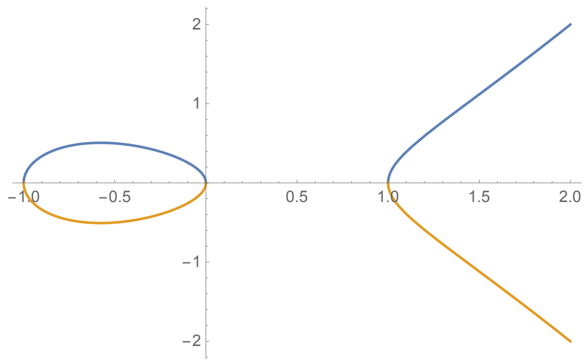

FIGURE 5. Graph of Hamiltonian with $c = -1, a = 1$

we get

$$\mathcal{B} = (a^2 + c^2 + ac)/3 \quad \text{and} \quad H(y, \eta) = ac(a + c)/3.$$

Therefore, the Hamiltonian system we are going to analyze is

$$\begin{cases} \dot{x}_0(t) = \xi_0(t), & x_0(0) = y \\ \dot{\xi}_0(t) = x_0^2(t) - (a^2 + c^2 + ac)/3, & \xi_0(0) = \eta, \end{cases} \tag{17}$$

with $y \in [c, -a - c]$. We assume without loss of generality that $2c + a < 0$, which entails $c < -a - c < a$. Now, rewriting the conservation of energy

$$H(y, \eta) = \dot{x}_0(t)^2/2 - x_0(t)^3/3 + \mathcal{B}x_0(t)$$

as

$$\dot{x}_0(t)^2/2 = x_0(t)^3/3 - \mathcal{B}x_0(t) + H(y, \eta)$$

we get

$$\dot{x}_0(t)^2/2 = (x_0(t) - a)(x_0(t) + a + c)(x_0(t) - c)/3, \quad x_0(0) = y$$

which in turn implies

$$\int_y^{x_0(t)} \frac{dv}{\sqrt{(v - a)(v + a + c)(v - c)}} = \sqrt{2/3}t. \tag{18}$$

The integral in equation (18) is related to elliptic integrals. It is in fact known (see for instance the book by Gradshteyn and Ryzhik [11]) that

$$\int_c^u \frac{dv}{\sqrt{(v - a)(v - b)(v - c)}} = \frac{2}{\sqrt{a - c}} \int_0^\gamma \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}} \tag{19}$$

whenever $c < u \le b < a$. Here γ and q are defined by the formulas

$$\gamma =: \arcsin \sqrt{\frac{u - c}{b - c}} \quad \text{and} \quad q := \sqrt{\frac{b - c}{a - c}}.$$

In the sequel we set

$$F(\gamma, q) := \int_0^\gamma \frac{d\alpha}{\sqrt{1 - q^2 \sin^2 \alpha}} \tag{20}$$

for the so-called elliptic integral of the first kind. We also recall that

$$u = F(\gamma, q) \quad \text{is equivalent to} \quad \mathrm{sn}(u, q) = \sin \gamma, \tag{21}$$

where $\operatorname{sn}(u,q)$ denotes the *Jacobi elliptic sine* function with modulus q. While referring to Gradshteyn and Ryzhik [11] or the website [24] for a complete exposition of the Jacobi elliptic functions, we sum up in the Appendix (A) some of the essential features we will be using in the following. Therefore, comparing (18) with (19) (where we set $b=-a-c$) we get

$$\sqrt{\frac{2}{3}}\,t=\frac{2}{\sqrt{a-c}}F\left(\arcsin\sqrt{\frac{c-x_0(t)}{2c+a}},q\right)-\frac{2}{\sqrt{a-c}}F\left(\arcsin\sqrt{\frac{c-y}{2c+a}},q\right),$$

with $q=\sqrt{\frac{2c+a}{c-a}}$. It is then easy to see that the last identity combined with (21) gives

$$x_0(t)=c-(a+2c)\operatorname{sn}^2\left(\sqrt{\frac{a-c}{6}}t+i_y,q\right), \tag{22}$$

where we denote

$$i_y:=F\left(\arcsin\sqrt{\frac{c-y}{2c+a}},q\right). \tag{23}$$

We have therefore proved the following.

Theorem 3.1. *The unique solution $\{x_0(t)\}_{t\geq 0}$ of the deterministic Hamiltonian system (17) where*

$$c<0<a,\quad 2c+a<0\quad \text{and}\quad y\in[c,-a-c]$$

is explicitly given by formula (22) with $q=\sqrt{\frac{2c+a}{c-a}}$ and i_y defined by (23).

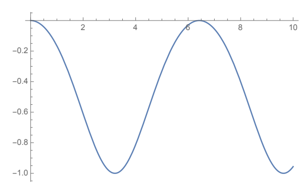

FIGURE 6. Graph of (22) with $c=-1,a=1$

Here the oval part in Figure 1, parametrised by $(x_0,\dot{x}_0)$:

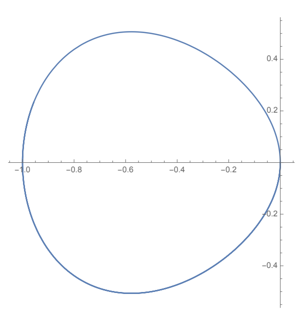

FIGURE 7. Graph of $H(y,\eta)=0$

3.2. **The equation for x_1: a stochastic Lamé's equation.** We now want to solve

$$\ddot{x}_1(t) = 2x_0(t)x_1(t) + \dot{Z}(t), \quad x_1(0) = 0 \quad \dot{x}_1(0) = 0 \tag{24}$$

which is equivalent to the system of stochastic differential equations

$$\begin{cases} dx_1(t) = \xi_1(t)dt, & x_1(0) = 0 \\ d\xi_1(t) = 2x_0(t)x_1(t)dt + dZ(t), & \xi_1(0) = 0 \end{cases}$$

with $x_0(t)$ given by (22). We first investigate the homogeneous equation

$$\ddot{w}(t) - 2x_0(t)w(t) = 0 \tag{25}$$

and search for two independent solutions. We observe that according to formula (22) equation (25) can be rewritten as

$$\ddot{w}(t) - 2\left(c - (a+2c)\mathrm{sn}^2\left(\sqrt{\frac{a-c}{6}}t + i_y, q\right)\right)w(t) = 0.$$

It is then equivalent to study equation

$$\ddot{u}(t) - \frac{12}{a-c}\left(c - (a+2c)\mathrm{sn}^2(t,q)\right)u(t) = 0 \tag{26}$$

and set

$$w(t) := u\left(\sqrt{\frac{a-c}{6}}t + i_y\right), \quad t \geq 0.$$

The first solution we are going to find is related to the Lamé's equation which we now briefly recall. Lamé's equation is usually given as

$$\ddot{u}(t) + (h - \nu(\nu+1)q^2\mathrm{sn}^2(t,q))u(t) = 0. \tag{27}$$

For fixed q and ν an eigenvalue of (27) is a value of h for which (27) has a nontrivial odd or even solution with period $2K$ or $4K$ where $K = K(q) = F(\frac{\pi}{2}, q)$ (recall equality (20)). Comparing (27) with (26) we see that

$$h = \frac{12c}{c-a} = 4 + 4q^2 \tag{28}$$

due to the equality $q = \sqrt{\frac{2c+a}{c-a}}$. Moreover, since $t \mapsto \mathrm{sn}(t,q)$ is periodic of period $4K$ and $t \mapsto \mathrm{sn}^2(t,q)$ is periodic of period $2K$, we conclude that h given in (28) is an eigenvalue of (27) corresponding to case (8) at pag. X in Arscott and Khabaza [5] and hence

$$u_1(t) = \mathrm{sn}(t,q)\mathrm{cn}(t,q)\mathrm{dn}(t,q), \quad t \geq 0 \tag{29}$$

is the first solution of (26) we are looking for. It is a special Lamé's polynomial of order three satisfying $u_1(0) = 0$ and $\dot{u}_1(0) = 1$.

We now need another independent solution. It is a very elementary fact that if u_1 solves the equation $\ddot{u}(t) + \alpha u(t) = 0$, then $u_2(t) := u_1(t) \cdot \int^t \frac{ds}{u_1^2(s)}$ solves the same equation. Since

$$\int \frac{dt}{[\mathrm{sn}(t,q)\mathrm{cn}(t,q)\mathrm{dn}(t,q)]^2} = \frac{C(t,q)}{D(t,q)}$$

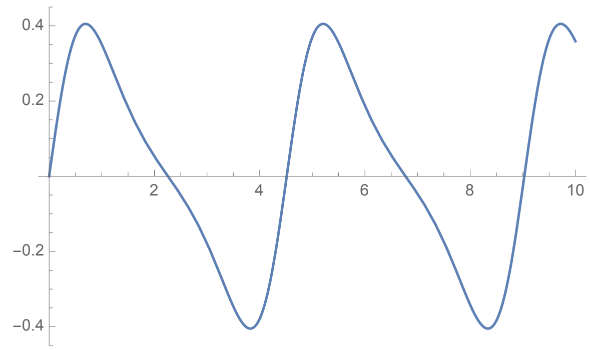

FIGURE 8. Graph of u_1 with $q = 2/\sqrt{5}$

with

$$\begin{aligned} C(t,q) := &- \mathrm{dn}(t,q)\Big[-1+q^2+\Big(2+q^2(-5+(3-2q^2)q^2)\Big)\mathrm{cn}^2(t,q) \\ &+ 2q^2\Big(1+(-1+q^2)q^2\Big)\mathrm{cn}^4(t,q) \\ &+ \Big((2-q^2)(-1+q^2)x+2(q^4-q^2+1)\mathcal{E}(t,q)\Big) \\ &\times \mathrm{sn}(t,q)\mathrm{cn}(t,q)\mathrm{dn}(t,q)\Big] \end{aligned}$$

and

$$D(t,q) := (-1+q^2)^2\mathrm{sn}(t,q)\mathrm{cn}(t,q)\mathrm{dn}^2(t,q)$$

we get that

$$u_2(t) = \mathcal{C}(t,q) + \mathcal{D}(t,q)u_1(t,q) \tag{30}$$

is a second independent solution of (26). Here

$$\begin{aligned} \mathcal{C}(t,q) &:= \alpha_0(q) + \alpha_1(q)\mathrm{cn}^2(t,q) + \alpha_2(q)\mathrm{cn}^4(t,q) \\ \mathcal{D}(t,q) &:= \beta_0(q)t + \beta_1(q)\mathcal{E}(t,q) \end{aligned}$$

and

$$\begin{aligned} \alpha_0(q) &:= -1+q^2 \\ \alpha_1(q) &:= -2q^6+3q^4-5q^2+2 \\ \alpha_2(q) &:= 2q^2(q^4-q^2+1) \\ \beta_0(q) &:= -q^4+3q^2-2 \\ \beta_1(q) &:= 2(q^4-q^2+1). \end{aligned}$$

The coefficient of t in (30) is given by

$$\mu(q) := \beta_0(q) + \beta_1(q)\frac{E(q)}{K(q)}$$

with $E(q)$ denotes the complete elliptic integral of the second kind. This coefficient behaves like this when $0 < q < 1$:

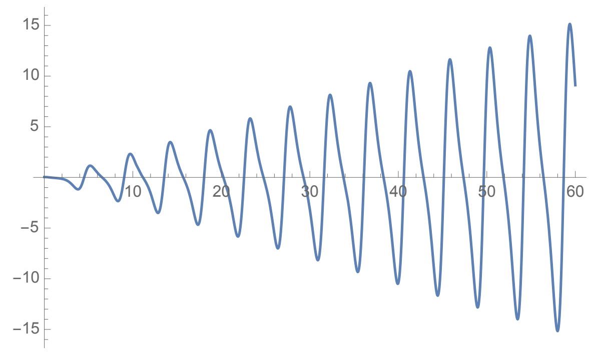

FIGURE 9. Graph of u_2 with $q = 2/\sqrt{5}$

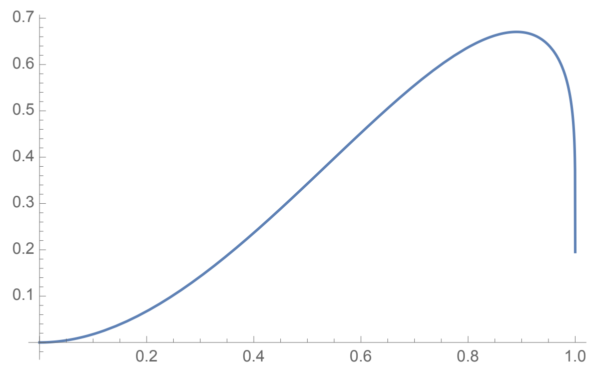

FIGURE 10. Graph of $\mu(q)$ with $q \in (0,1)$

Going back to equation (25) we have found two linearly independent solutions:

$$w_1(t) = u_1\left(\sqrt{\frac{a-c}{6}}t + i_y\right) \quad \text{and} \quad w_2(t) = u_2\left(\sqrt{\frac{a-c}{6}}t + i_y\right).$$

It is easy to verify that the Wronskian determinant of (u_1, u_2) is $-(1-q^2)^2 \neq 0$ and that of (w_1, w_2) is $-\sqrt{\frac{a-c}{6}}(1-q^2)^2$. Without loss of generality multiplying w_1 and w_2 by suitable constants we may assume that their Wronskian determinant is 1. Moreover, since u_1 is periodic of period $2K$ we get that w_1 is periodic of period $2\sqrt{6/(a-c)}K$. A simple application of standard Floquet-Lyapunov results, see e.g Yakubovich and Starzhinskii [23] page 97, tells us that (27) has one periodic solution (here w_1) and it is unstable, due to a double eigenvalue in the monodromy matrix. We omit the trivial details.

Theorem 3.2. *Equation (24) has a unique global strong solution adapted to the filtration $\{\mathcal{F}_t^Z\}_{t\geq 0}$. The solution is a continuous process which can be explicitly represented as*

$$x_1(t) = w_2(t)\int_0^t w_1(s)dZ(s) - w_1(t)\int_0^t w_2(s)dZ(s)$$

or equivalently

$$x_1(t) = \int_0^t \mathcal{K}(t,s)dZ(s) \quad \textit{with} \quad \mathcal{K}(t,s) := w_2(t)w_1(s) - w_1(t)w_2(s). \tag{31}$$

Proof. It is clear from (31) and the basic properties of the stochastic integrals that $\{x_1(t)\}_{t\geq 0}$ is continuous and adapted to the filtration $\{\mathcal{F}_t^Z\}_{t\geq 0}$. We have to verify that it solves equation (24) (uniqueness follows from the linearity of the equation). An application of the Itô formula gives

$$\begin{aligned} dx_1(t) &= \left(\dot{w}_2(t)\int_0^t w_1(s)dZ(s)\right)dt + w_2(t)w_1(t)dZ_t \\ &\quad - \left(\dot{w}_1(t)\int_0^t w_2(s)dZ(s)\right)dt - w_1(t)w_2(t)dZ_t \\ &= \left(\dot{w}_2(t)\int_0^t w_1(s)dZ(s) - \dot{w}_1(t)\int_0^t w_2(s)dZ(s)\right)dt \end{aligned}$$

and hence,

$$\dot{x}_1(t) = \dot{w}_2(t)\int_0^t w_1(s)dZ(s) - \dot{w}_1(t)\int_0^t w_2(s)dZ(s).$$

Then, via a second application of Itô formula (where we now use the facts that w_1 and w_2 solve the homogeneous system and have Wronskian equal to 1) we get

$$\begin{aligned} d\dot{x}_1(t) &= \left(\ddot{w}_2(t)\int_0^t w_1(s)dZ(s)\right)dt + \dot{w}_2(t)w_1(t)dZ(t) \\ &\quad - \left(\ddot{w}_1(t)\int_0^t w_2(s)dZ(s)\right)dt - \dot{w}_1(t)w_2(t)dZ(t) \\ &= \left(\ddot{w}_2(t)\int_0^t w_1(s)dZ(s) - \ddot{w}_1(t)\int_0^t w_2(s)dZ(s)\right)dt + dZ(t) \\ &= \left(2x_0(t)w_2(t)\int_0^t w_1(s)dZ(s) - 2x_0(t)w_1(t)(t)\int_0^t w_2(s)dZ(s)\right)dt + dZ(t) \\ &= 2x_0(t)x_1(t)dt + dZ(t). \end{aligned}$$

The proof is complete. □

Example 1. If we consider

$$\ddot{x}_0(t) = x_0(t)^2 - 1/3, \quad x_0(0) = 0 \quad \dot{x}_0(0) = 0$$

we see that

$$x_0(t) = -(6 - 3\ sn^2(t, 1/2))/2.$$

With such choice for the coefficients equation (24) becomes

$$\ddot{x}_1(t) = -(6 - 3\ sn^2(t,1/2))x_1(t) + \dot{Z}(t), \quad x_1(0) = 0 \quad \dot{x}_1(0) = 0. \tag{32}$$

The picture below gives a clear idea of the oscillatory nature of the solution process (32) for Z being a one dimensional standard Brownian motion.

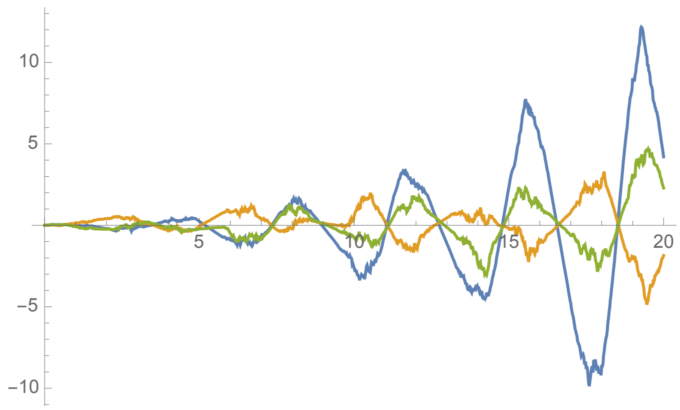

FIGURE 11. Graph of three paths of process (32) with $\{Z(t)\}_{t\geq 0}$ being a one dimensional standard Brownian motion

3.3. **The equations for x_n with $n \geq 2$: Lamé's equations with random coefficients.** We now want to solve

$$\ddot{x}_n(t) = 2x_0(t)x_n(t) + \sum_{j=1}^{n-1} x_j(t)x_{n-j}(t), \quad x_n(0) = 0 \quad \dot{x}_n(0) = 0 \tag{33}$$

which is equivalent to the system of differential equations with random coefficients

$$\begin{cases} \dot{x}_n(t) = \xi_n(t), & x_n(0) = 0 \\ \dot{\xi}_n(t) = 2x_0(t)x_n(t) + \sum_{j=1}^{n-1} x_j(t)x_{n-j}(t), & \xi_n(0) = 0 \end{cases}$$

with $x_0(t)$ given by (22). We remark that the non homogeneous term $\sum_{j=1}^{n-1} x_j(t)x_{n-j}(t)$ depends on the random processes $\{x_m(t)\}_{t\geq 0}$ for $m < n$. Therefore, equation (33) is described inductively by solving the linear equations associated to lower terms in the expansion (4). We have the following.

Theorem 3.3. *For every $n \geq 2$ equation (33) has a unique global strong solution adapted to the filtration $\{\mathcal{F}_t^Z\}_{t\geq 0}$. The solution can be explicitly represented as*

$$\begin{aligned} x_n(t) =& w_2(t) \int_0^t w_1(s) \left(\sum_{j=1}^{n-1} x_j(s)x_{n-j}(s) \right) ds \\ &- w_1(t) \int_0^t w_2(s) \left(\sum_{j=1}^{n-1} x_j(s)x_{n-j}(s) \right) ds \end{aligned}$$

or equivalently

$$x_n(t) = \int_0^t \mathcal{K}(t,s) \left(\sum_{j=1}^{n-1} x_j(s)x_{n-j}(s) \right) ds.$$

Proof. The proof is obtained via straightforward modifications of the proof of Theorem 3.2. □

4. **Estimates for the truncated series: the Brownian case.** In this section we assume $\{Z_t\}_{t\geq 0}$ to be a one dimensional standard Brownian motion which we denote by $\{B_t\}_{t\geq 0}$. This specification will allow us to get explicit estimates on the probabilistic behaviour of the truncated expansion. We start with the following.

Proposition 2. *For any $T>0$ and $n\geq 1$ we have*

$$\begin{aligned}\mathbb{P}&\left(|x_n(t)|\leq\frac{\gamma_n(t)}{\sigma^n}\text{ for all } t\in[0,T]\right)\\&\geq 1-\sigma 2^{(n-2)^+}\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])}+\|w_2\|_{L^2([0,T])}\right)\end{aligned}$$

where $(n-2)^+:=\max\{n-2,0\}$ while $\{\gamma_n\}_{n\geq 1}$ is defined recursively as

$$\gamma_1(t):=|w_1(t)|+|w_2(t)|,\quad t\geq 0 \tag{34}$$

and for $n\geq 2$,

$$\gamma_n(t):=\int_0^t|K(t,s)|\left(\sum_{j=1}^{n-1}\gamma_j(s)\gamma_{n-j}(s)\right)ds,\quad t\geq 0.$$

Proof. We will prove the theorem by induction dividing the proof in two steps.

Step One: $n=1$. We recall that

$$\begin{aligned}x_1(t)&=\int_0^t K(t,s)dB(s)\\&=w_2(t)\int_0^t w_1(s)dB(s)+w_1(t)\int_0^t w_2(s)dB(s).\end{aligned}$$

We fix a positive constant T and observe that

$$\begin{aligned}|x_1(t)|&\leq|w_2(t)|\left|\int_0^t w_1(s)dB(s)\right|+|w_1(t)|\left|\int_0^t w_2(s)dB(s)\right|\\&\leq|w_2(t)|\sup_{t\in[0,T]}\left|\int_0^t w_1(s)dB(s)\right|+|w_1(t)|\sup_{t\in[0,T]}\left|\int_0^t w_2(s)dB(s)\right|.\end{aligned}$$

We now denote

$$A_1:=\left\{\omega\in\Omega:\sup_{t\in[0,T]}\left|\int_0^t w_1(s)dB(s)\right|\leq 1/\sigma\right\}$$

and

$$A_2:=\left\{\omega\in\Omega:\sup_{t\in[0,T]}\left|\int_0^t w_2(s)dB(s)\right|\leq 1/\sigma\right\}.$$

On the set $A_1\cap A_2$ the inequality

$$|x_1(t)|\leq\frac{|w_1(t)|+|w_2(t)|}{\sigma}\qquad\text{for all } t\in[0,T]$$

holds true; we can therefore write recalling (34) that

$$\begin{aligned}\mathbb{P}\left(|x_1(t)|\le\frac{\gamma_1(t)}{\sigma}\text{ for all }t\in[0,T]\right)&\ge\mathbb{P}(A_1\cap A_2)\\&=1-\mathbb{P}\left(A_1^c\cup A_2^c\right)\\&\ge 1-\mathbb{P}\left(A_1^c\right)-\mathbb{P}\left(A_2^c\right).\end{aligned}\tag{35}$$

Now, according to Doob's maximal inequality for $i=1,2$ we have

$$\begin{aligned}\mathbb{P}\left(A_i^c\right)&=\mathbb{P}\left(\sup_{t\in[0,T]}\left|\int_0^t w_i(s)dB(s)\right|>1/\sigma\right)\\&\le\sigma\mathbb{E}\left[\left|\int_0^T w_i(s)dB(s)\right|\right]\\&=\sigma\sqrt{2/\pi}\|w_i\|_{L^2([0,T])}\end{aligned}\tag{36}$$

(the last equality is an explicit evaluation of the $L^1(\Omega)$-norm of the Gaussian random variable $\int_0^T w_i(s)dB(s)$). Hence, combining the upper bound (36) with the lower bound (35) we conclude that

$$\mathbb{P}\left(|x_1(t)|\le\frac{\gamma_1(t)}{\sigma}\text{ for all }t\in[0,T]\right)\ge 1-\sigma\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])}+\|w_2\|_{L^2([0,T])}\right).$$

Step two: $n\ge 2$. We now assume the statement to be true for any $i\le n-1$ and prove it for $i=n$. According to the representation

$$x_n(t)=\int_0^t K(t,s)\left(\sum_{j=1}^{n-1}x_j(s)x_{n-j}(s)\right)ds,\quad t\ge 0$$

we can bound $|x_n(t)|$ as follows

$$|x_n(t)|\le\int_0^t|K(t,s)|\left(\sum_{j=1}^{n-1}|x_j(s)||x_{n-j}(s)|\right)ds.$$

We now denote for $i\le n-1$

$$A_i:=\left\{\omega\in\Omega:|x_i(t)|\le\frac{\gamma_i(t)}{\sigma^i}\text{ for all }t\in[0,T]\right\}$$

and observe that according to the inductive hypothesis

$$\mathbb{P}(A_i)\ge 1-\sigma 2^{(i-2)^+}\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])}+\|w_2\|_{L^2([0,T])}\right)$$

or equivalently,

$$\mathbb{P}(A_i^c)\le\sigma 2^{(i-2)^+}\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])}+\|w_2\|_{L^2([0,T])}\right).\tag{37}$$

We also note that on the set $A_1 \cap \cdots \cap A_{n-1}$ we have

$$\begin{aligned}|x_n(t)| &\leq \int_0^t |K(t,s)| \left(\sum_{j=1}^{n-1} |x_j(s)||x_{n-j}(s)|\right) ds \\ &\leq \int_0^t |K(t,s)| \left(\sum_{j=1}^{n-1} \frac{\gamma_j(s)}{\sigma^j}\frac{\gamma_{n-j}(s)}{\sigma^{n-j}}\right) ds \\ &= \frac{\gamma_n(t)}{\sigma^n}\end{aligned}$$

for all $t \in [0,T]$. Therefore,

$$\begin{aligned}&\mathbb{P}\left(|x_n(t)| \leq \frac{\gamma_n(t)}{\sigma^n} \text{ for all } t \in [0,T]\right) \\ &\geq \mathbb{P}\left(A_1 \cap \cdots \cap A_{n-1}\right) \\ &= 1 - \mathbb{P}\left(A_1^c \cup \cdots \cup A_{n-1}^c\right) \\ &\geq 1 - \mathbb{P}\left(A_1^c\right) - \cdots - \mathbb{P}\left(A_{n-1}^c\right) \\ &= 1 - \sum_{i=1}^{n-1} \mathbb{P}(A_i^c) \\ &\geq 1 - \sigma\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])} + \|w_2\|_{L^2([0,T])}\right)\sum_{i=1}^{n-1} 2^{(i-2)^+} \\ &= 1 - \sigma 2^{(n-2)^+}\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])} + \|w_2\|_{L^2([0,T])}\right)\end{aligned}$$

where in the last inequality we utilized the bound (37). The proof is complete. □

We now prove a lower bound for the probability that the n-th order truncated expansion stays close to the solution of the deterministic equation during a given time interval $[0,T]$.

Theorem 4.1. *For $n \geq 1$ we let*

$$X_n^\sigma(t) := x_0(t) + \sigma x_1(t) + \cdots + \sigma^n x_n(t), \quad t \geq 0.$$

Then, for any $T > 0$ we have

$$\begin{aligned}&\mathbb{P}\left(|X_n^\sigma(t) - x_0(t)| \leq \Gamma_n(t) \text{ for all } t \in [0,T]\right) \\ &\quad\geq 1 - \sigma 2^{(n-1)^+}\sqrt{2/\pi}\left(\|w_1\|_{L^2([0,T])} + \|w_2\|_{L^2([0,T])}\right)\end{aligned}$$

where

$$\Gamma_n(t) := \sum_{i=1}^{n} \gamma_i(t), \quad t \geq 0$$

and $\{\gamma_n\}_{n\geq 1}$ is the sequence of functions defined in Theorem 2.

Proof. We proceed as before. For $i \leq n$ we introduce the events

$$A_i := \left\{\omega \in \Omega : |x_i(t)| \leq \frac{\gamma_i(t)}{\sigma^i} \text{ for all } t \in [0,T]\right\}$$

and observe that according to Theorem 2 we have

$$\mathbb{P}(A_i^c) \leq \sigma 2^{(i-2)^+} \sqrt{2/\pi} \left(\|w_1\|_{L^2([0,T])} + \|w_2\|_{L^2([0,T])} \right).$$

We also note that on the set $A_1 \cap \cdots \cap A_n$ we have

$$\begin{aligned} |X_n^\sigma(t) - x_0(t)| &\leq \sum_{i=1}^n \sigma^i |x_i(t)| \\ &\leq \sum_{i=1}^n \gamma_i(t) \\ &= \Gamma_n(t) \end{aligned}$$

for all $t \in [0,T]$. Therefore,

$$\begin{aligned} &\mathbb{P}\left(|X_n^\sigma(t) - x_0(t)| \leq \Gamma_n(t) \text{ for all } t \in [0,T]\right) \\ &\quad \geq \mathbb{P}\left(A_1 \cap \cdots \cap A_n\right) \\ &\quad = 1 - \mathbb{P}\left(A_1^c \cup \cdots \cup A_n^c\right) \\ &\quad \geq 1 - \mathbb{P}\left(A_1^c\right) - \cdots - \mathbb{P}\left(A_n^c\right) \\ &\quad = 1 - \sum_{i=1}^n \mathbb{P}(A_i^c) \\ &\quad \geq 1 - \sigma \sqrt{2/\pi} \left(\|w_1\|_{L^2([0,T])} + \|w_2\|_{L^2([0,T])} \right) \sum_{i=1}^n 2^{(i-2)^+} \\ &\quad = 1 - \sigma 2^{(n-1)^+} \sqrt{2/\pi} \left(\|w_1\|_{L^2([0,T])} + \|w_2\|_{L^2([0,T])} \right). \end{aligned}$$

□

5. **Estimates for the global series: the bounded case.** In this section we study the global behaviour of the series

$$X^\sigma(t) = x_0(t) + \sum_{n \geq 1} \sigma^n x_n(t) \tag{38}$$

under the additional assumption that for all $t \geq 0$ the random variable $Z(t)$ is bounded. The next theorem shows that in this case the series (38) converges almost surely for all t in a suitably small interval.

Theorem 5.1. *Let $\{Z_t\}_{t \geq 0}$ satisfy for all $t \geq 0$ the condition $Z(t) \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exists $T_\sigma > 0$ such that the series (38) converges almost surely for any $t \in [0, T_\sigma]$. More precisely, the uniform bound*

$$\sup_{t \in [0,T_\sigma]} |X^\sigma(t) - x_0(t)| \leq \frac{1}{2\mathcal{N}(T_\sigma)} \qquad \text{almost surely,} \tag{39}$$

or equivalently,

$$\sup_{t \in [0,T_\sigma]} |X^\sigma(t)| \leq \sup_{t \in [0,T_\sigma]} |x_0(t)| + \frac{1}{2\mathcal{N}(T_\sigma)} \qquad \text{almost surely}$$

holds with

$$\mathcal{N}(T) := \|w_2\|_{L^\infty([0,T])} \|w_1\|_{L^1([0,T])} + \|w_1\|_{L^\infty([0,T])} \|w_2\|_{L^1([0,T])}.$$

Proof. We start as before observing that

$$\begin{aligned} x_1(t) &= \int_0^t \mathcal{K}(t,s)dZ(s) \\ &= -\int_0^t \partial_2\mathcal{K}(t,s)Z(s)ds \end{aligned}$$

(recall the kernel $\mathcal{K}$ defined in (31)). This implies

$$\begin{aligned} |x_1(t)| &\leq \int_0^t |\partial_2\mathcal{K}(t,s)||Z(s)|ds \\ &= \sup_{s\in[0,t]}\|Z(s)\|_{L^\infty(\Omega)}\int_0^t |\partial_2\mathcal{K}(t,s)|ds \\ &= \sup_{s\in[0,t]}\|Z(s)\|_{L^\infty(\Omega)}\left(|\dot{w}_2(t)|\int_0^t |w_1(s)|ds + |\dot{w}_1(t)|\int_0^t |w_2(s)|ds\right) \\ &\leq \sup_{s\in[0,t]}\|Z(s)\|_{L^\infty(\Omega)}\left(\|\dot{w}_2\|_{L^\infty([0,t])}\|w_1\|_{L^1([0,t])} + \|\dot{w}_1\|_{L^\infty([0,t])}\|w_2\|_{L^1([0,t])}\right) \\ &= \mathcal{M}(t) \end{aligned}$$

where to ease the notation we set

$$\mathcal{M}(t) := \sup_{s\in[0,t]}\|Z(s)\|_{L^\infty(\Omega)}\left(\|\dot{w}_2\|_{L^\infty([0,t])}\|w_1\|_{L^1([0,t])} + \|\dot{w}_1\|_{L^\infty([0,t])}\|w_2\|_{L^1([0,t])}\right).$$

The first step is to prove by induction that for any fixed $T > 0$ we have

$$|x_n(t)| \leq c_n\mathcal{M}(T)^n\mathcal{N}(T)^{n-1} \quad \text{almost surely for all } n \geq 1 \text{ and } t \in [0,T] \tag{40}$$

where $\{c_n\}_{n\geq 1}$ denotes the sequence of *Catalan numbers* which are defined recursively as

$$c_1 = 1, \quad c_2 = 1, \quad c_n := \sum_{j=1}^{n-1} c_j c_{n-j}.$$

Inequality (40) is trivially true for $n = 1$ (note that the function $t \mapsto \mathcal{M}(t)$ is increasing). We now assume the property to be true for all $j \leq n-1$; then,

$$\begin{aligned} |x_n(t)| &= \left|\int_0^t \mathcal{K}(t,s)\left(\sum_{j=1}^{n-1} x_j(s)x_{n-j}(s)\right)ds\right| \\ &\leq \int_0^t |\mathcal{K}(t,s)|\left(\sum_{j=1}^{n-1}|x_j(s)||x_{n-j}(s)|\right)ds \\ &\leq \int_0^t |\mathcal{K}(t,s)|\left(\sum_{j=1}^{n-1} c_j\mathcal{M}(T)^j\mathcal{N}(T)^{j-1}c_{n-j}\mathcal{M}(T)^{n-j}\mathcal{N}(T)^{n-j-1}\right)ds \\ &= \mathcal{M}(T)^n\mathcal{N}(T)^{n-2}\int_0^t |\mathcal{K}(t,s)|\left(\sum_{j=1}^{n-1} c_jc_{n-j}\right)ds \\ &\leq c_n\mathcal{M}(T)^n\mathcal{N}(T)^{n-1} \end{aligned}$$

completing the proof of (40). Therefore, recalling that

$$\sum_{n\geq 1} c_n y^n = \frac{1-\sqrt{1-4y}}{2}, \quad y \in [0, 1/4]$$

we can write for $t \in [0, T]$ that

$$\begin{aligned} |X^\sigma(t) - x_0(t)| &\leq \sum_{n\geq 1} \sigma^n |x_n(t)| \\ &\leq \sum_{n\geq 1} \sigma^n c_n \mathcal{M}(T)^n \mathcal{N}(T)^{n-1} \\ &= \frac{1}{\mathcal{N}(T)} \sum_{n\geq 1} c_n \sigma^n \mathcal{M}(T)^n \mathcal{N}(T)^n \\ &= \frac{1-\sqrt{1-4\sigma\mathcal{M}(T)\mathcal{N}(T)}}{2\mathcal{N}(T)} \end{aligned} \tag{41}$$

provided that

$$\mathcal{M}(T)\mathcal{N}(T) \leq \frac{1}{4\sigma}.$$

Hence, since $\mathcal{M}(0) = \mathcal{N}(0) = 0$ and the functions $T \mapsto \mathcal{M}(T)$ and $T \mapsto \mathcal{N}(T)$ are increasing, continuous and unbounded (by the properties of w_1 and w_2), we deduce that the equation

$$\mathcal{M}(T)\mathcal{N}(T) = \frac{1}{4\sigma}$$

has a unique solution $T_\sigma > 0$ and that $\mathcal{M}(T)\mathcal{N}(T) \leq \frac{1}{4\sigma}$ for all $T \leq T_\sigma$. Therefore, choosing $T = T_\sigma$ in (41) we obtain

$$\sup_{t\in[0,T_\sigma]} |X^\sigma(t) - x_0(t)| \leq \frac{1}{2\mathcal{N}(T_\sigma)} \qquad \text{almost surely.}$$

□

Example 2. We may choose

$$Z(t) := \sin(B(t))e^t, \quad t \geq 0$$

where $\{B(t)\}_{t\geq 0}$ is a one dimensional standard Brownian motion. The process $\{Z(t)\}_{t\geq 0}$ is clearly continuous and starts at zero; moreover, using the Itô formula one verifies immediately the martingale property. In this case we have

$$\sup_{s\in[0,t]} \|Z(s)\|_{L^\infty(\Omega)} = e^t, \quad t \geq 0.$$

Appendix A. **Appendix.** Here we collect very briefly a number of elementary identities and formulas for some of the special functions needed. In particular, the function defined in (30) and the second solution of the deterministic Lamé equation relies on the Jacobi Epsilon function, which is detailed below. Of course a very comprehensive collection of results for Jacobian Elliptic Functions is contained in the website [24].

$$\begin{aligned}
\mathrm{sn}^2(x,q)+\mathrm{cn}^2(x,q)&=1\\
\mathrm{dn}^2(x,q)+q^2\mathrm{sn}^2(x,q)&=1\\
\mathrm{dn}^2(x,q)-q^2\mathrm{cn}^2(x,q)&=1-q^2=q'^2\\
\mathrm{am}(x,q)&=\int_0^x \mathrm{dn}(t,q)dt\\
\mathrm{sn}(x,q)&=\sin(\mathrm{am}(x,q))\\
E(\phi,q)&=\int_0^\phi \sqrt{1-q^2\sin^2\theta}d\theta.
\end{aligned}$$

Jacobi's Epsilon function:

$$\mathcal{E}(x,q)=\int_0^x \mathrm{dn}^2(t,q)dt\,(=E(\mathrm{am}(x,q),q)\ \text{ for } -K\le x\le K)\,.$$

Asymptotics of Jacobi's Epsilon function and relation to Theta functions:

$$\mathcal{E}(x,q)=\frac{E(q)}{K(q)}x+\frac{\frac{d\theta_4}{d\xi}((\xi,p))}{\theta_3^2(0,p)\theta_4(\xi,p)}\ , \tag{42}$$

with $\xi=x/\theta_3^2(0,p)$ and $p=\exp\{-\pi K'(q)/K(q)\}$, $K'(q)=K(q')$, $q'^2+q^2=1$. The logarithmic derivative in (42) can be expressed as:

$$\frac{\frac{d\theta_4}{dz}(z,p)}{\theta_4(z,p)}=4\sum_{n=1}^{\infty}\frac{p^n}{1-p^{2n}}\sin(2nz)$$

REFERENCES

[1] S. Albeverio, A. Hilbert and E. Zehnder, Hamiltonian systems with a stochastic force: nonlinear versus linear, and a Girsanov formula, *Stochastics Stochastics Rep.* **39** 2-3 (1992) 159-188

[2] S. Albeverio, A. Hilbert and V. Kolokoltsov, Estimates uniform in time for the transition probability of diffusions with small drift and for stochastically perturbed Newton equations, *J. Theoret. Probab.* **12** 2 (1999) 293-300

[3] J. Appleby, X. Maoand A. Rodkina, Stabilization and destabilization of nonlinear differential equations by noise, *IEEE Trans. Automat. Control* **53** 3 (2008) 683-691.

[4] F.M. Arscott, *Periodic Differential Equations*, The Macmillan Company, New York, 1964

[5] F.M. Arscott and I.M.Khabaza, *Table of Lamé's Polynomials*, Pergamon Press, Oxford, London, New York, Paris, 1962

[6] E. Bernardi and T. Nishitani, On the Cauchy problem for non-effectively hyperbolic operators, the Gevrey 5 well-posedness, *J. d'Analyse Math.* **105** (2008) 197–240.

[7] E. Bernardi and T. Nishitani, On the Cauchy problem for non-effectively hyperbolic operators, the Gevrey 4 well-posedness, *Kyoto J. Math.* **51** (2011) 767–810.

[8] E. Delabaere and D.T. Trinh, Spectral analysis of the complex cubic oscillator, *J. Phys. A: Math.gen.* **33** 2 (2000) 8771-8796

[9] E.M. Ferreira and J. Sesma, Global solution of the cubic oscillator, *J. of Phys. A: Math.* **47** (2014) 415306

[10] C.W. Gardiner, *Handbook of Stochastic Methods* , Third Edition, Springer-Verlag, New York, 2004.

[11] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Elsevier, 7th Edition, 2007

[12] L. Hörmander, The Cauchy problem for differential equations with double characteristics, *Journal D'Analyse Mathématique*, **32**, (1977), 118-196.

[13] N. Ikeda and S.Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North Holland, Amsterdam, New York, Oxford, Kodansha, 1981

[14] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Springer-Verlag, New York, 1991

[15] R. Khasminskii, *Stochastic Stability of Differential Equations*,Second Edition, Springer-Verlag, Berlin, 2012.

[16] L. Markus and A. Weerasinghe, Stochastic oscillators, *J. Differential Equations* **71** 2 (1988) 288-314

[17] L Markus and A. Weerasinghe, Stochastic nonlinear oscillators, *Adv. in Appl. Probab.* **25** 3 (1993) 649-666

[18] T.Nishitani, A simple proof of the existence of tangent bicharacteristics for noneffectively hyperbolic operators *Kyoto J. Math.*. **55** (2015) 2 281–297

[19] Frank W.J.Olver and Daniel W.Lozier, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010

[20] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Third Edition, Springer-Verlag, Berlin, 1999.

[21] H. Volker, Four Remarks on Eigenvalues of Lamé's Equation, *Analysis and Applications* **2** (2004) 161-175

[22] Weinan E, Tiejun Li, E. Vanden-Eijnden, *Applied Stochastic Analysis*, Graduate Studies in Mathematics, 199, American Mathematical Society, 2019.

[23] V.A. Yakubovich and V.M. Starzhinskii, Linear Differential Equations With Periodic Coefficients vol.1, John Wiley & Sons, New York, 1975

[24] `http://dlmf.nist.gov/22`

[25] `http://dlmf.nist.gov/29`

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