Santosh B. Joshi, Sayali S. Joshi, Haridas H. Pawar and Daniele Ritelli*

Connections between normalized Wright functions with families of analytic functions with negative coefficients

https://doi.org/10.1515/anly-2021-1027
Received December 20, 2021; revised February 1, 2022; accepted February 9, 2022

Abstract: In this article we present sufficient conditions that ensures that normalized Wright functions belong to certain subclasses of analytic univalent functions with negative coefficients in the unit disc $\mathcal{U}$. We also provide some geometric properties of integral transforms involving normalized Wright functions.

Keywords: Wright functions, Bessel functions, analytic functions

MSC 2010: 33E12 , 33C10, 30H99

1 Introduction

Let $\mathcal{A}$ denote the set all analytic normalized functions in the unit disc $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$, i.e., such that $f(0) = 0$ and $f'(0) = 1$. Therefore $f \in \mathcal{A}$ has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Also, let $\mathcal{S}$ denote the subset of $\mathcal{A}$ consisting of functions which are univalent in $\mathcal{U}$.

Definition 1. A function $f \in \mathcal{A}$ is said to be *starlike of order* α $(0 \leq \alpha < 1)$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U},$$

and is said to be *convex of order* α $(0 \leq \alpha < 1)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathcal{U}.$$

We denote these classes by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. Note that $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of *starlike and convex functions*, respectively (for details, see [3], [20], [21] and [24]).

Kulkarni in [9] introduced a subset of $\mathcal{S}$, say $\mathcal{D}(\alpha, \beta, \gamma)$, defined as follows:

Definition 2. We define $\mathcal{D}(\alpha, \beta, \gamma)$ to be the subset of $\mathcal{S}$ consisting of the functions f that for any $z \in \mathcal{U}$ satisfy the inequality

$$\left| \frac{f'(z) - 1}{2\gamma(f'(z) - \alpha) - (f'(z) - 1)} \right| < \beta,$$

where $0 < \beta \leq 1$, $0 \leq \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \leq \gamma \leq 1$.

***Corresponding author: Daniele Ritelli,** Department of Statistical Sciences, University of Bologna, Bologna, Italy, e-mail: daniele.ritelli@unibo.it. https://orcid.org/0000-0001-8805-8132
Santosh B. Joshi, Department of Mathematics, Walchand College of Engineering, Sangli, India, e-mail: santosh.joshi@walchandsangli.ac.in
Sayali S. Joshi, Department of Mathematics, Sanjay Bhokare Group of Institutes, Miraj, India, e-mail: joshiss@sbgimiraj.org
Haridas H. Pawar, Department of Mathematics, Sveri's College of Engineering, Pandharpur, India, e-mail: haridas_pawar007@yahoo.co.in

Remark 1. In [22] Silverman introduced the subset T of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0.$$

Kulkarni in [9] introduced and studied the subset $\mathcal{P}^*(\alpha, \beta, \gamma) = T \cap \mathcal{D}(\alpha, \beta, \gamma)$. For suitable specializations of α, β and γ, several notable subsets of T were previously studied by various researchers:

(i) For $\gamma = 1$, we obtain the class $\mathcal{P}^*(\alpha, \beta, 1) = P^*(\alpha, \beta)$ of functions $f(z) \in T$ satisfying the condition

$$\left|\frac{f'(z) - 1}{f'(z) + 1 - 2\alpha}\right| < \beta, \quad z \in \mathcal{U},$$

defined by Gupta and Jain [6].

(ii) For $\alpha = 0$ and $\gamma = 1$, we obtain the class $\mathcal{P}^*(0, \beta, 1) = D^*(\beta)$ of functions $f(z) \in T$ satisfying the condition

$$\left|\frac{f'(z) - 1}{f'(z) + 1}\right| < \beta, \quad z \in \mathcal{U},$$

introduced and studied by Kim and Lee [8].

The following lemma, due to [9]), is essential for our results:

Lemma 1. *A function $f \in T$ belongs to $\mathcal{P}^*(\alpha, \beta, \gamma)$ if and only if*

$$\sum_{n=2}^{\infty} [1 + \beta(1 - 2\gamma)] n a_n \leq 2\beta\gamma(1 - \alpha). \tag{1.1}$$

Moreover, inequality (1.1) *is sharp.*

It is well known that special functions play an significant role in geometric function theory. Several special functions, called recently special functions of fractional calculus, play a very important and interesting role as solutions of fractional order differential equations, such as the Mittag-Leffler function, Wright function with its auxiliary functions, and Fox's H-function (refer to [4] and [12] for a comprehensive treatment.)

Wright function plays an important role in the solution of linear partial fractional differential equations; in fact to solve the boundary-value problems for the fractional diffusion-wave equation, i.e., the linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first or second-order time derivative by a fractional derivative of order α with $0 < \alpha \leq 2$, it known that the corresponding Green functions can indeed be represented in terms of the Wright function (these aspects are presented in [5] and [7]).

We recall, for completeness, the definition of *Wright function* $W_{\lambda,\mu}(z)$ introduced by Wright in [25]:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n + \mu)} \frac{z^n}{n!}, \quad \lambda > -1,\ \mu, z \in \mathbb{C}. \tag{1.2}$$

Series (1.2) is absolutely convergent in $\mathbb{C}$ when $\lambda > -1$ and absolutely convergent in the open unit disc for $\lambda = -1$. Also, for $\lambda > -1$, the Wright function is an entire function.

If λ is a positive rational number, then $W_{\lambda,\mu}(z)$ can be represented in terms of the more familiar generalized hypergeometric functions (see [5, Section 2.1]). In particular, when $\lambda = 1$ and $\mu = p + 1$, the functions $W_{1,p+1}(-\frac{z^2}{4})$ are expressed in terms of the Bessel functions $J_p(z)$ as

$$J_p(z) = \left(\frac{z}{2}\right)^p W_{1,p+1}\left(\frac{-z^2}{4}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\frac{z}{2})^{2n+p}}{\Gamma(n + p + 1)}.$$

We point out that $W_{\lambda,p+1}(-z) \equiv J_p^{\lambda}(z)$ $(\lambda > 0,\ p > -1)$ is known as the generalized Bessel function (sometimes it is also called the Bessel–Wright function). We refer to [5] for a comprehensive exposition of Wright's functions.

Many researchers have investigated classes of analytic functions involving special function $F \subset \mathcal{A}$, to find different conditions such that the members of F have certain geometric properties such as starlikeness and convexity in $\mathcal{U}$. There is widespread literature dealing with various properties, generalizations and applications of different types of hypergeometric functions, especially for the generalized Gaussian, Kummer hypergeometric and Bessel functions (see [1], [10] and [15–18]).

Note that $W_{\lambda,\mu}(z)$ as defined by (1.2) does not belong to the class T. Thus it is natural to consider the following normalizations of the Wright function:

$$W^{(1)}(\lambda,\mu;z) := 2z - \Gamma(\mu) z W_{\lambda,\mu}(z) = 2z - \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \frac{z^{n+1}}{n!}, \quad z \in U,\ \lambda > -1,\ \mu > 0$$

and

$$W^{(2)}(\lambda,\mu;z) := 2z - \Gamma(\lambda+\mu)\Big[W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)}\Big] = 2z - \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda n+\lambda+\mu)} \frac{z^{n+1}}{(n+1)!}, \quad z \in U,\ \lambda > -1\ (\lambda+\mu) > 0.$$

From this, we can easily write

$$W^{(1)}(\lambda,\mu;z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{z^{n}}{(n-1)!}, \quad z \in U,\ \lambda > -1,\ \mu > 0, \tag{1.3}$$

and

$$W^{(2)}(\lambda,\mu;z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^{n}}{n!}, \quad z \in U,\ \lambda > -1\ (\lambda+\mu) > 0. \tag{1.4}$$

Furthermore, we observe that the normalized Wright functions $W^{(1)}(\lambda,\mu;z)$ and $W^{(2)}(\lambda,\mu;z)$ verify the following relations:

$$\begin{aligned} \lambda z(W^{(1)}(\lambda,\mu;z))' &= (\mu-1)W^{(1)}(\lambda,\mu-1;z) + (\lambda-\mu+1)W^{(1)}(\lambda,\mu;z), \\ \lambda z(W^{(2)}(\lambda,\mu;z))' &= (\lambda+\mu-1)W^{(2)}(\lambda,\mu-1;z) + (1-\mu)W^{(2)}(\lambda,\mu;z), \\ z(W^{(2)}(\lambda,\mu;z))' &= W^{(1)}(\lambda,\lambda+\mu;z). \end{aligned}$$

In [22] Silverman studied starlikeness and convexity properties for hypergeometric functions, giving necessary and sufficient conditions for hypergeometric functions $F(a,b;c;z)$ to be in the classes $S^*(\alpha)$ and $C(\alpha)$, examining also a linear operator acting on hypergeometric functions. Geometric properties of the normalized Wright functions were studied extensively by various authors like Baricz, Toklu and Kadioğlu [2], El-Shahed and Salem [4], Kilbas, Saigo and Trujillo [7], Mustafa [11], Mustafa and Altintas [12] and Prajapat [19]. In fact, the more generalized Fox–Wright hypergeometric functions were studied by Srivastava [23].

The contribution of the present article consists in establishing sufficient conditions so that the normalized Wright functions $W^{(1)}(\lambda,\mu;z)$ and $W^{(2)}(\lambda,\mu;z)$ belong to the class $\mathcal{P}^*(\alpha,\beta,\gamma)$. Furthermore, we have obtained sufficient conditions for the integral transforms involving the normalized Wright functions $W^{(1)}(\lambda,\mu;z)$ and $W^{(2)}(\lambda,\mu;z)$ belong to $\mathcal{P}^*(\alpha,\beta,\gamma)$.

2 Main results

Unless otherwise mentioned, we assume throughout this paper that $\mu > 0$, $\lambda > -1$, $0 < \beta \le 1$, $0 \le \alpha < \frac{1}{2\gamma}$ and $\frac{1}{2} \le \gamma \le 1$. Moreover, we recall the Pochhammer symbol $(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(n)} = \mu(\mu+1)(\mu+2)\cdots(\mu+n-1)$, $(\mu)_0 = 1$ defined in terms of the Euler gamma function.

2.1 Sufficient conditions for the Wright functions to be in the class $\mathcal{P}^*(\alpha,\beta,\gamma)$

Theorem 1. *Assume $\lambda \ge 1$, $\mu > 0$ and we assume also that*

$$2\mu\beta\gamma(1-\alpha) + (\mu+1)[1+\beta(1-2\gamma)] - [1+\beta(1-2\gamma)](\mu+2)e^{(\frac{1}{\mu+1})} \ge 0. \tag{2.1}$$

Then the normalized Wright function $W^{(1)}(\lambda,\mu;z)$ belongs to the class $\mathcal{P}^*(\alpha,\beta,\gamma)$.

Proof. Since

$$W^{(1)}(\lambda,\mu;z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{z^n}{(n-1)!},$$

according to Lemma 1, we need to show that

$$\sum_{n=2}^{\infty}n[1+\beta(1-2\gamma)]\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{1}{(n-1)!}\le 2\beta\gamma(1-\alpha). \tag{2.2}$$

For this purpose let

$$\sum_{n=2}^{\infty}n[1+\beta(1-2\gamma)]\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{1}{(n-1)!}=:T_1.$$

Hence

$$\begin{aligned}T_1&=[1+\beta(1-2\gamma)]\sum_{n=2}^{\infty}n\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{1}{(n-1)!}\\&=[1+\beta(1-2\gamma)]\left[\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{1}{(n-2)!}+\sum_{n=2}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\frac{1}{(n-1)!}\right].\end{aligned}$$

For every $n\in\mathbb{N}_2:=\mathbb{N}\setminus\{1\}=\{2,3,\dots\}$, the inequality $\Gamma(\lambda(n-1)+\mu)\ge\Gamma(n-1+\mu)$ holds true. Using the Pochhammer notations $\Gamma(n-1+\mu)=\Gamma(\mu)\,(\mu)_{n-1}$, we have

$$\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}\le\frac{1}{(\mu)_{n-1}},\qquad n\in\mathbb{N}_2. \tag{2.3}$$

Thus from (2.3), we get

$$T_1\le[1+\beta(1-2\gamma)]\left[\sum_{n=2}^{\infty}\frac{1}{(\mu)_{n-1}}\frac{1}{(n-2)!}+\sum_{n=2}^{\infty}\frac{1}{(\mu)_{n-1}}\frac{1}{(n-1)!}\right].$$

Also, from the inequality $(\mu)_{n-1}=\mu(\mu+1)(\mu+2)\cdots(\mu+n-2)\ge\mu(\mu+1)^{n-2}$, $n\in\mathbb{N}_2$, we arrive at

$$\frac{1}{(\mu)_{n-1}}\le\frac{1}{\mu(\mu+1)^{n-2}},\qquad n\in\mathbb{N}_2. \tag{2.4}$$

Now, using (2.4), we have

$$\begin{aligned}T_1&\le[1+\beta(1-2\gamma)]\left[\sum_{n=2}^{\infty}\frac{1}{\mu(\mu+1)^{n-2}}\frac{1}{(n-2)!}+\sum_{n=2}^{\infty}\frac{1}{\mu(\mu+1)^{n-2}}\frac{1}{(n-1)!}\right]\\&\le[1+\beta(1-2\gamma)]\left\{\frac{1}{\mu}e^{\left(\frac{1}{\mu+1}\right)}+\frac{\mu+1}{\mu}\left[e^{\left(\frac{1}{\mu+1}\right)}-1\right]\right\}.\end{aligned}$$

In view of (2.1) we can eventually write

$$[1+\beta(1-2\gamma)]\left\{\frac{(\mu+2)}{\mu}e^{\left(\frac{1}{\mu+1}\right)}-\frac{(\mu+1)}{\mu}\right\}\le 2\beta\gamma(1-\alpha),$$

which implies (2.2). This completes the proof. □

Theorem 2. *Let $\lambda\ge1$, $\mu>0$ and assume that the following condition is satisfied:*

$$\begin{aligned}2(\lambda+\mu)(\lambda+\mu+1)\beta\gamma(1-\alpha)-[1+\beta(1-2\gamma)](\lambda+\mu+1)^2(\lambda+\mu+2)e^{\left(\frac{1}{\lambda+\mu+1}\right)}&\\+[1+\beta(1-2\gamma)]((\lambda+\mu+1)^3+(\lambda+\mu+1)^2+(\lambda+\mu+1))\ge0.&\end{aligned} \tag{2.5}$$

Then the normalized Wright function $W^{(2)}(\lambda,\mu;z)$ belongs to the class $\mathcal{P}^*(\alpha,\beta,\gamma)$.

Proof. Since

$$W^{(2)}(\lambda,\mu;z)=z-\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{z^n}{n!},$$

according to Lemma 1, we need to show that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!} \le 2\beta\gamma(1-\alpha). \tag{2.6}$$

Let

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)]\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!} =: T_2.$$

Hence

$$\begin{aligned} T_2 &= [1+\beta(1-2\gamma)]\sum_{n=2}^{\infty} n\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!} \\ &= [1+\beta(1-2\gamma)]\left[\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{(n-1)!} + \sum_{n=2}^{\infty}\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)}\frac{1}{n!}\right]. \end{aligned}$$

For every $n \in \mathbb{N}_2 := \mathbb{N}\setminus\{1\}$, the inequality $\Gamma(\lambda(n-1)+\lambda+\mu) \ge \Gamma(n-1+\lambda+\mu)$ holds true. Since

$$\Gamma(n-1+\lambda+\mu) = \Gamma(\lambda+\mu)(\lambda+\mu)_{n-1},$$

we have

$$\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \le \frac{1}{(\lambda+\mu)_{n-1}}, \quad n \in \mathbb{N}_2. \tag{2.7}$$

Using (2.7), we get

$$T_2 \le [1+\beta(1-2\gamma)]\left[\sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)_{n-1}}\frac{1}{(n-1)!} + \sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)_{n-1}}\frac{1}{n!}\right].$$

Also, the inequality

$$(\lambda+\mu)_{n-1} = (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-2) \ge (\lambda+\mu)(\lambda+\mu+1)^{n-2}, \quad n \in \mathbb{N}_2,$$

holds and is equivalent to

$$\frac{1}{(\lambda+\mu)_{n-1}} \le \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}, \quad n \in \mathbb{N}_2. \tag{2.8}$$

From (2.8), we infer

$$\begin{aligned} T_2 &\le [1+\beta(1-2\gamma)]\left[\sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}\frac{1}{(n-1)!} + \sum_{n=2}^{\infty}\frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}\frac{1}{n!}\right] \\ &\le [1+\beta(1-2\gamma)]\left\{\frac{\lambda+\mu+1}{\lambda+\mu}\left(e^{(\frac{1}{\lambda+\mu+1})}-1\right) + \frac{(\lambda+\mu+1)^2}{(\lambda+\mu)}\left(e^{(\frac{1}{\lambda+\mu+1})} - \frac{1}{(\lambda+\mu+1)} - 1\right)\right\}. \end{aligned}$$

In view of (2.5) we can write

$$[1+\beta(1-2\gamma)]\left\{\frac{\lambda+\mu+1}{\lambda+\mu}\left(e^{(\frac{1}{\lambda+\mu+1})}-1\right) + \frac{(\lambda+\mu+1)^2}{(\lambda+\mu)}\left(e^{(\frac{1}{\lambda+\mu+1})} - \frac{1}{(\lambda+\mu+1)} - 1\right)\right\} \le 2\beta\gamma(1-\alpha),$$

which implies that (2.6) holds true. This completes the proof. □

By setting $\gamma = 1$ in Theorem 1 and Theorem 2, we obtain two corollaries.

Corollary 1. *The normalized Wright function $W^{(1)}(\lambda,\mu;z)$ belongs to the class $P^*(\alpha,\beta)$ if $\lambda \ge 1$, $\mu > 0$ and the following condition is satisfied:*

$$2\mu\beta(1-\alpha) + (\mu+1)[1-\beta] - [1-\beta](\mu+2)e^{(\frac{1}{\mu+1})} \ge 0.$$

Corollary 2. *The normalized Wright function $W^{(2)}(\lambda,\mu;z)$ belongs to the class $P^*(\alpha,\beta)$ if $\lambda \ge 1$, $\mu > 0$ and the following condition is satisfied:*

$$\begin{aligned} &2(\lambda+\mu)(\lambda+\mu+1)\beta(1-\alpha) - [1-\beta](\lambda+\mu+1)^2(\lambda+\mu+2)e^{(\frac{1}{\lambda+\mu+1})} \\ &\quad + [1-\beta]\big((\lambda+\mu+1)^3 + (\lambda+\mu+1)^2 + (\lambda+\mu+1)\big) \ge 0. \end{aligned}$$

Taking $\alpha = 0$ and $\gamma = 1$ in Theorem 1 and Theorem 2, we obtain two more corollaries.

Corollary 3. *The normalized Wright function $W^{(1)}(\lambda, \mu; z)$ belongs to the class $D^*(\beta)$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$2\mu\beta + (\mu + 1)[1 - \beta] - [1 - \beta](\mu + 2)e^{(\frac{1}{\mu+1})} \geq 0.$$

Corollary 4. *The normalized Wright function $W^{(2)}(\lambda, \mu; z)$ belongs to the class $D^*(\beta)$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$\begin{aligned}2(\lambda + \mu)(\lambda + \mu + 1)\beta - [1 - \beta](\lambda + \mu + 1)^2(\lambda + \mu + 2)e^{(\frac{1}{\lambda+\mu+1})}\\ + [1 - \beta]\big((\lambda + \mu + 1)^3 + (\lambda + \mu + 1)^2 + (\lambda + \mu + 1)\big) \geq 0.\end{aligned}$$

3 Sufficient conditions for the integrals involving normalized Wright functions

In recent years several interesting families of integral operators have been investigated rather extensively studied by various authors (see for example [13, 14]) in analytic function theory, including each of the following integral operators:

$$G_{p,q}(z) = \left\{p \int_0^z t^{p-1}\left(\frac{f(t)}{t}\right)^q dt\right\}^{\frac{1}{p}} \tag{3.1}$$

and

$$G_q(z) = \left\{p \int_0^z t^{p-1}(e^{f(t)})^q\, dt\right\}^{\frac{1}{p}}, \tag{3.2}$$

where the function $f(z)$ belongs to the class $\mathcal{A}$ and the parameters p, q are complex numbers such that the integrals in (3.1) and (3.2) exist.

In this section we establish sufficient conditions for the integral operators of type (3.1) and (3.2) when the function $f(z)$ is the normalized Wright functions to be univalent in the open unit disc U.

Let

$$G_1(\lambda, \mu; z) = \int_0^z \frac{W^{(1)}(\lambda, \mu; t)}{t}\, dt, \quad z \in U,$$

and

$$G_2(\lambda, \mu; z) = \int_0^z \frac{W^{(2)}(\lambda, \mu; t)}{t}\, dt, \quad z \in U,$$

where $W^{(1)}(\lambda, \mu; z)$ and $W^{(2)}(\lambda, \mu; z)$ are defined by (1.3) and (1.4), respectively. Note that $G_1(\lambda, \mu; z)$ and $G_2(\lambda, \mu; z)$ are in the class T.

Theorem 3. *Let $\lambda \geq 1$, $\mu > 0$ and assume that the following condition is satisfied:*

$$\begin{aligned}2(\mu)(\mu + 1)\beta\gamma(1 - \alpha) - [1 + \beta(1 - 2\gamma)](\mu + 1)^2(\mu + 2)e^{(\frac{1}{\mu+1})}\\ + [1 + \beta(1 - 2\gamma)]\big((\mu + 1)^3 + (\mu + 1)^2 + (\mu + 1)\big) \geq 0.\end{aligned}$$

Then the function $G_1(\lambda, \mu; z)$ belongs to the class $\mathcal{P}^*(\alpha, \beta, \gamma)$.

Proof. The proof of Theorem 3 is similar to the proof of Theorem 2. Indeed, from the definition of function $G_1(\lambda, \mu; z)$, we get

$$G_1(\lambda, \mu; z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1) + \mu)} \frac{z^n}{n!} = W^{(2)}(\lambda, \mu - \lambda; z).$$

Hence, the details of proof of Theorem 3 is omitted. □

Theorem 4. *Let $\lambda \geq 1$, $\mu > 0$ and assume that the following condition is satisfied:*

$$2(\lambda+\mu)\beta\gamma(1-\alpha) - [1+\beta(1-2\gamma)](\lambda+\mu+1)^2 e^{(\frac{1}{\lambda+\mu+1})} + [1+\beta(1-2\gamma)]((\lambda+\mu+1)(\lambda+\mu+2)) \geq 0. \tag{3.3}$$

Then the normalized Wright function $G_2(\lambda, \mu; z)$ belongs to the class $\mathcal{P}^*(\alpha, \beta, \gamma)$.

Proof. Since

$$G_2(\lambda, \mu; z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{z^n}{n.n!},$$

according to Lemma 1, we need to show that

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)] \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n.n!} \leq 2\beta\gamma(1-\alpha). \tag{3.4}$$

Let

$$\sum_{n=2}^{\infty} n[1+\beta(1-2\gamma)] \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n.n!} =: T_3.$$

Hence

$$\begin{aligned} T_3 &= [1+\beta(1-2\gamma)] \sum_{n=2}^{\infty} n \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n.n!} \\ &= [1+\beta(1-2\gamma)] \left[\sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n!} + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \frac{1}{n.n!} \right]. \end{aligned}$$

For every $n \in \mathbb{N}_2 := \mathbb{N} \setminus \{1\}$, the inequality $\Gamma(\lambda(n-1)+\lambda+\mu) \geq \Gamma(n-1+\lambda+\mu)$ holds true. Since

$$\Gamma(n-1+\lambda+\mu) = \Gamma(\lambda+\mu)(\lambda+\mu)_{n-1},$$

we have

$$\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda(n-1)+\lambda+\mu)} \leq \frac{1}{(\lambda+\mu)_{n-1}}, \quad n \in \mathbb{N}_2, \tag{3.5}$$

where $(\lambda+\mu)_n$ is again the Pochhammer symbol. Using (3.5), we get

$$T_3 \leq [1+\beta(1-2\gamma)] \left[\sum_{n=2}^{\infty} \frac{1}{(\lambda+\mu)_{n-1}} \frac{1}{n!} \right].$$

Also, we obtain the inequality

$$(\lambda+\mu)_{n-1} = (\lambda+\mu)(\lambda+\mu+1)\cdots(\lambda+\mu+n-2) \geq (\lambda+\mu)(\lambda+\mu+1)^{n-2}, \quad n \in \mathbb{N}_2,$$

which is equivalent to

$$\frac{1}{(\lambda+\mu)_{n-1}} \leq \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}}, \quad n \in \mathbb{N}_2. \tag{3.6}$$

From (3.6), we obtain

$$\begin{aligned} T_3 &\leq [1+\beta(1-2\gamma)] \left[\sum_{n=2}^{\infty} \frac{1}{(\lambda+\mu)(\lambda+\mu+1)^{n-2}} \frac{1}{n!} \right] \\ &\leq [1+\beta(1-2\gamma)] \left\{ \frac{(\lambda+\mu+1)^2}{(\lambda+\mu)} \left(e^{(\frac{1}{\lambda+\mu+1})} - \frac{1}{(\lambda+\mu+1)} - 1 \right) \right\}. \end{aligned}$$

In view of (3.3) we can write

$$[1+\beta(1-2\gamma)] \left\{ \frac{(\lambda+\mu+1)^2}{(\lambda+\mu)} \left(e^{(\frac{1}{\lambda+\mu+1})} - \frac{1}{(\lambda+\mu+1)} - 1 \right) \right\} \leq 2\beta\gamma(1-\alpha),$$

which implies that (3.4) holds true. This completes the proof of Theorem 4. □

By setting $\gamma = 1$ in Theorem 3 and Theorem 4, we obtain the following corollaries.

Corollary 5. *The normalized Wright function $G_1(\lambda, \mu; z)$ belongs to the class $P^*(\alpha, \beta)$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\mu)(\mu+1)\beta(1-\alpha) - [1-\beta](\mu+1)^2(\mu+2)e^{(\frac{1}{\mu+1})} + [1-\beta]\big((\mu+1)^3 + (\mu+1)^2 + (\mu+1)\big) \geq 0.$$

Corollary 6. *The normalized Wright function $G_2(\lambda, \mu; z)$ belongs to the class $P^*(\alpha, \beta)$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\lambda+\mu)\beta(1-\alpha) - [1-\beta](\lambda+\mu+1)^2 e^{(\frac{1}{\lambda+\mu+1})} + [1-\beta]\big((\lambda+\mu+1)(\lambda+\mu+2)\big) \geq 0.$$

Also, by taking $\alpha = 0$ and $\gamma = 1$ in Theorem 3 and Theorem 4, we obtain the following corollaries.

Corollary 7. *The normalized Wright function $G_1(\lambda, \mu; z)$ belongs to the class $D^*(\beta)$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\mu)(\mu+1)\beta - [1-\beta](\mu+1)^2(\mu+2)e^{(\frac{1}{\mu+1})} + [1-\beta]\big((\mu+1)^3 + (\mu+1)^2 + (\mu+1)\big) \geq 0.$$

Corollary 8. *The normalized Wright function $G_2(\lambda, \mu; z)$ belongs to the class $D^*(\beta)$ if $\lambda \geq 1$, $\mu > 0$ and the following condition is satisfied:*

$$2(\lambda+\mu)\beta - [1-\beta](\lambda+\mu+1)^2 e^{(\frac{1}{\lambda+\mu+1})} + [1-\beta]\big((\lambda+\mu+1)(\lambda+\mu+2)\big) \geq 0.$$

References

[1] A. Baricz and S. Ponnusamy, Starlikeness and convexity of generalized Bessel functions, *Integral Transforms Spec. Funct.* **21** (2010), no. 9–10, 641–653.
[2] A. Baricz, E. Toklu and E. Kadioğlu, Radii of starlikeness and convexity of Wright functions, *Math. Commun.* **23** (2018), no. 1, 97–117.
[3] P. L. Duren, *Univalent Functions*, Grundlehren Math. Wiss. 259, Springer, New York, 1983.
[4] M. El-Shahed and A. Salem, An extension of Wright function and its properties, *J. Math.* **2015** (2015), Article ID 950728.
[5] R. Gorenflo, Y. Luchko and F. Mainardi, Analytical properties and applications of the Wright function, *Fract. Calc. Appl. Anal.* **2** (1999), no. 4, 383–414.
[6] V. P. Gupta and P. K. Jain, Certain classes of univalent functions with negative coefficients. II, *Bull. Aust. Math. Soc.* **15** (1976), no. 3, 467–473.
[7] A. A. Kilbas, M. Saigo and J. J. Trujillo, On the generalized Wright function, *Fract. Calc. Appl. Anal.* **5** (2002), no. 4, 437–460.
[8] H. S. Kim and S. K. Lee, Some classes of univalent functions, *Math. Japon.* **32** (1987), no. 5, 781–796.
[9] S. R. Kulkarni, *Some problems connected with univalent functions*, Ph.D. Thesis. Shivaji University, Kolhapur, 1981.
[10] S. S. Miller and P. T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, *Proc. Amer. Math. Soc.* **110** (1990), no. 2, 333–342.
[11] N. Mustafa, Geometric properties of normalized Wright functions, *Math. Comput. Appl.* **21** (2016), no. 2, Paper No. 14.
[12] N. Mustafa and O. Altintaş, Normalized Wright functions with negative coefficients and some of their integral transforms, *TWMS J. Pure Appl. Math.* **9** (2018), no. 2, 190–206.
[13] V. Pescar, A new generalization of Ahlfors's and Becker's criterion of univalence, *Bull. Malays. Math. Soc. (2)* **19** (1996), no. 2, 53–54.
[14] V. Pescar, Univalence of certain integral operators, *Acta Univ. Apulensis Math. Inform.* **12** (2006), 43–48.
[15] S. Ponnusamy and F. Rø nning, Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane, *Integral Transform. Spec. Funct.* **8** (1999), no. 1–2, 121–138.
[16] S. Ponnusamy, V. Singh and P. Vasundhra, Starlikeness and convexity of an integral transform, *Integral Transforms Spec. Funct.* **15** (2004), no. 3, 267–280.
[17] S. Ponnusamy and M. Vuorinen, Univalence and convexity properties for confluent hypergeometric functions, *Complex Var. Theory Appl.* **36** (1998), no. 1, 73–97.
[18] S. Ponnusamy and M. Vuorinen, Univalence and convexity properties for Gaussian hypergeometric functions, *Rocky Mountain J. Math.* **31** (2001), no. 1, 327–353.
[19] J. K. Prajapat, Certain geometric properties of the Wright function, *Integral Transforms Spec. Funct.* **26** (2015), no. 3, 203–212.

[20] M. I. S. Robertson, On the theory of univalent functions, *Ann. of Math. (2)* **37** (1936), no. 2, 374–408.
[21] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51** (1975), 109–116.
[22] H. Silverman, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.* **172** (1993), no. 2, 574–581.
[23] H. M. Srivastava, Some Fox–Wright generalized hypergeometric functions and associated families of convolution operators, *Appl. Anal. Discrete Math.* **1** (2007), no. 1, 56–71.
[24] H. M. Srivastava and S. Owa, *Current Topics in Analytic Function Theory*, World Scientific, River Edge, 1992.
[25] E. M. Wright, On the coefficients of power series having exponential singularities, *J. Lond. Math. Soc.* **8** (1933), no. 1, 71–79.