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THE IMPORTANCE OF BEING MEAN

ALDO SCIMONE and DANIELE RITELLI and GIULIA SPALETTA

Abstract. Many historical loci can be obtained through approaches that differ from classic methods built on their own pure description as loci. Aim of this work is to show how various algebraic curves can be obtained as the locus for particular midpoints, and how the consideration of the latter can give rise to a succession of homotheties, namely, contractions or dilations, of the original curve. What is important to underline is that points generating a locus possess the particular property of being midpoints of other pairs of points, or segments, stemming from a particular configurations of points and lines. This is illustrated here, by various examples.

1. Introduction

Aim of this work is to broaden the existing and ongoing study on the representation of a locus [5, 6]. Through various examples of algebraic curves, geometric loci for some sets of points, we show how they can be defined not only in the historical way in which they were introduced in the mathematical literature, but also through the simple movement of some particular point in the two–dimensional plane.

Here, the focus is on the description of algebraic curves interpreted as the locus for particular *midpoints*. We further show how this approach leads to homothetic affine transformations of contractions or dilations. The geometric construction of various loci is outlined here, while the method for their analytic description and rendering was presented in [6] and is briefly recalled here for reading convenience.

In all the considered examples, the starting point is the intersection between a line r and a circle Γ, whose equations are given with respect to a Cartesian reference system xOy. Circle Γ is centered at the origin $O = O(0, 0)$ and has radius given by the positive parameter a, while r is any line through O and with non–negative angular coefficient m, namely:

$$(1) \qquad \Gamma : \quad x^2 + y^2 - a^2 = 0 \,, \qquad\qquad r : \quad m\,x - y = 0 \,.$$

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The assumption is made that each point P, involved in a locus generation, has coordinates that are rational functions of a and m, that is:

(2)
$$\begin{cases} x_P = \dfrac{x_{num}(a\,,m)}{x_{den}(a\,,m)} \\[2ex] y_P = \dfrac{y_{num}(a\,,m)}{y_{den}(a\,,m)} \end{cases} .$$

In particular, the coordinates of points $H\,, H' = \Gamma \cap r$, respectively in the first and third quadrants, are:

(3)
$$\begin{cases} x_H = 2\,\mathcal{K}_1 \\ y_H = 2\,m\,\mathcal{K}_1 \end{cases} , \qquad \begin{cases} x_{H'} = -2\,\mathcal{K}_1 \\ y_{H'} = -2\;m\,\mathcal{K}_1 \end{cases} ,$$

where

(4)
$$\mathcal{K}_1 = \mathcal{K}_1(a\,,m) = \frac{a}{2\sqrt{m^2\,+\,1}} .$$

To simplify the notation, the following factor is also employed:

(5)
$$\mathcal{K}_2 = \mathcal{K}_2(a\,,m) = \frac{2\,\mathcal{K}_1}{m^2\,+\,1} .$$

We further denote as $x-$parallel, or $y-$parallel, a line that is parallel to the respective coordinate axis. Similarly, we call $x-$symmetric, or $y-$symmetric, a point that is symmetric to a given point through the respective coordinate axis. Finally, we employ the short nomenclature $P_1 - P_2-$midpoint to indicate the midpoint between points P_1 and P_2.

1.1. **Loci construction.** Using (2), two univariate polynomials in the variable m are formed:

(6)
$$\begin{cases} x_{den}(a\,,m)^w \quad x^w \;=\; x_{num}(a\,,m)^w \\ y_{den}(a\,,m)^w \quad y^w \;=\; y_{num}(a\,,m)^w \end{cases} ,$$

where the exponent has the role of eliminating any irrational term, and is set as $w = 2$, here.

The *resultant* of the two polynomials (6), namely, the determinant of their associated *Sylvester* matrix [1], is then computed, factorized and simplified, by eliminating factors that are constant or depend on the parameter a only; integer powers of factors containing linear combinations of $x\,, y$ and a can be simplified out too.

The above procedure can be applied in an automated way, within a computer algebra environment, to all the loci considered in this paper, and yields an equation in the variables x and y that, evaluated at some value of a (here, $a = 1$), represents the analytic cartesian description of the desired locus:

(7)
$$p(x\,,y) \;=\; \sum_{k=0}^{d} \alpha_k\; x^k\; y^{d-k} ,$$

where α_k are rational constant coefficients, while d is some integer degree. Curve (7) can be finally rendered as a contour line of zero level value.

2. Bernoulli Quartic

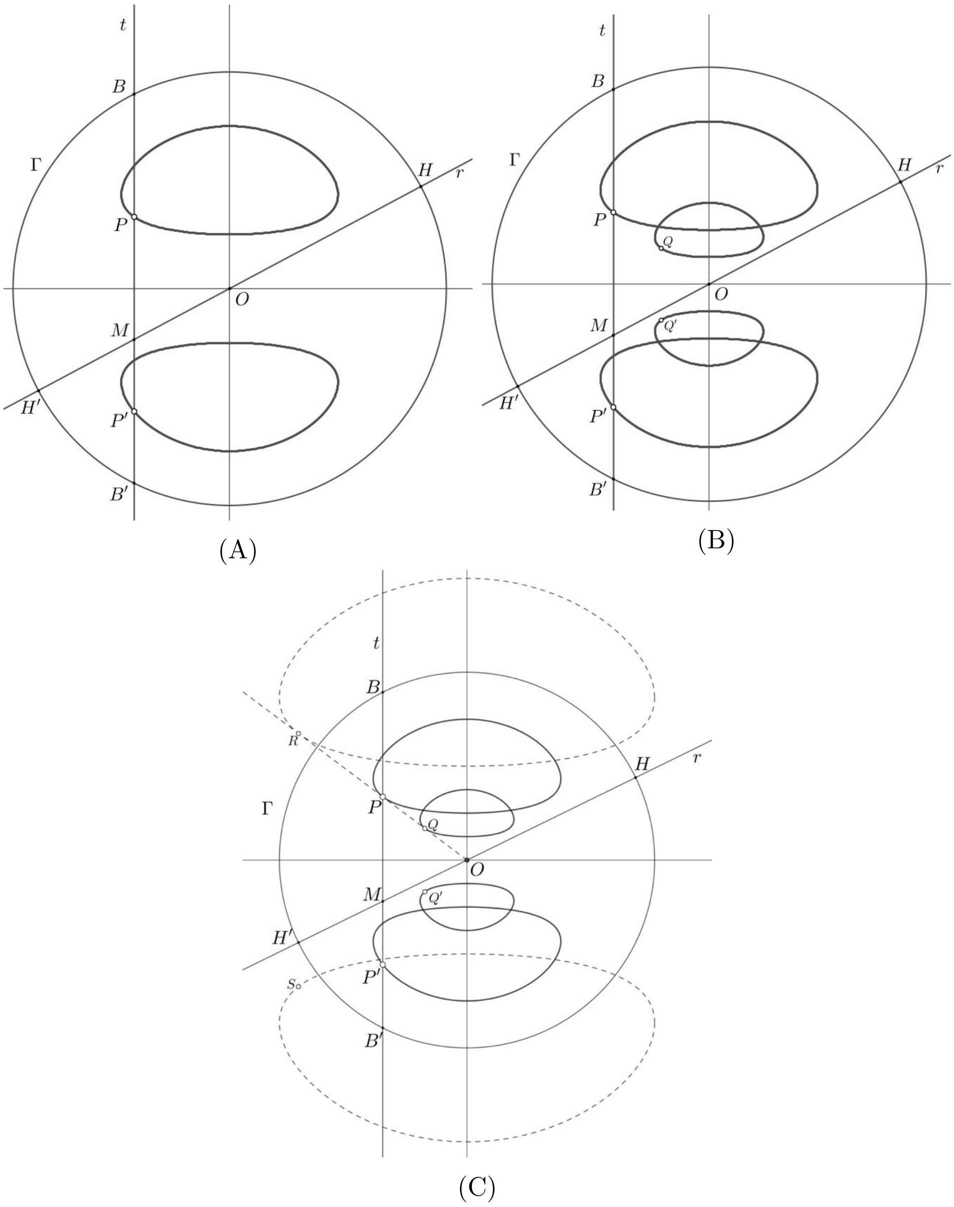

FIGURE 1. Quartic of Bernoulli (1A) with its contraction (1B) and dilation (1C).

Given $H\,,H'$ in (3), let M be the $H'-O-$midpoint. Then, consider the $x-$parallel line t, with $M \in t$, and construct points $B\,,B' = \Gamma \cap t$:

$$\begin{cases} x_M = -\mathcal{K}_1 \\ y_M = -m\,\mathcal{K}_1 \end{cases} , \qquad t: \quad x + \mathcal{K}_1 = 0\,,$$

$$\begin{cases} x_B = -\mathcal{K}_1 \\ y_B = \mathcal{K}_1\sqrt{4\,m^2+3} \end{cases} , \quad \begin{cases} x_{B'} = -\mathcal{K}_1 \\ y_{B'} = -\mathcal{K}_1\sqrt{4\,m^2+3} \end{cases} ,$$

where $\mathcal{K}_1$ is given in (4). Let P be the $B' - M-$midpoint:

$$\begin{cases} x_P = -\mathcal{K}_1 \\ y_P = \mathcal{K}_1\,(-m + \sqrt{4\,m^2+3}) \end{cases} . \tag{8}$$

As H varies along Γ, point P describes the so–called *Bernoulli Quartic* given in Figure 1A and having equation:

$$256\,y^2\,(x^2+y^2) = a^2\,(160\,y^2 - 9\,a^2)\,. \tag{9}$$

This algebraic curve derives its name from Jacob Bernoulli [1], who studied it in 1687 [4]; the quartic was later studied by Leibniz [2] in 1696; historically, its construction is far more complicated than our midpoint interpretation. If we further construct the $P-O-$midpoint Q and its $x-$symmetric point Q', then we can describe another Bernoulli Quartic, homothetic to (9) as shown in Figure 1B; the origin O is the homothety center. With this procedure, it is possible to obtain a family of *contractions* of the Bernoulli Quartic. It is also possible to create *dilations*, each of which is built by considering a point R, along the half–line joining O and P, such that the distance $d(O\,,R)$ is an integer multiple of $d(O\,,P)$; as an example, in Figure 1C, it is $d(O\,,R) = 2\,d(O\,,P)$.

3. Dürher Folium

Given H as in (3), let Q be its $x-$symmetric point, and build point R as the symmetric of Q w.r.t. line r :

$$\begin{cases} x_R = \mathcal{K}_2\,(1 - 3\,m^2) \\ y_R = m\,\mathcal{K}_2\,(3 - m^2) \end{cases} .$$

where $\mathcal{K}_2$ is given in (5). Let P be the $R - H-$midpoint:

$$\begin{cases} x_P = \mathcal{K}_2\,(1 - m^2) \\ y_P = 2\,m\,\mathcal{K}_2 \end{cases} . \tag{10}$$

As H varies along Γ, point P describes the so–called *Dürer*[3] *Folium*, a special case of a *rhodonea* (or simply *rose*) curve with intertwined leaves; it is illustrated in Figure 2A and its equation is:

$$4\,(x^2+y^2)^3 = a^2\left(4\,(x^2+y^2)^2 - a^2\,y^2\right)\,. \tag{11}$$

This algebraic curve is sextic, rational, symmetric w.r.t. to the reference axes, and tangent to the $x-$axis. The origin O is a double contact–point for the curve two branches, while each of the points $C(0\,,\frac{a}{\sqrt{2}})$ and $C'(0\,,-\frac{a}{\sqrt{2}})$ is a double contact–point for the $y-$axis, with distinct tangents.
If we further construct the $P-O-$midpoint P', we can obtain a *contraction* of the original locus, as shown in Figure 2B; pursuing this procedure further, it is possible to obtain a family of homotetic loci. The same procedure also yields *dilations*, by considering points like P'', along the half–line joining

[1]Jacob Bernoulli (1654–1705), Swiss mathematician, one of the many notable academics in the Bernoulli family.

[2]Gottfried Wilhelm Leibniz (1646–1716), famous German polymath.

[3]Albrecht Dürer (1471–1528), German painter and theorist.

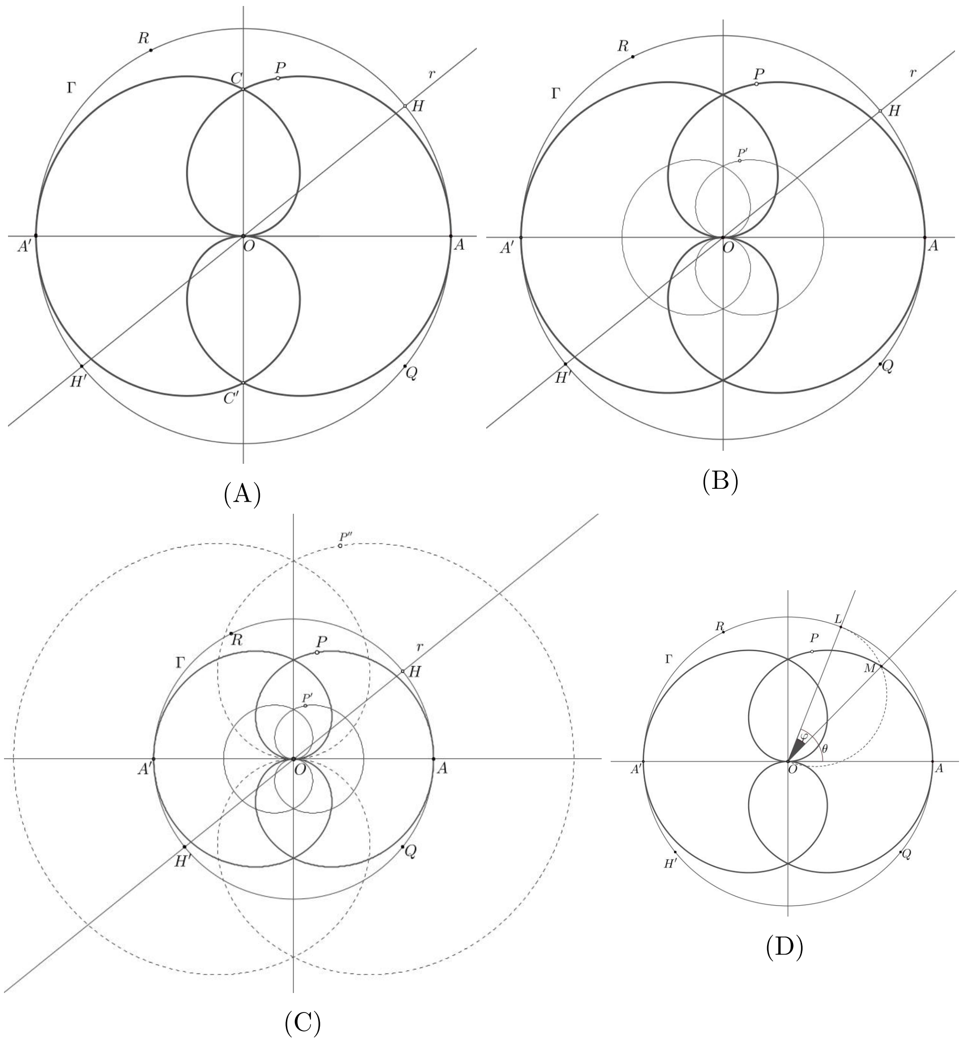

FIGURE 2. Folium of Dürer (2A), with its contraction (2B), dilations (2C) and trisectrix property (2D).

O and P, such that the ratio of the two distances $d(O\,,P'')$ and $d(O\,,P)$ is integer; for example, in Figure 2C it is $d(O\,,P'') = 2\;d(O\,,P)$.
Finally, let us mention that the Dürer Folium enjoys the beautiful property of being a *trisectrix*: with reference to Figure 2D, in fact, if θ is the angle $A\hat{O}L$, and M is the intersection point between the curve and the semicircle insisting on diameter OL, it then holds $\theta = 3\,\phi$, where ϕ is the angle $M\hat{O}L$.

4. Maltese Cross

Let H in (3) have Q as $y-$symmetric point. Then, consider both line s through Q and tangent to Γ, and line $t \perp s$ such that $H \in t$, and build

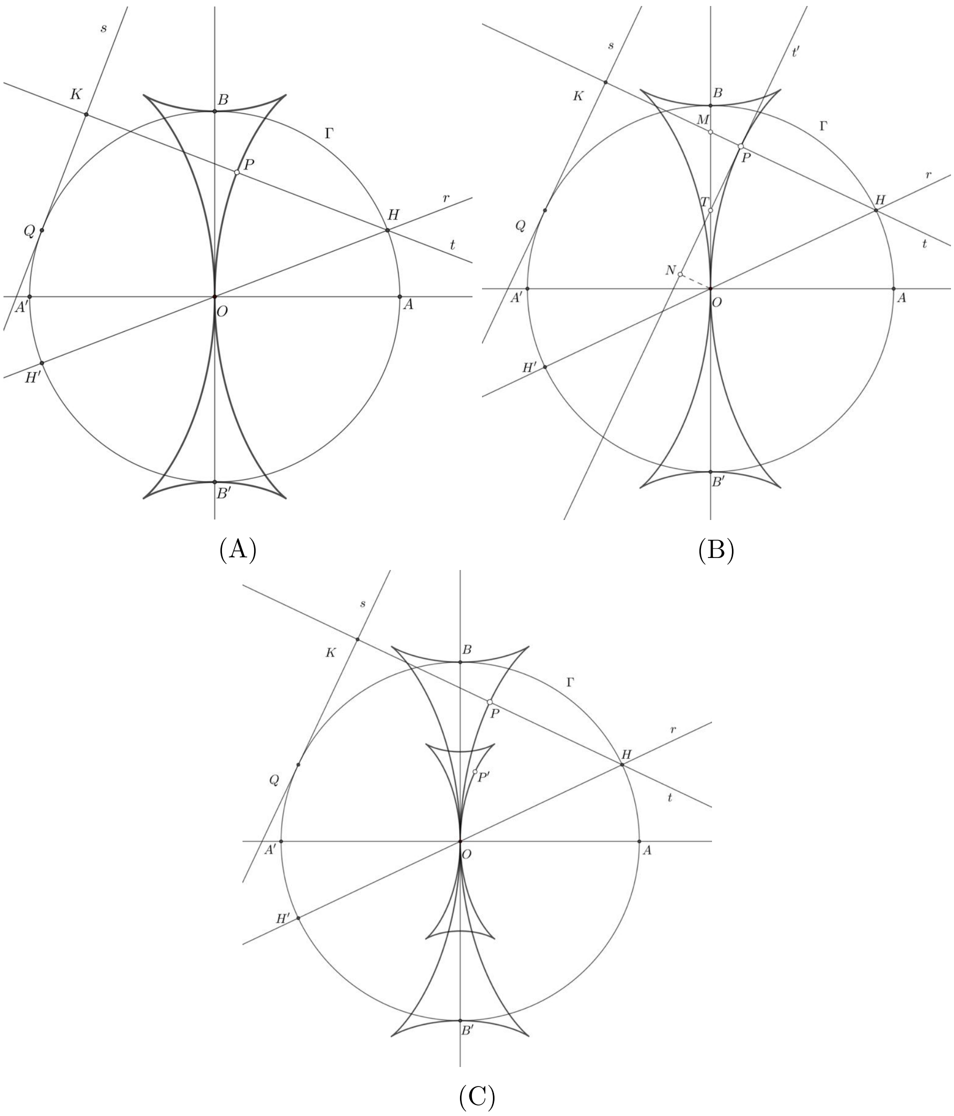

FIGURE 3. Maltese Cross (3A), its related mid–point property of point T (3B), and its homoteties (3C).

their intersection point $K = s \cap t$:

$$(12) \qquad \begin{aligned} s: &\quad x - m\,y + a\sqrt{m^2+1} = 0\,, \\ t: &\quad m\,x + y - 4\,m\,\mathcal{K}_1 = 0\,, \end{aligned} \qquad \begin{cases} x_K = \mathcal{K}_2\,(m^2-1) \\ y_K = m\,\mathcal{K}_2\,(m^2+3) \end{cases} .$$

Finally, let P be the $K-H-$midpoint:

$$(13) \qquad \begin{cases} x_P = m^2\,\mathcal{K}_2 \\ y_P = m\,\mathcal{K}_2\,(m^2+2) \end{cases} .$$

As H varies along Γ, point P describes one arm of the so–called *Maltese Cross*, or *Bow–Tie*, rendered in Figure 3A and whose equation is:

$$(14) \qquad (x^2+y^2)^3 = a^2\,(y^4 + 20\,x^2\,y^2 - 8\,x^4 - 16\,a^2\,x^2)\,.$$

The curve was studied by Besant[4], d'Ocagne[5] and Gaedecke[6], respectively in 1870, 1884 and 1917. It is a sextic, rational curve that has a few nice properties.
Referring to Figure 3B, let T be the point of intersection between the $x-$parallel line joining Q and H, i.e., T is $Q-H-$midpoint. Since P is $K-H-$midpoint, by the properties of triangles (here applied to the right triangle QKH), it follows that line t' through P and T is parallel to line s; thus, $t' \perp t$ at P and t' is tangent to the Maltese Cross curve at P. Now, let t intersect the $y-$axis at point M and let N be the projection of O on t'. The right triangles TPH and TNQ are congruent, since they have same angle measures and $d(Q,T)=d(T,H)$; thus, T is $P-N-$midpoint. By congruency of the right triangles TPM and TNO, point T is also $M-O-$midpoint.
Another beautiful property that the Maltese Cross curve enjoys is that its *orthoptic* (i.e., set of points for which two tangents to a given curve meet at a right angle) is the *Cornoid* discussed in § 5.
Finally, referring to Figure 3C, if we consider the $P-O-$midpoint P' and trace down the locus described by P' as H varies, then, it can be proved that such a locus is always homothetic to (14). In this way, a family of curves can be obtained, all homothetic to each other, with the origin O being their homothety center.

5. Cornoid

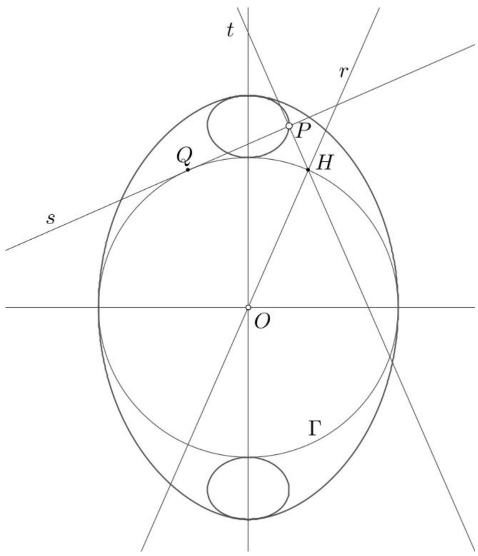

Figure 4. Cornoid.

The *Cornoid* is the orthoptic of the Maltese Cross, that is to say, it is formed by those points from which the Maltese Cross is seen under a right

[4]William Henry Besant (1828–1917) British mathematician.
[5]Philibert Maurice d'Ocagne (1862–1938), French mathematician.
[6]Werner Gaedecke, German mathematician, active in the 1920s.

angle; in other words, the Cornoid is the locus of the points from which two tangents to the Maltese Cross can be conducted perpendicularly to each other. Its generator point (as H varies along Γ) is $K = s \cap t$, where the two lines $t \perp s$ are built as described in § 4; the s and t equations and the K coordinates are given in (12). As shown in Figure 4, the Cornoid is a rational sixth degree curve, with two axes of symmetry; its equation is:

$$(x^2 + y^2)^3 = a^2\,(5\,y^4 + 6\,x^2\,y^2 - 3\,x^4 - 8\,a^2\,y^2 + 4\,a^4)\,. \tag{15}$$

6. Cross Curve

Consider the intersection $\Sigma = s \cap r$ bewteen lines r and s, given in (1) and (12) respectively:

$$\begin{cases} x_\Sigma = \dfrac{a\sqrt{m^2+1}}{m^2-1} \\ y_\Sigma = m\,x_\Sigma \end{cases} . \tag{16}$$

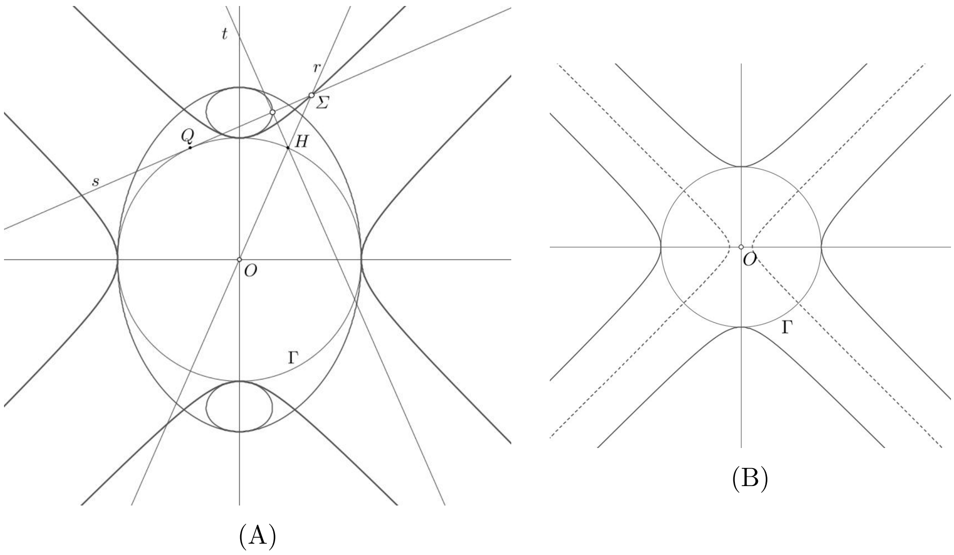

Figure 5. Cross Curve (5A) and its asymptotic hyperbola (5B).

Recall circle Γ in (1) and point H in (3). As H varies along Γ, point Σ generates the so-called *Cross Curve*:

$$(y^2 - x^2)^2 = a^2\,(y^2 + x^2)\,. \tag{17}$$

The quartic (17) is illustrated in Figure 5A, while Figure 5B shows that the Cross Curve is asymptotic to the hyperbola:

$$x^2 - y^2 = \frac{1}{a}\,. \tag{18}$$

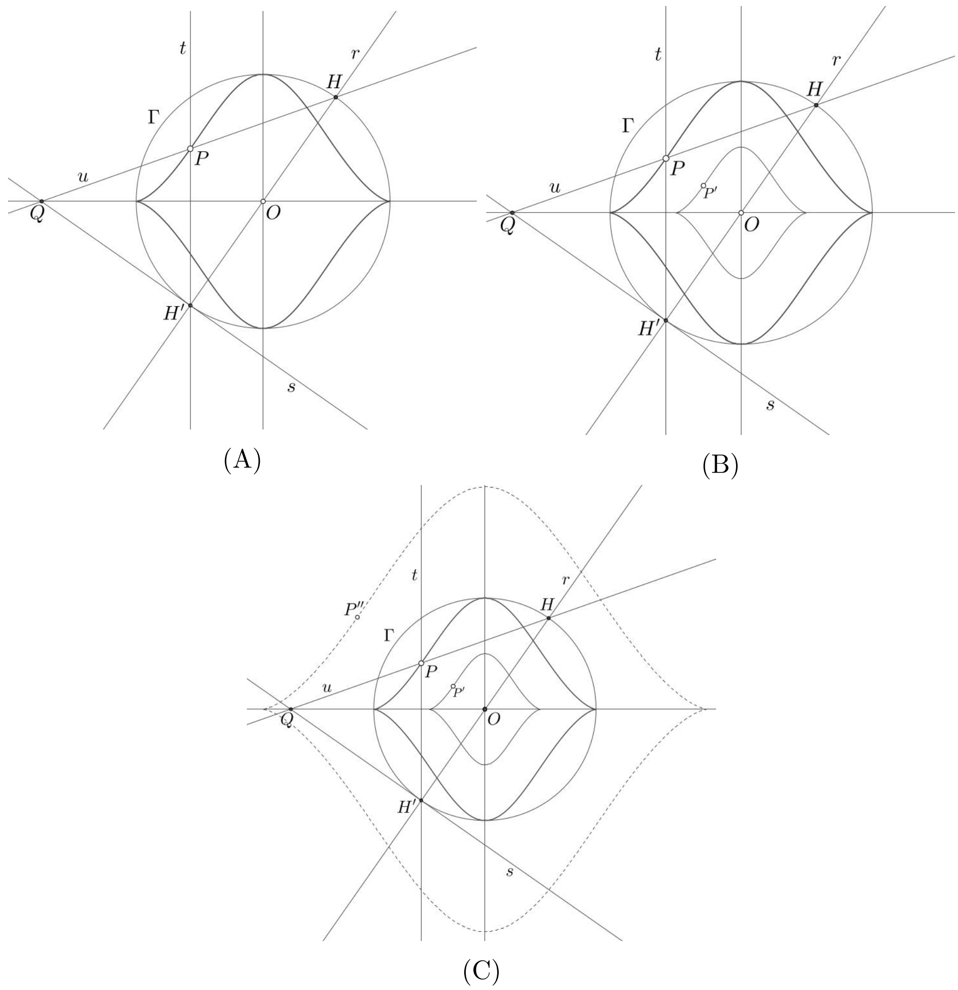

Figure 6. Kiss Curve (6A), with its contraction (6B) and dilation (6C).

7. Kiss Curve

Consider circle Γ and point H', respectively given in (1) and (3). Construct line s, tangent to Γ at H', and its intersection Q with the $x-$axis.

$$s: \quad x + m\,y = -a\sqrt{m^2+1}\,, \qquad \begin{cases} x_Q = -a\sqrt{m^2+1} \\ y_Q = 0 \end{cases} .$$

Then, draw line u through points Q and H, and the $y-$parallel line t such that $H' \in t$:

$$u: \quad m\,x - (m^2+2)\,y = a\,m\sqrt{m^2+1} \qquad t: \quad x = -2\,\mathcal{K}_1\,.$$

The intersection $P = t \cap u$, whose coordinates are:

$$\begin{cases} x_P = -2\,\mathcal{K}_1 \\ y_P = \dfrac{2\,m^3\mathcal{K}_1}{m^2+2} \end{cases} , \tag{19}$$

generates, as H varies along Γ, the so–called *Kiss* curve:

$$y^2\,(a^2+x^2)^2=(a^2-x^2)^3\,. \tag{20}$$

The geometric locus (20) is a rational sextic curve, with two cusps at the intersections between Γ and the $x-$axis, as illustrated in Figure 6A:
The $P-O-$midpoint P' describes another Kiss curve, homothetic to (20) with homothety ratio equal to $1/2$, thus a contraction, as shown in Figure 6B.
To obtain a dilation of the Kiss curve, it sufficies to consider a point P'', along the line joining P with the origin O, such that the homothety ratio $d(P''\,,O)/d(P\,,O)$ is integer and greater than 1; as an example, in Figure 6C, it is $d(P''\,,O)/d(P\,,O)=2$.

8. Watt Curve

Recall again circle Γ and point H, respectively given in (1) and (3). Consider line t, tangent to Γ at point Q, which is $y-$symmetric w.r.t. H, and let T be the $x-$axis intersection of line t :

$$\begin{cases} x_Q=-2\,\mathcal{K}_1 \\ y_Q=2\,\mathcal{K}_1 \end{cases}\,,\quad t:\ \ x-m\,y=-a\,\sqrt{m^2+1}\,,\quad \begin{cases} x_T=-a\,\sqrt{m^2+1} \\ y_T=0 \end{cases}\,.$$

Now, given r in (1), draw line $u\perp r$ and such that $T\in u$, and construct line s through points H and T. Finally, let $R=u\cap s$. Therefore:

$$\begin{aligned} u:&\quad x+m\,y=2\,\mathcal{K}_1\,(m^2-1)\,, \\ s:&\quad m\,x-(m^2+2)\,y=-a\,m\,\sqrt{m^2+1}\,, \end{aligned}\qquad \begin{cases} x_R=-\mathcal{K}_2 \\ y_R=m^3\,\mathcal{K}_2 \end{cases}\,.$$

The $R-H-$midpoint P, whose coordinates are:

$$\begin{cases} x_P=\dfrac{m^2\,\mathcal{K}_2}{2} \\[2ex] y_P=\dfrac{m\,\mathcal{K}_2\,(2\,m^2+1)}{2} \end{cases}\,. \tag{21}$$

generates, as H varies along Γ, the so–called *Watt*[7] *Curve*, or *Handlebar Curve*:

$$4\,(x^2+y^2)^3=a^2\,(4\,x^4-19\,x^2\,y^2+4\,y^4-a^2\,x^2)\,. \tag{22}$$

Figure 7A represents the Watt Curve (22), while Figures 7B and 7C respectively illustrate a contraction and a dilation, that can be obtained by considering appropriate homethety ratios, with the same procedure described in the previous sections.

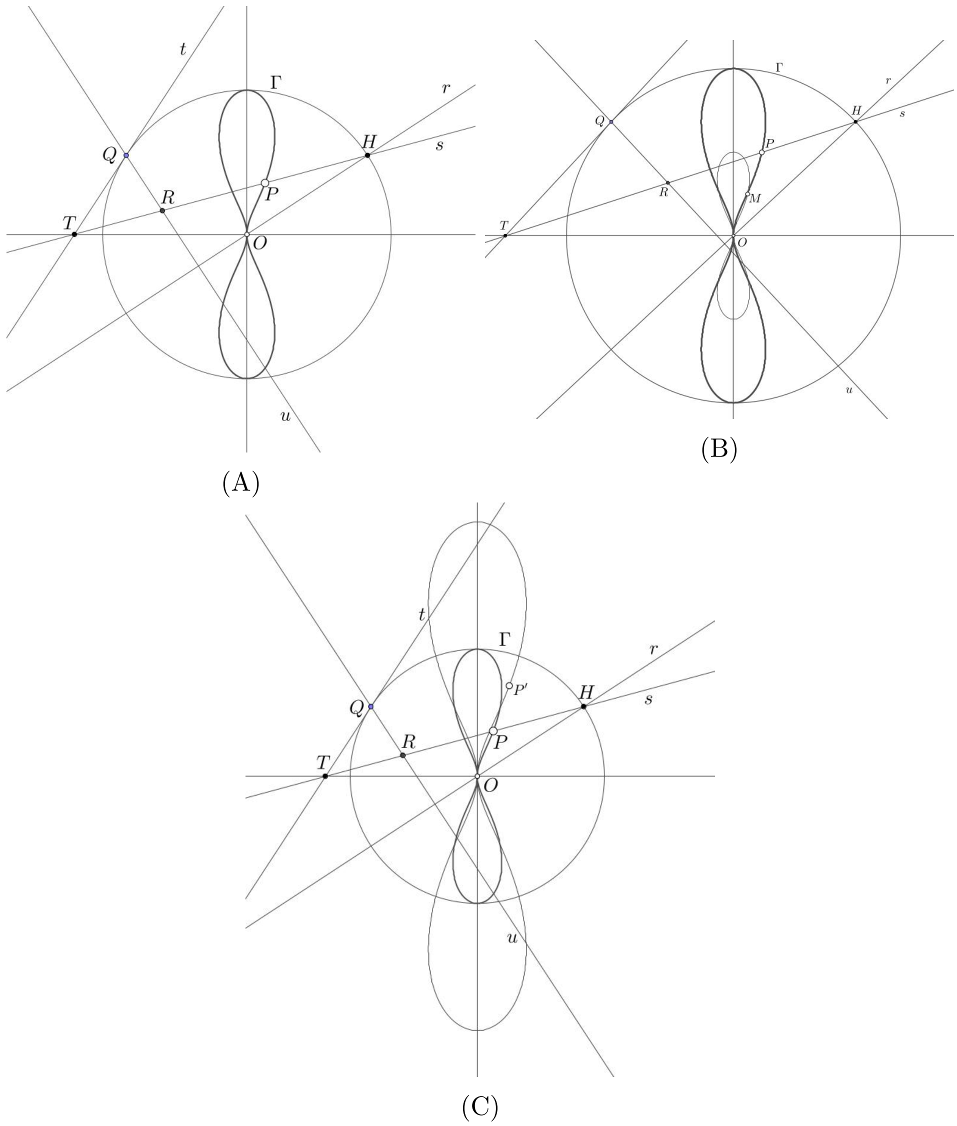

FIGURE 7. Watt Curve (7A), with its contraction (7B) and dilation (7C).

9. Nicomede Concoid

Consider the intersection $C = s \cap r$, where s is the line through $A(a,0)$ and $B(0,a)$ and r is as in (1):

$$s: \quad x + y = a, \qquad\qquad \begin{cases} x_C = \dfrac{a}{m+1} \\ y_C = m \;\; x_C \end{cases} .$$

[7]James Watt (1736–1819), Scottish inventor, engineer and chemist, whose 1776 improved version of the 1712 steam-engine, by the English inventor Thomas Newcomen (1664-1729), was fundamental to the Industrial Revolution.

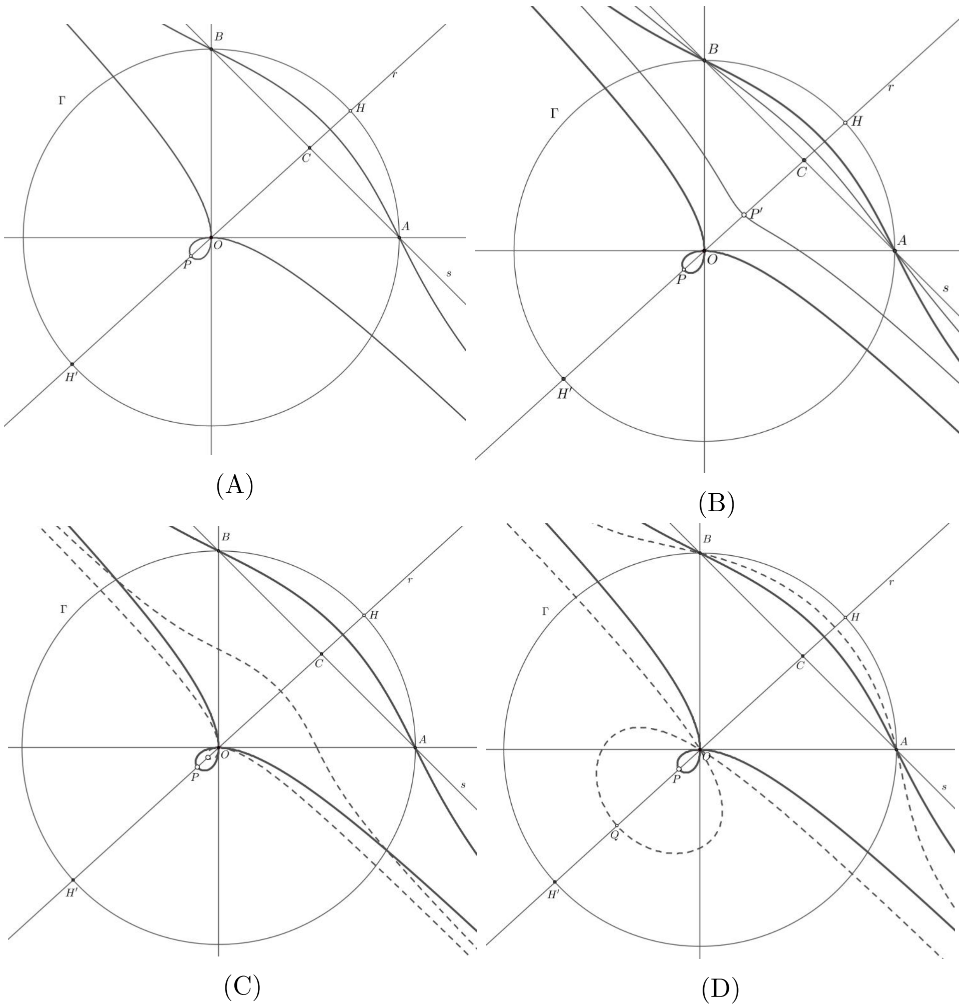

FIGURE 8. Concoid of Nicomede with a node (8A) or no nodes (8B), and its contraction (8C) or dilation (8D).

Recalling H' in (3), the $C-H'-$midpoint P has coordinates:

$$\begin{cases} x_P = \dfrac{\mathcal{K}_1\,(-m-1+\sqrt{m^2+1})}{m+1} \\ y_P = m\,x_P \end{cases} \quad . \tag{23}$$

As H' varies along Γ given by (1), point P describes the so-called *Concoid of Nicomede*[8], illustrated in Figure 8A; it is a quartic rational curve, passing through A and B, having a *node* at the origin O, and whose equation is:

$$2\,(x^2+y^2)\,(x+y)^2 = a^2\,x\,y + 2\,a\,(x^2+y^2)\,(x+y)\,. \tag{24}$$

By considering various midpoints, we can obtain contractions or dilations of (24), or a *nodeless* concoid, as listed below.

[8]Nicomede (280–210 B.C.), ancient Greek mathematician.

- The $P-O-$midpoint describes the reduced concoid, drawn as the dashed curve in Figure 8C, homothetic to (24) through a homotethy ratio equal to $1/2$.
- The $P-H'-$midpoint Q describes the dilated concoid, drawn as the dashed curve in Figure 8D, homothetic to (24) through an integer homotethy ratio greater than 1.
- The $P-C-$midpoint P' describes the nodeless concoid, drawn as the thin curve in Figure 8B.

10. Clairaut Curve

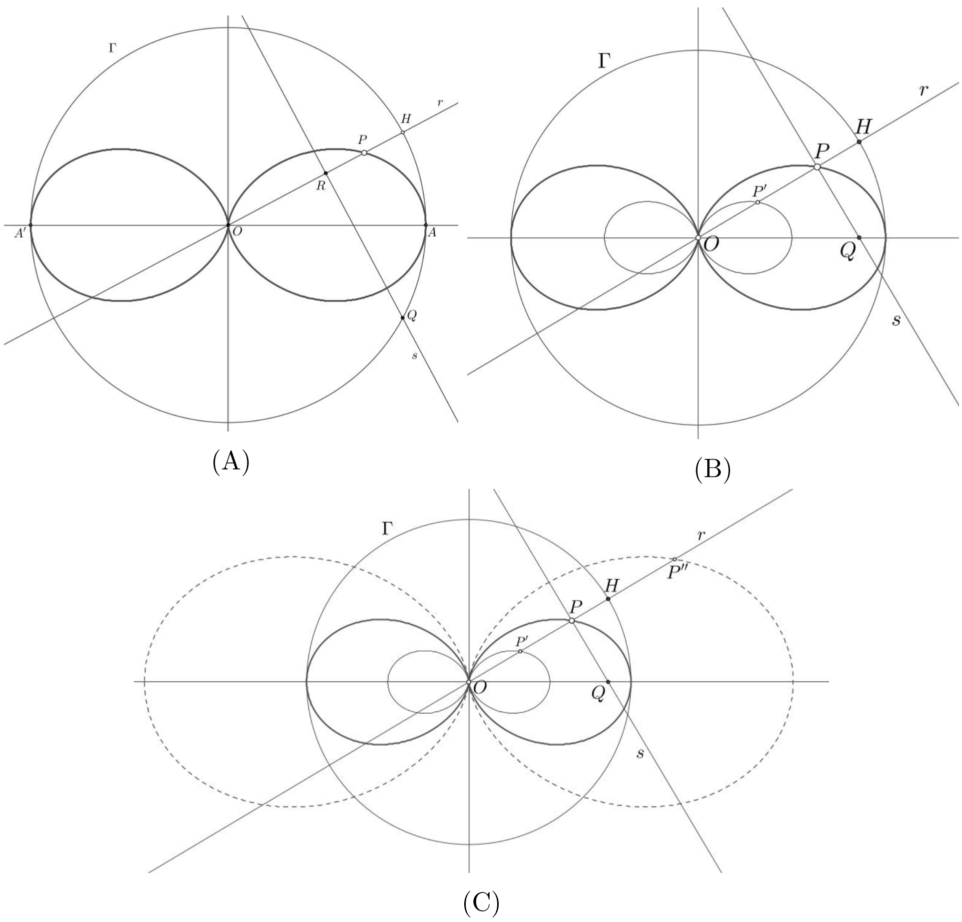

Figure 9. Double Egge Curve (9A), with its contraction (9B) and dilation (9C).

Consider again Γ, r in (1) and H in (3). Let H have Q as $x-$symmetric point, and draw line $s \perp r$ such that $Q \in s$. Then, consider $R = s \cap r$. Therefore:

$$\begin{cases} x_Q = 2\,\mathcal{K}_1 \\ y_Q = -m\,x_Q \end{cases} \quad , \quad s: \;\; x + m\,y = 2\,\mathcal{K}_1\,(1-m^2)\,, \quad \begin{cases} x_R = \mathcal{K}_2\,(1-\,m^2) \\ y_R = m\,x_R \end{cases} \quad .$$

The $R-H-$midpoint P has coordinates:

$$\begin{cases} x_P = \mathcal{K}_2 \\ y_P = m\,x_P \end{cases} . \tag{25}$$

As H varies along Γ, point P describes the so-called *Clairaut*[9] *Curve* or *Double Egg Curve*, shown in Figure 9A and whose equation is:

$$(x^2+y^2)^3 = a^2\,x^4 . \tag{26}$$

The locus (26) is a rational sextic curve, of type zero. To obtain its contraction by a homothety factor equal to $1/2$, we can use the $P-O-$midpoint P', as illustrated in Figure 9B. Dilations of (26) are achieved by considering a point P'' such that $d(P'',O) = k\ d(P,0)$, where factor $k>1$ is integer, as described in Figure 9C.

Here, we linked (26) to the motion of a particular midpoint. In the mathematical literature, the Clairaut Curve is instead constructed through other procedures, such as via the inverse of the *Eudoxus*[10] *Kampyle* (*curved line*) w.r.t. its center, or by rolling an ellipse along a four–leaf clover.

Locus (26) finds application in Physics: the magnetic field lines, created by a magnetic dipole, are Clairaut curves.

11. Delanges Trisectrix

Given Γ, r in (1) and H, H' in (3), consider the $H'-O-$midpoint M and draw line $t\perp r$ such that $M\in t$. Then, denote Q the point of intersection of t with the $x-$axis. Therefore:

$$\begin{cases} x_M = -\,\mathcal{K}_1 \\ y_M = -m\,\mathcal{K}_1 \end{cases} , \quad t:\ x+m\,y = -\mathcal{K}_1\,(m^2+1)\,, \quad \begin{cases} x_Q = -\mathcal{K}_1\,(m^2+1) \\ y_Q = 0 \end{cases} .$$

The $Q-H-$midpoint P has coordinates:

$$\begin{cases} x_P = \dfrac{\mathcal{K}_1\,(1-m^2)}{2} \\ \\ y_P = m\,\mathcal{K}_1 \end{cases} . \tag{27}$$

As H varies along Γ, point P describes the so-called *Delanges*[11] *Trisectrix*, which is a quartic curve, circular and symmetric w.r.t. the coordinate axes, as illustrated in in Figure 10A. Its equation is:

$$64\,y^2\,(x^2+y^2) = 16\,a^2\,(x^2+y^2) - a^4 . \tag{28}$$

Curve (28) intersects the $x-$axis at the points of coordinates $A(\pm a/4\,,0)$, while it intersects the $y-$axis at the points of coordinates $B(0\,,\pm a\sqrt{2}/4)$. The latter points B are *double points* of the curve itself; furthermore, the *point-at-infinity* of the $x-$axis is also a *double point* for the Delanges Curve, which is thus rational. Lines $y=\pm a/2$ are its asymptotes.

[9] Alexis Claude Clairaut or Clairault (1713–1765) French mathematician and astronomer.

[10] Eudoxus of Cnidos (400–350 B.C. circa), ancient Greek mathematician and astronomer.

[11] Paolo Delanges (1750 ca.–1810), Italian engineer and mathematician.

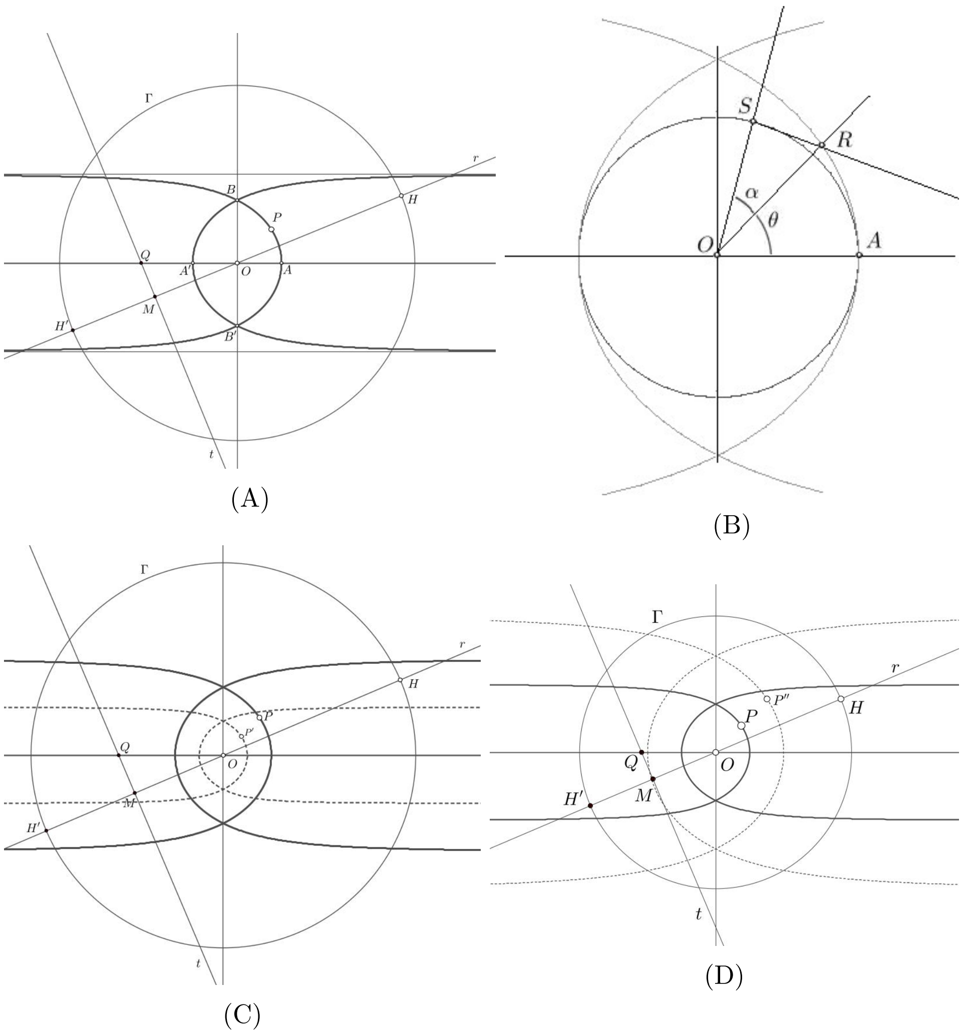

FIGURE 10. Trisectrix of Delanges (10A), its angle trisection property (10B), and its contraction (10C) and dilation (10D).

Figures 10C and 10D respectively illustrates a contraction and a dilation, that can be obtained by considering appropriate homethety ratios, with the same procedure described in the previous sections of this paper.
Curve (28) is named after Delanges [2], who studied it in 1783 and used it to trisect an angle. A century later, in 1878, it was rediscovered by the American mathematician Hillhouse [3], who also devised a tool to trace it mechanically. The angle trisection property is illustrated in Figure 10B. Consider a circle centered at the origin $O(0\,,0)$, with radius given by $d(O\,,A)$, where $A(a/4\,,0)$. Choose any point S along the considered circle, and let $\theta = A\hat{O}S$ be the angle to be trisected. The line through point S, tangent to the circle at S, intersects the Delanges Curve at point R, forming an angle $\alpha = R\hat{O}S$. It can then be proved that $\theta = 3\,\alpha$.
The Delanges Curve also holds the beautiful property of having Dürer Folium (see § 3) as inverse curve w.r.t. the origin O. Furthermore, it is

a special case of the *Epispiral*, the latter being a plane curve, whose equation can be given in polar form as $\rho = a\,|sec(n\,\theta)|$, where ρ and θ are, obviously, the polar variables.

12. Conclusions

The examples of algebraic curves, considered in this paper, have shown that they can be defined not only in the historical way in which they were introduced in the mathematical literature, but also through the simple movement of some particular points of the plane. This applies to various other loci, which may be the subject for additional investigation. Further work may also involve consideration of particular points, other than the midpoints we focused on here.

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PRIVATE ADDRESS
PALERMO, VIA NIGRA 30
PALERMO, I–90141, ITALY
E-mail address: aldoscimone@outlook.it

DEPARTMENT OF STATISTICAL SCIENCES
BOLOGNA UNIVERSITY
BOLOGNA, I–40126, ITALY
E-mail address: daniele.ritelli@unibo.it

DEPARTMENT OF STATISTICAL SCIENCES
BOLOGNA UNIVERSITY
BOLOGNA, I–40126, ITALY
E-mail address: giulia.spaletta@unibo.it
Member of INDAM–GNCS