A BRIEF INTRODUCTION TO MULTIDIMENSIONAL PERSISTENT BETTI NUMBERS

ANDREA CERRI AND PATRIZIO FROSINI

ABSTRACT. In this paper we propose a brief overview about multidimensional persistent Betti numbers (PBNs) and the metric that is usually used to compare them, i.e. the multidimensional matching distance. We recall the main definitions and results, mainly focusing on the 2-dimensional case. An algorithm to approximate n-dimensional PBNs with arbitrary precision is described.

1. INTRODUCTION

Persistent topology and homology are the main tools in Topological Data Analysis. They study how the topology and homology of the sublevel set X_u of a continuous function $f : X \to \mathbb{R}^n$ change when u varies in $\mathbb{R}^n$. The case $n = 1$ has been considered in many papers, starting from the beginning of the '90s (see [3] for historical notes). The case $n > 1$ (i.e. multidimensional persistence) was firstly investigated in [19] as regards homotopy groups, while multidimensional persistence modules were considered in [7, 8] and subsequently studied in other papers including [21, 22, 23]. In particular, the *interleaving distance* between multidimensional persistence modules has been formally introduced and discussed in [21]. Another useful tool in persistence theory is given by *multidimensional persistent Betti number functions* (briefly, n-dimensional PBNs) [9], also called *rank invariants* [8]. They have been studied in [2] by means of the so-called *foliation method*. Focusing on the 0th homology, that paper proved that for $n > 1$ a foliation in half-planes can be given, such that the restriction of the n-dimensional PBNs to these half planes turns out to be 1-dimensional. Each plane in the foliation corresponds to a positive slope line r in $\mathbb{R}^n$ and to the 1-dimensional filtration X_p of X, where X_p is the set of points of X whose images by f are both under and on the left of the point $p \in r$. This approach leads to an algorithm to approximate with arbitrary precision the multidimensional persistent Betti number functions. Furthermore, a stable matching distance between n-dimensional PBNs is available, namely the *n-dimensional matching distance* ([2, 5, 9]). The interest in the n-dimensional matching distance between PBNs derives from the fact that, while its computation is pretty simple, the computation of the interleaving distance between persistence modules is NP-hard [4]. This survey paper illustrates the main results concerning n-dimensional PBNs and the n-dimensional matching distance, with particular reference to the case $n = 2$. Finally, we present a recent variant of this last metric, called *coherent matching distance* [11]. For each result, the paper where the interested reader can find the corresponding proof and further details is reported.

---

2. PBNs: Definitions and first properties

In this section, we recall some basic definitions and properties in persistent homology and topology. For further information we refer the interested reader to the surveys [3, 17, 20, 24]. We will assume that the considered filtering functions are *continuous*, and make use of Čech homology. Although different from the more usual setting of tame functions and simplicial or singular homology, our choice is motivated by the following facts:

- the reduction of multidimensional persistence to the 1-dimensional setting is not possible in the setting of tame functions, as observed in [5], but it luckily does in the wider setting of continuous functions;
- using the continuity axiom of Čech homology, it is possible to prove the Representation Theorem 2.5, stating that the PBNs of a scalar-valued filtering function can be completely described by a persistence diagram.

Hereafter, X is a finitely triangulable topological space. The symbol Δ^+ denotes the half-plane $\{(u,v) \in \mathbb{R} : u < v\}$, while Δ^* is the set $\Delta^+ \cup \{(u, \infty) :\in \mathbb{R}\}$.

2.1. **1-dimensional PBNs.** We first consider the case when the filtering function f is real-valued. Indeed, our approach to the multidimensional setting of PBNs is based on a reduction to the 1-dimensional situation. We can consider the sublevel sets of f to define a family of subspaces $X_u = f^{-1}((-\infty, u])$, $u \in \mathbb{R}$, nested by inclusion, i.e. a *filtration of X*. Homology may be applied to derive some topological information about the filtration of X induced by f. The first step is to define persistent homology groups as follows. For $u < v \in \mathbb{R}$, we consider the inclusion of X_u into X_v, which induces a homomorphism of homology groups $H_k(X_u) \to H_k(X_v)$ for every $k \in \mathbb{Z}$. Its image consists of the k-homology classes that live at least from $H_k(X_u)$ to $H_k(X_v)$: It is called the *kth persistent homology group of (X, f) at (u, v)*, denoted by $H_k^{(u,v)}(X, f)$. By assuming that coefficients are chosen in a field $\mathbb{K}$, we get that homology groups are vector spaces. Therefore, they can be completely described by their dimension, leading to the following definition [18].

Definition 2.1 (Persistent Betti Numbers)**.** The *persistent Betti numbers function* of f in degree k, briefly PBN, is the function $\beta_f : \Delta^+ \to \mathbb{N}$ defined as

$$\beta_f(u, v) = \dim H_k^{(u,v)}(X, f).$$

Since X is finitely triangulable, we have that $\beta_f(u, v) < \infty$ for every $(u, v) \in \Delta^+$. Hereafter, we will assume that a degree $k \in \mathbb{Z}$ has been chosen.

Persistence diagrams and Representation Theorem. One of the main properties of 1-dimensional PBNs is that they admit a very simple and compact representation. Precisely, under our assumptions on X and f, and making use of Čech homology, it is possible to prove that each 1-dimensional PBNs can be compactly described by a multiset of points, proper and at infinity, of the real plane. We call them *proper cornerpoints* and *cornerpoints at infinity (or cornerlines)*, respectively.

Definition 2.2 (Proper cornerpoint)**.** For every point $p = (u, v) \in \Delta^+$, the number $\mu(p)$ is the minimum over all the positive real numbers ε, with $u + \varepsilon < v - \varepsilon$, of

$$\beta_f(u + \varepsilon, v - \varepsilon) - \beta_f(u - \varepsilon, v - \varepsilon) - \beta_f(u + \varepsilon, v + \varepsilon) + \beta_f(u - \varepsilon, v + \varepsilon).$$

The number $\mu(p)$ will be called the *multiplicity* of p for β_f. Any point $p \in \Delta^+$ such that the number $\mu(p)$ is strictly positive is said to be a *proper cornerpoint for β_f*.

Definition 2.3 (Cornerpoint at infinity)**.** For every vertical line r, with equation $u = \bar{u}$, $\bar{u} \in \mathbb{R}$, we identify r with $(\bar{u}, \infty) \in \Delta^*$, and define the number $\mu(r)$ as the minimum over all the positive real numbers ε, with $\bar{u} + \varepsilon < 1/\varepsilon$, of

$$\beta_f\left(\bar{u} + \varepsilon, 1/\varepsilon\right) - \beta_f\left(\bar{u} - \varepsilon, 1/\varepsilon\right).$$

The number $\mu(r)$ will be called the *multiplicity* of r for β_f. When this finite number is strictly positive, r is said to be a *cornerpoint at infinity for β_f*.

The concept of cornerpoint finds application in providing a representation of PBNs [9, 15]. Set $\bar{\Delta}^* = \Delta^* \cup \partial\Delta^+$.

Definition 2.4 (Persistence diagram)**.** The *persistence diagram* $\mathrm{Dgm}(f) \subset \bar{\Delta}^*$ is the multiset of all cornerpoints (both proper and at infinity) for β_f, counted with their multiplicity, union the points of Δ, counted with infinite multiplicity.

The key role of persistence diagrams is shown in the following Representation Theorem 2.5 [9, 15], claiming that they uniquely determine 1-dimensional PBNs (the converse also holds by definition of persistence diagram).

Theorem 2.5 (Representation Theorem)**.** *For every $(\bar{u}, \bar{v}) \in \Delta^+$, we have*

$$\beta_f(\bar{u}, \bar{v}) = \sum_{\substack{(u,v) \in \Delta^* \\ u \leq \bar{u},\, v > \bar{v}}} \mu((u, v)).$$

In practice, Theorem 2.5 states that the value assumed by β_f at a point $(\bar{u}, \bar{v}) \in \Delta^+$ equals the number of cornerpoints lying above and on the left of $(\bar{u}, \bar{v})$. By means of this theorem, 1-dimensional PBNs can be compactly represented as multisets of cornerpoints and cornerpoints at infinity, i.e. as persistence diagrams.

Stability of 1-dimensional PBNs. The Representation Theorem 2.5 implies that any distance between persistence diagrams induces a distance between 1-dimensional PBNs. This justifies the following Definition 2.6 [9, 15, 16]. Before proceeding, we need to introduce the extended metric $\tilde{d}(p, q) := \|p - q\|_{\widetilde{\infty}}$ on Δ^*. For every $p = (u, v), q = (u', v') \in \Delta^*$, we define

$$(2.1) \quad \|p - q\|_{\widetilde{\infty}} = \min\left\{\max\left\{|u - u'|, |v - v'|\right\}, \max\left\{(v - u)/2, (v' - u')/2\right\}\right\},$$

with the convention about points at infinity that $\infty - c = \infty$ and $c - \infty = -\infty$ when $c \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\pm\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$. In plain words, $\tilde{d}(p, q)$ measures the pseudo-distance between two points p and q as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that any two points of the diagonal have vanishing pseudo-distance (we recall that a pseudo-distance d is just a distance missing the condition $d(X, Y) = 0 \Rightarrow X = Y$, i.e. two distinct elements may have vanishing distance with respect to d). When the number of cornerpoints is finite, the matching of persistence diagrams is related to the bottleneck transportation problem, and the matching distance reduces to the bottleneck distance [15]. However, this is not always the case when working with continuous filtering functions, as the number of cornerpoints may be countably infinite.

Definition 2.6 (Matching distance). Let $f, g : X \to \mathbb{R}$ be two continuous functions. For any bijection σ between $\mathrm{Dgm}(f)$ and $\mathrm{Dgm}(g)$, set $\mathrm{cost}(\sigma) := \max_{p \in \mathrm{Dgm}(f)} \|p - \sigma(p)\|_{\widetilde{\infty}}$. The *matching distance d_{match}* between β_f and β_g is defined as

$$(2.2) \qquad d_{match}(\beta_f, \beta_g) = \min_{\sigma} \mathrm{cost}(\sigma),$$

where σ ranges over all bijections between $\mathrm{Dgm}(f)$ and $\mathrm{Dgm}(g)$.

We remark that the matching distance is stable with respect to perturbations of the filtering functions, as the following Matching Stability Theorem states:

Theorem 2.7 (1-Dimensional Stability Theorem). *If $f, g : X \to \mathbb{R}$ are two continuous functions, then $d_{match}(\beta_f, \beta_g) \leq \|f - g\|_{\infty}$.*

For a proof of the previous theorem and more details about the matching distance the reader is referred to [9, 16] (see also [14, 15] for the bottleneck distance).

2.2. **The Foliation Method.** We now review the so-called *foliation method*, leading to the definition of a stable distance for multidimensional PBNs [9].

If the considered filtering function is vector-valued, i.e. $f : X \to \mathbb{R}^n$, providing the multidimensional analogue of PBNs is straightforward. For $u, v \in \mathbb{R}^n$, with $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, we say $u \preceq v$ (resp. $u \prec v$) if and only if $u_i \leq v_i$ (resp. $u_i < v_i$) for every index $i = 1, \ldots, n$. We also endow $\mathbb{R}^n$ with the max-norm $\|(u_1, u_2, \ldots, u_n)\|_{\infty} = \max_{1 \leq i \leq n} |u_i|$, and use the symbol Δ_n^+ to denote the open set $\{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u \prec v\}$.

Given $u \prec v$, the *multidimensional kth persistent homology group of (X, f) at (u, v)* is defined as the image $H_k^{(u,v)}(X, f)$ of the homomorphism $H_k(X_u) \to H_k(X_v)$ induced in homology by the inclusion of $H_k(X_u)$ into $H_k(X_v)$, with $X_u = \{x \in X : f(x) \preceq u\}$.

Definition 2.8 (Persistent Betti Numbers). The *multidimensional persistent Betti numbers function* of $f : X \to \mathbb{R}^n$ in degree k, briefly PBN, is the function $\beta_f : \Delta_n^+ \to \mathbb{N} \cup \{\infty\}$ defined as

$$\beta_f(u, v) = \dim H_k^{(u,v)}(X, f).$$

Since X is finitely triangulable, we have that $\beta_f(u, v) < \infty$ for every $(u, v) \in \Delta_n^+$ (cf. [6, 9]). The key idea underlying the foliation method is that a collection of half-planes in Δ_n^+ can be given, such that the restriction of the multidimensional PBNs to these half-planes turns out to be a 1-dimensional PBNs function in two scalar variables. This approach implies that the comparison of two multidimensional PBNs can be performed half-plane by half-plane by measuring the distance of appropriate 1-dimensional PBNs. Therefore, the stability of multidimensional PBNs is a consequence of the 1-dimensional PBNs' stability.

We start by recalling that the following parameterized family of half-planes in $\mathbb{R}^n \times \mathbb{R}^n$ is a *foliation* of Δ_n^+ (cf. [2, Prop. 1] and [13]).

Definition 2.9 (Linearly admissible pairs). For every $m = (m_1, \ldots, m_n)$ of $\mathbb{R}^n$ such that $m_i > 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n m_i = 1$, and for every $b = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$ such that $\sum_{i=1}^n b_i = 0$, we shall say that the pair (m, b) is *linearly admissible*. We denote the set of all linearly admissible pairs in $\mathbb{R}^n \times \mathbb{R}^n$ by $Ladm_n$. Given a linearly admissible pair (m, b), we define the half-plane $\pi_{(m,b)}$ of $\mathbb{R}^n \times \mathbb{R}^n$ by the

following parametric equations:

$$\begin{cases} u = s \cdot m + b \\ v = t \cdot m + b \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

The set $Ladm_n$ is a set whose closure is $(2n-2)$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ with boundary. The collection of half-planes $\pi_{(m,b)}$ constitute a foliation of Δ_n^+, implying that for each $(u,v) \in \Delta_n^+$ there exists one and only one $(m,b) \in Ladm_n$ such that $(u,v) \in \pi_{(m,b)}$. Observe that m and b only depend on (u,v).

A first property of this foliation is that the restriction of β_f to each leaf can be seen as a particular 1-dimensional PBNs. Intuitively, on each half-plane $\pi_{(m,b)}$ one can find the PBNs corresponding to the filtration of X obtained by sweeping the line through u and v parameterized by $\gamma_{(m,b)} : \mathbb{R} \to \mathbb{R}^n$, with $\gamma_{(m,b)}(\tau) = \tau \cdot m + b$. Each set X_τ in this filtration is given by the points of X that are taken by f into the quadrant $\{u \in \mathbb{R}^n : u \preceq \gamma_{(m,b)}(\tau)\}$.

A second property is that this filtration is equivalent to the one given by the lower level sets of a certain real-valued continuous function. Both these properties are stated in the next theorem, proved in [9, Thm. 4.2], and are intuitively shown in Figure 1.

Theorem 2.10 (Reduction Theorem). *For every $(u,v) \in \Delta_n^+$, let (m,b) be the only linear admissible pair such that $(u,v) = (s \cdot m + b, t \cdot m + b) \in \pi_{(m,b)}$. Setting $m_* = \min_i m_i$, let moreover $f_{(m,b)} : X \to \mathbb{R}$ be the continuous filtering function defined by setting*

$$f_{(m,b)}(x) = m_* \cdot \max_i \left\{ \frac{f_i(x) - b_i}{m_i} \right\}.$$

Then it holds that

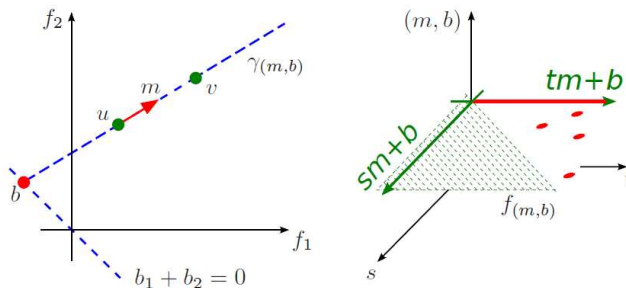$$\beta_f(u,v) = \beta_{\frac{f_{(m,b)}}{m_*}}(s,t).$$

FIGURE 1. 1-dimensional reduction of 2-dimensional PBNs. Left: a 1-dimensional filtration is constructed sweeping the line through u and v. A unit vector m and a point b are used to parameterize this line as $\gamma_{(m,b)}(\tau) = \tau \cdot m + b$. Right: the persistence diagram of this filtration can be found on the half-plane $\pi_{(m,b)}$.

The Reduction Theorem 2.10 implies that in the multidimensional case we can obtain an analogue D_{match} of the distance d_{match}. The metric D_{match} has a particularly simple form, but yet yields the desired stability properties [9].

Definition 2.11 (Multidimensional matching distance). Let $f, g : X \to \mathbb{R}^n$ be continuous functions. If $(m, b) \in Ladm_n$, set $d_{(m,b)}(\beta_f, \beta_b) = d_{match}(\beta_{f_{(m,b)}}, \beta_{g_{(m,b)}})$. The *multidimensional matching distance* D_{match} between β_f and β_g is defined as

$$D_{match}(\beta_f, \beta_g) = \sup_{(m,b)\in Ladm_n} d_{(m,b)}(\beta_f, \beta_g).$$

3. Evaluating the distance between multidimensional PBNs

Definition 2.11 implies that, in general, a direct computation of $D_{match}(\beta_f, \beta_g)$ is not feasible, as we should compute the value $d_{(m,b)}(\beta_f, \beta_g)$ for an infinite number of pairs $(m, b) \in Ladm_n$. On the other hand, taking a non-empty, finite subset $A \subseteq Ladm_n$ and replacing $\sup_{(m,b)\in Ladm_n}$ by $\max_{(m,b)\in A}$ in Definition 2.11, we get a stable and computable pseudo-distance between multidimensional PBNs, say $\tilde{D}_{match}(\beta_f, \beta_g)$, which is an approximation of D_{match} to be used in applications.

Computing $\tilde{D}_{match}(\beta_f, \beta_g)$ requires the definition of a subset $A \subseteq Ladm_n$ striking a balance between computational cost and approximation accuracy. In fact, it is reasonable that the larger the set A, the smaller the approximation error. On the other hand, the smaller the set A, the faster the computation of $\tilde{D}_{match}(\beta_f, \beta_g)$. In this perspective, the goal is to find a set A representing a compromise between these two situations. Additionally, given an arbitrary real value $\varepsilon > 0$ as an error threshold, we might want A depending on ε in a way that $\tilde{D}_{match}(\beta_f, \beta_g)$ accomplishes the inequality $\left| D_{match}(\beta_f, \beta_g) - \tilde{D}_{match}(\beta_f, \beta_g) \right| \le \varepsilon$.

In what follows we review the procedure proposed in [1, 12] to develop an algorithm resulting in an approximation $\tilde{D}_{match}(\beta_f, \beta_g)$ of the multidimensional matching distance $D_{match}(\beta_f, \beta_g)$, up to an input error threshold ε.

3.1. **Underlying theoretical results.** The first result stems from the fact that, at least in a wide subset of $Ladm_n$, the functions $f_{(m,b)}$ defined in the Reduction Theorem 2.10 do not depend on all the components of f. To see this, we first fix $c = \max\{\max_{x\in X} \|f(x)\|_\infty, \max_{x\in X} \|g(x)\|_\infty\}$. Given two indexes $\bar{\imath}, \bar{\jmath} \in \{1, \ldots, n\}$, with $\bar{\imath} \ne \bar{\jmath}$, it is quite easy to choose a linear admissible pair $(m, b) \in Ladm_n$ such that $f_{\bar{\imath}}(x) - b_{\bar{\imath}} \le 0$ and $f_{\bar{\jmath}}(x) - b_{\bar{\jmath}} \ge 0$ for every $x \in X$, thus implying that $f_{(m,b)} = m_* \cdot \max_{i\ne\bar{\imath}} \frac{f_i - b_i}{m_i}$. The simplest example is when $n = 2$: In such a case, the elements of $Ladm_2$ are given by $(m, b) = ((m_1, 1 - m_1), (b_1, -b_1))$, with $0 < m_1 < 1$ and $b_1 \in \mathbb{R}$. It is easy to check that, whenever $b_1 \ge c$ (respectively $b_1 \le -c$) it holds that $f_{(m,b)}(x) = m_* \cdot \frac{f_2(x)+b_1}{1-m_1}$ (resp. $f_{(m,b)}(x) = m_* \cdot \frac{f_1(x)-b_1}{m_1}$) for every $x \in X$. Similar arguments hold for $g_{(m,b)}$, so that we can write

$$(3.1) \qquad d_{(m,b)}(\beta_f, \beta_g) = \begin{cases} \frac{m_*}{m_1} \cdot d_{match}(\beta_{f_1}, \beta_{g_1}), & \text{if } b_1 \le -c; \\ \frac{m_*}{1-m_1} \cdot d_{match}(\beta_{f_2}, \beta_{g_2}), & \text{if } b_1 \ge c, \end{cases}$$

the equality in (3.1) coming from the properties of the matching distance d_{match} (see also [13, Prop. 2.3]).

Based on the above reasonings, the next result [12] states how and when it is possible to reduce the computation of $d_{(m,b)}(\beta_f, \beta_g)$ to a $(n-1)$-dimensional setting. Set $Ladm_n^+ = \{(m, b) \in Ladm_n : \|b\|_\infty \ge (n-1) \cdot c\}$. For every index

$i \in \{1, \ldots, n\}$, we denote by f^i (respectively g^i) the $\mathbb{R}^{n-1}$-valued function obtained from f (resp. g) by removing the i-th component. Similarly, the symbol m^i (resp. b^i) will be used for the element of $\mathbb{R}^{n-1}$ obtained from m (resp. b) by removing the i-th component.

Theorem 3.1. *Assume that* $(m, b) \in Ladm_n^+$. *Then an index* $\bar{\imath} \in \{1, \ldots, n\}$ *exists such that*

$$(3.2) \qquad d_{(m,b)} \left(\beta_f, \beta_g\right) = \frac{m_*}{\min_{i \neq \bar{\imath}} m_i} \cdot d_{(\hat{m}, \hat{b})} \left(\beta_{f^{\bar{\imath}}}, \beta_{g^{\bar{\imath}}}\right),$$

with $\left(\hat{m}, \hat{b}\right) \in Ladm_{n-1}$ *given by* $\hat{m} = \frac{m^{\bar{\imath}}}{(1 - m_{\bar{\imath}})}$ *and* $\hat{b} = b^{\bar{\imath}} + \hat{m} \cdot b_{\bar{\imath}}$.

It is also possible to bound the variation of $d_{(m,b)} \left(\beta_f, \beta_g\right)$ when moving from one half-plane to another in $Ladm_n \setminus Ladm_n^+$. To do this, it is useful to introduce a distance $d : Ladm_n \times Ladm_n \to \mathbb{R}^+$ on the set of admissible pairs [12]. For $(m, b), (m', b') \in Ladm_n$, we set

$$(3.3) \qquad d\left((m, b), (m', b')\right) = \max \left\{ \max_{i=1,\ldots,n} \left| \frac{m_*}{m_i} - \frac{m'_*}{m'_i} \right|, \|b - b'\|_\infty \right\}.$$

Based on the above distance, it is possible to prove the following result [12].

Theorem 3.2. *Let* $(m, b) \in Ladm_n \setminus Ladm_n^+$ *and* $(m', b') \in Ladm_n$, *and assume that* $d\left((m, b), (m', b')\right) \leq \delta$. *Then* $\left| d_{(m,b)} \left(\beta_f, \beta_g\right) - d_{(m',b')} \left(\beta_f, \beta_g\right) \right| \leq 2\delta(n \cdot c + 1)$.

Remark 3.3. Note that $d_{(m,b)} \left(\beta_f, \beta_g\right) \leq 2c$ for every $(m, b) \in Ladm_n$ (this is a trivial consequence of Theorem 2.7); thus we have $\left| d_{(m,b)} \left(\beta_f, \beta_g\right) - d_{(m',b')} \left(\beta_f, \beta_g\right) \right| \leq 2c$. Now, if $\delta \geq \frac{1}{n}$ then $2c \leq 2\delta (nc + 1)$. Consequently, the inequality claimed by Theorem 3.2 is trivial when $\delta \geq \frac{1}{n}$.

4. An algorithm for approximating D_{match}

The above Theorems 3.1 and 3.2 can be used to derive an algorithm for approximating the multidimensional matching distance $D_{match} \left(\beta_f, \beta_g\right)$.

The 2-dimensional case. We start by providing a detailed treatment of the case $n = 2$, since our approach for higher dimensions is based on a reduction to the 2-dimensional situation. We list the steps in the algorithm described in [12]. For a previous version of the algorithm in the case $n = 2$, the reader is referred to [1].

(a). Fix a threshold error ε. By rescaling appropriately both f and g (and consequently ε), we can assume without loss of generality that $c = 1$. For every $\delta > 0$, we can consider the concept of *regular δ-grid over a subset L of* $Ladm_2$, i.e. a collection of points $G = \{p = (m, b) \in Ladm_2\}$ such that, denoting by $B_\delta(p)$ the open ball centered at p having radius δ according to the distance d introduced by equality (3.3), the following statements hold:

  (1) $B_\delta(p) \cap B_\delta(p') = \emptyset$ for every $p, p' \in G$;
  (2) $L \subseteq \cup_{p \in G} \bar{B}_\delta(p)$, with $\bar{B}_\delta(p)$ the closure of $B_\delta(p)$.

(b). We need to fix δ. Because of Remark 3.3 we take δ smaller than $\frac{1}{2}$, say $\delta = \frac{1}{4}$. We also define a finite, regular δ-grid G on $L = Ladm_2 \setminus Ladm_2^+$, see Figure 2 for some examples. To display the grid, we use the fact that $Ladm_2$ can be identified with the product space $M_2 \times N_2$, with $M_2 = \{m = (m_1, 1 - m_1), 0 < m_1 < 1\}$ and $N_2 = \{b = (b_1, -b_1), b_1 \in \mathbb{R}\}$. Therefore we can represent $Ladm_2$ as the subset of the real plane given by $I \times \mathbb{R}$, I the open interval $\{m_1 \in \mathbb{R} : 0 < m_1 < 1\}$. In this perspective, the set $Ladm_2 \setminus Ladm_2^+ = \{(m,b) : \|b\|_\infty < 1\}$ is displayed as $I \times \{b \in \mathbb{R} : |b| < 1\}$. We refer the reader to [12] for a practical construction of G.
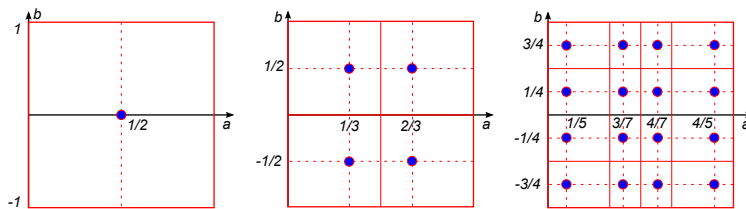

FIGURE 2. Regular grids on $Ladm_2 \setminus Ladm_2^+$ for $\delta = 1$ (left), $\delta = 1/2$ (center), and $\delta = 1/4$ (right). The grids are regular with respect to the distance d defined by the equality (3.3).

(c). Our goal is to compute the largest value for $d_{(m,b)}(\beta_f, \beta_g)$ on $Ladm_2^+$ and on $Ladm_2 \setminus Ladm_2^+$. Equality (3.1) allows us to simplify the computation of $d_{(m,b)}(\beta_f, \beta_g)$ on $Ladm_2^+$. Indeed, it implies that $d_{(m,b)}(\beta_f, \beta_g) \le d_{match}(\beta_{f_1}, \beta_{g_1})$ if $b = (b_1, -b_1)$ is such that $b_1 \le -c$, while $d_{(m,b)}(\beta_f, \beta_g) \le d_{match}(\beta_{f_2}, \beta_{g_2})$ if $b_1 \ge c$. Moreover, in the first case the value $d_{match}(\beta_{f_1}, \beta_{g_1})$ is achieved when $m = (m_1, 1 - m_1)$ is such that $m_1 \le \frac{1}{2}$, while in the second case the value $d_{match}(\beta_{f_2}, \beta_{g_2})$ is achieved when $m_1 \ge \frac{1}{2}$. Thus, it is sufficient to consider the maximum between $d_{match}(\beta_{f_1}, \beta_{g_1})$ and $d_{match}(\beta_{f_2}, \beta_{g_2})$ in order to know the value $\max_{Ladm_2^+} d_{(m,b)}(\beta_f, \beta_g)$. We denote such a maximum by D_{ext}.

(d). Theorem 3.2 allows us to control the variation of $d_{(m,b)}(\beta_f, \beta_g)$ in each set $(Ladm_2 \setminus Ladm_2^+) \cap \bar{B}_\delta(p)$, and hence in $Ladm_2 \setminus Ladm_2^+$. For every $p = (m,b) \in G$, we compute the value $d_{(m,b)}(\beta_f, \beta_g)$ and set $D_{int} = \max_{p \in G} d_{(m,b)}(\beta_f, \beta_g)$.

(e). The number $D_{tot} = \max\{D_{ext}, D_{int}\}$ is then a first approximation of the matching distance $D_{match}(\beta_f, \beta_g)$. We briefly describe how to refine the value D_{tot} to obtain an approximation of $D_{match}(\beta_f, \beta_g)$ up to the error threshold ε.

- If the inequality $2\delta \cdot (2c + 1) \le \varepsilon$ holds, by Definition 2.11 and by applying Theorem 3.2 it follows that $|D_{match}(\beta_f, \beta_g) - D_{tot}| \le \varepsilon$. Therefore we stop, having as output D_{tot};
- Otherwise, we delete each point $p = (m, b) \in G$ such that the inequality $D_{tot} - d_{(m,b)}(\beta_f, \beta_g) > 2\delta \cdot (2c + 1)$ holds. Indeed, Theorem 3.2 ensures that D_{tot} will not be achieved (or exceeded) by computing the values $d_{(m,b)}(\beta_f, \beta_g)$ over the sets $\bar{B}_\delta(p)$. Moreover, the grid G is refined as follows: Each p still in G is replaced by four suitable points $p_1, \ldots, p_4$, such that $\{p_j, j = 1, \ldots, 4\}$ is a regular $\frac{\delta}{2}$-grid on $B_\delta(p)$ based on the four balls $B_{\frac{\delta}{2}}(p_j)$. Finally, D_{int} and D_{tot} are updated according to the new grid G',

δ is replaced by $\frac{\delta}{2}$, and the algorithm restarts by checking if the inequality $2\delta \cdot (2c + 1) \leq \varepsilon$ holds.

The n-dimensional case. We can now show how the above procedure can be generalized to the n-dimensional setting, with $n > 2$. Such an extension is partially based on a reduction to the case $n = 2$.

Similarly to what happens in the case $n = 2$, we need to compute the largest value for $d_{(m,b)}(\beta_f, \beta_g)$ on $Ladm_n^+$ and on $Ladm_n \setminus Ladm_n^+$. We fix a threshold error ε. By appropriately rescaling both f and g (and consequently ε), we can assume without loss of generality that $c = 1$, so that $Ladm_n^+ = \{(m, b) \in Ladm_n : \|b\|_\infty \geq n - 1\}$. In $Ladm_n^+$, Theorem 3.1 allows us to reduce the computation of $d_{(m,b)}(\beta_f, \beta_g)$ to a $(n - 1)$-dimensional situation. Indeed, it implies that, for every $(m, b) \in Ladm_n^+$, there exists $(\hat{m}, \hat{b}) \in Ladm_{n-1}$ such that $d_{(m,b)}(\beta_f, \beta_g) \leq d_{(\hat{m}, \hat{b})}(\beta_{f^{\bar{\imath}}}, \beta_{g^{\bar{\imath}}})$ for a suitable index $\bar{\imath} \in \{1, \dots, n\}$. On the other hand, it is possible to prove that, for every $\bar{\imath} \in \{1, \dots, n\}$ and every $(\hat{m}, \hat{b}) \in Ladm_{n-1}$, there always exists $(m, b) \in Ladm_n^+$ such that $d_{(m,b)}(\beta_f, \beta_g) = d_{(\hat{m}, \hat{b})}(\beta_{f^{\bar{\imath}}}, \beta_{g^{\bar{\imath}}})$. As a consequence, the computation of $d_{(m,b)}(\beta_f, \beta_g)$ over the set $Ladm_n^+$ can be reduced to the one of the $(n-1)$-dimensional matching distances $D_{match}(\beta_{f^i}, \beta_{g^i})$, for $i = 1, \dots, n$.

Obviously, we can recursively repeat the same reasonings to progressively decrease the dimensionality of the problem. It turns out that computing the largest value for $d_{(m,b)}(\beta_f, \beta_g)$ on $Ladm_n^+$ can be reduced to the 2-dimensional case, by considering the $\binom{n}{2}$ 2-dimensional matching distances $D_{match}(\beta_{f_{ij}}, \beta_{g_{ij}})$, with $f_{ij} = (f_i, f_j)$ and $g_{ij} = (g_i, g_j)$ for every $i \neq j$.

Similarly to what happens in the 2-dimensional case, Theorem 3.2 allows us to control the variation of $d_{(m,b)}(\beta_f, \beta_g)$ on the set $Ladm_n \setminus Ladm_n^+$. Also in this case, we can define a regular grid G on $Ladm_n \setminus Ladm_n^+$ by extending the above reasonings for the 2-dimensional setting, see [12] for more details.

5. Beyond the multidimensional matching distance D_{match}

In Definition 2.11 we have seen that the multidimensional matching distance $D_{match}(\beta_f, \beta_g)$ depends on the comparison of the two collections $\{Dgm(f_{(m.b)})\}$ and $\{Dgm(g_{(m.b)})\}$, with (m, b) varying in $Ladm_n$. This is done by computing the supremum of the 1-dimensional matching distances $d_{(m,b)}(\beta_f, \beta_g)$ over (m, b). Note that, in principle, a small change of the pair (m, b) can cause a large change in the "optimal" matching, that is, the matching $\sigma : Dgm(f_{(m,b)}) \to Dgm(g_{(m,b)})$ whose cost is equal to the distance $d_{(m,b)}(\beta_f, \beta_g)$. In other words, the definition of $D_{match}(\beta_f, \beta_g)$ is based on a family of optimal matchings that is not required to change continuously with respect to the pair (m, b). This is due to the intrinsically discontinuous definition of $D_{match}(\beta_f, \beta_g)$, which in turn makes studying its properties difficult.

5.1. **The coherent matching distance for 2-dimensional persistent Betti numbers.** For these reasons, in [10, 11] a new matching distance between multidimensional PBNs has been introduced, called *coherent matching distance* and initially investigated in the 2-dimensional setting. The definition of the coherent matching distance is based on matchings that change "coherently" with the persistence diagrams of the 1-dimensional filtering functions that we take into account. In other words, the basic idea consists of considering only matchings between the

persistence diagrams $\mathrm{Dgm}(f_{(m,b)})$ and $\mathrm{Dgm}(g_{(m,b)})$ that change continuously with respect to the pair (m, b).

The idea of "coherent matching" leads to the discovery of an interesting phenomenon of *monodromy*. While requiring that the matchings change continuously, one has to avoid the pairs (m, b) at which the persistence diagram contains double points, called *singular pairs*. This is done by choosing a connected open set $U \subseteq Ladm_2$ of regular (i.e., non-singular) pairs, and assuming that $(m, b) \in U$. In doing this, it is possible to preserve the "identity" of points in the persistence diagram and follow them when moving in U. From this easily arises the concept of a family of matchings that is continuously changing. Interestingly, turning around a singular pair can produce a permutation in the considered persistence diagram, so that the considered filtering function is associated with a monodromy group. An example of this phenomenon can be found in [11].

Therefore, the definition of "coherent matching" must take a monodromy group into account. In [11], this is done by defining a transport operator $T_\gamma^{(f,g)}$, which continuously transports each matching $\sigma_{(m,b)}$ between the persistence diagrams $\mathrm{Dgm}(f_{(m,b)})$, $\mathrm{Dgm}(g_{(m,b)})$ to a matching $\sigma_{(m',b')}$ between the persistence diagrams $\mathrm{Dgm}(f_{(m',b')})$, $\mathrm{Dgm}(g_{(m',b')})$ along a path γ from (m, b) to (m', b') in the set U. The existence of monodromy implies that the transport of $\sigma_{(m,b)}$ depends not only on the pairs (m, b), (m', b') but also on the path γ.

Having introduced the transport operator $T_\gamma^{(f,g)}$, we can define the *coherent cost of a matching* $\sigma_{(m,b)}$ by considering the usual cost of all the matchings that are obtained by transportation of $\sigma_{(m,b)}$:

$$(5.1) \qquad \mathrm{cohcost}_U(\sigma_{(m,b)}) = \sup_\gamma \mathrm{cost}\left(T_\gamma^{(f,g)}\left(\sigma_{(m,b)}\right) \right),$$

where γ ranges over the set of all continuous paths from $[0, 1]$ to U starting at (m, b), while $\mathrm{cost}(\sigma)$ is the cost of a matching σ between persistence diagrams (see Definition 2.6).

This done, the definition of the coherent matching distance CD_U is straightforward: If two filtering functions $f, g : X \to \mathbb{R}^2$ are given and U does not contain singular pairs neither for f nor for g, then $CD_U(\beta_f, \beta_g)$ is the infimum of the coherent costs of the matchings between the persistence diagrams associated with an admissible pair $(m, b) \in U$ arbitrarily fixed:

$$(5.2) \qquad CD_U(\beta_f, \beta_g) = \inf_{\sigma_{(m,b)}} \mathrm{cohcost}_U\left(\sigma_{(m,b)}\right),$$

with $\sigma_{(m,b)}$ varying in the set of all matchings from $\mathrm{Dgm}(f_{(m,b)})$ to $\mathrm{Dgm}(g_{(m,b)})$.

It is important to remark that the definition of CD_U does not depend on the choice of the pair (m, b) [11, Prop. 4.14]. Moreover, under suitable conditions for the functions f and g, it is possible to prove that, if $\|f - g\|_\infty < c$ for a non-negative real value c sufficiently small, then $CD_U(\beta_f, \beta_g) \le |f - g\|_\infty$ [11, Thm. 4.18].

Another key point here is that the function $\mathrm{cost}(T_\gamma^{(f,g)}(\sigma_{(m,b)}))$ takes its global maximum over γ when the endpoint of γ belongs either to the vertical line $m = \frac{1}{2}$ or to the boundary of U [11, Thm. 5.4]. This result casts new light on the abundance of examples where the supremum defining the usual matching distance D_{match} is taken for the pairs $(m, b) \in Ladm_2$ with $m \approx \frac{1}{2}$ [1, 11]. Nevertheless, it suggests that the coherent matching distance CD_U could be used in place of the matching

distance D_{match} both in theory and applications, as it allows one to manage the parameter space $Ladm_2$ more efficiently.

Acknowledgement

References

1. S. Biasotti, A. Cerri, P. Frosini, and D. Giorgi, *A new algorithm for computing the 2-dimensional matching distance between size functions*, Pattern Recogn. Lett. **32** (2011), no. 14, 1735–1746.
2. S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, and C. Landi, *Multidimensional size functions for shape comparison*, J. Math. Imaging Vis. **32** (2008), no. 2, 161–179.
3. S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo, and M. Spagnuolo, *Describing shapes by geometrical-topological properties of real functions*, ACM Comput. Surv. **40** (2008), no. 4, 1–87.
4. H.B. Bjerkevik and M.B. Botnan, *Computational complexity of the interleaving distance*, 34th International Symposium on Computational Geometry (SoCG 2018) (Dagstuhl, Germany) (Bettina Speckmann and Csaba D. Tóth, eds.), Leibniz International Proceedings in Informatics (LIPIcs), vol. 99, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018, pp. 13:1–13:15.
5. F. Cagliari, B. Di Fabio, and M. Ferri, *One-dimensional reduction of multidimensional persistent homology*, Proc. Amer. Math. Soc. **138** (2010), no. 8, 3003–3017.
6. F. Cagliari and C. Landi, *Finiteness of rank invariants of multidimensional persistent homology groups*, Applied Mathematics Letters **24** (2011), no. 4, 516–518.
7. G. Carlsson and A.J. Zomorodian, *The theory of multidimensional persistence*, SCG '07: Proceedings of the twenty-third annual Symposium on Computational Geometry (New York, NY, USA), ACM, 2007, pp. 184–193.
8. _____, *The theory of multidimensional persistence*, Discrete Comput. Geom. **42** (2009), no. 1, 71–93.
9. A. Cerri, B. Di Fabio, M. Ferri, P. Frosini, and C. Landi, *Betti numbers in multidimensional persistent homology are stable functions*, Math. Method. Appl. Sci. **36** (2013), no. 12, 1543–1557.
10. A. Cerri, M. Ethier, and P. Frosini, *The coherent matching distance in 2d persistent homology*, Lecture Notes in Computer Science, Proceedings of the 6th International Workshop on Computational Topology in Image Context, Marseille, France, June 15-17, 2016, Springer International Publishing Switzerland, A. Bac and J.-L. Mari (Eds.) **9667** (2016), 216–227.
11. A. Cerri, M. Ethier, and P. Frosini, *On the geometrical properties of the coherent matching distance in 2D persistent homology*, ArXiv e-prints (2018), arXiv:1801.06636.
12. A. Cerri and P. Frosini, *A new approximation algorithm for the matching distance in multidimensional persistence*, Journal of Computational Mathematics, To Appear, (preprint available at http://amsacta.cib.unibo.it/2971/).
13. _____, *Invariance properties of the multidimensional matching distance in Persistent Topology and Homology*, Tech. Report no. 2765, Università di Bologna, 2010, Available at http://amsacta.cib.unibo.it/2765/.
14. F. Chazal, D. Cohen-Steiner, M. Glisse, L.J. Guibas, and S.Y. Oudot, *Proximity of persistence modules and their diagrams*, SCG '09: Proceedings of the 25th annual Symposium on Computational Geometry (New York, NY, USA), ACM, 2009, pp. 237–246.
15. D. Cohen-Steiner, H. Edelsbrunner, and J. Harer, *Stability of persistence diagrams*, Discrete Comput. Geom. **37** (2007), no. 1, 103–120.
16. M. d'Amico, P. Frosini, and C. Landi, *Natural pseudo-distance and optimal matching between reduced size functions*, Acta Appl. Math. **109** (2010), 527–554.
17. H. Edelsbrunner and J. Harer, *Computational topology: an introduction*, American Mathematical Society, Providence (R.I.), 2010.
18. H. Edelsbrunner, D. Letscher, and A.J. Zomorodian, *Topological persistence and simplification*, Discrete Comput. Geom. **28** (2002), no. 4, 511–533.

19. P. Frosini and M. Mulazzani, *Size homotopy groups for computation of natural size distances*, B. Belg. Math. Soc.-Sim. **6** (1999), no. 3, 455–464.

20. R. Ghrist, *Barcodes: the persistent topology of data*, Bull. Amer. Math. Soc. **45** (2008), no. 1, 61–75.

21. M. Lesnick, *The theory of the interleaving distance on multidimensional persistence modules*, Found. Comput. Math. **15** (2015), no. 3, 613–650.

22. M. Lesnick and M. Wright, *Interactive Visualization of 2-D Persistence Modules*, ArXiv e-prints (2015), arXiv:1512.00180.

23. M. Scolamiero, W. Chachólski, A. Lundman, R. Ramanujam, and S. Öberg, *Multidimensional persistence and noise*, Found. Comput. Math. **17** (2017), no. 6, 1367–1406.

24. A.J. Zomorodian, *Topology for computing*, Cambridge Monographs on Applied and Computational Mathematics, vol. 16, Cambridge University Press, Cambridge, 2005.

FST - Fom Software Technology, Italia
E-mail address: `acerri@fomsoftware.com, andrea.cerri1@gmail.com`

Dipartimento di Matematica, Università di Bologna, Italia
E-mail address: `patrizio.frosini@unibo.it`