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Additive logistic processes in option pricing

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Abstract
In option pricing, it is customary to first specify a stochastic underlying model and then extract valuation equations from it. However, it is possible to reverse this paradigm: starting from an arbitrage-free option valuation formula, one could derive a family of risk-neutral probabilities and a corresponding risk-neutral underlying asset process. In this paper, we start from two simple arbitrage-free valuation equations, inspired by the log-sum-exponential function and an ℓ^p vector norm. Such expressions lead respectively to logistic and Dagum (or "log-skew-logistic") risk-neutral distributions for the underlying security price. We proceed to exhibit supporting martingale processes of additive type for underlying securities having as time marginals two such distributions. By construction, these processes produce closed-form valuation equations which are even simpler than those of the Bachelier and Samuelson–Black–Scholes models. Additive logistic processes provide parsimonious and simple option pricing models capturing various important stylised facts at the minimum price of a single market observable input.

Keywords Logistic distribution · Additive processes · Derivative pricing · Dagum distribution · Generalised z-distributions

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1 Introduction

The classic early asset pricing models by Bachelier [2] and Samuelson–Black–Scholes (SBS) (Samuelson [29], Black and Scholes [5]) set at least two paradigms in deriva-

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tive pricing research. Firstly, they quickly imposed themselves as universal benchmarks, thereby placing the normal distribution and related formulae (e.g. for implied volatility computations) at the center of the stage. Later alternative models that became established were, and still are, assessed according to which shortcoming of the normal distribution they resolve, and to which extent they do so. Secondly, on a methodological level, they paved the way to the standard research practice of *first* introducing a risk-neutral process and *then* extracting from it valuation formulae. This way of proceeding is rather logical, since it is intrinsic in the fundamental theorem of asset pricing that specifying directly a risk-neutral distribution for the underlying must lead to a no-arbitrage valuation formula. The flip side of this approach is that such a formula is typically rather cumbersome, if available at all. It is generally accepted that the simplest option pricing equation is that from the SBS model, originally offered by Black and Scholes [5].

On the other hand, it is known since Kellerer [20] that for a given family of marginals satisfying a certain property (increase in convex order), there exists a Markovian martingale fitting those marginals; Madan and Yor [25] show various different ways of constructing one such martingale. What is more is that by virtue of the celebrated Breeden and Litzenberger [6] remark, each set of observed option prices uniquely determines a family of risk-neutral distributions. More precisely, as shown in Carr and Madan [9] and Davis and Hobson [12], providing a call option valuation formula which is increasing in maturity and decreasing and convex in strike, with slope in strike bounded below by -1, is sufficient for the application of Kellerer's argument, which in turn guarantees the existence of an underlying martingale security supporting the given formula.

Therefore, it is possible to specify a no-arbitrage option pricing formula, use theoretical arguments to establish the existence of supporting martingale(s), and then maybe try to provide an explicit representation along the lines of [25] (which include the classic Dupire [13] PDE argument). Of course, there is no guarantee that the resulting supporting martingales will have a simple expression. Two papers taking this approach are Figlewski [14] and Henderson et al. [19].

In this paper, we present two extremely simple no-arbitrage option valuation formulae that produce, in the modelling approach described above, risk-neutral distributions of *logistic* type. As it turns out, there exists a class of infinitely divisible distributions, the *generalised z-distributions* (GZD) introduced by Grigelionis [17], whose associated processes retain a simple and yet rich structure, able to naturally accommodate logistic marginals. Such associated processes turn out to be *additive* Markov processes, that is, stochastically continuous Markovian semimartingales with independent, but time-inhomogeneous, increments. Recent financial research has been focusing on additive processes as a promising alternative to classic Lévy models; see e.g. Madan and Wang [24].

The two option valuation formulae we introduce are inspired respectively by the log-sum-exponential (LSE) function, popular in computer science, and the ℓ^p-norm of a two-dimensional vector. The former has as support the whole real line and corresponds to a *logistic* distribution; in option pricing, the use of a logistic distribution has been advocated before by Levy and Levy [21]. The second is supported on the positive half-line and determines a *Dagum* risk-neutral law, which after taking a logarithmic transformation yields a *skew-logistic* distribution for the log-price. The two underlying processes we develop in correspondence of such distributions turn out to be martingales, and thus viable option pricing models in full accordance with the risk-neutral theory of option pricing.

The models obtained can be thought of as logistic analogues of the Bachelier and SBS models. However, unlike the normal models, they can reproduce financial stylised facts such as return kurtosis and skewness, self-similarity, semi-heavy tails and a realistic cumulant term structure which proves to be flexible enough to capture several shapes of the volatility surface. We also determine some potentially useful closed-form formulae for exotic derivative pricing and provide stochastic time-changed model representations of fairly general type. Moreover, after an appropriate measure change, we are able to present physical (non-martingale) dynamics for the involved processes which, although not logistic, still belong to the GZD class. This ideally concludes our "reverse trek" in stochastic modelling starting from, rather than leading to, no-arbitrage option prices.

The paper is organised in the following way. In Sect. 2, we introduce the pricing formulae and explain their connection with the logistic, Dagum and skew-logistic distributions. In Sect. 3, we detail some properties of the distribution classes we require for the analysis of the general framework. A theory of additive processes supporting the LSE and ℓ^p pricing formulae is presented in Sect. 4. In Sect. 5, we discuss the distributional properties of the models, in particular their cumulant term structure and its implications on the volatility surface. Some considerations and formulae for exotic derivative valuations are provided in Sect. 6. Section 7 illustrates the time-changed representation of GZD additive processes, and Sect. 8 the measure transformation taking returns of the logistic and Dagum martingales to some equivalent physical GZD process. Numerical comparisons in terms of calibration performance with some popular Lévy models are offered in Sect. 9. In Sect. 10, we conclude. The proofs are in the Appendix.

2 Option valuation in a logistic framework

On a filtered probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and representing a financial market, we assume that there exists an equivalent martingale measure $\mathbb{Q} \approx \mathbb{P}$ under which all discounted asset prices are martingales. For simplicity, **throughout the paper**, we assume that a zero risk-free interest rate is paid.

We consider two risk-neutral underlying asset price dynamics $S^R = (S_t^R)_{t\geq 0}$ and $S^P = (S_t^P)_{t\geq 0}$ producing two different valuation formulae for two distinct derivative securities. The superscripts R and P stand respectively for "real-valued" and "positive"; the terminology will be justified in what follows. In many respects, the two models we introduce represent logistic-world analogues of the Bachelier and SBS models, although superior to these under various aspects, as we hope to make clear in this paper.

Let $\mathbb{R}_+ = (0, +\infty)$. A call option written on S^R with strike $K \in \mathbb{R}$ and maturing at $T > 0$ is valued at time zero by the log-sum-exponential function

$$C_0^R(K, S_0^R, T) = s(T) \ln\left(1 + \exp\left(\frac{S_0^R - K}{s(T)}\right)\right), \qquad S_0^R \in \mathbb{R}, \tag{2.1}$$

where $s:\mathbb{R}_+\to\mathbb{R}_+$ is an increasing continuous function with $\lim_{t\to 0}s(t)=0$. Notice that enforcing call–put parity produces a put value equal to

$$P_0^R(K,S_0^R,T)=s(T)\ln\left(1+\exp\left(\frac{K-S_0^R}{s(T)}\right)\right),\qquad S_0^R\in\mathbb{R}.$$

The requirement on S^P is instead that the married put portfolio of long one put option of strike $K>0$ and maturity $T>0$ and long one unit of the underlying, whose payoff is the max-expression

$$M_T=\max\{S_T^P,K\},$$

has a price given by the formula

$$M_0^P(K,S_0^P,T)=\left((S_0^P)^{1/b(T)}+K^{1/b(T)}\right)^{b(T)},\qquad S_0^P>0.\tag{2.2}$$

This formula coincides with that of the ℓ^p-norm in $\mathbb{R}^2$, for $p=1/b(T)$, of the vector (S_0^P,K). We require the function $b:\mathbb{R}_+\to(0,1]$ to be increasing with $\lim_{t\to 0}b(t)=0$. From the call–put parity, we can derive from M_0^P the call and put option values C_0^P and P_0^P on S^P by using the relations $C_0^P=M_0^P-K$ and $P_0^P=M_0^P-S_0^P$.

As a consequence of Carr and Madan [9] and Davis and Hobson [12], a (differentiable) pricing formula for a call (respectively put) option is arbitrage-free if and only if the following requirements are fulfilled: (a) the formula is convex and decreasing (resp. increasing) in strike with derivative uniformly bounded from below by -1 (resp. from above by $+1$), avoiding static arbitrage; (b) it is increasing in maturity, avoiding calendar arbitrage; (c) the limit of the formula for maturity tending to zero is the option intrinsic value (payoff). Equation (2.1) clearly satisfies this set of requirements because of the assumptions on s. For (2.2), we observe that requirement (a) implies that a married put must be increasing in strike, which is easily checked in the expression. Furthermore, convexity and boundedness by -1 of the derivative in K follows from the condition $b\le 1$, and monotonicity in T is clear by recalling the properties of the ℓ^p-norms. Finally, it is easy to see that

$$\lim_{T\to 0}M_0^P(K,S_0^P,T)=M_0$$

for all $K,S_0^P>0$. These elementary no-arbitrage pricing formulae enjoy a surprising amount of empirically consistent properties.

We begin by analysing the functional expression of (2.1) and (2.2). By simply dividing, (2.1) is equivalent to

$$\frac{C_0^R(K,S_0^R,T)}{s(T)}=\ln\left(1+\exp\left(\frac{S_0^R-K}{s(T)}\right)\right),\tag{2.3}$$

whereas (2.2) can be rewritten as

$$\ln\frac{M_0^P(K,S_0^P,T)^{1/b(T)}}{S_0^P}=\ln\left(1+\exp\left(\frac{\ln(K/S_0^P)}{b(T)}\right)\right).\tag{2.4}$$

After appropriate normalisation, the option values depend on only one dimensionless market input parameter, namely the respective normalised moneynesses $\frac{S_0^R-K}{s(T)}$ and $\frac{\ln(K/S_0^P)}{b(T)}$. The functions s and b determine the prices term structure, although they have different unit measures: s is expressed in monetary terms, whereas b is a pure number. One can also explicitly relate s and b to the implied volatility at-the-money term structure, which we do in Sect. 5.

We denote by $C_0^R(K, S_0^R, s(T))$ and $M_0^P(K, S_0^P, b(T))$ the valuation equations in (2.1) and (2.2) with explicit dependence on the term functions. For all $\lambda > 0$, we have

$$
\begin{aligned}
C_0^R\big(K+\lambda, S_0^R+\lambda, s(T)\big) &= C_0^R\big(K, S_0^R, s(T)\big),\\
C_0^R\big(\lambda K, \lambda S_0^R, \lambda s(T)\big) &= \lambda C_0^R\big(K, S_0^R, s(T)\big),\\
M_0^P\big(\lambda K, \lambda S_0^P, b(T)\big) &= \lambda M_0^P\big(K, S_0^P, b(T)\big),\\
M_0^P\big(K^\lambda, (S_0^P)^\lambda, \lambda b(T)\big) &= M_0^P\big(K, S_0^P, b(T)\big)^\lambda.
\end{aligned}
$$

The value of a call option on S^R is translation-invariant and scale-invariant, whereas a married put on S^P is scale-invariant and enjoys a form of power-invariance. For comparison, the Bachelier call valuation formula is translation-invariant, but not scale-invariant; in the SBS model, the married put price is scale-invariant, but not power- or translation-invariant, nor can it be reduced to a function of a single moneyness input. It is also straightforward to show that the value $M_0^R(K, S_0^R, s(T))$ of a married put on S^R is given by

$$
M_0^R\big(K, S_0^R, s(T)\big) = s(T)\ln\left(\exp\Big(\frac{K}{s(T)}\Big) + \exp\Big(\frac{S_0^R}{s(T)}\Big)\right), \qquad S_0^R \in \mathbb{R},
$$

from which we can derive the striking equality

$$
\exp\Big(M_0^R\big(K, S_0^R, b(T)\big)\Big) = M_0^P\big(e^K, e^{S_0^R}, b(T)\big) \tag{2.5}
$$

connecting married put prices under S^R and S^P, which is evidence of the close relationship between the two models.

2.1 The associated distributions

Once valuation formulae are provided, we can use the classic Breeden and Litzenberger [6] argument to solve the inverse problem of finding the risk-neutral implied price distributions. The risk-neutral densities implied in a set of quoted option prices can be recovered as the second derivative of the values with respect to strike. Correspondingly we can obtain the risk-neutral cumulative distribution function (CDF) of the terminal underlying prices S_T^R and S_T^P as

$$
\mathbb{Q}[S_T^R < K] = 1 + \frac{\partial C_0^R}{\partial K} = 1 - \frac{e^{\frac{S_0^R-K}{s(T)}}}{1+e^{\frac{S_0^R-K}{s(T)}}} = \frac{1}{1+e^{-\frac{K-S_0^R}{s(T)}}}
$$

and

$$\mathbb{Q}[S_T^P < K] = \frac{\partial M_0^P}{\partial K} = \left(1 + \left(\frac{K}{S_0^P}\right)^{-1/b(T)}\right)^{b(T)-1}.$$

Recall that a real-valued random variable L has the *logistic distribution* $\mathrm{L}(\sigma, \mu)$, $\sigma > 0$, $\mu \in \mathbb{R}$, if its CDF F_L is given by

$$F_L(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}}, \qquad x \in \mathbb{R},$$

with corresponding probability density function (PDF)

$$f_L(x) = \frac{e^{-\frac{x-\mu}{\sigma}}}{\sigma(1 + e^{-\frac{x-\mu}{\sigma}})^2}, \qquad x \in \mathbb{R}.$$

A random variable D follows the *Dagum distribution* $\mathrm{D}(a, b, c)$ with $a, b, c > 0$ if it is positively supported with CDF

$$F_D(x) = \left(1 + \left(\frac{x}{b}\right)^{-a}\right)^{-c}, \qquad x \geq 0,$$

and PDF

$$f_D(x) = \frac{ac}{b}\left(1 + \left(\frac{x}{b}\right)^{-a}\right)^{-c-1}\left(\frac{x}{b}\right)^{-a-1}, \qquad x \geq 0. \tag{2.6}$$

We then see that the centred/normalised terminal distributions of S^R and S^P follow respectively

$$S_T^R - S_0^R \sim \mathrm{L}\big(s(T), 0\big), \qquad \frac{S_T^P}{S_0^P} \sim \mathrm{D}\big(1/b(T), 1, 1 - b(T)\big) \tag{2.7}$$

for all $T > 0$. Notice that the values $a = 1/b(T)$ and $c^{-1} = 1/(1 - b(T))$ are Hölder conjugates, that is, $a^{-1} + c = 1$, a property that will be relevant for martingale relations.

Let us further introduce the *skew-logistic* distribution class SL. We write for a random variable $\mathrm{SL} \sim \mathrm{SL}(\alpha, \sigma, \mu)$ with $\alpha, \sigma > 0$, $\mu \in \mathbb{R}$ if its CDF is such that

$$F_{\mathrm{SL}}(x) = \frac{1}{(1 + e^{-\frac{x-\mu}{\sigma}})^{\alpha}}, \qquad x \in \mathbb{R}. \tag{2.8}$$

Clearly, $\mathrm{SL}(1, \sigma, \mu) \equiv \mathrm{L}(\sigma, \mu)$. The skew-logistic distribution has negative skewness for $\alpha < 1$ and positive skewness for $\alpha > 1$. We notice that this CDF can be recovered by simply raising that of an $\mathrm{L}(\sigma, \mu)$ random variable to the power α. Therefore $x \mapsto x^{\alpha}$ acts as a distortion function taking logistic to skew-logistic. This can be seen as the analogous relationship at the CDF level (or, financially, for digital put prices/greeks) of (2.5).

Notice that by applying a log-transform to the price ratios S_T^P/S_0^P, we have for all $x \in \mathbb{R}$ the log-price probabilities

$$\mathbb{Q}[\ln(S_T^P/S_0^P) < x] = (1 + e^{-x/b(T)})^{b(T)-1}. \tag{2.9}$$

Comparing (2.8) with (2.9), we also see that

$$\ln(S_T^R/S_0^R) \sim \mathrm{SL}\big(1 - b(T), b(T), 0\big). \tag{2.10}$$

The problem of identifying risk-neutral price distributions consistent with the valuation formulae (2.1) and (2.2) is therefore completely solved. The centered distributions of S^R are logistic, allow negative values, are symmetric and leptokurtic with semi-heavy tails, making them better suited to capture real market distributions than its normal counterpart, the Bachelier model. In full analogy, the normalised skew-logistic distribution for the logarithm of the positive model S^P exhibits leptokurtosis, semi-heavy tails and negative skewness, features which the normal distribution for the log-price in the SBS model is lacking. What (2.7) and (2.10) also make clear is that the term functions s and b appearing in the valuation equations coincide with the scale functions of the underlying risk-neutral distribution, so that they play the same role of the term volatility in the normal asset pricing models. In order to build a fully consistent valuation theory based on logistic processes, we must seek two $\mathbb{Q}$-*martingales* S^R and S^P such that S_t^R and S_t^P satisfy (2.7) for all $t > 0$. The rest of this paper is devoted to identifying some suitable such processes and discussing their properties.

3 Generalised z-distributions and Lévy processes

In order to determine martingale dynamics for S^P and S^R with the appropriate logistic marginals, it is convenient to broaden the scope of our investigation and consider more general distribution families, to which both the logistic and skew-logistic distributions belong.

Let us begin by introducing the family of *z-distributions* $\mathrm{ZD}(\sigma, c_1, c_2, \mu)$ characterised by the PDF

$$f_{\mathrm{ZD}}(x) = \frac{1}{\sigma B(c_1, c_2)} \frac{e^{(x-\mu)c_1/\sigma}}{(1 + e^{(x-\mu)/\sigma})^{c_1+c_2}}, \qquad x \in \mathbb{R}, \tag{3.1}$$

where $B(\cdot, \cdot)$ is the Beta function. The ZD class has been prominently studied in Barndorff-Nielsen et al. [4]. The constant $\sigma > 0$ is the distribution scale, the value $\mu \in \mathbb{R}$ a location parameter, while $c_1, c_2 > 0$ represent respectively left and right asymmetry parameters. This can be appreciated by noticing that the distribution has log-linear tails: taking the limits to $\pm\infty$ in (3.1), we have limiting logarithmic slopes c_1/σ and $-c_2/\sigma$, showing that if $c_1 < c_2$ (resp. $c_2 < c_1$), the distribution is negatively (resp. positively) skewed. If $c_1 = c_2$, the distribution is symmetric. The CDF of a general SL law is not known in closed form. However, the characteristic function

$\hat{f}_{\mathrm{ZD}}$ of a ZD random variable is given by (see [4, Eq. (3.3)])

$$\hat{f}_{\mathrm{ZD}}(z)=\frac{B(c_1+iz\sigma,c_2-iz\sigma)}{B(c_1,c_2)}e^{i\mu z},\qquad z\in\mathbb{R}. \tag{3.2}$$

At this point, we notice that $SL\sim\mathrm{SL}(\alpha,\sigma,\mu)$ has the PDF

$$f_{\mathrm{SL}}(x)=\frac{\alpha}{\sigma}\frac{e^{-\frac{x-\mu}{\sigma}}}{(1+e^{-\frac{x-\mu}{\sigma}})^{\alpha+1}}=\frac{\alpha}{\sigma}\frac{e^{\alpha\frac{x-\mu}{\sigma}}}{(1+e^{\frac{x-\mu}{\sigma}})^{\alpha+1}},\qquad x\in\mathbb{R}, \tag{3.3}$$

so that the SL distribution family is a subclass of the ZD family.

The ZD class has been further extended by Grigelionis [17] who introduced the *generalised z-distribution* $\mathrm{GZD}(\sigma,c_1,c_2,\delta,\mu)$. A random variable GZD has a GZD distribution if its characteristic function is given by

$$\hat{f}_{\mathrm{GZD}}(z)=\left(\frac{B(c_1+iz\sigma,c_2-iz\sigma)}{B(c_1,c_2)}\right)^{\delta}e^{i\mu z},\quad z\in\mathbb{R}, \tag{3.4}$$

for some shape parameter $\delta>0$, with all the remaining parameters retaining the same interpretation as in the ZD case. The PDFs and CDFs of the GZD distributions are not known analytically, but we have the relations

$$\begin{aligned}\mathrm{GZD}(\sigma,c_1,c_2,1,\mu)&\equiv\mathrm{ZD}(\sigma,c_1,c_2,\mu),\\ \mathrm{GZD}(\sigma,\alpha,1,1,\mu)&\equiv\mathrm{ZD}(\sigma,\alpha,1,\mu)\equiv\mathrm{SL}(\alpha,\sigma,\mu).\end{aligned} \tag{3.5}$$

Another distribution class which can be embedded in the GZD family is the *Meixner* distribution class, which arises when $c_{1,2}=1/2\pm\beta$, $|\beta|<1/2$. Associated processes have been used in finance by Schoutens [33].

A property which allows us to canonically generate processes from an assigned distribution is *infinite divisibility*. We recall that a random variable X is infinitely divisible if for all $n\in\mathbb{N}$, there exists a family $\{X_k^n\}_{k=1,\dots,n}$ of independent identically distributed (i.i.d.) random variables such that in law

$$X\overset{d}{=}\sum_{k=1}^{n}X_k^n.$$

For a given infinitely divisible random variable X, standard theory (e.g. Sato [31, Theorem 7.10]) establishes the existence of a *Lévy process*, that is, a stochastically continuous process with i.i.d. increments, whose time-1 marginal has the same distribution as X.

Let us first investigate if infinite divisibility is helpful for the identification of a logistic process. Grigelionis [17, Proposition 1] proves that the $\mathrm{GZD}(\sigma,c_1,c_2,\delta,\mu)$ class is *self-decomposable*, a result which is well known for the SL and L distributions. A random variable S is said to be self-decomposable if for all $0<\alpha<1$, there exists a random variable R_α independent of S such that in law

$$S\overset{d}{=}\alpha S+R_\alpha.$$

Self-decomposability implies infinite divisibility so that we are in a familiar Lévy setup. From (3.4), we obtain that the characteristic exponent Ψ of a Lévy process $Z=(Z_t)_{t\geq 0}$, where Z_1 has a GZD distribution, is

$$\Psi(z) := \ln \hat{f}_{\mathrm{GZD}}(z) = \delta \ln \frac{B(c_1+iz\sigma, c_2-iz\sigma)}{B(c_1,c_2)} + i\mu z, \qquad z\in\mathbb{R}, \tag{3.6}$$

so that the Lévy marginals Z_t have characteristic function $\hat{f}_{Z,t}$ given by

$$\hat{f}_{Z,t}(z) = \exp\big(t\Psi(z)\big) = \left(\frac{B(c_1+iz\sigma, c_2-iz\sigma)}{B(c_1,c_2)}\right)^{\delta t} e^{it\mu z}, \qquad z\in\mathbb{R}.$$

Hence

$$Z_t \sim \mathrm{GZD}(\sigma, c_1, c_2, \delta t, \mu t), \tag{3.7}$$

and Z is called a *generalised-z Lévy process*.

Assume now we are given a ZD law from the SL or L family, and we wish to build some Lévy process Z such that Z_1 has that law. For this to happen, by (3.5), we need to set $\delta=1$ in the GZD specification, and from (3.7), we conclude that Z_t will not be ZD-distributed unless $t=1$. In particular, Z_t cannot have an L or SL distribution at all times. A logistic Lévy process seems then not to be obtainable along these lines. The reason for the introduction of GZD processes is exactly that of determining a class of infinitely divisible distributions of z-type closed under convolution, so that the associated Lévy processes – unlike those arising from L, SL and more general ZD distributions – have marginals in the same class. We need instead to bind Z_t to be (skew) logistic at all times, and in the next section, we shall see how this can be done.

4 The additive logistic framework

As observed, a logistic Lévy process seems not to be available. We could then try and relax the Lévy structure to see if considering a larger set of processes could accommodate one with the required laws. Removing the assumption of time-homogeneity of the increments leads to considering the so-called family of *additive* processes, i.e., processes with independent but non-stationary increments. The marginals of an additive process are still infinitely divisible random variables. For a given additive process, we call the Lévy triplet of the marginal distributions the *Lévy characteristic triplet* of the process; note that this depends on t in a usually nonlinear way. As illustrated in Sato [31, Chap. 2], there exists a canonical way of building additive processes from a time-dependent family of infinitely divisible distributions. Applying this technique to GZD laws allows us to establish the following general result.

Proposition 4.1 *Let $\sigma,\delta,c_1,c_2:\mathbb{R}_+\to\mathbb{R}_+$ and $\mu:\mathbb{R}_+\to\mathbb{R}$ be continuous functions. Then the random variables*

$$Z_t \sim \mathrm{GZD}\big(\sigma(t), c_1(t), c_2(t), \delta(t), \mu(t)\big) \tag{4.1}$$

are a self-decomposable family with Lévy characteristic triplet $(a, 0, v dx)$ *given by*

$$a_t = \delta(t)\sigma(t)\int_0^{1/\sigma(t)} \frac{e^{-c_2(t)x} - e^{-c_1(t)x}}{1-e^{-x}}dx, \tag{4.2}$$

$$v(t,x) = \begin{cases} \delta(t)\dfrac{e^{-x\frac{c_2(t)}{\sigma(t)}}}{x(1-e^{-\frac{x}{\sigma(t)}})}, & x > 0, \\ \delta(t)\dfrac{e^{x\frac{c_1(t)}{\sigma(t)}}}{|x|(1-e^{\frac{x}{\sigma(t)}})}, & x < 0. \end{cases} \tag{4.3}$$

If in addition δ, σ *are nondecreasing with* $\lim_{t\to 0}\sigma(t) = 0$ *and* c_1, c_2 *are bounded around zero and such that the functions* $c_1/\sigma, c_2/\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ *are nonincreasing, there exists a unique in law additive process* $Z = (Z_t)_{t\geq 0}$ *null at zero whose marginals are given by* (4.1).

Whenever v is absolutely continuous in t, we have that ν defined by

$$\nu(t,x) := \frac{d}{dt}v(t,x)$$

is such that the measure $\nu dt dx$ represents the compensating measure of the jumps of Z. Since the Lévy densities (i.e. the Radon–Nikodým derivatives of the Lévy measures with respect to Lebesgue measure) are $O(x^{-2})$ around zero for all $t > 0$, the corresponding GZD additive processes are of infinite variation. Although (4.2) and (4.3) are valid for any GZD law, GZD Lévy processes cannot be obtained as a particular case of the second statement of Proposition 4.1, consistently with the discussion at the end of the previous section, since a positive constant σ obviously does not meet the requirements of Proposition 4.1.

By an appropriate choice of parameters, it is easy to single out from the class of processes that can be built around Proposition 4.1 a pair with L and SL distributions, which also determine the martingale models S^P and S^R.

Proposition 4.2 *Let* s *and* b *be the functions appearing in the valuation formulae* (2.1) *and* (2.2). *There exist unique in law additive processes* $X = (X_t)_{t\geq 0}$ *and* $Y = (Y_t)_{t\geq 0}$ *null at zero such that for all* $t > 0$,

$$X_t \sim \mathrm{L}\big(s(t), 0\big),$$
$$Y_t \sim \mathrm{SL}\big(1-b(t), b(t), 0\big).$$

The processes X *and* Y *have respective Lévy characteristic triplets* $(0, 0, v^X dx)$ *and* $(a^Y, 0, v^Y dx)$, *where* $a_0^Y = v^Y(0,x) = v^X(0,x) = 0$ *and for* $t > 0$,

$$v^X(t,x) = \begin{cases} \dfrac{e^{-\frac{x}{s(t)}}}{x(1-e^{-\frac{x}{s(t)}})}, & x > 0, \\ \dfrac{e^{\frac{x}{s(t)}}}{|x|(1-e^{\frac{x}{s(t)}})}, & x < 0, \end{cases} \tag{4.4}$$

and

$$a_t^Y = b(t) \int_0^{1/b(t)} \frac{e^{-x} - e^{-(1-b(t))x}}{1 - e^{-x}} dx, \tag{4.5}$$

$$v^Y(t,x) = \begin{cases} \dfrac{e^{-\frac{x}{b(t)}}}{x(1 - e^{-\frac{x}{b(t)}})}, & x > 0, \\ \dfrac{e^{x\frac{1-b(t)}{b(t)}}}{|x|(1 - e^{\frac{x}{b(t)}})}, & x < 0. \end{cases} \tag{4.6}$$

Furthermore, the asset price processes S^R *and* S^P *defined respectively by*

$$S^R = X + S_0^R, \qquad S_0^R \in \mathbb{R}, \tag{4.7}$$

and

$$S^P = S_0^P \exp(Y), \qquad S_0^P > 0, \tag{4.8}$$

are martingales. In particular, for all K, T, *the respective values* $C_0^R(K, S_0^R, T)$ *and* $M_0^P(K, S_0^P, T)$ *of a call option written on* S^R *and a married put written on* S^P *are given by* (2.1) *and* (2.2), *respectively.*

In view of the discussion in Sect. 2, Proposition 4.2 naturally implies that we have $S_t^P/S_0^P \sim \mathrm{D}(1/b(t), 1, 1-b(t))$ for the positive model price ratios. Motivated by this result, we call the martingale underlying asset models S^R and S^P respectively the *symmetric logistic additive* (SLA) model and the *conjugate-power Dagum additive* (CPDA) model, following the remark that the parameters a and c^{-1} in the law of S_t^P/S_0^P are Hölder conjugates.

The martingale property is naturally featured by these processes since the time marginals reflect the fact that (2.1) and (2.2) are proper no-arbitrage valuation equations. For the CPDA model, this has an interesting implication on the exponential nature of the process. The commonest and simplest way of generating a positive equity model is to apply an exponential transformation to some given real-valued process. However, martingale relations are needed after such a transformation. Typically, in option pricing based on Itô diffusions, one starts with a martingale which can be both positive or negative, and then performs a *stochastic* exponentiation to end up with a positive martingale. Alternatively – usually for Lévy processes – one could perform a natural exponentiation, but then some form of drift adjustment of the base process is normally required to achieve the martingale property. Here, the situation is different. The risk-neutral valuation formula of the married put leading to the option prices (2.2) produces asset log-price processes which are not martingales. However, its natural exponential – having the required Dagum time marginals – is a martingale, and this is without any need of drift adjustment. This is a direct consequence of having worked out the log-returns Y_t directly from a valuation equation, instead of having supplied them as a model input.

4.1 The logistic self-similar additive pricing model

As observed, being members of the GZD distribution class, logistic distributions are self-decomposable. As demonstrated by Sato [30], starting from a self-decomposable distribution D and for all $H > 0$, one can determine a family of additive processes S^H such that S_1^H equals D in distribution. These processes retain the additional property of being *self-similar* of index H, i.e., for all $a, t > 0$, they satisfy the equality in law

$$S_{at} \overset{d}{=} a^H S_t.$$

The self-similar process S^H is in general not the same as the Lévy process associated with D (which exists, since self-decomposability implies infinite divisibility), unless D is a stable distribution; see [30] for more details. The analysis of self-similarity as a statistical property of asset returns is a well-established line of research since the work of Mandelbrot [26].

In our logistic additive framework, there exists a specification for the function s in the SLA model which coincides with the self-similar additive model that can be constructed from a given logistic distribution. We have the following corollary to Proposition 4.2.

Corollary 4.3 *Let $\sigma > 0$ and $H > 0$. The additive process X from Proposition 4.2 with the specification $s(t) = \sigma t^H$ is the self-similar additive process associated with a self-decomposable random variable $D \sim \mathrm{L}(\sigma, 0)$.*

In accordance with the introduced terminology, we refer to the SLA model under the specification $s(t) = \sigma t^H$ as the *self-similar logistic additive* model (SSLA) and denote it by $S^{R,H}$ to emphasise the self-similarity exponent H.

5 Distributional and term structure properties

The logistic distribution L is a leptokurtic distribution symmetric about the mean and thus has all odd moments zero, while the SL distribution is skewed. Both distributions have moments of all orders. The similarity between the logistic and normal distribution is a well-known fact, and in a real-valued option pricing context is reflected in a similarity of the SLA model with the Bachelier model. Additionally, the logistic distribution features excess kurtosis, an observed statistical property of (risk-neutral) financial returns. A comparison between an SSLA PDF and a normal one is provided in Fig. 1. Kurtosis and symmetry of the logistic distribution determine a symmetric smile in the normal implied volatility surface, as observed in Fig. 2. When comparing the risk-neutral Dagum distribution and its normal counterpart, the lognormal distribution, the similarity is less stringent. This has in part to do with the discussion on the nature of the exponential transform of the log-returns determining the security price, and we illustrate it further in the following.

More insight on the nature of the two martingale underlying price processes is provided by their return cumulant structure. Cumulants are related to the distribution

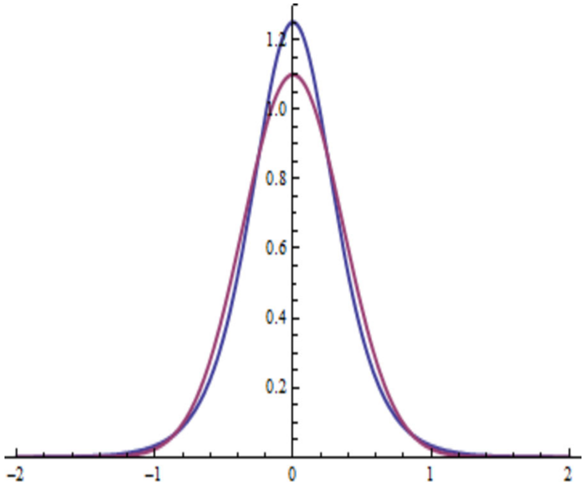


Fig. 1 PDF of $S_1^{R,1/2}$ with $\sigma = 0.2$ (blue) compared to a normal PDF from the Bachelier model with the same variance (purple). The excess kurtosis is slight but visible

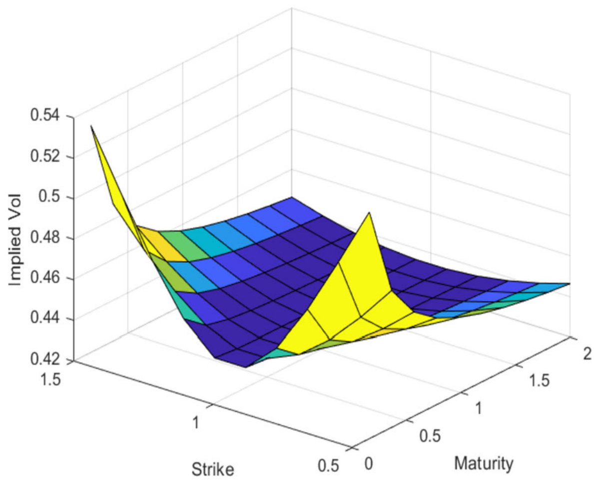


Fig. 2 Normal implied volatility surface of the $S^{R,1/2}$ model

shape and symmetry, which in turn connect to the volatility smile and skew, as well as the volatility term structure. As is well known, for $n \in \mathbb{N}$, the cumulants $\kappa_n^Y(t)$ of $Y_t = \ln(S_t^P/S_0^P)$ can be found by direct differentiation of the Lévy–Khintchine representation of the characteristic exponent: taking into account (4.5) and (4.6), we have

$$\kappa_n^Y(t) = b(t)^n \int_0^\infty x^{n-1} \frac{e^{-x} + (-1)^n e^{-x(1-b(t))}}{1-e^{-x}} dx =: b(t)^n I_n(t). \tag{5.1}$$

Using the integral representation of the digamma function $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$ given by

$$\psi(z) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-zx}}{1-e^{-x}} \right) dx, \tag{5.2}$$

we have the explicit expression

$$I_1(t) = \psi\big(1 - b(t)\big) - \psi(1) = \psi\big(1 - b(t)\big) + \gamma, \tag{5.3}$$

where $\gamma = -\psi(1)$ is the Euler–Mascheroni constant. Since ψ is increasing on $\mathbb{R}_+$, we see that Y_t has negative expectation, consistently with the property that the natural exponential of Y is a martingale. If $n > 1$, using the integral forms of the Riemann zeta function ζ and the polygamma function $\psi^{(m-1)} = \frac{d^m}{dz^m} \ln \Gamma(z)$, $m > 0$, we see that

$$\begin{aligned} I_n(t) &= \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx + (-1)^n \int_0^\infty x^{n-1} \frac{e^{-x(1-b(t))}}{1 - e^{-x}} dx \\ &= \zeta(n)(n-1)! + \psi^{(n-1)}\big(1 - b(t)\big). \end{aligned} \tag{5.4}$$

The above cumulant formulae recover and generalise those in Gupta and Kundu [18, Sect. 4.3]. The analogous differentiation for $X_t = S_t^R - S_0^R$ produces

$$\kappa_n^X(t) = s(t)^n \int_0^\infty x^{n-1} \frac{e^{-x} + (-1)^n e^{-x}}{1 - e^{-x}} dx,$$

which is zero for n odd. For n even, it holds that

$$\kappa_n^X(t) = 2s(t)^n \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx = 2s(t)^n \zeta(n)(n-1)!. \tag{5.5}$$

From the above, we recover the well-known variance and kurtosis values of the logistic distribution, namely $\mathrm{Var}(X_t) = s(t)^2\pi^2/3$ and $\mathrm{Kurt}(X_t) = \kappa_4^X(t)/(\kappa_2^X(t))^2 = 6/5$, indicating constant slight excess kurtosis in the (S)SLA model.

Focusing on Y_t, recall that we say that the tails of a PDF f are *semi-heavy* if

$$f(x) \sim C_\pm x^{\rho_\pm} e^{\beta_\pm x} \quad \text{as } x \to \pm\infty.$$

From (3.3), we see that the PDF of Y_t has semi-heavy tails with

$$\rho_\pm = 0, \quad C_\pm = \beta_- = \frac{1}{b(t)} - 1, \quad \beta_+ = -\frac{1}{b(t)}. \tag{5.6}$$

From these values, we can infer that the distributional asymmetry is minimal when $t \approx 0$ and increases as t gets larger. Note that this is not a property of the lognormal distributions in the SBS model, which are symmetric at all times. An illustration is given in Fig. 3.

A related aspect is the term structure of the return cumulants. In Lévy models, return cumulants increase linearly in t, and hence skewness and kurtosis are of the respective orders $O(t^{-1/2})$ and $O(t^{-1})$, which is at odds with market data. In an additive model, due to the time-inhomogeneity of the returns, the cumulants' time decay can in principle be different. We already observed constant intertemporal kurtosis for the (S)SLA model. For the CPDA model, we can calculate

$$\mathrm{Skew}(Y_t) = \frac{I_3(t)}{I_2(t)^{3/2}}, \quad \mathrm{Kurt}(Y_t) = \frac{I_4(t)}{I_2(t)^2}.$$

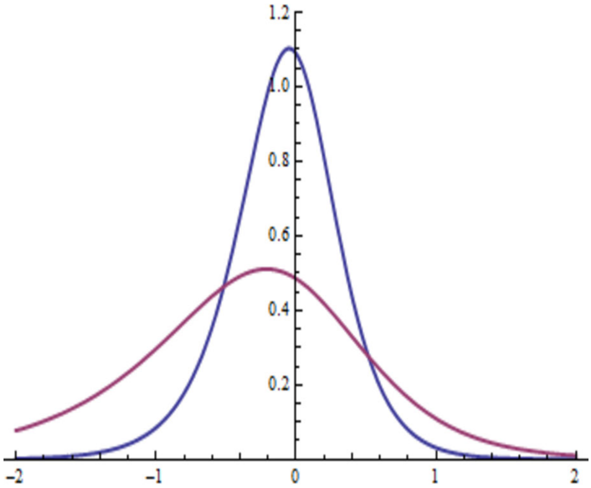


Fig. 3 PDFs of $Y_{0.5}$ (blue) and Y_2 (purple). We used b as in (5.7) with $\sigma = 0.3$. The asymmetry increases with t

The time evolution of the return cumulants is provided in Figs. 4 and 5. As we can see, for a set of maturities including those commonly traded, negative skewness and excess kurtosis increase with time, consistently with what can be deduced from (5.6). The function b chosen here is

$$b(t) = \sqrt{1 - e^{-\sigma^2 t}}, \tag{5.7}$$

which has the property that $b(t) \sim \sigma\sqrt{t}$, the familiar normal accrued volatility, when $t \approx 0$.

We can use cumulants to further investigate the analogy between the logistic and normal pricing paradigms. Having in mind that the Bachelier and SBS option values are similar at short maturities (see e.g. Schachermayer and Teichmann [32]), in order to replicate such a similarity for the SSLA and the CPDA models, given an exponent $H > 0$, we choose $b = b_H$ as

$$b_H(t) = (1 - e^{-t\,\sigma^{1/H}})^H. \tag{5.8}$$

Indeed, from (5.5) with $s(t) = \sigma t^H$ and (5.8), using that the integrand is dominated and $b_H(t) \sim \sigma t^H$ when $t \approx 0$, we obtain the estimates

$$\begin{aligned} \kappa_n^Y(t) &= b_H(t)^n \int_0^\infty x^{n-1} \frac{e^{-x} + (-1)^n e^{-(1-b_H(t))x}}{1 - e^{-x}} dx \\ &\sim \mathbb{1}_{\{n=2k, k\in\mathbb{N}\}} \sigma^n t^{nH} 2 \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx = \kappa_n^H(t), \end{aligned}$$

where $\kappa_n^H(t)$ is the nth cumulant of $S_t^{R,H}$. This highlights how at small times under the specification b_H, the distributions of S_t^P and $S_t^{R,H}$ are similar.

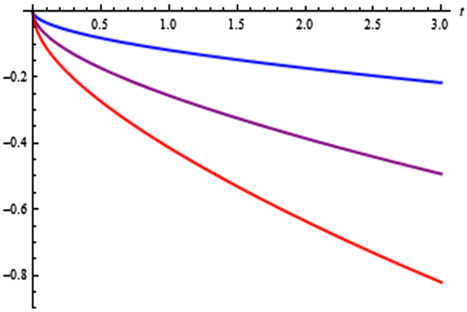


Fig. 4 Skew(Y_t) with the parametrisation (5.7), using $\sigma = 0.1$ (blue), $\sigma = 0.2$ (purple) and $\sigma = 0.3$ (red)

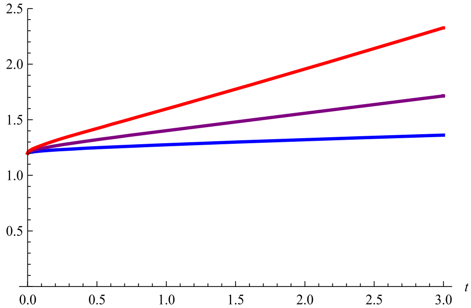


Fig. 5 Kurt(Y_t) with the parametrisation (5.7), using $\sigma = 0.1$ (blue), $\sigma = 0.2$ (purple) and $\sigma = 0.3$ (red)

Another element of interest is the phenomenon of moment explosion in the CPDA model. Using (2.6), for all $n > 0$, the nth moment of S_t^P is given by

$$\begin{aligned}\mathbb{E}[(S_t^P)^n] &= \frac{1-b(t)}{b(t)} \int_0^\infty x^n \frac{x^{-\frac{1}{b(t)}-1}}{(x^{-1/b(t)}+1)^{2-b(t)}} dx \\ &= \big(1-b(t)\big) \int_0^\infty \frac{y^{-nb(t)}}{(y+1)^{2-b(t)}} dy. \end{aligned} \tag{5.9}$$

The last integral converges at zero whenever $b(t) < 1/n$, and in that case, using Gradshteyn and Ryzhik [16, 8.380.3], we can verify that

$$\mathbb{E}[(S_t^P)^n] = \big(1-b(t)\big) B\big(1+(n-1)b(t), 1-nb(t)\big),$$

which coincides with the characteristic function of an SL$(1-b(t), b(t), 0)$ random variable calculated in $z = -in$, as it must. However, the integral in (5.9) is divergent whenever $b(t) \geq 1/n$; since $b(t)$ is increasing, this means that for all $n > 1$, the root t_n of $b(t) = 1/n$ is the explosion time for the nth moment of the asset, i.e., $\mathbb{E}[(S_t^P)^n] = \infty$ for all $t \geq t_n$. As an implication, if we require for the market analysis a moment of some given order n to exist, we need to choose b such that the equation $b(t) = 1/n$ has no roots. This can be achieved by modifying (5.8) to

$$b_{H,n}(t) = \frac{1}{n}\big(1-e^{-t\,(n\sigma)^{1/H}}\big)^H.$$

Fig. 6 Implied volatility surface of the S^P model using b_H with $H = 0.5$ and $\sigma = 0.25$

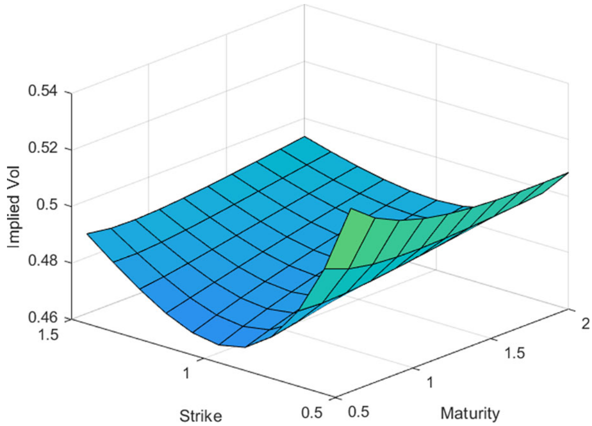
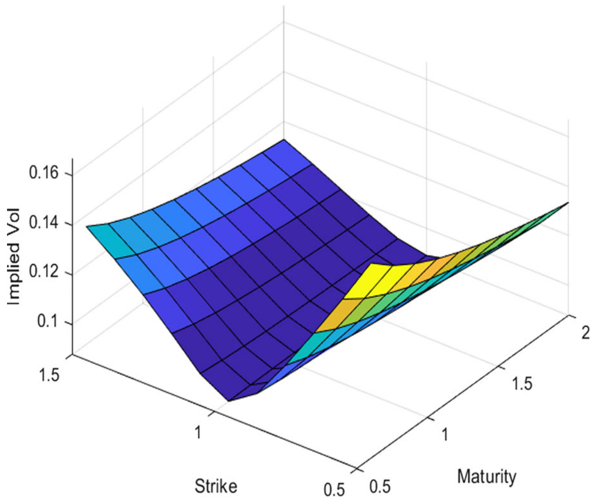


Fig. 7 Implied volatility surface of the S^P model using b_H with $H = 0.5$ and $\sigma = 0.05$



This function is always bounded by $1/n$ and maintains the asymptotic regime σt^H when $t \approx 0$. For example, if the asset manager wishes to perform minimum variance hedging on S^P, she or he must take into consideration $b_{H,2}$, with H and σ being free parameters.

Finally we can connect the term functions s and b with the at-the-money (ATM) implied volatility term structure σ_{ATM} as follows. Denoting by C^R_{ATM} and C^P_{ATM} the ATM call prices respectively in the SLA and CPDA model and equating them to the ATM call prices from respectively the Bachelier and Black–Scholes call pricing formulae, we have

$$C^R_{\text{ATM}}(T) = \sigma^R_{\text{ATM}}(T)\sqrt{\frac{T}{2\pi}}, \quad C^P_{\text{ATM}}(T) = S_0\bigg(2\Phi\Big(\frac{\sigma^P_{\text{ATM}}(T)\sqrt{T}}{2}\Big) - 1\bigg), \quad (5.10)$$

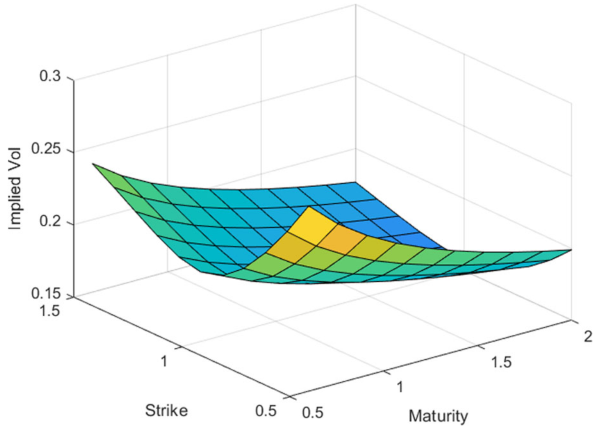


Fig. 8 Implied volatility surface of the S^P model using b_H with $H = 0.3$ and $\sigma = 0.1$

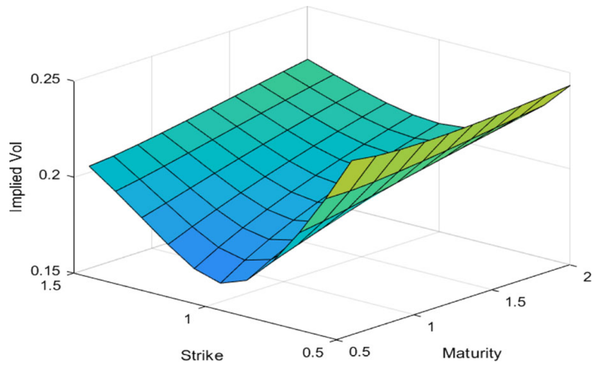


Fig. 9 Implied volatility surface of the S^P model using b_H with $H = 0.7$ and $\sigma = 0.1$

where Φ is the standard normal CDF. Now by (2.1) and (2.2), $C^R_{\mathrm{ATM}}(T) = s(T)\ln 2$ and $C^P_{\mathrm{ATM}}(T) = S_0(2^{b(T)} - 1)$, so that substituting in (5.10) yields the relations

$$\sigma^R_{\mathrm{ATM}}(T) = s(T)\sqrt{\frac{2\pi}{T}}\ln 2, \qquad s(T) = \frac{\sigma^R_{\mathrm{ATM}}(T)}{\ln 2}\sqrt{\frac{T}{2\pi}},$$

$$\sigma^P_{\mathrm{ATM}}(T) = \frac{2}{\sqrt{T}}\Phi^{-1}(2^{b(T)-1}), \qquad b(T) = \frac{\ln\Phi(\frac{\sigma^P_{\mathrm{ATM}}(T)\sqrt{T}}{2})}{\ln 2} + 1.$$

These equations reveal that the functions s and b can also be interpreted as a transformation of the ATM implied volatility term structure, which can be helpful to calibrate the (S)SLA and CPDA models to market option prices. In particular, s is nothing but a rescaling of σ^R_{ATM}. By inspection, we see that in the SLA model, we can generate an upward, downward or constant ATM volatility term structure according to whether $H > 1/2$, $H < 1/2$ or $H = 1/2$. Similar sensitivity patterns can be observed in the CPDA model volatility surface, which we illustrate in Figs. 6, 7, 8 and 9. Decreasing σ in (5.8) increases the volatility skew, while changing H primarily acts on the slope of the term structure.

6 Exotic derivative pricing

One of the benefits of an additive framework is that a more realistic intertemporal behaviour of price distributions comes at almost no cost in terms of added complexity for derivative pricing. Pricing techniques for additive models typically consist of minor modifications of those used for Lévy models; see e.g. Cont and Tankov [11, Chap. 14] for an account. In addition, in our setting, the explicit knowledge of the underlying probability densities and the properties of the logistic distributions contribute to an even higher degree of tractability.

It is well known (see e.g. Lewis [22]) that whenever the characteristic function Φ_T of a terminal log-price distribution $\ln S_T$ is known and F is a European-style contingent claim maturing at T and satisfying some minimal regularity assumptions, the time-0 value V_0 of $F(S_T) = G(\ln S_T)$ is given by the complex Parseval integral

$$V_0 = \mathbb{E}[F(S_T)] = \int_{\mathcal{C}} \Phi_T(-z)\hat{G}(z)dz. \tag{6.1}$$

Here $\widehat{\cdot}$ denotes the Fourier transform, and the integration contour $\mathcal{C}$ is a line contained in the region of analyticity of Φ_T. This formula is very useful if the probability densities of the underlying model are not known, as virtually everywhere in the literature. However, in the (S)SLA and CPDA models, while characteristic functions are available, we also know the explicit distributions and hence have the plain representations

$$\mathbb{E}[F(S_T^R)] = \int_{\mathbb{R}} F(S_0^R + x) f_L(x)dx, \qquad \mathbb{E}[F(S_T^P)] = \int_0^\infty F(S_0^P e^x) f_{\mathrm{SL}}(x)dx,$$

where $L \sim \mathrm{L}(s(T), 0)$ and $SL \sim \mathrm{SL}(1 - b(T), b(T), 0)$. These equations are simple real-valued integrals which do not suffer from the complications surrounding complex integration (e.g. branch cuts or loss of analyticity in $\mathcal{C}$) and are clearly preferable to (6.1). As a consequence, in some cases, closed-form formulae are available. For example, valuing the log-contract $F(x) = \ln x$ in the CPDA model involves just the calculation of the expectation of a skew-logistic random variable, which is known from Sect. 5. Using (5.1)–(5.3), we have the time-0 value

$$\mathbb{E}[\ln S_T^P] = \ln S_0^P + b(T)\Big(\psi\big(1 - b(T)\big) + \gamma\Big). \tag{6.2}$$

As is well known (see Neuberger [27], Carr and Madan [8]), the log-contract is intrinsically linked to volatility derivatives. For instance, since its 2003 revision, the continuously monitored volatility index VIX can be defined in terms of a continuum of traded vanilla options on the S&P 500 index S synthesising the log-contract. This leads for the theoretical VIX at time zero to the formula

$$\mathrm{VIX}_0 = \sqrt{\mathbb{E}\left[\frac{-2\ln(S_\tau/S_0)}{\tau}\right]},$$

where $\tau = 30/365$. Using (6.2), we then have that the VIX based on a CPDA model is valued by

$$\mathrm{VIX}_0 = \sqrt{\mathbb{E}\left[\frac{-2\ln(S^P_\tau/S^P_0)}{\tau}\right]} = \sqrt{-2\frac{b(\tau)(\psi(1-b(\tau))+\gamma)}{\tau}}.$$

In CPDA models, variance swap valuation is also straightforward. A continuously monitored variance swap pays at maturity T the quadratic variation $[Y]_T$ of the log-returns of the underlying. The semimartingale jump characteristic $\nu^Y(t,x)dtdx$ of Y is known by differentiating (4.6); hence basic martingale theory and making use of (5.1)–(5.4) gives

$$\begin{aligned}\mathbb{E}\big[[Y]_T\big] &= \mathbb{E}\bigg[\sum_{0<s\le T}(Y_s - Y_{s-})^2\bigg] = \mathbb{E}\bigg[\int_{\mathbb{R}}\int_0^T x^2 \upsilon^Y(t,x)dtdx\bigg] \\ &= \int_{\mathbb{R}} x^2 \upsilon^Y(T,x)dx = \kappa_2^Y(T) = b(T)^2\bigg(\frac{\pi^2}{6} + \psi^{(1)}\big(1-b(T)\big)\bigg). \qquad (6.3)\end{aligned}$$

When path-dependent options are considered, general additive models are well suited for numerical PIDE valuation, and this applies to the (S)SLA and CPDA models as well. For example, consider a down-and-out barrier option paying off at maturity T

$$(S_T - K)^+ \mathbb{1}_{\{\inf_{t\in[0,T]} S_t > D\}}$$

for some $D > 0$. Assuming some technical conditions are met, the time-0 value V_0 of this option in the CPDA case is $V_0 = V(0,S)$, where $V = V(t,S)$ is the solution of the PIDE

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,S) + \int_0^\infty \bigg(V(t,Se^x) - V(t,x) - S(e^x-1)\frac{\partial V}{\partial S}(t,S)\bigg)\upsilon^Y(t,x)dx = 0 \\ &\qquad\qquad \text{for } (t,S) \in (0,T] \times (D,\infty), \\ &V(t,S) = 0 \qquad \text{for } (t,S) \in (0,T] \times [0,D], \\ &V(T,S) = (S-K)^+,\end{aligned}$$

with υ^Y given by (4.6). The solution can be numerically approximated using e.g. finite difference methods. As an another example, American options can be valued by studying the linear complementarity problem associated with our models. For a comprehensive treatment, see Cont and Tankov [11, Chap. 12].

A context in which techniques for pricing under an additive process come together with the analytic properties of the GZD world is the valuation of forward-starting options. Assume we want to find the time-0 value of a put option on S^P with expiration T_2 and strike set at $0 < T_1 < T_2$ to be a multiple $m > 0$ of $S^P_{T_1}$. Indicating by $\mathbb{E}_{T_1}[\,\cdot\,]$

the conditional expectation at time T_1, using the independence of increments and the martingale property, we have that the time-0 value V_0 of this payoff equals

$$\begin{aligned} V_0 &= \mathbb{E}[(mS^P_{T_1} - S^P_{T_2})^+] = \mathbb{E}\big[S^P_0 e^{Y_{T_1}} \mathbb{E}_{T_1}[(m - e^{Y_{T_2}-Y_{T_1}})^+]\big] \\ &= S^P_0 \mathbb{E}[(m - e^{Y_{T_2}-Y_{T_1}})^+]. \end{aligned}$$

Now (6.1) can be used since again by the independence of increments, the characteristic function Φ_{T_2,T_1} of $Y_{T_2} - Y_{T_1}$ satisfies $\Phi_{T_2,T_1}(z) = \Phi_{T_2}(z)/\Phi_{T_1}(z)$. Here the characteristic function Φ_t of Y_t is known from (3.4) with CPDA parameters; hence Φ_{T_2,T_1} has in terms of the Beta function and the term function b the expression

$$\Phi_{T_2,T_1}(z) = \frac{1-b(T_2)}{1-b(T_1)} \frac{B(1+(iz-1)b(T_2), 1-izb(T_2))}{B(1+(iz-1)b(T_1), 1-izb(T_1))}.$$

Finally, the logistic marginals of our models find use in the valuation of certain exchange options. Assume for instance that S^1 and S^2 are two independent identical copies of an SLA asset of term function s. An exchange option with maturity T is the right of exchanging S^1 for S^2 at time T. The risk-neutral value V_0 of the corresponding payoff is thus

$$V_0 = \mathbb{E}[(S^1_T - S^2_T)^+].$$

Recall that the L-scale λ_2 of a random variable X with finite mean is the transformation of the expectations of the order statistics $X_{k:2}$ given by

$$\lambda_2 = \frac{\mathbb{E}[X_{2:2}] - \mathbb{E}[X_{1:2}]}{2}.$$

It is well known that when X follows the logistic distribution, λ_2 equals the scale parameter. This implies that when $X = S^R_T$, then $\lambda_2 = s(T)$. Furthermore, it is easy to show that for two i.i.d. copies X_1, X_2 of X, we have

$$(X_1 - X_2)^+ = \frac{X_{2:2} - X_{1:2}}{2}.$$

By taking expectations in the above, we can then conclude that

$$V_0 = s(T),$$

which also provides a third interpretation of the logistic term function s. Order statistics properties of the logistic distribution may play a role in valuing more general types of exchange options or options depending on more than one underlying security, such as basket options.

7 Time-changed representation of GZD processes

Subordination, or more generally *time-changing*, is the operation of stochastically changing the time evolution of a stochastic process by using a second, increasing process almost surely diverging at infinity (the time change). The theory of stochastic time changes in option pricing is well established. There are multiple benefits linked to subordination and time-changing, concerning the availability of closed-form formulae, mathematical tractability, the possibility of recovering normality of returns in business time, and incorporating stochastic volatility in jump processes (see e.g. Geman et al. [15], Carr and Wu [10], Ané and Geman [1], Carr et al. [7]).

In order to devise a time-changed representation for GZD processes, and thus in particular for the logistic and skew-logistic processes X and Y, we need the class of *generalised Gamma convolutions* $\mathrm{GGC}(a,u)$ or *Thorin distributions*, introduced in Thorin [35]. A GGC random variable G is characterised by the Laplace transform

$$\mathbb{E}[e^{-sG}] = \exp\big(-\phi(s)\big), \qquad \mathrm{Re}(s) > 0, \tag{7.1}$$

with

$$\phi(s) = as + \int_0^\infty \ln\left(1 + \frac{s}{y}\right) u(dy),$$

where $a \geq 0$ and u is a positive measure (the *Thorin measure*) such that

$$\int_0^1 |\ln y| u(dy) < \infty, \quad \int_1^\infty \frac{u(dy)}{y} < \infty.$$

Using the Frullani integral and Fubini's theorem, one has

$$\begin{aligned}\int_0^\infty \ln\left(1 + \frac{s}{y}\right) u(dy) &= \int_0^\infty \left(\int_0^\infty \frac{e^{-yx} - e^{-(s+y)x}}{x} dx\right) u(dy)\\ &= \int_0^\infty \frac{1 - e^{-sx}}{x} \left(\int_0^\infty e^{-yx} u(dy)\right) dx.\end{aligned}$$

Therefore G is a positively supported infinitely divisible (and self-decomposable) distribution with Laplace exponent ϕ, drift a and Lévy measure

$$\sigma(dx) := \frac{1}{x}\left(\int_0^\infty e^{-yx} u(dy)\right) \mathbb{1}_{\{x>0\}} dx.$$

Conversely, all Lévy measures of the form $k(x)\mathbb{1}_{\{x>0\}} dx/x$, where k is a completely monotone function, define a GGC distribution whose Thorin measure u coincides with the measure in the Laplace-integral representation of k, i.e.,

$$k(x) = \int_0^\infty e^{-yx} u(dy),$$

and such a representation is unique. Some examples are the following.

Example 7.1 Setting $a = 0$ and $u_{\lambda,\kappa} = \kappa\delta_\lambda$ for $\kappa, \lambda > 0$, with δ_λ the Dirac delta concentrated in λ, we obviously have $\sigma(dx) = \frac{e^{-\lambda x}}{x}\mathbb{1}_{\{x>0\}}dx$, which is the Lévy measure of a Gamma random variable $\Gamma(\lambda, \kappa)$. Therefore $\mathrm{GGC}(0, u_{\lambda,\kappa}) \equiv \Gamma(\lambda, \kappa)$ and we can verify that

$$\phi(s) = \int_0^\infty \kappa(1 - e^{-sx})\frac{e^{-\lambda x}}{x}dx = \ln\left(\left(1 + \frac{s}{\lambda}\right)^\kappa\right).$$

Example 7.2 More generally, if $a = 0$ and

$$u_n = \sum_{j=0}^n \kappa_j \delta_{\lambda_j}$$

for $n > 0$ and some $\kappa_j, \lambda_j > 0$, $j = 0, \dots, n$, then

$$\phi_n(s) = \int_0^\infty (1 - e^{-sx})\frac{\sum_{j=0}^n \kappa_j e^{-\lambda_j x}}{x}dx = \ln\left(\prod_{j=0}^n \left(1 + \frac{s}{\lambda_j}\right)^{\kappa_j}\right) \tag{7.2}$$

and thus

$$G = \sum_{j=0}^n \Gamma_j$$

with $\Gamma_j \sim \Gamma(\lambda_j, \kappa_j)$, $j = 0, \dots, n$, i.e., $G \sim \mathrm{GGC}(0, u_n)$ is a sum of $n + 1$ independent Gamma random variables.

Example 7.3 According to Thorin [36, Theorem 5.2], the lognormal distribution with parameters μ, σ is a $\mathrm{GGC}(0, u_\ell dy)$ law, where

$$u_\ell(y) = \frac{1}{\pi}\arctan\left(\frac{\mathrm{Im}(\lambda(y))}{\mathrm{Re}(\lambda(y))}\right),$$

$$\lambda(y) = \frac{e^{\frac{\pi^2}{2\sigma^2}}}{\sqrt{2\pi}}\int_{-\infty}^\infty \exp\left(-ye^{-\sigma x+\mu} - \frac{x^2}{2} + \frac{i\pi x}{\sigma}\right)dx.$$

This is how infinite divisibility of the logistic distribution was originally proved.

Assume now $a = 0$ and let $(\kappa_j)_{j\geq 0}$, $(\lambda_j)_{j\geq 0}$ be two sequences of positive real numbers. For $\mathrm{Re}(s) > 0$, let G^∞ be the random variable whose Laplace transform is given by

$$\mathbb{E}[e^{-sG^\infty}] = \prod_{j=0}^\infty \left(1 + \frac{s}{\lambda_j}\right)^{-\kappa_j}, \tag{7.3}$$

assuming further that $(\kappa_j)_{j\geq 0}$, $(\lambda_j)_{j\geq 0}$ are such that the limit on the right-hand side exists. Analogously to Example 7.2 above, by setting

$$u_\infty = \sum_{j=0}^{\infty} \kappa_j \delta_{\lambda_j},$$

we see that G^∞ is $\mathrm{GGC}(0, u_\infty)$, since this choice produces

$$\phi_\infty(s) = \ln\left(\prod_{j=0}^{\infty}\left(1+\frac{s}{\lambda_j}\right)^{\kappa_j}\right) < \infty$$

in the representation (7.1). We have the formal expression

$$G^\infty = \sum_{j=0}^{\infty} \Gamma_j,$$

with $\Gamma_j \sim \Gamma(\lambda_j, \kappa_j)$, $j \geq 0$, i.e., G^∞ can be seen as an infinite sum (a weak limit) of Gamma random variables. Finally, as in (7.2), we observe that the Lévy density corresponding to u is the convergent series

$$\Sigma(x) = \sum_{j=0}^{\infty} \kappa_j \frac{e^{-\lambda_j x}}{x} \mathbb{1}_{\{x>0\}}. \tag{7.4}$$

We now show that there exists a GGC additive subordinator with time marginals of the form G^∞ above, providing a time-changed representation of a certain class of GZD additive processes. This way of specifying a process starting from some given marginals has also been explored in Madan and Yor [25], and nicely ties in with our results in the previous section. The following proposition is inspired by Barndorff-Nielsen et al. [4] who give a Gaussian mixing formula for the ZD class.

Proposition 7.4 *Let $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions with β nondecreasing, α nonincreasing and $\alpha > \beta$. Assume Z is a* GZD *additive process as in Proposition 4.1, whose marginals can be written in the form*

$$Z_t \sim \mathrm{GZD}\big(\sigma(t), \alpha(t)-\beta(t), \alpha(t)+\beta(t), \kappa(t), \mu(t)\big).$$

Then

$$Z \stackrel{d}{=} B_\Gamma + \mu,$$

where $B = (B_t)_{t\geq 0}$ is a Gaussian continuous additive process with variance $\sigma(t)$ and mean $-\beta(t)$, whereas $\Gamma = (\Gamma_t)_{t\geq 0}$ is a driftless additive GGC *subordinator independent of B whose Lévy densities $\Sigma(t,x)$ are given by the right-hand side of (7.4) with*

$$\kappa_j = \kappa(t), \qquad \lambda_j(t) = \frac{(\alpha(t)+j)^2 - \beta(t)^2}{2\sigma(t)}.$$

From the above result, the CPDA log-returns process Y is recovered by setting $\sigma(t)=b(t)$, $\alpha(t)=1-b(t)/2$, $\beta(t)=b(t)/2$, $\kappa(t)=1$, $\mu(t)=0$, while the process X for the SLA returns is obtained by choosing $\sigma(t)=s(t)$, $\alpha(t)=\kappa(t)=1$ and $\beta(t)=\mu(t)=0$. Actually, for these processes, we have an alternative, more familiar, time-changed representation: the parent Gaussian process can be taken to be a Brownian motion.

Proposition 7.5 *Let X, Y be the processes in Proposition 4.2 and Γ^X, Γ^Y additive* GGC *subordinators having Lévy densities $\Sigma^X(t,x)$, $\Sigma^Y(t,x)$ with specifications in* (7.4) *given respectively by*

$$\kappa_j^X=1, \qquad \lambda_j^X(t)=\frac{(1+j)^2}{2s(t)^2}, \tag{7.5}$$

$$\kappa_j^Y=1, \qquad \lambda_j^Y(t)=\frac{(1+j)^2-(1+j)b(t)}{2b(t)^2}.$$

Then for a standard Brownian motion W independent of Γ^X and Γ^Y, we have

$$X \stackrel{d}{=} W_{\Gamma^X},$$

$$Y \stackrel{d}{=} W_{\Gamma^Y}-\frac{\Gamma^Y}{2}. \tag{7.6}$$

When Γ is a GGC Lévy process having Lévy measure $\Sigma(x)$ in (7.4) with $\kappa=1$ and λ_j as in (7.5), where $s(t)\equiv r/\pi$, $r>0$ (a *hyperbolic subordinator*), the process W_Γ was discussed by Pitman and Yor [28]. The representation of the logistic distribution as a normal mixture is known since Stefanski [34]. Time-changed representations for S^R and S^P can be easily obtained from Propositions 7.4 and 7.5 by applying the necessary transformations taking returns to prices.

8 Physical dynamics

The underlying asset processes that we gave in this paper are naturally defined in terms of their risk-neutral dynamics $\mathbb{Q}$, as they must conform to price distributions implied by a valuation formula. However, for GZD additive processes, an equivalent measure change theory exists, relying on additive Esscher density transforms, which makes it possible to explicitly determine equivalent physical dynamics. As we shall illustrate, when starting from risk-neutral dynamics, physical equivalent processes will no longer be logistic but still retain ZD marginals. In other words, logistic additive processes do not constitute a class closed under equivalent measure changes, but ZD additive processes do. This supports the idea that logistic processes are better looked at as members of the class of the (G)ZD additive processes. That GZD and ZD Lévy processes are closed under Esscher transforms is shown in Grigelionis [17, Proposition 8]. Similar arguments apply in our additive framework.

Proposition 8.1 *Fix $T > 0$ and let X, Y be the additive processes of Proposition 4.2 with respective Laplace cumulants ψ^X and ψ^Y.*

(i) *For $\theta \in \Theta = (0, 1)$, the additive Esscher transform*

$$\left.\frac{d\mathbb{P}^\theta}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left(\theta Y_t - \psi_t^Y(\theta)\right), \qquad t \in [0, T], \tag{8.1}$$

induces an equivalent measure change $\mathbb{P}^\theta \approx \mathbb{Q}$ on $\mathcal{F}_T$, and on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}^\theta)$, the process Y is an additive ZD *process with distribution*

$$Y_t^\theta \sim \mathrm{ZD}\big(b(t), c_1(\theta, t), c_2(\theta, t), 0\big), \tag{8.2}$$

where

$$c_1(\theta, t) = 1 - (1-\theta)b(t), \tag{8.3}$$

$$c_2(\theta, t) = 1 - \theta b(t). \tag{8.4}$$

(ii) *For all $\theta \in \Theta = (-1/s(T), 1/s(T))$, setting*

$$\left.\frac{d\mathbb{P}^\theta}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left(\theta X_t - \psi_t^X(\theta)\right), \qquad t \in [0, T], \tag{8.5}$$

induces an equivalent measure change $\mathbb{P}^\theta \approx \mathbb{Q}$ on $\mathcal{F}_T$, and on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}^\theta)$, the process X is an additive ZD *process with distribution*

$$X_t^\theta \sim \mathrm{ZD}\big(s(t), s_1(\theta, t), s_2(\theta, t), 0\big), \tag{8.6}$$

where

$$s_1(\theta, t) = 1 + \theta s(t), \tag{8.7}$$

$$s_2(\theta, t) = 1 - \theta s(t). \tag{8.8}$$

Proposition 8.1 identifies for each θ a physical (log-)returns process, which in turn determines θ-physical underlying models $S^{R,\theta}$ and $S^{P,\theta}$ for the (S)SLA and CPDA model respectively. Conversely, since the transformations (8.3), (8.4) and (8.7), (8.8) are invertible in θ, for any fixed specification of the dynamics of Y^θ and X^θ under $\mathbb{P}^\theta$, there exist unique additive Esscher equivalent martingale measure changes with densities of the form (8.1)–(8.5) determining the risk-neutral dynamics of Y and X.

Proposition 8.1 is consistent with the facts observed in Sect. 2 related to the logistic distributions. Changing measure in the logistic (and more generally, in the GZD) world acts on the whole distribution symmetry, not just on the mean/location as in the normal case. Therefore measure changing does not merely reduce to a drift change (which is, however, present), but modifies the whole moment structure. This also means that the usual rule thumb of an "invariant variance" upon changing measure breaks down, the measure change invariants being instead the functions s and b relating to the whole cumulant term structure. Clearly, the subordination representation of Sect. 7 also holds for the processes X^θ and Y^θ.

Table 1 VG model calibration to near-the-money S&P 500 call option prices, April 16th, 2021. In the first four columns, the results of the single maturity calibrations are listed, in the last two those for multiple maturities

Parameters / T	3 months	6 months	9 months	12 months	3 and 12 months	all maturities
κ	0.4818	0.7666	1.3197	1.745	0.2970	0.2473
σ	0.1598	0.1709	0.1851	0.1861	0.1470	0.1000
θ	−0.1301	−0.1115	−0.0925	−0.0877	−0.1575	−0.2673
Error	1.07%	0.23%	0.10%	0.03%	4.79%	7.71%

Table 2 NIG model calibration to near-the-money S&P 500 call option prices, April 16th, 2021. In the first four columns, the results of the single maturity calibrations are listed, in the last two those for multiple maturities

Parameters / T	3 months	6 months	9 months	12 months	3 and 12 months	all maturities
κ	0.8701	1.0797	1.7065	1.6241	0.4836	0.2198
σ	0.1601	0.1600	0.1658	0.1504	0.1464	0.1085
θ	−0.1399	−0.1385	−0.1272	−0.1388	−0.1650	−0.2890
Error	0.51%	0.23%	0.10%	0.03%	4.54%	7.54%

9 Empirical comparisons

In the calibration exercise of this section, we test the logistic models' option pricing performance and compare it to that of popular Lévy models. We considered four maturity time sections of near-the-money S&P 500 call options with times to maturity of 3, 6, 9 and 12 months. Moneynesses are about ±6% strike to spot, with 20 options for each maturity. We calibrated four models: the CPDA and SSLA models and the variance gamma (VG, Madan et al. [23]) and normal inverse Gaussian (NIG, Barndorff-Nielsen [3]) exponential Lévy models. For the VG and NIG models, we used the reduced-form time-changed specification in Cont and Tankov [11, Chap. 4] with parameters κ, σ, θ. Here σ and θ are the parent arithmetic Brownian motion volatility and drift, and κ represents the unit time variance of the subordinator. In general terms, the parameter σ mostly picks up the risk-neutral variance, θ the skewness and κ the excess kurtosis. For the CPDA model, we chose the parametrisation (5.7), which requires calibration of the single parameter σ; the alternative two-parameter term function b_H in (5.8) does not appear to lead to a major improvement of the calibration quality. To be consistent with this choice, the SSLA was taken in its $S^{R,1/2}$ specification. We employed a differential evolution global optimisation algorithm with the mean relative pricing error (MRE) as objective function; the calibration error is thus given by the in-sample MRE. The call option pricing formulae for the Lévy models are implemented according to the integral valuation equation (6.1). The results are shown in Tables 1, 2, 3 and 4.

As we can observe in the first four columns of each table, the VG and NIG models calibrate very accurately to any given individual maturity cross-section. The ac-

Table 3 SSLA model calibration to near-the-money S&P 500 call option prices, April 16th, 2021. In the first four columns, the results of the single maturity calibrations are listed, in the last two those for multiple maturities

Parameters / T	3 months	6 months	9 months	12 months	3 and 12 months	all maturities
σ	344.6777	370.5770	393.8968	401.0870	356.2414	354.1769
Error	7.26%	5.94%	4.84%	4.08%	8.78%	10.25%

Table 4 CPDA model calibration to near-the-money S&P 500 call option prices, April 16th, 2021. In the first four columns, the results of the single maturity calibrations are listed, in the last two those for multiple maturities

Parameters / T	3 months	6 months	9 months	12 months	3 and 12 months	all maturities
σ	0.0812	0.0862	0.0902	0.0927	0.0831	0.0825
Error	8.37%	7.32%	6.25%	5.52%	9.35%	10.98%

curacy increases with time-to-maturity, as the implied volatility skew/smile relaxes over time. The one-parameter SSLA and CPDA models show the same error reduction pattern when moving the calibration sample forward in time, but feature a substantially larger overall error. This was expected: notwithstanding any consideration on the model structure, a one-parameter model can hardly provide a better fit to empirical data than a three-parameter one. In our view, that single-parameter models are able to achieve calibration results as those listed in Tables 3 and 4 is already quite remarkable.

The data in the last two columns provide instead more insight on the structural difference between reduced-form Lévy and additive logistic models. Here we calibrate the models with respect to multiple maturity cross-sections: the 3- and 12-month maturity sections in the fifth column, and across the whole range of maturities in the sixth column. As theory predicts, the calibration of the Lévy models noticeably deteriorates, up to a tenfold increase from the maximum single section calibration error. This is due to the well-recognised fact that the time rate of statistical dispersion of homogeneous models does not match the one implied by the rate of volatility skew flattening observed in market prices. In contrast, the CPDA and SSLA models do not suffer that much when calibrated to multiple time sections: compared to the 3-month single maturity calibration, the error increase after adding further time sections is only about 2–3 percentage points. In other words, owing to the additive structure, the single parameter in the logistic models is able to interpolate across different volatility skews at different maturities in a way comparable to models with a richer distributional parametrisation, but a less flexible term structure.

When comparing the SSLA and CPDA models, we notice that the SSLA model performs slightly better. A possible heuristic explanation may be the following. The SSLA is a symmetric distribution so that when extracting the lognormal volatility (a convex decreasing function of strike), a skew naturally arises. However, as we observed, the SSLA excess kurtosis is small and constant across maturities. Thus for any given maturity, the SSLA model reproduces the implied volatility skew, but less so its convexity. In contrast, the CPDA model has a more pronounced time-varying kurtosis, but small skewness at the origin (see Figs. 4 and 5). Therefore, it may fit well the short-term implied volatility smile, but not its asymmetry. Apparently, when calibrating to a single maturity, for the set of prices considered, the trade-off between these two complementary features favors the SSLA model.

We finally observe that as expected, the calibration run-time for the logistic models is much lower than that of Lévy models. In our implementation, the SSLA and CPDA models take about 0.5 seconds on average to calibrate to a single maturity cross-section (20 options), while the VG and NIG models take approximately 11 seconds. With respect to the whole set of prices considered (80 options), the values are instead about 1.5 seconds for the logistic models and between 20 and 30 seconds for the Lévy ones. This is a clear consequence of having fully analytic valuation equations and only one parameter to calibrate.

10 Conclusions and future research

In this paper, we have demonstrated that simple no-arbitrage valuation formulae can produce risk-neutral distributions fully supported by additive processes. By assuming option valuation formulae to be a log-sum-exponential and an ℓ^p-norm, we have derived logistic-type risk-neutral distributions for which a theory of Markov additive processes can be devised. The overall take is that logistic laws for asset returns correspond to simple and parsimonious additive underlying asset pricing models, which nonetheless adequately capture several market stylised facts and are viable for derivative pricing. Ultimately, by starting to model underlying security prices from option valuation formulae rather than stochastic processes, it is to a certain extent possible to challenge the common wisdom that realism must necessarily come at the cost of mathematical tractability. In parallel work, we are currently investigating the properties of a class of diffusion processes supporting the pricing formulae (2.1) and (2.2).

Appendix: Proofs

Proof of Proposition 4.1 The Lévy characteristic triplets of general GZD distributions and their self-decomposability have been determined in Grigelionis [17, Proposition 1] by matching the derivatives of the digamma function with the Lévy cumulants. An alternative proof in our setup goes as follows.

Let $\Psi_t(z) = \ln \mathbb{E}[e^{izZ_t}]$ be the Fourier cumulant function associated to the variables Z_t. After expanding (3.6), we have for $z \in \mathbb{R}$ that

$$\begin{aligned}\Psi_t(z) = \delta(t)\Big(&\ln \Gamma\big(c_1(t) + iz\sigma(t)\big) - \ln \Gamma\big(c_1(t)\big) \\ &+ \ln \Gamma\big(c_2(t) - iz\sigma(t)\big) - \ln \Gamma\big(c_2(t)\big)\Big). \qquad \text{(A.1)}\end{aligned}$$

We recall that the integral representation for $\ln\Gamma(z)$, $\mathrm{Re}(z)>0$, is (see Gradshteyn and Ryzhik [16, 8.341.3])

$$\ln\Gamma(z)=\int_0^\infty\left(\frac{e^{-zx}-e^{-x}}{x(1-e^{-x})}-(z-1)\frac{e^{-x}}{x}\right)dx.$$

Therefore for all $z\in\mathbb{R}$, we can use the above in (A.1). For the last two terms, we obtain

$$\begin{aligned}&\ln\Gamma\big(c_2(t)-iz\sigma(t)\big)-\ln\Gamma(c_2(t))\\&=\int_0^\infty\left(\frac{e^{-(c_2(t)-iz\sigma(t))x}-e^{-c_2(t)x}}{x(1-e^{-x})}-iz\sigma(t)\frac{e^{-x}}{x}\right)dx,\end{aligned}$$

which after the substitution $x\sigma(t)\mapsto x$ becomes

$$\begin{aligned}&\ln\Gamma\big(c_2(t)-iz\sigma(t)\big)-\ln\Gamma\big(c_2(t)\big)\\&=\int_0^\infty\left(\frac{(e^{izx}-1)e^{-x\frac{c_2(t)}{\sigma(t)}}}{x(1-e^{-x/\sigma(t)})}-iz\sigma(t)\frac{e^{-x/\sigma(t)}}{x}\right)dx\\&=\int_0^\infty\left(\frac{(e^{izx}-1-izx\mathbb{1}_{\{x<1\}})e^{-x\frac{c_2(t)}{\sigma(t)}}}{x(1-e^{-x/\sigma(t)})}-iz\sigma(t)\mathbb{1}_{\{x>1\}}\frac{e^{-x/\sigma(t)}}{x}\right)dx\\&\quad+iz\int_0^1\left(\frac{e^{-x\frac{c_2(t)}{\sigma(t)}}}{1-e^{-x/\sigma(t)}}-\sigma(t)\frac{e^{-x/\sigma(t)}}{x}\right)dx.\end{aligned}\tag{A.2}$$

An analogous calculation for the first two terms of (A.1) produces

$$\begin{aligned}&\ln\Gamma\big(c_1(t)+iz\sigma(t)\big)-\ln\Gamma\big(c_1(t)\big)\\&=\int_{-\infty}^0\left(\frac{(e^{izx}-1)e^{x\frac{c_1(t)}{\sigma(t)}}}{-x(1-e^{x/\sigma(t)})}+iz\sigma(t)\frac{e^{x/\sigma(t)}}{x}\right)dx\\&=\int_{-\infty}^0\left(\frac{(e^{izx}-1-izx\mathbb{1}_{\{x>-1\}})e^{x\frac{c_1(t)}{\sigma(t)}}}{-x(1-e^{x/\sigma(t)})}+iz\sigma(t)\mathbb{1}_{\{x<-1\}}\frac{e^{x/\sigma(t)}}{x}\right)dx\\&\quad+iz\int_{-1}^0\left(-\frac{e^{x\frac{c_1(t)}{\sigma(t)}}}{1-e^{x/\sigma(t)}}+\sigma(t)\frac{e^{x/\sigma(t)}}{x}\right)dx.\end{aligned}\tag{A.3}$$

Summing expressions (A.2) and (A.3), multiplying by $\delta(t)$ and using the obvious integral substitution in the second term yields

$$\Psi_t(z)=\int_{-\infty}^\infty(e^{izx}-1-iz\mathbb{1}_{\{|x|<1\}})\upsilon(t,x)dx+iza_t,$$

which is the Lévy–Khintchine representation we require.

To show the second part, by virtue of Sato [31, Theorem 9.8], all we need to verify to prove the existence of Y is that a_t is continuous for all $t\geq0$ and that the Lévy measures $\mu_t(dx) = v(t,x)dx$ are such that $\mu_t(B)$ is a continuous and nondecreasing function of t for all Borel sets $B \subseteq \mathbb{R}$ not containing 0.

Set $a_0 = v(0,x) = 0$ for all $x \neq 0$. From the assumptions on $\sigma, c_1/\sigma$, c_2/σ and δ, we see that for all x, the Lévy densities $v(t,x)$ are nondecreasing in t; hence by positivity of the Lévy densities, so is $\mu_t(B)$. Continuity of a_t for all $t \geq 0$ and that of $v(t,x)$ at $t \neq 0$ is clear. When $t = 0$, the assumptions on c_1, c_2 and δ ensure that all the involved right limits in zero exist, and setting $y = x/\sigma(t)$, we have for $x > 0$ that

$$\lim_{t\to 0} v(t,x) = \lim_{y\to+\infty} \sigma(0+)\delta(0+)\frac{e^{-c_2(0+)y}}{y(1-e^{-y})} = 0,$$

and analogously for $x < 0$ that

$$\lim_{t\to 0} v(t,x) = -\lim_{y\to-\infty} \sigma(0+)\delta(0+)\frac{e^{c_1(0+)y}}{y(1-e^{y})} = 0.$$

Let now $s_n \uparrow t$; by the previous part, for all Borel sets B as above, we have

$$\lim_{n\to\infty} \mu_{s_n}(B) = \lim_{n\to\infty} \int_B v(s_n,x)dx = \int_B \lim_{n\to\infty} v(s_n,x)dx = \int_B v(t,x)dx = \mu_t(B)$$

due to monotone convergence. This shows $\mu_s(B) \to \mu_t(B)$ when $s \to t^-$. Similarly, if $s_n \to t$ and since $v(s_1,x)$ has finite integral on B, monotone convergence shows that $\mu_s(B) \to \mu_t(B)$ as $s \to t^+$. □

Proof of Proposition 4.2 Define X_t and Y_t as in (4.1) with respectively

$$\sigma(t) = s(t), \quad c_1(t) = 1, \quad c_2(t) = 1, \quad \delta(t) = 1, \quad \mu(t) = 0 \tag{A.4}$$

and

$$\sigma(t) = b(t), \quad c_1(t) = 1 - b(t), \quad c_2(t) = 1, \quad \delta(t) = 1, \quad \mu(t) = 0. \tag{A.5}$$

In view of (3.5), X_t and Y_t have the required distributions. Furthermore, both sets of functions above satisfy the assumptions of Proposition 4.1 so that the existence of X and Y and their uniqueness in law is established. Using (A.4) and (A.5) in (4.3), we obtain (4.4) and (4.6). Moreover, (4.5) follows from (A.5) and (4.2), while the analogous substitution for X shows that the drift is zero.

To show the martingale property of X (and hence that of S^R), recall that v^X is the jump compensator of the pure jump process X which implies that

$$X_t - \int_{\mathbb{R}} x v^X(t,x)dx$$

must be a local martingale. But v^X is symmetric about zero for all t, and therefore

$$\int_{\mathbb{R}} x v^X(t,x)dx = 0$$

so that X is a mean-zero local martingale. Furthermore, proceeding as in (6.3), we have for all $t \geq 0$ that

$$\mathbb{E}\big[[X]_t\big] = \int_{\mathbb{R}} x^2 \upsilon^X(t,x)dx = \kappa_2^X(t) < \infty,$$

and therefore X is a martingale.

Regarding the process Y, we recall from general theory that if $\Psi(z) = (\Psi_t(z))_{t\geq 0}$, $z \in \mathbb{C}$, is the Fourier cumulant process of Y, then $\exp(Y - \Psi(-i))$ is a local martingale. For additive processes, Fourier cumulants are deterministic functions of time, and the cumulant function of the infinitely divisible distribution $\mathrm{SL}(1-b(t), b(t), 0)$ can be recovered from (3.6) as

$$\Psi_t(z) = \ln \frac{B(1+(iz-1)b(t), 1-izb(t))}{B(1-b(t),1)}.$$

But then

$$\Psi_t(-i) = \ln \frac{B(1, 1-b(t))}{B(1-b(t),1)} = 0,$$

and therefore $S^P = S_0^P \exp(Y)$ is a positive local martingale and thus also a martingale, being a supermartingale with constant expectation S_0^P. Now comparing (4.7) and (4.8) with (2.7), the last statement follows from the strike differentiation argument for (2.3) and (2.4) illustrated in Sect. 2. □

Proof of Corollary 4.3 Let $D \sim \mathrm{GZD}(\sigma, 1, 1, 0) \equiv \mathrm{L}(\sigma, 0)$. Since D is self-decomposable, the existence and uniqueness of a self-similar process X is established in Sato [30, Theorem 3.2]. Furthermore, from that theorem, the Lévy characteristic triplet of X has zero drift and Lévy measure given by

$$\upsilon^H(t,x)dx := t^{-H}\upsilon^D(xt^{-H})dx, \tag{A.6}$$

where υ^D is the Lévy density of D which by the first part of Proposition 4.1 can be recovered from (4.3) using $c_1 = c_2 = 1$, $\delta(t) = 1$, $\mu = 0$, $\sigma(t) = \sigma$. It is straightforward to check that under these choices, (A.6) equals (4.4) with $s(t) = \sigma t^H$, and this proves the corollary. □

Proof of Proposition 7.4 By the assumptions, the function $\lambda_j(t)$ is continuous, positive and decreasing in t for all j, with $\lambda_j(0+) = +\infty$. Therefore the densities $\Sigma(t,x)$ are strictly increasing and continuous in $t > 0$ with $\Sigma(0+, x) = 0$ for all $x > 0$. Completing $\Sigma(t,x)$ with the value zero for $t = 0$, we can use Sato [31, Theorem 9.8] as in Proposition 4.1 to establish the existence of Γ. The existence of B is another straightforward application of that theorem. To prove the claim, we then only need to show the identity of the characteristic functions. The calculation of the characteristic function of $Z_t = B_{\Gamma_t} + \mu(t)$ can be performed using (7.3). Denoting by ϕ the Laplace exponent of Γ, the familiar argument on conditioning under independence leads to

$$
\begin{aligned}
&\mathbb{E}\Big[\exp\Big(iz\big(B_{\Gamma_t}+\mu(t)\big)\Big)\Big]\\
&=e^{iz\mu(t)}\mathbb{E}\bigg[\exp\bigg(-\Big(\frac{z^2\sigma(t)}{2}+iz\beta(t)\Big)\Gamma_t\bigg)\bigg]\\
&=e^{iz\mu(t)}\exp\bigg(-\phi\Big(\frac{z^2\sigma(t)}{2}+iz\beta(t)\Big)\bigg)\\
&=e^{iz\mu(t)}\prod_{j=0}^{\infty}\bigg(1+\frac{z^2\sigma^2(t)+2iz\sigma(t)\beta(t)}{(\alpha(t)+j)^2-\beta^2(t)}\bigg)^{-\kappa(t)}\\
&=e^{iz\mu(t)}\prod_{j=0}^{\infty}\bigg(\Big(1+\frac{iz\sigma(t)}{\alpha(t)-\beta(t)+j}\Big)\Big(1-\frac{iz\sigma(t)}{\alpha(t)+\beta(t)+j}\Big)\bigg)^{-\kappa(t)}\\
&=e^{iz\mu(t)}\bigg(\frac{\Gamma(\alpha(t)-\beta(t)+iz\sigma(t))\Gamma(\alpha(t)+\beta(t)-iz\sigma(t))}{\Gamma(\alpha(t)-\beta(t))\Gamma(\alpha(t)+\beta(t))}\bigg)^{\kappa(t)}\\
&=e^{iz\mu(t)}\bigg(\frac{B(\alpha(t)-\beta(t)+iz\sigma(t),\alpha(t)+\beta(t)-iz\sigma(t))}{B(\alpha(t)-\beta(t),\alpha(t)+\beta(t))}\bigg)^{\kappa(t)},
\end{aligned}
\tag{A.7}
$$

which matches the required GZD characteristic function. In the second to last equality, we have used Gradshteyn and Ryzhik [16, Eq. 8.325.1]. □

Proof of Proposition 7.5 The existence and properties of Γ are shown exactly as in Proposition 7.4. When analysing the characteristic function of e.g. (7.6), we can proceed as in (A.7). Denoting by ϕ^Y the Laplace exponent of Γ^Y, we obtain

$$
\begin{aligned}
&\mathbb{E}\big[\exp\big(iz(W_{\Gamma_t^Y}-\Gamma_t^Y/2)\big)\big]\\
&=\mathbb{E}\bigg[\exp\bigg(-\frac{z^2+iz}{2}\Gamma_t^Y\bigg)\bigg]\\
&=\exp\bigg(-\phi^Y\Big(\frac{z^2+iz}{2}\Big)\bigg)=\prod_{j=0}^{\infty}\bigg(1+\frac{b(t)^2(z^2+2iz)}{(1+j)^2-(1+j)b(t)}\bigg)^{-1}\\
&=\prod_{j=0}^{\infty}\bigg(\Big(1+\frac{izb(t)}{1-b(t)+j}\Big)\Big(1-\frac{izb(t)}{1+j}\Big)\bigg)^{-1}\\
&=\frac{B(1+(iz-1)b(t),1-izb(t))}{B(1-b(t),1)},
\end{aligned}
$$

which is the $\mathrm{SL}(1-b(t),b(t),0)$ characteristic function. A similar but simpler calculation yields the representation of X as a subordinated Brownian motion (see also Pitman and Yor [28, Eq. (78)]). □

Proof of Proposition 8.1 (i) Since $b(t) \in (0,1]$, we have for all $\theta \in \Theta$, $t > 0$ that $c_1(\theta,t), c_2(\theta,t) > 0$, and the density process (8.1) is a local martingale. Moreover, because Y is additive, (8.1) is a supermartingale with expectation one and hence a martingale, so that the equivalent change of measure $\mathbb{P}^\theta \approx \mathbb{Q}$ is well defined.

By general theory, the claim is equivalent to $\psi_t^\theta(z) = \psi_t(z+\theta) - \psi_t(\theta)$, where $\psi_t^\theta(z)$ denotes the Laplace cumulant of the additive GZD process from Proposition 4.1 with the specification (8.2)–(8.4). Using (3.2) yields

$$\begin{aligned}\psi_t(z+\theta) - \psi_t(\theta) &= \ln \frac{B(1-b(t)+(z+\theta)b(t), 1-(z+\theta)b(t))}{B(1-b(t),1)} \\ &\quad - \ln \frac{B(1-b(t)+\theta b(t), 1-\theta b(t))}{B(1-b(t),1)} \\ &= \ln \frac{B(c_1(\theta,t)+zb(t), c_2(\theta,t)-zb(t))}{B(c_1(\theta,t), c_2(\theta,t))} \\ &= \psi_t^\theta(z) \end{aligned} \tag{A.8}$$

for all t. Now note that as functions of t, $c_1(\theta,t)$, $c_2(\theta,t)$ are bounded in zero for all θ and $c_1(\theta,t)/b(t)$, $c_2(\theta,t)/b(t)$ are decreasing in t. Therefore an application of Proposition 4.1 proves part (i).

(ii) We notice that since s is increasing, its maximum for fixed T is $s(T)$ so that if $\theta \in \Theta \neq \{0\}$, then $s_1(\theta,t), s_2(\theta,t) > 0$ for all t, and the Esscher transform (8.5) exists for all $t \leq T$ and is a martingale. The verification of (8.6) is performed in exactly the same way as in (A.8), and the existence of an additive process follows again from Proposition 4.1 after observing that as functions of t, $s_1(\theta,t)$ and $s_2(\theta,t)$ are bounded around zero for all θ and $s_1(\theta,t)/s(t)$ and $s_2(\theta,t)/s(t)$ are decreasing in t. □

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