Exact and Approximate Symmetries and Approximate Conservation Laws of Differential Equations with a Small parameter

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Abstract

The frameworks of Baikov-Gazizov-Ibragimov (BGI) and Fushchich-Shtelen (FS) approximate symmetries have proven useful for many examples where a small perturbation of an ordinary or partial differential equation (ODE, PDE) destroys its local exact symmetry group. For the perturbed model, some of the local symmetries of the unperturbed equation may (or may not) re-appear as approximate symmetries. Approximate symmetries are useful as a tool for systematic construction of approximate solutions. While for algebraic and first-order differential equations, to every point symmetry of the unperturbed equation, there corresponds an approximate point symmetry of the perturbed equation, for second and higher-order ODEs, this is not the case: a point symmetry of the original ODE may be unstable, that is, not have an analogue in the approximate point symmetry classification of the perturbed ODE. We show that such unstable point symmetries correspond to higher-order BGI approximate symmetries of the perturbed ODE, and can be systematically computed. We present a relation between BGI and FS approximate point symmetries for perturbed ODEs. Multiple examples of computations of exact and approximate point and local symmetries are presented, with two detailed examples that include a fourth-order nonlinear Boussinesq ODE reduction. Examples of the use of higher-order approximate symmetries and approximate integrating factors to obtain approximate solutions of higher-order ODEs, including Benjamin-Bona-Mahony ODE reduction are provided.

The frameworks of BGI and FS approximate symmetries are used to study symmetry properties of partial differential equations with a small parameter. In general, we show that unlike in the ODE case, unstable point symmetries of an unperturbed PDE do not necessarily yield local approximate symmetries for the perturbed equation. We classify stable point symmetries of a one-dimensional wave model in terms of BGI and FS frameworks. We find a connection between BGI and FS approximate local symmetries for a PDE family. We classify approximate point symmetries for a family of one-dimensional wave equations with a small nonlinear term, and construct a physical approximate solution for a family that includes a one-dimensional wave equation describing the wave motion in a hyperelastic material with a single family of fibers. For this model, we find wave breaking times numerically and using the approximate solution. A complete classification of exact and approximate point symmetries of the two-dimensional wave equation with a general small nonlinearity is presented.

We investigate approximate conservation laws of systems of perturbed PDEs. We apply the direct multiplier method to obtain new approximate conservation laws for perturbed PDEs including nonlinear heat and wave equations. We show that the direct method generalizes the Noether's theorem for construction of approximate conservation laws by proving that an approximate multiplier corresponds to an approximate local symmetry of an approximately variational problem. We present two formulas relating to construct additional approximate conservation laws for a system of perturbed PDEs. We illustrate these formulas using perturbed wave equation and nonlinear telegraph system. An application for using approximate conservation laws to construct potential systems and approximate potential symmetries is provided.

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List of Abbreviations

ODE	Ordinary differential equation
PDE	Partial differential equation
BGI	Baikov-Gazizov-Ibragimov
FS	Fushchich-Shtelen
BBM	Benjamin-Bona-Mahony

1 Introduction to Exact and Approximate Lie Symmetries, Conservation Laws

1.1 Introduction

In the nineteenth century, Sophus Lie introduced the notion of continuous groups, known as Lie groups. The original Lie's aim was that of setting a general theory for the integration of ordinary differential equations (ODEs) [1]. He pointed out that the order of an ODE could be reduced by one if it is invariant under a one-parameter Lie group of point transformations. A symmetry of a system of algebraic or differential equations is a transformation that maps solutions of the system to other solutions. One important type of symmetries is local symmetries, where the components of the dependent variables in their generators depend at most on finite number of derivatives of the dependent variables. Local symmetries include point, contact and higher-order symmetries. Lie symmetry groups have seen significant development over the last century, relating to symmetry reduction and solution of differential equations, integrating factors, conserved quantities and local conservation laws, integrability, nonlocal extensions, invertible and non-invertible mappings between different classes of differential equations, and more [2–7].

The important feature of the Lie group theory lies in the possibility of replacing the nonlinear conditions of invariance of the system of differential equations under a group of transformations by linear conditions that reflects the infinitesimal invariance of the system under the generators of the Lie group of transformations. This infinitesimal criterion leads to an over-determined system of linear partial differential equations (PDEs) satisfied by the components of the infinitesimal generators. So, Lie groups of symmetries can be systematically computed. Several symbolic softwares have been developed for this task [8, 9].

Perturbed equations are equations differing from some canonical or otherwise well understood model by extra term(s) with involving a small parameter ϵ. This small perturbation disturbs the symmetry group properties of the unperturbed equations in the sense that the perturbed model admits fewer point and local symmetries than the unperturbed model since a perturbed equation cannot have more symmetries holding for all values of ϵ than a given value of ϵ, including $\epsilon = 0$. In fact, one usually needs to solve a perturbed model, which is more complex than the unperturbed one. In particular, exact solutions to the perturbed model may not be known and there are not enough symmetries to obtain them. Another important aspect for considering the perturbed models is that the mathematical properties of perturbed and unperturbed models can be very different. For example, Euler system is hyperbolic but Navier-Stokes system, that can be considered a perturbation of Euler equations, is parabolic. A class of Lie symmetry-like transformations which are useful in studying the symmetry properties of the perturbed differential equations and/or provide new symmetries for these equations are called approximate Lie symmetries.

Several approximate Lie symmetry methods have been developed to study symmetry properties of perturbed models, and relate and compare them to symmetry structure of the unperturbed equations. An approximate symmetry transformation method (referred to here as the *BGI method*) was introduced by Baikov, Gazizov and Ibragimov [10–12], where the approximate symmetry generator is expanded in a perturbation series. A different approach to approximate symmetries, developed by Fushchich and Shtelen (*FS method*) [13], combines a perturbation technique with the symmetry group method by expanding the dependent variables in a Taylor series in the small parameter, and approximately replacing the original equations by a system of equations that are coefficients at different powers of the parameter. The classical Lie symmetry method is applied to obtain symmetries of the new system. The BGI and Fushchich-Shtelen approaches are not equivalent. They have been compared and used to obtain approximate symmetries and approximate solutions for several PDE models [14–16]. Burde [17] developed a new approach for approximate symmetries by constructing equations that could be reduced by exact transformations to an unperturbed equation and at the same time would coincide approximately with the perturbed equation. In this thesis, we consider the BGI and FS approaches.

Through seeking approximate symmetries for perturbed differential equations, different kinds of approximate symmetries can appear: trivial approximate symmetries, exact symmetries inherited from the unperturbed equations as they stand and *new (genuine)* approximate symmetries. The latter are the most important type that are useful in constructing new approximate solutions for the perturbed models [18–22]. The algorithm for the computation of approximate symmetries for a perturbed model consists of two steps starts by finding the exact local symmetry of the unperturbed equation. Then in the following step, one finds the approximate symmetry components through set of determining equations. These equations may have some constraints on the exact symmetry components leads to some exact local symmetries disappear from the approximate symmetry classification of the perturbed model. BGI and FS approximate symmetries have been found for many models, including ODEs and PDEs (e.g., [10–13, 22–26]). However, it has not been clarified under what conditions point or local symmetries of unperturbed equations can become unstable, disappearing from the classification of approximate symmetries of a perturbed system of the same differential order, and what form they therefore take.

The majority of differential equations involve arbitrary parameters or arbitrary functions. These parameters have physical meanings and assume values in some ranges or belong to certain classes. For example, if the viscosity in Navier–Stokes equations is zero, then one obtains the Euler equations which have different symmetry properties. Thus, to study the symmetry properties of system of differential equations involving arbitrary elements, one needs to investigate what happens to symmetries as these parameters assume special values. Namely, one needs to find the symmetry group classification for a class of system of differential equations depending on the values of the arbitrary elements (see, e.g., [23, 27–32]). Equivalence transformations, which are transformations that map each system from the class to another system of the same class, are useful in simplifying the group symmetry classification by considering only forms of the arbitrary elements that are not related by an equivalence transformation [33–35]. While BGI and FS approximate symmetry classifications have been found for some perturbed models (see,e.g., [23, 36] and references therein), the exact and approximate (BGI and FS) symmetry classification for perturbed models has not been considered and compared in details in the literature.

In the study of differential equations, conservation laws have many important uses. They describe the familiar physical conserved quantities like energy, mass, momentum and so on. They are used to study the basic properties of the solutions and to develop numerical methods. They are also used in construction of potential systems [4, 37, 38]. Emmy Noether proved that for a system of differential equations arising from a variational principle, every conservation law of the system corresponds to a variational symmetry of the system [39]. Another method for finding conservation laws is the direct method [5, 6], which it is more general than Noether's theorem since it is applicable to any system of differential equations and includes all conservation laws obtained from the Noether's theorem. The symmetry action on a known conservation law for any system of differential equations could yield new conservation laws for the given system [4, 40]. In the case of system of ODEs, a conservation law for a system is equivalent to the first integral or constant of the motion of the system [41].

The study of approximate symmetries led naturally to extension to approximate variational symmetries [42], approximate Noether's theorem and approximate conservation laws for system of perturbed differential equations that has an approximate Lagrangian [23, 43]. It also led to study of approximate conservation laws for any perturbed differential equation without resource to variational principle. In particular, the approximate invariance condition under the approximate symmetry generator was used to construct approximate conservation laws [44, 45]. Similar to the exact case, the direct method is the most efficient way to construct approximate conservation laws for differential equations with a small parameter since it is not assumed that any approximate symmetries are known, nor that the equations are approximately equivalent to the Euler-Lagrange equations of a variational problem. Following the direct method, approximate multipliers and approximate conservation laws were found for some perturbed models [46].

In this thesis we focus on exact and approximate (BGI and FS) symmetries for perturbed differential equations, and we study approximate conservation laws for perturbed PDEs. The thesis is organized as follows.

In the current chapter, we recall the definitions of Lie groups and Lie algebras [4, 47], including Lie groups of point transformations [3]. We review the framework of Lie point and local symmetries, in comparison with the BGI [10–12] and Fushchich-Shtelen [13] approximate symmetry frameworks for differential equations involving a small parameter. Finally, we overview the direct method and Noether's theorem for constructing conservation laws [37].

In Chapter 2, we consider exact and approximate local symmetries of algebraic and ordinary differential equations with a small parameter. In particular, we show that for algebraic and first-order differential equations, to every point symmetry of the unperturbed equation, there corresponds an approximate point symmetry of the perturbed equation. We find a connection between BGI and FS approximate symmetries of a perturbed first-order ODE. We investigate the BGI and FS approximate symmetries of the perturbed higher-order ODEs. We construct general formulas for the determining equations of BGI and FS approximate symmetries of a perturbed ODE that simplify the calculations of approximate symmetry components and help in studying the stability of the exact point symmetries of the unperturbed model. It is shown that point symmetries of the unperturbed equation may indeed disappear from the classification of approximate point symmetries of the perturbed model, and conditions for that are given. We consider point and higher-order local exact and approximate symmetries of second and higher-order ODEs in evolutionary form, and show that a point or local symmetry of the unperturbed equation usually yields a higher-order (generally, of order $n-1$) BGI approximate symmetry of the perturbed model and we present a systematic way to find approximate symmetry components for approximate symmetries that correspond to *every* point and local symmetry of the unperturbed equation. We show for a family of perturbed higher-order ODEs that a genuine BGI approximate point symmetry yields a genuine FS approximate point symmetry for the same family. Relations between exact and BGI approximate symmetries are considered in detail for two examples, including a nonlinearly perturbed second-order ODE, and a fourth-order ODE arising as a traveling wave reduction of the Boussinesq partial differential equation modeling shallow water wave propagation [48]. Finally, we determine the approximate integrating factors of perturbed first-order ODEs using BGI approximate point symmetries and we show that the components of an approximate integrating factor of a perturbed first-order ODE defines a BGI approximate point symmetry of the same ODE. We find the determining equations of approximate integrating factors, and show how these determining equations and higher-order approximate symmetries can be used to obtain approximate solutions of perturbed Boussinesq and Benjamin-Bona-Mahony (BBM) ODEs [49].

In Chapter 3, we study exact and approximate local symmetries of perturbed partial differential equations. We show that in general, unlike the higher-order ODEs, an unstable point symmetry of the unperturbed PDE does not yield a higher-order approximate symmetry of the perturbed equation. An illustration of unstable point symmetry of a one dimensional wave equation is given. We classify stable point symmetries in the sense of BGI and FS frameworks for a wave equation according to the form of the arbitrary function in the perturbation term. BGI and FS frameworks are different approaches which give different approximate symmetry structures however we find some connection between the two frameworks for a general PDE in solved form. For some stable point symmetries as BGI, there is a corresponding higher-order FS approximate symmetry.

The study of wave propagation in nonlinear elastic materials has numerous applications in the study of complex materials [50], medical imaging [51], and other areas [52]. Of particular interest are hyperelastic solids, a class of materials that act as ideal elastic solids. In particular, the stress within a hyperelastic solid is related to the deformation through a strain energy density function. The displacements in hyperelastic materials in one and two dimensions are modeled respectively by the nonlinear wave equations

$$u_{tt} = R(u_x)u_{xx}, \quad u = u(x,t), \tag{1.1}$$

$$u_{tt} = \left(u_x K(u_x^2 + u_y^2)\right)_x + \left(u_y K(u_x^2 + u_y^2)\right)_y, \quad u = u(t,x,y), \tag{1.2}$$

where R and K are related to the stored energy functions [53, 54], (here and below subscripts denote partial derivatives). First, we study the symmetry properties of the family (1.2). Then, having exact symmetries computed for the one- and two-dimensional linear wave equations, we classify the exact and the approximate symmetries of the perturbed models

$$u_{tt} = (c^2 + \epsilon T(u_x))u_{xx}, \quad u = u(x,t), \tag{1.3}$$

$$u_{tt} = \left(u_x[c^2 + \epsilon Q(u_x^2 + u_y^2)]\right)_x + \left(u_y[c^2 + \epsilon Q(u_x^2 + u_y^2)]\right)_y, \quad u = u(t,x,y), \tag{1.4}$$

where $c > 0$ and ϵ is a small positive parameter that stands for a combination of the Mooney–Rivlin and the standard reinforcement terms (anisotropy material parameter). The above perturbed models include equations that describe the motion in the fiber-reinforced elastic solids such as biological membranes when fiber effects are relatively small (see, e.g. [53, 55]). We construct a general approximate solution of the perturbed one-dimensional wave equation (1.3) with $T(u_x) = u_x^s$. In particular, for $s = 2$, we find the finite-time singularities for a certain initial-boundary value problem using the approximate solution and numerical methods, respectively.

In Chapter 4, we consider approximate conservation laws for perturbed PDEs. We review the definitions of approximate conservation laws and approximate multipliers and Noether's theorem for finding approximate conservation laws. Following the direct method, we find approximate conservation laws for some perturbed PDEs including perturbed wave and heat equations. We show that new approximate conservation laws, which do not arise from the exact conservation laws of the perturbed equation, can be obtained. We show, like in the case of exact conservation laws, that the direct method includes Noether's theorem for construction of conservation laws in sense that the approximate local multipliers correspond to approximate Noether symmetries of the variational problem. We derive two formulas to construct additional approximate conservation laws from known approximate conservation laws using the action of an invertible approximate transformation. In the first formula, we show that if an invertible approximate transformation maps a given perturbed PDE system to another perturbed PDE system, then each conservation law of the first PDE system is mapped to a conservation law for the transformed PDE system. The second formula uses the action of the approximate point symmetry of a system of perturbed PDEs on a given set of approximate multipliers of a known approximate conservation law for this system to find new set of approximate multipliers. If the transformed set of approximate multipliers are independent of the given set of approximate multipliers, then one gets a new approximate conservation law for the given system. We apply these formulas to construct new approximate conservation laws for perturbed wave equations and nonlinear telegraph system [56]. As an application, we find potential systems and approximate conservation laws for a nonlinear wave equation. Chapter 5 contains a discussion and some open problems.

In Appendix A, we present `Maple` code for computations of approximate local symmetries and approximate conservation laws. This code can be used to find approximate local symmetries and approximate conservation laws for perturbed differential equations.

A part of Chapter 2 is published jointly with my supervisor Alexey Shevyakov [57]. Another part of Chapter 2 has been submitted for publication. Sections 3.7.2 and 3.7.3 are based on the contribution by Brian Pitzel.

1.2 Exact Lie symmetries

1.2.1 Notation

Let $f(x), g(x) : \mathbb{R} \to \mathbb{R}$. We say $f = O(g)$ as $x \to \infty$ if there exist a positive real number k and a real number x_0 such that

$$|f(x)| \leq k|g(x)|$$

for all $x \geq x_0$. For some real number b, we write $f = O(g)$ as $x \to b$ if there exists positive real numbers λ and k such that for all x with $|x - b| < \lambda$,

$$|f(x)| \leq k|g(x)|.$$

In other words, if $f = O(g)$ as $x \to b$, then $|f(x)/g(x)|$ is bounded in a neighbourhood of b, where f/g is defined [58].

We say that $f = o(g)$ as $x \to b$ if

$$\lim_{x \to b} \frac{f(x)}{g(x)} = 0.$$

1.2.2 Lie groups and Lie algebras

A function $f : E \to M$ between two topological spaces is continuous if, whenever A is an open subset in M, $f^{-1}(A)$ is open in E. If f is one-to-one and both f and f^{-1} are continuous, then f is called a *homeomorphism* and E and M are said to be *homeomorphic* [59].

Definition 1.2.1. An *n-dimensional manifold* is a set E, together with a countable collection of subsets P_i, called coordinate charts, and one-to-one functions $\tau_i : P_i \to Q_i$ onto connected open subsets Q_i of $\mathbb{R}^n$, called local coordinate maps, satisfying [4]:

1. The coordinate charts cover E

$$\bigcup_i P_i = E.$$

2. The composite map

$$\tau_j \circ \tau_i^{-1} : \tau_i(P_i \cap P_j) \to \tau_j(P_i \cap P_j)$$

   is a smooth function.

3. If $p \in P_i$, $\tilde{p} \in P_j$ are distinct points of E, then there exist open subsets M of $\tau_i(p)$ in Q_i and $\tilde{M}$ of $\tau_j(\tilde{p})$ in Q_j such that

$$\tau_i^{-1}(M) \cap \tau_j^{-1}(\tilde{M}) = \phi.$$

Example 1.2.1. The simplest n-dimensional manifold is $\mathbb{R}^n$. There is a single coordinate chart $P = \mathbb{R}^n$, with local coordinate map given by the identity: $\tau = id : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.2.2. A *Lie group* is an abstract group and a smooth n-dimensional manifold so that multiplication $G \times G \to G$: $(a, b) \to ab$ and inverse $G \to G$: $a \to a^{-1}$ are smooth.

Example 1.2.2. (a) $(\mathbb{R}^n, +)$ is an abelian Lie group.
(b) Consider the general linear group

$$GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) : \det A \neq 0\}.$$

Since the determinant map is continuous, $GL(n, \mathbb{R})$ is an open subset of the space $M(n, \mathbb{R})$ of all $n \times n$ matrices. But $M(n, \mathbb{R})$ is isomorphic to $\mathbb{R}^{n^2}$. Thus $GL(n, \mathbb{R})$ is an n^2-dimensional manifold. Matrix multiplication and inversion are smooth maps. Hence, $GL(n, \mathbb{R})$ is a Lie group.

Let $f : \mathtt{M} \to \mathtt{S}$ be a smooth map between the smooth manifolds $\mathtt{M}$ and $\mathtt{S}$. f maps each parameterized curve $C = \{\Theta(t) : t \in I\}$ on $\mathtt{M}$ to a parameterized curve $\tilde{C} = \{\tilde{\Theta}(t) = f(\Theta(t)) : t \in I\}$ on $\mathtt{S}$. Hence f induces a map from the tangent vector $d\Theta(t)/dt$ to C at $x = \Theta(t)$ to the corresponding tangent vector $d\tilde{\Theta}(t)/dt$ to $\tilde{C}$ at $f(x) = f(\Theta(t)) = \tilde{\Theta}(t)$. This map is called the differential of f and denoted by [4]

$$df\,(d\Theta/dt) = \frac{d}{dt}\left(f\left(\Theta(t)\right)\right).$$

Let G be a Lie group. For any $g \in G$, the right multiplication map

$$R_g : G \to G$$

defined by $R_g(h) = hg$ is a diffeomorphism. A vector field X on G is called *right invariant* if

$$dR_g(X(h)) = X(hg)$$

for all g and h in G, where dR_g is the differential of R_g.

Definition 1.2.3. The *Lie algebra* of a Lie group G is the vector space of all right invariant vector fields on G.

Remark 1.2.1. Any right invariant vector field is completely determined by its value at the identity e because

$$X(g) = dR_g(X(e))$$

for all $g \in G$. Conversely, any tangent vector X to G at the identity uniquely determines a right invariant vector field on G since

$$dR_g(X(h)) = dR_g(dR_h(X(e))) = d(R_g \circ R_h)(X(e)) = dR_{hg}(X(e)) = X(hg).$$

Therefore the Lie algebra $\mathfrak{g}$ of the Lie group G can be identified by the tangent space G at the identity e

$$\mathfrak{g} \simeq T_e G$$

Definition 1.2.4. A *Lie algebra* is a vector space $\mathfrak{g}$ over $\mathbb{F}$($\mathbb{R}$ or $\mathbb{C}$) with a skew-symmetric $\mathbb{F}$-bilinear map (the Lie bracket, or commutator) $[\,,\,]$:$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad [X, Y] = XY - YX. \tag{1.5}$$

Example 1.2.3. The space of all $n \times n$ matrices $\mathfrak{gl}(n, \mathbb{R})$ with the Lie bracket $[A, B] = AB - BA$ being the matrix commutator is the *Lie algebra* of $GL(n, \mathbb{R})$ with dimension n^2.

Commutator table

It is useful to arrange commutators in a table, where $[X_j, X_k]$ is the entry of the intersection of j^{th} row with k^{th} column. The commutator table is represented by a skew-symmetric matrix with zeros on its diagonal.

Example 1.2.4. Consider the special linear group

$$SL(2, \mathbb{R}) = \{A \in GL(2, \mathbb{R}) : \det A = 1\}.$$

Its Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$: the Lie algebra of all 2×2 matrices with trace 0. We use the basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.6}$$

for $\mathfrak{sl}(2, \mathbb{R})$, then the commutation of the above matrices yields

$$[e, f] = h, \quad [h, f] = -2f, \quad [h, e] = 2e.$$

In table form, one has

	e	f	h
e	0	h	$-2e$
f	$-h$	0	$2f$
h	$2e$	$-2f$	0

Table 1.1: Commutator table for the matrices (1.6).

1.2.3 One-parameter Lie group of point transformations

We now consider Lie groups of transformations acting on $\mathbb{R}^n$.

Definition 1.2.5. Let $x = (x^1, x^2, ..., x^n) \in D \subset \mathbb{R}^n$. The set of transformations

$$(x^i)^* = T^i(x; a), \quad i = 1, .., n \tag{1.7}$$

defined for each $x \in D$ and parameter a in a set $M \subset \mathbb{R}$, with $\phi(a, b)$ defining a law of composition of parameters a and b in M, forms a one-parameter group of transformations on D if the following conditions hold:

1. The transformations are one-to-one and onto D,
2. (M, ϕ) is a group,
3. Each $T^i(x; a_0) = x^i$ for the identity element a_0 and for each x in D,
4. If $(x^i)^* = T^i(x; a), (x^i)^{**} = T^i(x^*, b)$, then $(x^i)^{**} = T^i(x, \phi(a, b))$.

Definition 1.2.6. The one-parameter group of transformations (1.7) is a one-parameter Lie group of transformations if:

1. M is an interval in $\mathbb{R}$,
2. T^i is infinitely differentiable with respect to x in D and an analytic function of $a \in M$,
3. $\phi(a, b)$ is an analytic function of a and b.

Example 1.2.5. An example of the Lie group of point transformation is the group of translations in the plane

$$\begin{aligned} x^* &= T^1(x, y; a) = x + a, \\ y^* &= T^2(x, y; a) = y. \end{aligned} \tag{1.8}$$

Repeating the transformations

$$x^{**} = T^1(x^*, y^*; b) = x + a + b, \quad y^{**} = T^2(x^*, y^*; b) = y.$$

Here $\phi(a, b) = a + b$, and $a = 0$ corresponds to the identity element.

Remark 1.2.2. A multi-parameter Lie group of point transformations is given by [60]

$$x^* = T(x; \mathbf{a}),$$

with $x = (x^1, ..., x^n)$ and parameters $\mathbf{a} = (a_1, ..., a_r)$. Each parameter leads to an infinitesimal generator with infinitesimals given by

$$\xi^{ki}(x) = \left.\frac{\partial (x^i)^*}{\partial a_k}\right|_{\mathbf{a}=0}, \quad k = 1, ..., r, \quad i = 1, ..., n.$$

If an n^{th} order ODE admits an r-parameter Lie group of transformations, $2 \le r \le n$, with an r-dimensional solvable Lie algebra, then the order of the given ODE can be reduced by r. Throughout this thesis, we only consider one-parameter Lie groups of transformations.

1.2.4 Infinitesimal transformations

Let

$$(x^i)^* = T^i(x; a) \tag{1.9}$$

be a one-parameter Lie group of transformations. A Taylor expansion of it about $a = 0$ is given by

$$\begin{aligned} (x^i)^* &= x^i + a\left[\left.\frac{\partial T^i(x,a)}{\partial a}\right|_{a=0}\right] + \frac{1}{2}a^2\left[\left.\frac{\partial^2 T^i(x,a)}{\partial a^2}\right|_{a=0}\right] + ... \\ &= x^i + a\left[\left.\frac{\partial T^i(x,a)}{\partial a}\right|_{a=0}\right] + O(a^2). \end{aligned}$$

One denotes

$$\xi^i(x) = \left[\left.\frac{\partial T^i(x,a)}{\partial a}\right|_{a=0}\right].$$

The components of $\xi(x) = (\xi^1(x), \xi^2(x), ..., \xi^n(x))$ are called the infinitesimals of the Lie group of transformations (1.9). The transformation $x + a\xi(x)$ is called the infinitesimal transformation of (1.9).

Definition 1.2.7. Let $\xi^1(x), \xi^2(x), ..., \xi^n(x)$ be the infinitesimals of the Lie group of transformations (1.9), then the infinitesimal generator (operator) of (1.9) is given by

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{1.10}$$

where (as well as below, where appropriate) summation in repeated indices is assumed.

Example 1.2.6. For the group of translations (1.8), one has

$$\xi^1(x,y) = \left.\frac{\partial x^*}{\partial a}\right|_{a=0} = 1, \quad \xi^2(x,y) = \left.\frac{\partial y^*}{\partial a}\right|_{a=0} = 0.$$

Hence, the infinitesimal generator for (1.8) has the form

$$X = \xi^1(x,y)\frac{\partial}{\partial x} + \xi^2(x,y)\frac{\partial}{\partial y} = \frac{\partial}{\partial x}.$$

The following theorem shows that one can reconstruct the global Lie group (1.9) from its infinitesimals [60].

Theorem 1.2.1 (Lie's First Theorem)**.** *The one-parameter Lie group of point transformations* (1.9) *is equivalent to the solution of the initial-value problem*

$$\frac{d(x^i)^*}{da} = \xi^i\left(x^*\right), \quad (x^i)^*\big|_{a=0} = x^i. \tag{1.11}$$

Example 1.2.7. Consider the infinitesimal generator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{1.12}$$

For this generator, the Lie's equations (1.11) simplifies to

$$\begin{aligned} &\frac{dx^*}{da} = y^*, \quad \frac{dy^*}{da} = -x^*, \\ &x^*\big|_{a=0} = x, \quad y^*\big|_{a=0} = y. \end{aligned} \tag{1.13}$$

Differentiating the first equation of (1.13) with respect to a, one gets a second-order ODE

$$\frac{d^2x^*}{da^2} + x^* = 0,$$

which has a solution

$$x^* = C_1 \sin a + C_2 \cos a. \tag{1.14}$$

Using the initial condition $x^*\big|_{a=0} = x$, one finds $C_2 = x$. Differentiate (1.14) and then substitute $a = 0$, one has

$$C_1 = \frac{dx^*}{da}\big|_{a=0} = y^*\big|_{a=0} = y.$$

Consequently, x^* has the form

$$x^* = y \sin a + x \cos a.$$

Similarly, one can find that

$$y^* = y \cos a - x \sin a.$$

Hence, the Lie group of point transformations that corresponds to the generator (1.12) is the *rotation group* in the plane by the angle a.

1.2.5 Point symmetries

Before proceeding to symmetry of differential equations, we consider a simpler case: symmetries of algebraic equations.

Point symmetries of algebraic equations

A system of algebraic equations is given by

$$F^\sigma(x) = 0, \quad \sigma = 1, ..., N, \tag{1.15}$$

where $F^1, ..., F^N$ are smooth real-valued functions defined for each $x \in \mathbb{R}^n$.

Definition 1.2.8. A real-valued function $R(x)$ is called an invariant function of the Lie group of transformations (1.9) (or invariant under (1.9)) if for all $x \in D \subset \mathbb{R}^n$ and $a \in M \subset \mathbb{R}$ such that (1.9) is defined,

$$R(x^*) \equiv R(x). \tag{1.16}$$

An important feature of Lie group theory is the possibility of replacing the nonlinear condition for the invariance of a function under the Lie group of transformations (1.9) by linear condition of infinitesimal invariance under the corresponding infinitesimal generator of (1.9). This infinitesimal criterion is the key for the determination of the symmetry groups of the differential equations. Starting from the simpler case of invariant functions, the following theorem holds [4].

Theorem 1.2.2. *An infinitely differentiable function $R(x)$ is invariant under the Lie group of transformations* (1.9) *if and only if*

$$XR(x) \equiv 0 \tag{1.17}$$

for all $x \in \mathbb{R}^n$, where (1.17) *is defined.*

Example 1.2.8. Consider the one-parameter Lie group of translations

$$x^* = x + a, \quad t^* = t - a \tag{1.18}$$

with the corresponding infinitesimal generator

$$X = \frac{\partial}{\partial x} - \frac{\partial}{\partial t}. \tag{1.19}$$

Then the function $R(x,t) = x + t$ is invariant under (1.18) since

$$R(x^*, t^*) = R(x + a, t - a) = R(x, t).$$

Using the infinitesimal generator (1.19), one has

$$XR = 1 - 1 = 0.$$

In fact, every invariant of the group (1.18) is of the form $\tilde{R}(x,t) = r(x+t)$, where r is an arbitrary function of its argument.

Remark 1.2.3. If $F = (F^1, ..., F^N)$ is an invariant function under the Lie group of transformations (1.9), then clearly every level set of F is invariant under (1.9). However it is not true that if the solution set $\{x : F(x) = 0\}$ is invariant under (1.9), then F is invariant under (1.9).

For example, the set $\{(x,y), xy = 0\}$ is invariant under the Lie group of scaling transformations

$$x^* = ax, \quad y^* = ay, \quad a > 0. \tag{1.20}$$

Whereas, $F(x,y) = xy$ is not invariant under (1.20) since $F(ax, ay) = a^2xy \neq F(x,y)$ for $a \neq 1$. However, if every level set of F is invariant under the Lie group of transformations (1.9), then F itself is invariant under (1.9).

Remark 1.2.4. Consider a system of algebraic equations (1.15). The invariance of this system under a transformation of x is the invariance of the solution set (in this case, a hypersurface in $\mathbb{R}^n$), which does not require each function F^σ to be an invariant function. In particular, the following infinitesimal criterion of invariance of the solution set of (1.15) holds [4] .

Theorem 1.2.3. *Suppose that the system of algebraic equations* (1.15) *is of maximal rank, meaning that the Jacobian matrix* $\left(\partial F^\sigma/\partial x^i\right)$ *is of rank* N *at every solution* x *of the system. Then, the Lie group of transformations* (1.9) *is a symmetry of the system* (1.15) *if and only if*

$$XF^\sigma = 0 \tag{1.21}$$

when $F^\sigma = 0, \quad \sigma = 1, ..., N.$

The maximal rank condition in Theorem 1.2.3 is important. For instance, consider the function $g(x, y) = (y-1)^2$. The solution set for g is the line $\{y = 1\}$ which is not invariant under the rotation group given in Example 1.2.7 with symmetry generator $X = y\,\partial/\partial x - x\,\partial/\partial y$. However,

$$Xg = 2x(y-1) = 0$$

when $g(x, y) = 0$, hence the infinitesimal condition (1.21) is satisfied but the maximal rank condition does not hold since $\nabla g = (0, 2y-2)$ vanishes on the solution set.

Point symmetries of differential equations

In analogy with the infinitesimal criterion of Theorem 1.2.3 for system of algebraic equations, we review the infinitesimal criterion of invariance of a system of differential equation that enable us to check whether or not a given Lie group is a symmetry for the system of differential equations and also to find the global symmetry group of the given system.

Let $x = (x^1, ..., x^n)$, $n \geq 1$, and $u(x) = (u^1(x), ..., u^m(x))$, $m \geq 1$ denote respectively independent and dependent variables of a given problem. We denote by $\partial^k u$ the set of coordinates

$$u^\mu_{i_1 i_2 \dots i_k} = \frac{\partial^k u^\mu}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

that correspond to all k^{th}-order partial derivatives of u with respect to x for $\mu = 1, ..., m$, $i_j = 1, ..., n$, $j = 1, ..., k$.

Definition 1.2.9. The total derivative operator with respect to the independent variable x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u^\mu_i \frac{\partial}{\partial u^\mu} + u^\mu_{ij} \frac{\partial}{\partial u^\mu_j} + \ldots + u^\mu_{i i_1 i_2 \dots i_n} \frac{\partial}{\partial u^\mu_{i_1 i_2 \dots i_n}} + \ldots . \tag{1.22}$$

A general system of N differential equations is given by

$$F^\sigma[u] \equiv F^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0, \quad k \geq 1, \quad \sigma = 1, 2, ..., N. \tag{1.23}$$

In (1.23) and below, $F^\sigma[u]$ and similar notation denotes differential functions (i.e., functions depending on x, u, and derivatives of u up to some prescribed order s), defined in a domain of the jet space $J^s(x|u)$. (The latter is viewed as a multi-dimensional space with coordinates $x, u, \partial u, \ldots, \partial^s u$.)

A smooth solution of the system (1.23) is a smooth function $u = f(x)$ such that f and its derivatives $\partial^j f, j = 1, ..., k$ satisfy the constraints of (1.23), i.e.

$$F^\sigma(x, f, \partial f, \ldots, \partial^k f) = 0, \quad \sigma = 1, 2, ..., N.$$

Definition 1.2.10. A one-parameter Lie group of transformations (1.9) in the space of the problem variables (x, u) is given by

$$\begin{aligned} (x^i)^* &= f^i(x, u; a) = x^i + a\xi^i(x, u) + O(a^2), \quad i = 1, 2, \ldots, n, \\ (u^\mu)^* &= g^\mu(x, u; a) = u^\mu + a\eta^\mu(x, u) + O(a^2), \quad \mu = 1, 2, \ldots, m, \end{aligned} \tag{1.24}$$

with the group parameter a, and the corresponding infinitesimal generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\mu(x, u)\frac{\partial}{\partial u^\mu}, \tag{1.25}$$

where

$$\xi^i(x, u) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \qquad \eta^\mu(x, u) = \left.\frac{\partial g^\mu}{\partial a}\right|_{a=0}$$

denote the infinitesimal components (see, e.g., [4, 37, 61] and references therein).

Prolongation

Consider the one-parameter Lie group of transformations (1.24) with one independent variable x and one dependent variable $u(x)$:

$$\begin{aligned} x^* &= f(x, u; a) = x + a\xi(x, u) + O(a^2), \\ u^* &= g(x, u; a) = u + a\eta(x, u) + O(a^2). \end{aligned} \tag{1.26}$$

The group of transformations (1.26) can be prolonged to $(x, u, u', ..., u^{(k)})$-space, $k \geq 1$ by requiring that the transformations (1.26) preserve the contact conditions

$$du = u'dx, \quad du^{(k)} = u^{(k+1)}dx.$$

The first-prolongation of (1.26) can be found as follows:

$$du^* = dg(x, u; a) = g_x dx + g_u du, \quad dx^* = df(x, u; a) = f_x dx + f_u du.$$

Consequently,

$$(u')^* = \frac{du^*}{dx^*} = g_1(x, u, u'; a) = \frac{g_x + u'g_u}{f_x + u'f_u} = \frac{Dg}{Df}$$

Similarly, the k^{th} prolongation of (1.26) is given by

$$(u^{(k)})^* = g_k(x, u, u', ..., u^{(k)}; a) = \frac{Dg_{k-1}(x, u, u', ..., u^{(k-1)}; a)}{Df}, \quad k \geq 2.$$

From equation (1.26), one has

$$(u')^* = \frac{a\eta_x + u'(1+a\eta_u) + O(a^2)}{1 + a\xi_x + au'\xi_u + O(a^2)} = u' + a\left(\eta_x + (\eta_u - \xi_x)u' - u'^2\xi_u\right) + O(a^2).$$

The function $\eta^{(1)}(x,u,u') = \eta_x + (\eta_u - \xi_x)u' - u'^2\xi_u$ is the first prolongation of the infinitesimal $\eta(x,u)$. In the same way, one can find the extended infinitesimals $\eta^{(k)}, k \geq 2$:

$$\begin{aligned}
(u^{(k)})^* &= \frac{D\left(u^{(k-1)} + a\eta^{(k-1)} + O(a^2)\right)}{D(x + a\xi + O(a^2))} \\
&= \frac{u^{(k)} + aD\eta^{(k-1)} + O(a^2)}{1 + aD\xi + O(a^2)} \\
&= u^{(k)} + a\left(D\eta^{(k-1)} - u^{(k)}D\xi\right) + O(a^2) \\
&= u^{(k)} + a\eta^{(k)}(x,u,u',...,u^{(k)}) + O(a^2).
\end{aligned}$$

Definition 1.2.11. The k^{th} prolongation of the Lie group of transformations (1.24) is the following group of transformations

$$\begin{aligned}
&(x^i)^* = x^i + a\xi^i(x,u) + O(a^2), \\
&(u^\mu)^* = u^\mu + a\eta^\mu(x,u) + O(a^2), \\
&(u_i^\mu)^* = u_i^\mu + a\eta_i^{(1)\mu}(x,u,\partial u) + O(a^2), \\
&\vdots \\
&(u^\mu_{i_1 i_2 \ldots i_k})^* = u^\mu_{i_1 i_2 \ldots i_k} + a\eta^{(k)\mu}_{i_1 i_2 \ldots i_k}(x,u,\partial u,\ldots,\partial^k u) + O(a^2)
\end{aligned} \tag{1.27}$$

acting on the $(x,u,\partial u,\ldots,\partial^k u)$ jet space.

The extended infinitesimals $\eta_i^{(1)\mu}$, $\eta^{(k)\mu}_{i_1 i_2 \ldots i_k}$ appearing above are given by the prolongation formulas [3]

$$\eta_i^{(1)\mu} = D_i\eta^\mu - u_j^\mu D_i\xi^j, \qquad \eta^{(s)\mu}_{i_1 i_2 \ldots i_k} = D_{i_s}\eta^{(s-1)\mu}_{i_1 i_2 \ldots i_{s-1}} - u^\mu_{i_1 i_2 \ldots i_{s-1} j} D_{i_s}\xi^j, \tag{1.28}$$

$\mu = 1,2,\ldots,m,\ i,i_j = 1,2,\ldots,n$ for $s = 1,2,\ldots,k$. D_i is the total derivative operator given by (1.22).

Similar to the Lie group of transformations (1.24), the corresponding infinitesimal generator is prolonged to an infinitesimal generator acting on the $(x,u,\partial u,\ldots,\partial^k u)$ jet space.

Definition 1.2.12. The k^{th} prolongation of the infinitesimal generator (1.25) is

$$X^{(k)} = \xi^i \frac{\partial}{\partial x^i} + \eta^\mu \frac{\partial}{\partial u^\mu} + \eta_i^{(1)\mu}\frac{\partial}{\partial u_i^\mu} + \ldots + \eta^{(k)\mu}_{i_1 i_2 \ldots i_k}\frac{\partial}{\partial u^\mu_{i_1 i_2 \ldots i_k}},\ \ k \geq 1. \tag{1.29}$$

Now we review the definition and the infinitesimal criterion for invariance of system of differential equations (1.23) under the one-parameter Lie group of transformations (1.24).

Definition 1.2.13. The system of differential equations (1.23) is invariant under the Lie group of point transformations (1.24) if and only if its k^{th} prolongation (1.27) leaves invariant the solution manifold of the system (1.23). In this case, the one-parameter Lie group of point transformations (1.23) is called a *point symmetry* of the system (1.23).

To apply Theorem 1.2.3 to the system of differential equations (1.23), we need a corresponding maximal rank condition for (1.23).

Definition 1.2.14. The system of differential equations (1.23) is of *maximal rank* if the Jacobian matrix

$$J[u] = \left(\frac{\partial F^\sigma}{\partial x^i}, \frac{\partial F^\sigma}{\partial u^\mu_{i_1 i_2 \ldots i_k}}\right)$$

of (1.23) with respect to $(x, u, \partial u, \ldots, \partial^k u)$ is of rank N whenever $F^\sigma = 0, \sigma = 1, ..., N$.

Theorem 1.2.4. *Suppose* (1.23) *is a system of differential equations of maximal rank. Let* (1.25) *be the infinitesimal of the one parameter Lie group of point transformations* (1.24) *and* (1.29) *be its* k^{th} *prolongation. If for each* $\sigma = 1, 2, ..., N$,

$$X^{(k)} F^\sigma(x, u, \partial u, \ldots, \partial^k u) = 0 \tag{1.30}$$

when $F^\sigma = 0$. *Then the transformation* (1.24) *is admitted by the system* (1.23), *or is a* point symmetry *of the system* (1.23).

Proof. [4]. □

Example 1.2.9. Consider the Lie group of rotations

$$x^* = y \sin a + x \cos a, \quad y^* = y \cos a - x \sin a. \tag{1.31}$$

The corresponding infinitesimal generator is

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{1.32}$$

The first prolongation of (1.32) is given by

$$X^{(1)} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \eta^{(1)}(x, y, y_x)\frac{\partial}{\partial y_x}, \tag{1.33}$$

where $\eta^{(1)}$ can be found using the formula (1.28):

$$\eta^{(1)} = D_x\eta - y_x D_x \xi = -(y_x^2 + 1).$$

Consider now the first-order ODE

$$F(x, y, y') = (y - x)y' + y + x = 0. \tag{1.34}$$

Here and below, we use primes to denote derivatives. The Jacobian matrix of F is given by

$$J = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y'}\right) = (1 - y', 1 + y', y - x)$$

which is of rank 1. Applying the prolonged generator (1.33) to the ODE (1.34), one has

$$\begin{aligned} X^{(1)}F &= y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} - (y'^2 + 1)\frac{\partial F}{\partial y'} \\ &= y(1 - y') - x(1 + y') - (y'^2 + 1)(y - x) \\ &= -y'\left((y - x)y' + y + x\right) \\ &= 0 \end{aligned}$$

when $F = 0$. Hence the infinitesimal criterion (1.30) in Theorem 1.2.4 is satisfied. It follows that the ODE (1.34) admits the rotation group (1.31).

To construct the most general symmetry group of the system of differential equations (1.23) using the infinitesimal method given in Theorem 1.2.4, it is necessary to add a nondegeneracy condition [4].

Definition 1.2.15. The system of differential equations (1.23) is *locally solvable* at the point

$$(x_0, u_0, \partial u_0, \dots, \partial^k u_0) \in S = \{(x, u, \partial u, \dots, \partial^k u) : F[u] = 0\} \subset J^s(x|u)$$

if there exists a smooth function $u = f(x)$ defined in a neighbourhood of x_0 which has the initial conditions $\partial^k u_0 = f^{(k)}(x_0)$. The system is *locally solvable* if it is locally solvable at every point of S. A system of differential equations (1.23) is *nondegenerate* if at every point $(x_0, u_0, \partial u_0, \dots, \partial^k u_0) \in S$, it is both of maximal rank and locally solvable.

Theorem 1.2.5. *Let* (1.23) *be a nondegenerate system of differential equations. Let* (1.25) *be the infinitesimal of the one parameter Lie group of point transformations* (1.24) *and* (1.29) *be its* k^{th} *prolongation. Then the transformation* (1.24) *is a* point symmetry *of the system* (1.23)*, if and only if for each* $\sigma = 1, 2, ..., N$,

$$X^{(k)} F^\sigma[u] = 0 \tag{1.35}$$

holds on solutions of (1.23)*.*

Proof. [4]. □

For the one-parameter Lie group of transformations (1.24), the *evolutionary (characteristic) form* providing the same mapping between solutions is the one-parameter family of transformations given by

$$\begin{aligned} (x^i)^* &= x^i, \quad i = 1, 2, \dots, n, \\ (u^\mu)^* &= u^\mu + a\zeta^\mu[u] + O(a^2), \quad \mu = 1, 2, \dots, m, \end{aligned} \tag{1.36}$$

with the evolutionary component $\zeta^\mu[u] = \eta^\mu(x, u) - u_i^\mu \xi^i(x, u)$ and infinitesimal generator

$$\hat{X} = \zeta^\mu[u] \frac{\partial}{\partial u^\mu}. \tag{1.37}$$

The infinitesimal generator (1.37) is the *characteristic* form (or the *evolutionary form*) of the infinitesimal generator (1.25) [37].

Computation of Lie point symmetries

Finding the Lie point symmetries for the system of differential equations (1.23) consists of the following steps.

1. Find the prolonged infinitesimal generator $X^{(k)}$ in terms of arbitrary functions $\xi^i(x, u)$ and $\eta^\mu(x, u)$.
2. Apply the extended generator $X^{(k)}$ to the N differential equations (1.23). Then substitute the N differential equations and their differential consequences into the N equations (1.30).

3. The resulting equation from the previous step is a polynomial in the remaining derivatives of u. Setting to zero the coefficients of the derivatives of u leads to a system of linear PDEs in $\xi^i(x,u)$ and $\eta^\mu(x,u)$ called the *set of determining equations for the point symmetries* of $F^\sigma[u]$(1.23).

4. Solve the determining equations for the infinitesimals ξ^i and η^μ.

Remark 1.2.5. The set of determining equations is an over-determined system of linear PDEs in ξ^i and η^μ. In solving the determining equations, the following cases can arise.

- The only solution for the determining equations is the trivial solution $\xi^i = \eta^\mu = 0$. In this case, the system of differential equations (1.23) has no point symmetries.

- The general solution of the determining equations has finite number s of arbitrary constants. Then the system (1.23) admits s-dimensional Lie algebra of point symmetry generators.

- The general solution of the determining equations contains an infinite number of arbitrary constants or arbitrary functions of (x,u). In this case, the system (1.23) admits an infinite set of point symmetry generators.

When the symmetry components ξ^i and η^μ are found, one can construct the global Lie group of point transformations using the system of ODEs (1.11) (see, e.g., [3, 4, 62, 63]).

Example 1.2.10. Consider the linear heat equation

$$u_t = u_{xx}. \tag{1.38}$$

The infinitesimal generator for the PDE (1.38) has the form

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{1.39}$$

The determining equations (1.30) to find exact point symmetries of (1.38) reads

$$X^{(2)}(u_t - u_{xx})\Big|_{u_t=u_{xx}} = (\eta_t^{(1)} - \eta_{xx}^{(2)})\Big|_{u_t=u_{xx}} = 0, \tag{1.40}$$

where the prolonged infinitesimals $\eta_t^{(1)}$ and $\eta_{xx}^{(2)}$ are given by (1.28). Equation (1.40) leads to a split system of linear PDEs in ξ^1, ξ^2 and η given by

$$\begin{aligned} &\xi^1_u = 0, \quad \xi^2_x = 0, \quad \xi^2_u = 0, \quad \eta_{uu} = 0, \\ &2\xi^1_x - \xi^2_t = 0, \quad 2\eta_{xu} - \xi^1_{xx} + \xi^2_t = 0, \quad \eta_t - \eta_{xx} = 0. \end{aligned} \tag{1.41}$$

Solving the determining equations (1.41), one finds that the heat equation admits infinite number of point symmetries given by

$$X_\infty = \alpha(x,y)\frac{\partial}{\partial u}$$

with $\alpha_t = \alpha_{xx}$, and six additional point symmetries given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}, \quad X_4 = t\frac{\partial}{\partial x} - \frac{1}{2}xu\frac{\partial}{\partial u}, \\ X_5 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \quad X_6 = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{1}{2}tu + \frac{1}{4}x^2\right)u\frac{\partial}{\partial u}. \end{aligned} \tag{1.42}$$

The global Lie group of point transformations corresponds to the symmetry generator X_6 can be found by solving the ODEs

$$\frac{dx^*}{da} = x^*t^*, \quad \frac{dt^*}{da} = (t^*)^2, \quad \frac{du^*}{da} = -\left(\frac{t^*}{2} + \frac{(x^*)^2}{4}\right)u^*$$

with $x^* = x$, $t^* = t$ and $u^* = u$ when $a = 0$. Solving the above system leads to the one-parameter Lie group of point transformations

$$x^* = \frac{x}{1-at}, \quad t^* = \frac{t}{1-at}, \quad u^* = \left(\sqrt{1-at}\, e^{\frac{-ax^2}{4(1-at)}}\right)u$$

admitted by the heat equation (1.38).

1.2.6 Contact and higher-order symmetries

A significant generalization of the point symmetry group (1.36) can be obtained by allowing the evolutionary infinitesimal components $\zeta^\mu[u]$ to depend on higher derivatives of u.

Definition 1.2.16. Consider the case of n independent variables $x = (x^1, ..., x^n)$ and one dependent variable $u(x)$. A *contact transformation* is a transformation given by

$$\begin{aligned} (x^i)^* &= f^i(x, u, \partial u), \quad i = 1, 2, \ldots, n, \\ u^* &= g(x, u, \partial u), \\ (u_i)^* &= h_i(x, u, \partial u), \end{aligned} \tag{1.43}$$

which is one-to-one in some domain D in $(x, u, \partial u)$ and preserves the contact condition $du = u_i dx^i$:

$$du^* = (u_i)^* d(x^i)^*. \tag{1.44}$$

It is assumed that f^i, g depend essentially on the first derivative of u. Otherwise, a contact transformation is a point transformation.

Definition 1.2.17. A *one-parameter Lie group of contact transformations* is given by

$$\begin{aligned} (x^i)^* &= x^i + a\xi^i(x, u, \partial u) + O(a^2), \quad i = 1, 2, \ldots, n, \\ (u)^* &= u + a\eta(x, u, \partial u) + O(a^2), \\ (u_i)^* &= u_i + \eta_i^{(1)}(x, u, \partial u) + O(a^2), \end{aligned} \tag{1.45}$$

with infinitesimal generator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \eta_i^{(1)}\frac{\partial}{\partial u_i}. \tag{1.46}$$

Theorem 1.2.6. *The transformations* (1.45) *with infinitesimal generator* (1.46) *define a one-parameter Lie group of contact transformations if and only if*

$$\frac{\partial \eta}{\partial u_i} - u_j \frac{\partial \xi^j}{\partial u_i} = 0, \quad i = 1, ..., n.$$

Proof. [37]. □

Higher-order local transformations generalize the Lie group of point transformations (1.36) by allowing the infinitesimal components to depend on higher derivatives of u.

Definition 1.2.18. A one-parameter *higher-order evolutionary local transformation* is a transformation of the form

$$\begin{aligned} &(x^i)^* = x^i, \quad i = 1, 2, \ldots, n, \\ &(u^\mu)^* = u^\mu + a\zeta^\mu(x, u, \partial u, ..., \partial^s u) + O(a^2), \quad \mu = 1, 2, \ldots, m \end{aligned} \tag{1.47}$$

acting on the space of functions $u = u(x)$. The corresponding infinitesimal generator is given by

$$\hat{X} = \zeta^\mu[u] \frac{\partial}{\partial u^\mu}, \tag{1.48}$$

where each $\zeta^\mu[u] = \zeta^\mu(x, u, \partial u, ..., \partial^s u)$ is a certain differential function.

To compute the higher-order terms in (1.47), one needs to extend the infinitesimal generator (1.48) to act on derivatives of u by requiring that the contact conditions (1.44) are preserved.

Definition 1.2.19. The prolongation of (1.48) is defined by [4]

$$\hat{X}^\infty = \zeta^\mu \frac{\partial}{\partial u^\mu} + \zeta_i^{(1)\mu} \frac{\partial}{\partial u_i^\mu} + \ldots + \zeta_{i_1 i_2 \ldots i_p}^{(p)\mu} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_p}^\mu} + \ldots, \tag{1.49}$$

where the higher-order components are computed using

$$\zeta_i^{(1)\mu} = D_i \zeta^\mu, \qquad \zeta_{i_1 i_2 \ldots i_p}^{(p)\mu} = D_{i_p} \zeta_{i_1 i_2 \ldots i_{p-1}}^{(p-1)\mu}, \tag{1.50}$$

for $\mu = 1, \ldots, m$; $i, i_j = 1, \ldots, n$, $p = 2, 3, \ldots$.

The system of differential equations (1.23) is *invariant* under a one-parameter local transformation (1.47) if and only if its k^{th} extension

$$\hat{X}^{(k)} = \zeta^\mu \frac{\partial}{\partial u^\mu} + \zeta_i^{(1)\mu} \frac{\partial}{\partial u_i^\mu} + \ldots + \zeta_{i_1 i_2 \ldots i_k}^{(k)\mu} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}^\mu} \tag{1.51}$$

leaves invariant the solution manifold of (1.23) in the space $(x, u, \partial u, ..., \partial^k u)$. In this case, we call the one-parameter local transformation a *local symmetry* of the system (1.23).

The infinitesimal criterion for the invariance of the system of differential equations (1.23) under a one-parameter local transformation (1.47) is given in the following theorem [37].

Theorem 1.2.7. *Let* (1.48) *be the infinitesimal generator for a one-parameter local transformation* (1.47) *of order* $s \geq 0$*, and* (1.51) *be its prolongation. Then the local transformation* (1.47) *is a local (point, contact or higher-order) symmetry of the system* (1.23) *if and only if for each* $\sigma = 1, 2, ..., N$,

$$\hat{X}^{(k)} F^{\sigma}(x, u, \partial u, \ldots, \partial^k u) = 0 \tag{1.52}$$

holds on solutions of (1.23) *and their differential consequences up to order* s.

Example 1.2.11. The infinitesimal generator

$$\hat{X} = \left(u_{xxx} + 3u_x u_{xx} + u_x^3\right) \frac{\partial}{\partial u}$$

corresponds to a third-order symmetry for the potential Burgers' equation,

$$u_t = u_{xx} + u_x^2. \tag{1.53}$$

For more details about contact and higher-order transformations, see [64–66].

1.2.7 Solutions of differential equations using symmetries

One of the most important applications of Lie symmetries is the integration of ODEs and the construction of invariant solutions for the PDEs [4, 37].

Reduction of order of ODEs

Consider the first-order ODE

$$\frac{dy}{dx} = F(x, y). \tag{1.54}$$

Assume that the ODE (1.54) admits a one-parameter Lie group of point transformations

$$\begin{aligned} x^* &= x + a\xi(x, y) + O(a^2), \\ y^* &= y + a\eta^{\mu}(x, y) + O(a^2), \end{aligned} \tag{1.55}$$

with the corresponding symmetry generator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{1.56}$$

The general solution of the ODE (1.54) can be found from the infinitesimals $\xi(x, y), \eta(x, y)$ using *canonical coordinates* or determination of an *integrating factor* [3].

For any Lie group of point transformations (1.55), there exist canonical coordinates $r(x, y)$, $s(x, y)$ satisfying $Xr = 0$ and $Xs = 1$ so that (1.55) becomes a translation group

$$r^* = r, \quad s^* = s + a. \tag{1.57}$$

In terms of the canonical coordinates, the ODE (1.54) becomes

$$\frac{ds}{dr} = \frac{s_x + s_y F(x, y)}{r_x + r_y F(x, y)} = G(r, s). \tag{1.58}$$

The invariance of (1.58) under the Lie group of transformations (1.57) yields the ODE

$$\frac{ds}{dr} = M(r),$$

with general solution

$$s(x,y) = \int M(r)dr + C$$

for some constant C.

The ODE (1.54) can be rewritten in the form

$$A(x,y)\,dx + B(x,y)\,dy = 0. \tag{1.59}$$

This equation is exact if $A_y = B_x$. In this case the solution can be found implicitly. Otherwise, we can multiply it by an integrating factor $\mu(x,y)$:

$$\mu(x,y) = \frac{1}{A\,\xi + B\,\eta} \tag{1.60}$$

to get an exact equation. Conversely, if μ (1.60) is an integrating factor for (1.54), then ξ and η are the infinitesimals of the point symmetry X (1.56) for the ODE (1.54).

The method of canonical coordinates extends to the integration of higher order ODEs [60].

Theorem 1.2.8. *Suppose a one-parameter Lie group of point transformations* (1.55) *is a point symmetry of a higher-order ODE*

$$y^{(n)} = F(x,y,y',...,y^{(n-1)}), n > 1. \tag{1.61}$$

Let $r(x,y), s(x,y)$ be the corresponding canonical coordinates. Then the ODE (1.61) *reduces to an $(n-1)$-order ODE*

$$\frac{d^{n-1}z}{dr^{n-1}} = M\left(r, z, \frac{dz}{dr}, ..., \frac{d^{n-2}z}{dr^{n-2}}\right),$$

where $z = ds/dr$.

An alternative method to reduce the order of the ODE (1.61) is the method of *differential invariants*. Indeed, suppose that the ODE (1.61) admits the Lie group of transformations (1.55). Then

$$X^{(n)}(y^{(n)} - F)\big|_{y^{(n)}=F} = 0,$$

where $X^{(n)}$ is the prolongation of (1.56) given by

$$X^{(n)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y')\frac{\partial}{\partial y'} + ... + \eta^{(n)}(x,y,y',...,y^{(n)})\frac{\partial}{\partial y^{(n)}}.$$

The solution of the corresponding characteristic system

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{dy'}{\eta^{(1)}(x,y,y')} = \cdots = \frac{dy^{(n)}}{\eta^{(n)}(x,y,y',...,y^{(n)})}$$

yields $n+1$ invariants

$$u(x,y), v_1(x,y,y'), ..., v_n(x,y,y',...,y^{(n)}), \tag{1.62}$$

which satisfy $Xu = 0$, $X^{(k)}v_k(x, y, y', ..., y^{(k)}) = 0$, $k \geq 1$. Since $u(x, y)$ and $v_1(x, y, y') = v(x, y, y')$ are invariants under the k^{th} prolongation of (1.55), $k \geq 1$. It follows that dv/du is invariant under the $(k+1)^{\text{th}}$ prolongation of (1.55) since $(dv/du)^* = dv^*/du^* = dv/du$. By induction, one finds that

$$\frac{dv}{du}, \frac{d^2v}{du^2}, \dots, \frac{d^{n-1}v}{du}$$

are invariants under the n^{th} prolongation of (1.55). These invariants are called differential invariants of n^{th} prolongation of (1.55). Using these invariants, the ODE (1.61) reduces to an $(n-1)$-order ODE

$$R\left(u, v, \frac{dv}{du}, ..., \frac{d^{n-1}v}{du}\right) = 0.$$

Remark 1.2.6. The method of differential invariants using point symmetries of the ODE (1.61), generalizes naturally to using admitted contact symmetries and higher-order symmetries (e.g., [3]).

Invariant solutions of PDEs

A surface (or curve) $F(x) = 0$ is an invariant surface (or curve) of the one-parameter Lie group of transformations (1.9) if and only if $F(x^*) = 0$ when $F(x) = 0$.

Definition 1.2.20 (*Classical method*)**.** $u = f(x)$ is an *invariant solution* of the PDE system (1.23) corresponding to the point symmetry (1.24) admitted by (1.23) if and only if

1. $u^\mu = f^\mu, \quad \mu = 1, ..., m$ is an invariant surface of the Lie group of transformations (1.24),

2. $u = f(x)$ solves the system of PDEs (1.23).

It follows that, $u = f(x)$ is an *invariant solution* of the PDE system (1.23) resulting from the point symmetry X (1.25) or, equivalently, $\hat{X}$ (1.37) if and only if

1.
$$X\left(u^\mu - f^\mu\right)\big|_{u=f(x)} = 0, \quad \mu = 1, ..., m, \tag{1.63}$$

2.
$$F^\sigma[u]\big|_{u=f(x)} = 0. \tag{1.64}$$

One can find the invariant solution $u = f(x)$ by solving the system of equations (1.63) and (1.64) through two different ways.

(I) Invariant form method

The general solution of (1.63) is found by solving the corresponding characteristic system of ODEs

$$\frac{dx^1}{\xi^1(x, u)} = \cdots = \frac{dx^n}{\xi^n(x, u)} = \frac{du^1}{\eta^1(x, u)} = \cdots = \frac{du^m}{\eta^m(x, u)}. \tag{1.65}$$

If

$$R^1(x,u), ..., R^{n-1}(x,u), q^1(x,u), ..., q^m(x,u)$$

are functionally independent invariants of (1.65) with Jacobian

$$J = \frac{\partial(q^1, ..., q^m)}{\partial(u^1, ..., u^m)} \neq 0.$$

Then the general solution of the invariant surface condition (1.63) is given by

$$q^\mu(x,u) = Q^\mu\left(R^1, ..., R^{n-1}\right), \tag{1.66}$$

where Q^μ is an arbitrary function of its arguments, $\mu = 1, ..., m$. The variables $R^1, ..., R^{n-1}$ are called the *similarity variables*. If the PDE system (1.23) is transformed by the corresponding invertible point transformation into a PDE system with independent variables $R = (R^1, ..., R^n)$, and dependent variables $q = (q^1, ..., q^m)$, then the transformed PDE system admits the translation point symmetry

$$(R^i)^* = R^i, \quad (R^n)^* = R^*, \quad (q^\mu)^* = q^\mu. \quad i = 1, ..., n-1, q = 1, ..., m.$$

It follows that the variable R^n does not appear explicitly in the transformed PDE system, and hence the transformed PDE system has particular solutions given by (1.66). Consequently, the PDE system (1.23) has invariant solutions implicitly given by the invariant form (1.66). These invariant solutions are found by solving a reduced system of differential equations with $n-1$ independent variables $R^1, ..., R^{n-1}$, and m dependent variables $q = (q^1, ..., q^m)$.

(II) Direct substitution method

This method can be used when one is unable to solve the characteristic system (1.65). Assume, without loss of generality, that $\xi^n \neq 0$. Then the PDE system (1.63) can be written as

$$\frac{\partial u^\mu}{\partial x^n} = \frac{\eta^m(x,u)}{\xi^n(x,u)} - \sum_{i=1}^{n-1} \frac{\partial u^\mu}{\partial x^i}, \quad \mu = 1, ..., m. \tag{1.67}$$

The substitution of (1.67) in the PDE system (1.23) leads to a reduced system of differential equations in the dependent variables $u^1, ..., u^m$, the independent variables $x^1, ..., x^{n-1}$, and the parameter x^n. A solution $u = \Theta(x^1, ..., x^{n-1}; x^n)$ of the reduced system yields the invariant solution of the given PDE system (1.23) provided that equations (1.63) are satisfied.

Example 1.2.12. Consider the linear wave equation

$$u_{tt} = u_{xx} \tag{1.68}$$

that admits a scaling symmetry

$$x^* = e^a x, \quad t^* = e^a t, \quad u^* = u, \tag{1.69}$$

with generator

$$X = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}. \tag{1.70}$$

The solution of the corresponding characteristic equations

$$\frac{dx}{x} = \frac{dt}{t} = \frac{du}{0}$$

leads to the invariant solution form

$$u = \phi(\alpha) = \phi\left(\frac{x}{t}\right). \tag{1.71}$$

The substitution of (1.71) into (1.68) reduces the wave equation (1.68) to an ODE

$$(1-\alpha^2)\frac{d^2\phi}{d\alpha^2} - 2\alpha\frac{d\phi}{d\alpha} = 0, \tag{1.72}$$

which has a general solution

$$\phi = C_1 \ln\left(\frac{\alpha-1}{\alpha+1}\right) + C_2. \tag{1.73}$$

Hence, one obtains the solution

$$u(x,t) = C_1(\ln(x-t) - \ln(x+t)) + C_2 \tag{1.74}$$

for the linear wave equation (1.68), invariant with respect to (1.69).

More examples of invariant solutions of PDEs appear in [3, 60, 67–70]. Another method of obtaining solutions of PDEs is the *nonclassical method* introduced by Bluman [71]. Here, one finds $\xi^i(x,u), \eta^\mu(x,u), i = 1,...,n, \mu = 1,...,m$ so that (1.25) is a *nonclassical symmetry* of the augmented PDE system consisting of the given PDE system (1.23), the invariant surface condition equations

$$\eta^\mu(x,u) - \xi^i(x,u)\frac{\partial u^\mu}{\partial x^i} = 0, \quad \mu = 1,...,m, \tag{1.75}$$

and the differential consequences of (1.75). A solution of a given system of PDEs (1.23) is a *nonclassical solution* if it is an invariant solution of the augmented system and does not arise as an invariant solution for (1.23) from its local symmetries. Nonclassical solutions have been obtained for some PDE models (e.g., [72, 73]).

Remark 1.2.7. Lie symmetries help in finding the general solution for an ODE. Whereas for a PDE, we only get symmetry-invariant solution, which is a small subset of the general solution.

1.2.8 Equivalence transformations

For PDE/ODE models that include arbitrary constitutive functions and/or constant parameters, one is interested in classifying their Lie point/local symmetries. At the first step towards the classification of symmetries of a system of differential equations involving arbitrary elements, it is essential to find the *equivalence transformations* for this system. An equivalence transformation maps the given differential equation to another differential equation from the same general class [33–35].

Consider a system of differential equations

$$F^{\sigma}[u;Q] = F^{\sigma}(x,u,\partial u,...,\partial^k u,Q) = 0, \quad \sigma = 1,...,N \tag{1.76}$$

contains q constitutive functions and/or parameters $Q = (Q^1,...,Q^q)$. These functions may depend on particular independent and dependent variables and derivatives of dependent variables.

Definition 1.2.21. A one-parameter Lie group of equivalence transformations of the system (1.76) is given by

$$\begin{aligned} (x^i)^* &= f^i(x,u;a), \quad i = 1,...,n, \\ (u^{\mu})^* &= g^{\mu}(x,u;a), \quad \mu = 1,...,m, \\ (Q^{\nu})^* &= h^{\nu}(x,u,Q;a), \quad \nu = 1,...,q \end{aligned} \tag{1.77}$$

which maps a system (1.76) into another system of differential equations $F^{\sigma}[u^*;Q^*]$ in the same family.

Example 1.2.13. Consider the one-dimensional heat equation

$$u_t = Q(u_x)u_{xx}. \tag{1.78}$$

The equivalence transformations for the PDE (1.78) have the form [74]

$$t^* = C_1 t + C_2, \quad x^* = C_3 x + C_4, \quad u^* = C_5 u, \quad Q^*(u^*) = \frac{C_3^2}{C_1}Q(u), \tag{1.79}$$

where $C_1,...,C_5$ are arbitrary constants and $C_1 > 0$. The corresponding infinitesimal generators are given by

$$X_1 = t\frac{\partial}{\partial t} - \frac{1}{2}Q\frac{\partial}{\partial Q}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x\frac{\partial}{\partial x} + Q\frac{\partial}{\partial Q}, \quad X_4 = \frac{\partial}{\partial x}, \quad X_5 = u\frac{\partial}{\partial u}. \tag{1.80}$$

A *symmetry classification problem* of a system of differential equations (1.76) is to classify the family (1.76) into subfamilies with the property that all differential equations in the same subfamily have the same symmetries. Equivalence transformations can be used to simplify the symmetry classification by finding the classification of the family (1.76) modulo the group of the equivalence transformations admitted by (1.76).

1.3 Approximate Lie symmetries

Here we give an introduction to approximate symmetry methods for regularly perturbed differential equations.

A general system of N algebraic or differential equations is given by

$$F_0^{\sigma}[u] \equiv F_0^{\sigma}(x,u,\partial u,\ldots,\partial^k u) = 0, \quad k \geq 0, \quad \sigma = 1,2,...,N, \tag{1.81}$$

and its first-order perturbation in terms of a small parameter ϵ is written as:

$$F^{\sigma}[u;\epsilon] = F_0^{\sigma}(x,u,\partial u,\ldots,\partial^k u) + \epsilon F_1^{\sigma}(x,u,\partial u,\ldots,\partial^k u) = o(\epsilon). \tag{1.82}$$

1.3.1 Exact local symmetries of perturbed equations

The exact point symmetry generator of the system (1.82) is given by

$$Y = \alpha^i(x,u;\epsilon)\frac{\partial}{\partial x^i} + \beta^\mu(x,u;\epsilon)\frac{\partial}{\partial u^\mu}, \tag{1.83}$$

and an exact point or higher-order local symmetry generator in evolutionary form is written as

$$\hat{Y} = \zeta^\mu(x,u,\partial u,...,\partial^s u;\epsilon)\frac{\partial}{\partial u^\mu}. \tag{1.84}$$

Solving the determining equations (1.30), one finds exact symmetries of (1.82), holding for an arbitrary ϵ. Since (1.82) is a family that includes an arbitrary element ϵ, the dimension of Lie algebra of point or local symmetries holding for a general ϵ cannot exceed that for some fixed ϵ, including $\epsilon = 0$. Therefore the family (1.82) of perturbed differential equations will admit the same or smaller number of local symmetries than its unperturbed version (1.81).

Example 1.3.1. Consider an ODE

$$y'' = \epsilon(y')^{-1}, \tag{1.85}$$

which is a perturbed version of

$$y''(x) = 0. \tag{1.86}$$

Let

$$X^0 = \xi^0(x,y)\frac{\partial}{\partial x} + \eta^0(x,y)\frac{\partial}{\partial y}$$

denote the point symmetry generator admitted by the ODE (1.86). The prolongation of X^0 to the higher order of (1.86) is given by

$$X^{0^{(2)}} = X^0 + \eta^{0^{(2)}}(x,y,y',y'')\frac{\partial}{\partial y''}.$$

The split determining equations (1.30) yield the general solution

$$\xi^0 = C_1x^2 + C_3\frac{xy}{2} + C_7x + C_6y + C_8, \tag{1.87a}$$

$$\eta^0 = C_1xy + C_2x + C_3\frac{y^2}{2} + C_4y + C_5, \tag{1.87b}$$

where C_i are arbitrary constants [4]. The resulting eight-parameter Lie group of point symmetries of (1.86) is spanned by the generators

$$\begin{array}{l} X_1^0 = xy\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial x}, \quad X_2^0 = x\frac{\partial}{\partial y}, \quad X_3^0 = \frac{y^2}{2}\frac{\partial}{\partial y} + \frac{xy}{2}\frac{\partial}{\partial x}, \\ X_4^0 = y\frac{\partial}{\partial y}, \quad X_5^0 = \frac{\partial}{\partial y}, \quad X_6^0 = y\frac{\partial}{\partial x}, \quad X_7^0 = x\frac{\partial}{\partial x}, \quad X_8^0 = \frac{\partial}{\partial x}. \end{array} \tag{1.88}$$

It can be shown that the only symmetries of (1.86) that are also symmetries of (1.85), holding for an arbitrary ϵ, are the translations

$$Y_1 = X_5^0 = \frac{\partial}{\partial y}, \quad Y_2 = X_8^0 = \frac{\partial}{\partial x}. \tag{1.89}$$

Small perturbation in a differential equation destroys many useful symmetries, and this limits the applicability of exact Lie group methods to perturbed problems. To overcome this inconvenient, some approximate symmetry methods have been proposed in order to deal with differential equations involving small terms, and the notion of approximate invariance has been introduced.

1.3.2 BGI approximate symmetries

Approximate symmetries present a tool to seek additional symmetry structure of the system of perturbed equations (1.82) that are not its exact symmetries but rather preserve (1.82) *approximately*, up to $o(\epsilon)$ [23,75].

Definition 1.3.1. A one-parameter family of Baikov-Gazizov-Ibragimov (BGI) approximate point transformations with the parameter a, acting on the (x, u)-space, is given by

$$\begin{aligned} (x^i)^* &= f^i(x,u;a,\epsilon) = f_0^i(x,u;a) + \epsilon f_1^i(x,u;a) + o(\epsilon), \quad i = 1,...,n, \\ (u^\mu)^* &= g^\mu(x,u;a,\epsilon) = g_0^\mu(x,u;a) + \epsilon g_1^\mu(x,u;a) + o(\epsilon), \quad \mu = 1,...,m, \end{aligned} \tag{1.90}$$

where f_j^i, g_j^μ are sufficiently smooth functions.

The infinitesimal generator of the family of transformations (1.90) is given by

$$\begin{aligned} X &= X^0 + \epsilon X^1 \\ &= \left(\xi_0^i(x,u) + \epsilon \xi_1^i(x,u)\right) \frac{\partial}{\partial x^i} + \left(\eta_0^\mu(x,u) + \epsilon \eta_1^\mu(x,u)\right) \frac{\partial}{\partial u^\mu}, \end{aligned} \tag{1.91}$$

where

$$\xi_j^i = \left.\frac{\partial f_j^i(x,u;a)}{\partial a}\right|_{a=0}, \quad \eta_j^\mu = \left.\frac{\partial g_j^\mu(x,u;a)}{\partial a}\right|_{a=0}, \quad j = 0,1, \quad i = 1,...,n, \quad \mu = 1,...,m$$

are the infinitesimal components.

Remark 1.3.1. In certain cases, such as for differential equations involving several terms involving different orders of the small parameter ϵ, one can seek approximate symmetries with generators of the form

$$X = \left(\xi_0^i(x,u) + \epsilon \xi_1^i(x,u) + ... + \epsilon^p \xi_p^i(x,u)\right) \frac{\partial}{\partial x^i} + \left(\eta_0^\mu(x,u) + \epsilon \eta_1^\mu(x,u) + ... + \epsilon^p \eta_p^\mu(x,u)\right) \frac{\partial}{\partial u^\mu}$$

for an arbitrary order $p \geq 1$ [23]. In this thesis, we consider the first-order of precision in ϵ, $p = 1$.

Similarly to the exact symmetry groups, one can reconstruct the global family of approximate transformations from its generator [23].

Theorem 1.3.1. *For any BGI approximate generator* (1.91), *the solution of the approximate Cauchy problem*

$$
\begin{aligned}
&\frac{df_0^i}{da} = \xi_0^i(f_0, g_0), \quad \frac{df_1^i}{da} = \sum_{k=1}^{n} \frac{\partial \xi_0^i}{\partial x^k}\bigg|_{(x,u)=(f_0,g_0)} f_1^k + \xi_1^i(f_0, g_0), \\
&f_0^i\big|_{a=0} = x^i, \quad f_1^i\big|_{a=0} = 0, \; i = 1, ..., n, \\
&\frac{dg_0^\mu}{da} = \eta_0^\mu(f_0, g_0), \quad \frac{dg_1^\mu}{da} = \sum_{k=1}^{m} \frac{\partial \eta_0^\mu}{\partial u^k}\bigg|_{(x,u)=(f_0,g_0)} f_1^k + \eta_1^\mu(f_0, g_0), \\
&g_0^\mu\big|_{a=0} = u^\mu, \quad g_1^\mu\big|_{a=0} = 0, \; \mu = 1, ..., m.
\end{aligned}
\tag{1.92}
$$

determines the BGI approximate transformation (1.90).

Example 1.3.2. Let $n = 1$, and consider a generator

$$X = (1 + \epsilon x)\frac{\partial}{\partial x}. \tag{1.93}$$

If (1.93) is treated as a generator of an approximate transformation (1.90), with $\xi_0(x) = 1$ and $\xi_1(x) = x$, the Lie's equations (1.92) become

$$\frac{df_0}{da} = 1, \quad f_0\big|_{a=0} = x, \quad \frac{df_1}{da} = f_0, \quad f_1\big|_{a=0} = 0,$$

with the solution $f_0 = x + a$, $f_1 = ax + a^2/2$, leading to the global approximate transformation

$$x^* = x + a + \epsilon\left(ax + \frac{a^2}{2}\right). \tag{1.94}$$

If (1.93) is considered as an exact generator of a Lie group, then solving the Lie's equation (1.11) yields the global group

$$x^* = xe^{a\epsilon} + \frac{e^{a\epsilon} - 1}{\epsilon} = x + a + \epsilon\left(ax + \frac{a^2}{2}\right) + \epsilon^2\left(\frac{a^2}{2}x + \frac{a^3}{6}\right) + \dots, \tag{1.95}$$

where the Taylor expansion of the transformed x in the small parameter contains the approximate transformation (1.94) as the first three terms.

Determining equations. Stable symmetries

Let G be a BGI approximate point transformation (1.90). A system of perturbed algebraic or differential equations (1.82) is *approximately invariant* with respect to G if

$$F^\sigma(x^*, u^*, \partial u^*, \dots, \partial^k u^*; \epsilon) = o(\epsilon), \quad \sigma = 1, ..., N,$$

whenever $F^\alpha(x, u, \partial u, \dots, \partial^k u; \epsilon) = 0$ for $\alpha = 1, ..., N$.

Definition 1.3.2. The family (1.90) of BGI approximate point transformations defines a *BGI approximate point symmetry* of the PDE (1.82) if it satisfies the *approximate invariance condition* of (1.82) under the action of (1.91):

$$(X^{0\,(k)} + \epsilon X^{1\,(k)})(F_0^\sigma[u] + \epsilon F_1^\sigma[u])\bigg|_{F_0[u]+\epsilon F_1[u]=0} = o(\epsilon), \quad \sigma = 1, ..., N. \tag{1.96}$$

In (1.96), $O(1)$ and $O(\epsilon)$ terms must vanish independently. It is easy to see that the $O(1)$ term yields the determining equation (1.30) for the invariance of the unperturbed equation (1.81) under a point transformation X^0 (1.51). Hence the following result holds [23].

Theorem 1.3.2. *Let the equations* (1.82) *be approximately invariant under the approximate point transformation* (1.90) *with the generator* (1.91) *such that* $\xi^0, \eta^0(x, u) \neq 0$. *Then the infinitesimal operator*

$$X^0 = \xi_0^i(x, u)\frac{\partial}{\partial x^i} + \eta_0^\mu(x, u)\frac{\partial}{\partial u^\mu} \tag{1.97}$$

is a generator of an exact symmetry group for the unperturbed equations (1.81).

The converse of the above result does not always hold. Indeed, as it will be seen in examples below, if X^0 (1.97) generates an exact point symmetry group of the unperturbed PDE (1.81), there may be no corresponding BGI transformation (1.90) that approximately preserved the perturbed PDE (1.82). The following definition is important.

Definition 1.3.3. Suppose X^0 (1.97) is a generator of an exact point symmetry group of the unperturbed PDE (1.81). If the perturbed PDE (1.82) admits an approximate generator X (1.91) with its $O(1)$ part given by X^0, then X^0 corresponds to a *stable* point symmetry of the unperturbed PDE (1.81) (in the BGI sense). Otherwise, it corresponds to an *unstable* point symmetry of (1.81).

Below in this thesis, Definition 1.3.3 will be used not only for BGI approximate point symmetries, but more generally, for BGI and FS approximate point and local symmetries.

Remark 1.3.2. Solving the determining equations (1.96) to calculate first-order BGI approximate point symmetry components for equations (1.82) with a small parameter can be subdivided in the following steps:

1. Compute an exact point/local symmetry generator X^0 of the unperturbed equations (1.81) using determining equations (1.52) for exact local or point symmetries.

2. Find the corresponding first-order deformation (the part X^1 of the generator (1.91)) using the equation

$$X^{1^{(k)}} F_0^\sigma \Big|_{F_0^\sigma = 0} = H[u], \tag{1.98}$$

where H is obtained from the coefficients of ϵ in

$$-X^{0^{(k)}}(F_0^\sigma + \epsilon F_1^\sigma)\Big|_{F_0^\sigma + \epsilon F_1^\sigma = 0}, \quad \sigma = 1, ..., N. \tag{1.99}$$

The first-order condition (1.98) may (or may not) contain additional conditions on the components ξ_0^i, η_0 of the unperturbed symmetry generator X^0 (1.97). This leads to the symmetry generated by X^0 being unstable (or respectively, stable). If all symmetries of the equations (1.81) are stable, the perturbed equations (1.82) are said to *inherit* the symmetry structure of the unperturbed equations [23].

Higher-order BGI approximate symmetries

Similarly to exact local transformations with generators of the form (1.48), one can define more general *local approximate BGI transformations* with generators in evolutionary form given by

$$\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = (\zeta_0^\mu[u] + \epsilon \zeta_1^\mu[u]) \frac{\partial}{\partial u^\mu} . \tag{1.100}$$

Approximate local (including point, contact, and higher-order) BGI symmetries of the perturbed PDE (1.82) can be found using the same procedure as described above for BGI approximate point symmetries. In particular, the analog of the first-order condition (1.98) takes the form

$$\left(\zeta_1^\mu \frac{\partial}{\partial u} + (\zeta_1^{(1)\mu})_i \frac{\partial}{\partial u_i} + \ldots + (\zeta_1^{(p)\mu})_{i_1 i_2 \ldots i_p} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_p}} \right) F_0^\sigma \Big|_{F_0 = 0} = H[u], \tag{1.101}$$

where the higher-order components are computed using the equations (1.50).

Theorem 1.3.2, the stability definition 1.3.3 concerning stability conditions of approximate symmetries directly carry over to the case of general local BGI symmetries.

Approximate commutator

Let $\xi_j^i(x,u)$, $\eta_j^i(x,u)$, $j = 0,1$, $i = 1,...,n$ and $\mu = 1,...,m$ be smooth functions, the approximate operator is a differential operator given by

$$X = \left(\xi_0^i(x,u) + \epsilon \xi_1^i(x,u)\right) \frac{\partial}{\partial x^i} + \left(\eta_0^\mu(x,u) + \epsilon \eta_1^\mu(x,u)\right) \frac{\partial}{\partial u^\mu}.$$

Definition 1.3.4. The approximate commutator for the approximate operators X_1, X_2 is given by

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 + o(\epsilon). \tag{1.102}$$

Similar to an exact operator, the approximate operator satisfies

1. linearity: $[c_1 X_1 + c_2 X_2, X_3] = c_1 [X_1, X_3] + c_2 [X_2, X_3] + o(\epsilon)$,
2. skew-symmetric: $[X_1, X_2] = -[X_2, X_1] + o(\epsilon)$,
3. Jacobi identity: $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = o(\epsilon)$

for any approximate operators X_q and arbitrary constants c_j [76, 77].

Example 1.3.3. Consider the approximate operators

$$X_1 = \frac{\partial}{\partial x} + \epsilon x \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t} + \epsilon t \frac{\partial}{\partial x}.$$

One finds

$$[X_1, X_2] = \epsilon^2 \left(x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} \right).$$

The linear span of X_1, X_2 is not an exact Lie algebra. However, up to $o(\epsilon)$, these operators are approximately commuted.

1.3.3 Fushchich-Shtelen approximate symmetries

Unlike the BGI approach where the symmetry generator is expanded in a power series in terms of the small parameter, the Fushchich-Shtelen method [13] applies the perturbation technique to the solution $u(x)$ and the given system of differential equations. In particular, the solution is written as

$$u(x;\epsilon) = v(x) + \epsilon w(x) + O(\epsilon^2) \tag{1.103}$$

with components $u^\mu(x) = v^\mu(x) + \epsilon w^\mu(x), \mu = 1, ..., m$. Substituting (1.103) into the system of perturbed equations (1.82) with a small parameter, expanding the result, and setting to zero the $O(1)$ and $O(\epsilon)$ terms independently, one obtains a system of $2N$ equations on $v(x)$ and $w(x)$ without the small parameter, given by

$$G_1^\sigma[v,w] \equiv F_0^\sigma[v] = 0, \tag{1.104a}$$

$$G_2^\sigma[v,w] \equiv (F_0^\sigma)_v \cdot w + (F_0^\sigma)_{v_i} \cdot w_i + (F_0^\sigma)_{v_{ij}} \cdot w_{ij} + ... + (F_0^\sigma)_{v_{i_1 i_2 \dots i_k}} \cdot w_{i_1 i_2 \dots i_k} + F_1^\sigma[v] = 0. \tag{1.104b}$$

It is clear that the first equations (1.104a) are independent of w, and the second equations (1.104b) are linear in w, with the linear operator being the Frèchet derivative of the $F_0^\sigma[v]$. We refer to equations (1.104) as the *Fushchich-Shtelen system* for the system (1.82). The system (1.104) approximates the given system of differential equations (1.82), in the sense that each exact solution pair $(v(x), w(x))$ of (1.104) yields an approximate solution (1.103) of the given system (1.82) up to the order $o(\epsilon)$.

Definition 1.3.5. The Lie group of point transformations with the group parameter a

$$\begin{aligned} (x^i)^* &= f^i(x,v,w;a) = x^i + a\lambda^i(x,v,w) + O(a^2),\ i = 1,...,n, \\ (v^\mu)^* &= g^\mu(x,v,w;a) = v^\mu + a\phi_1^\mu(x,v,w) + O(a^2),\ \mu = 1,...,m, \\ (w^\mu)^* &= h^\mu(x,v,w;a) = w^\mu + a\phi_2^\mu(x,v,w) + O(a^2) \end{aligned} \tag{1.105}$$

with the generator

$$Z = \lambda^i(x,v,w)\frac{\partial}{\partial x^i} + \phi_1^\mu(x,v,w)\frac{\partial}{\partial v^\mu} + \phi_2^\mu(x,v,w)\frac{\partial}{\partial w^\mu} \tag{1.106}$$

defines a *FS approximate point symmetry* of the perturbed equations (1.82) if it is an exact Lie point symmetry group of the Fushchich-Shtelen system (1.104).

In a similar manner, a generalized local (point or higher-order) transformation group in the evolutionary form

$$\begin{aligned} (x^i)^* &= x^i, \qquad i = 1,...,n, \\ (v^\mu)^* &= v^\mu + a\psi_1^\mu[v,w] + O(a^2), \\ (w^\mu)^* &= w^\mu + a\psi_2^\mu[v,w] + O(a^2), \mu = 1,...,m \end{aligned} \tag{1.107}$$

with the generator

$$\hat{Z} = \psi_1^\mu[v,w]\frac{\partial}{\partial v^\mu} + \psi_2^\mu[v,w]\frac{\partial}{\partial w^\mu} \tag{1.108}$$

defines a *local (point or higher-order) FS approximate symmetry* of the PDE (1.82) if it is a local symmetry of the Fushchich-Shtelen system (1.104).

It is important to know whether the FS approximate symmetry structure of a PDE (1.82) with a small parameter is in some sense *inherited* from exact local symmetries of the unperturbed PDE (1.81). Similarly to the BGI case, one can define stable and unstable symmetries in the Fushchich-Shtelen framework.

Definition 1.3.6. Suppose $\hat{X}_0 = \zeta_0^\mu[u]\,\partial/\partial u^\mu$ (1.100) is a generator of an exact local symmetry group of the unperturbed equations (1.81). If the perturbed equations (1.82) admit an approximate FS symmetry with generator (1.108) where the v-part $\psi_1^\mu[v,w] \equiv \zeta_0^\mu[v]$, then $\hat{X}_0$ corresponds to a *stable* point symmetry of the unperturbed PDE (1.81) (in the FS sense). Otherwise, it corresponds to an *unstable* point symmetry of (1.81).

Similarly to the case for BGI approximate symmetries, a FS approximate symmetry of a system of equations (1.82) given by (1.108) may be unstable because the second symmetry determining equation for the Fushchich-Shtelen system (1.104)

$$\hat{Z}^{(k)} G_2^\sigma[v,w]\bigg|_{G_1[v,w]=G_2[v,w]=0} = 0, \sigma = 1, ..., N$$

could contain additional conditions on the v-components $\psi_1^\mu[v,w]$ in (1.108).

1.3.4 Trivial approximate symmetries

Trivial BGI approximate symmetries

Consider a local BGI approximate transformation with the evolutionary generator (1.100):

$$\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = (\zeta_0^\mu[u] + \epsilon \zeta_1^\mu[u])\,\frac{\partial}{\partial u^\mu}, \quad \mu = 1, ..., m.$$

The determining equations (1.96) for the generator (1.100) to define an approximate local symmetry of the perturbed equations (1.82) with a small parameter split into the $O(1)$ part (1.30) and $O(\epsilon)$ part (1.98) with H defined by (1.99). Suppose that the $O(1)$ part of the generator vanishes: $\hat{X}^0 = 0$. In that case, the $O(1)$ part (1.30) of the approximate symmetry determining equations is satisfied identically, and (1.99) yields $H = 0$. Consequently, the $O(\epsilon)$ part (1.98) of the determining equations (1.96) becomes

$$\hat{X}^{1\,(k)} F_0^\sigma[u]\bigg|_{F_0[u]=0} = 0. \quad \sigma = 1, ..., N, \tag{1.109}$$

which means that such $\hat{X}^1$ must be a local symmetry generator of the unperturbed equations (1.81). The opposite is also true: if $\hat{X}^0$ is a local symmetry generator of the unperturbed equations (1.81), then

$$\hat{X} = \epsilon \hat{X}^0 \tag{1.110}$$

is a BGI approximate symmetry generator of the perturbed equations (1.82). In the light of the above, we call a BGI approximate symmetry that has a generator with vanishing $O(1)$ part

$$\hat{X} = \epsilon \hat{X}^1 = \epsilon \zeta_1^\mu[u] \frac{\partial}{\partial u^\mu} \tag{1.111}$$

a *trivial BGI approximate symmetry.* This triviality relates not to the trivial action of such symmetries but rather to the fact that *every* local symmetry $\hat{X}^0$ of the unperturbed equations (1.81) is guaranteed to yield a BGI approximate symmetry of the perturbed equations (1.82) having the form (1.110). The local action of a trivial BGI approximate symmetry in the evolutionary form defined by (1.111) is given by

$$\begin{aligned} (x^i)^* &= x^i, \quad i = 1, 2, \dots, n, \\ (u^\mu)^* &= u^\mu + a\epsilon\zeta_1^\mu[u] + O(a^2), \quad \mu = 1, ..., m, \end{aligned} \tag{1.112}$$

with the first Taylor term of the transformation having the order of smallness $\sim a\epsilon = o(a, \epsilon)$.

Trivial FS approximate symmetries

In a parallel fashion, one can define a *trivial FS approximate symmetry* of the perturbed PDE (1.82) as one for which the local generator (1.108) has a special form with the vanishing transformation component of the $O(1)$ part of the solution $\psi_1^\mu = 0$, and $\psi_2^\mu[v, w] = \psi_2^\mu[v]$:

$$\hat{Z} = 0 + \psi_2^\mu[v] \frac{\partial}{\partial w^\mu}\,. \tag{1.113}$$

For FS local symmetries with the generator of the form (1.113), it is straightforward to show that $\psi_2^\mu[u]$ are the evolutionary components of the local symmetry of the unperturbed equations (1.81) generated by

$$\hat{X}^0 = \psi_2^\mu[u] \frac{\partial}{\partial u^\mu}. \tag{1.114}$$

Indeed, the action of (1.113) on the first equations (1.104a) of Fushchich-Shtelen system is trivial, and the action on the linear equations (1.104b) is equivalent to the local symmetry determining equation (1.52) of the unperturbed equations (1.81).

1.3.5 Types of approximate symmetries

In the computation of BGI approximate symmetries of a PDE (1.82) with a small parameter, the following three types of symmetries can arise.

1. BGI approximate symmetries with generators (1.100) having $\hat{X}^0 \neq 0$, $\hat{X}^1 = 0$ correspond to exact local symmetries of the perturbed equation (1.82) (see Section 1.3.1).

2. BGI approximate symmetries with generators having $\hat{X}^0 = 0$, $\hat{X}^1 \neq 0$ correspond to trivial BGI approximate symmetries.

3. Genuine BGI approximate symmetries have generators with both $\hat{X}^0$ and $\hat{X}^1 \neq 0$.

For FS approximate symmetries, the following types can arise.

1. Symmetries with the same action on $O(1)$ solution part v and $O(\epsilon)$ solution part w correspond to exact local symmetries of the perturbed equation (1.82). For example, an exact scaling symmetry with the generator $u\,\partial/\partial u$ admitted by the perturbed equation (1.82) is equivalent to a FS scaling symmetry with the generator $v\,\partial/\partial v + w\,\partial/\partial w$.

2. Trivial FS approximate symmetries.

3. Genuine FS approximate symmetries.

Genuine BGI and FS approximate symmetries are the main focus of the approximate symmetry study.

Example 1.3.4. Consider the second-order ODE

$$y'' = \epsilon(y')^{-1} \tag{1.115}$$

which is a perturbed version of the ODE (1.86). The latter has eight exact point symmetries given by (1.88). Let

$$X = X^0 + \epsilon X^1 \left(\xi^0(x,y) + \epsilon\xi^1(x,y)\right)\frac{\partial}{\partial x} + \left(\eta^0(x,y) + \epsilon\eta^1(x,y)\right)\frac{\partial}{\partial y}$$

be the approximate symmetry generator of (1.3.4), where X^0 is an exact symmetry generator of the unperturbed ODE (1.86). The determining equations (1.96) for approximate symmetries yield

$$\eta^1_{xx} + (2\eta^1_{xy} - \xi^1_{xx})y' + (\eta^1_{yy} - 2\xi^1_{xy}){y'}^2 - \xi^1_{yy}{y'}^3 = (3\xi^0_x - 2\eta^0_y)y'^{-1} + 4\xi^0_y - \eta^0_x y'^{-2}, \tag{1.116}$$

where ξ^0, η^0 are exact symmetry components (1.87) computed in Example 1.3.1. The determining equations (1.116) splits into the PDEs

$$\eta^1_{xx} = 4C_6, \quad 2\eta^1_{xy} - \xi^1_{xx} = 0, \quad \eta^1_{yy} - 2\xi^1_{xy} = 0, \quad \xi^1_{yy} = 0, \tag{1.117}$$

for ξ^1, η^1, and the additional conditions

$$3\xi^0_x - 2\eta^0_y = 0, \quad \eta^0_x = 0$$

on the unperturbed symmetry components ξ^0, η^0 (1.87). These provide restrictions on free constants in (1.87):

$$C_1 = C_2 = C_3 = 0, \quad C_4 = \frac{3}{2}C_7.$$

The remaining space of exact symmetry components ξ^0, η^0 reduces to

$$\xi^0 = \frac{2C_4}{3}x + C_6 y + C_8, \quad \eta^0 = C_4 y + C_5.$$

The approximate components are found from (1.117) and have the form

$$\begin{aligned} \xi^1(x,y) &= a_1x^2 + \frac{a_2}{2}xy + a_3x + a_4y + a_5, \\ \eta^1(x,y) &= 2C_6x^2 + a_1xy + \frac{a_2}{2}y^2 + a_6x + a_7y + a_8. \end{aligned} \tag{1.118}$$

Since the constants $a_1 \dots a_8$ and C_4, C_5, C_6, C_8 are free, the ODE (1.115) admits 12 approximate point symmetries. These approximate symmetries can be divided into the following classes:

1. Exact symmetries inherited from the unperturbed ODE (1.86), involving only $O(\epsilon^0)$ components

$$X_9 = X_4^0 + \frac{2}{3}X_7^0, \quad X_{10} = X_5^0, \qquad X_{12} = X_8^0. \tag{1.119a}$$

2. A genuine approximate symmetry

$$X_{11} = X_6^0 + 2\epsilon x^2 \frac{\partial}{\partial y} \tag{1.119b}$$

   with $O(\epsilon^0)$ part inherited from the stable symmetry X_6^0 of the unperturbed ODE (1.86) (see (1.88)).

3. Eight trivial symmetries $X_j = \epsilon X_j^0$, $j = 1, 2, ..., 8$, given by

$$\begin{aligned} X_1 &= \epsilon\left(xy\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial x}\right), \quad X_2 = \epsilon\left(x\frac{\partial}{\partial y}\right), \quad X_3 = \epsilon\left(\frac{y^2}{2}\frac{\partial}{\partial y} + \frac{xy}{2}\frac{\partial}{\partial x}\right), \\ X_4 &= \epsilon y\frac{\partial}{\partial y}, \quad X_5 = \epsilon\frac{\partial}{\partial y}, \quad X_6 = \epsilon y\frac{\partial}{\partial x}, \quad X_7 = \epsilon x\frac{\partial}{\partial x}, \quad X_8 = \epsilon\frac{\partial}{\partial x}, \end{aligned} \tag{1.119c}$$

   corresponding to the free constants $a_1 \ldots a_8$ in (1.118), having only $O(\epsilon)$ components, and arising from each exact point symmetry (1.88) of the unperturbed ODE (1.86).

Concerning the “fate” of the exact point symmetries (1.88) of the unperturbed ODE (1.86) in the approximate symmetry classification (1.119) of the perturbed ODE (1.3.4), it turns out that only four exact symmetries are stable: these are X_5^0, X_6^0, X_8^0 and the linear combination

$$X_s^0 = X_4^0 + \frac{2}{3}X_7^0$$

that is contained in X_9 of (1.119a). The other four symmetries of the unperturbed ODE (1.86) are *unstable*, including the generators X_1^0, X_2^0, X_3^0 in (1.88), and the transverse linear combination of X_4^0 and X_7^0:

$$X_u^0 = X_4^0 - \frac{3}{2}X_7^0. \tag{1.120}$$

Now, we proceed to compute FS approximate symmetries for the perturbed ODE (1.115). Substituting $y(x) = v(x) + \epsilon w(x)$ into the ODE (1.115) leads to the FS system

$$\begin{aligned} v'' &= 0, \\ w'' &= (v')^{-1}. \end{aligned} \tag{1.121}$$

Since the first equation of the system (1.121) is equivalent to the unperturbed ODE (1.86), the exact symmetry generator for the system the system (1.121) can be sought in the form

$$Z = \xi^0(x, v)\frac{\partial}{\partial x} + \eta^0(x, v)\frac{\partial}{\partial v} + \eta^w(x, v, w)\frac{\partial}{\partial w},$$

where ξ^0, η^0 are the unperturbed symmetry components (1.87). The determining equation (1.30) applied to the second ODE in (1.121) yields the following system of PDEs in η^w:

$$\eta^w_{xx} = \frac{3}{2}C_3 x + 3C_6, \quad \eta^w_{xv} = 0, \quad \eta^w_{vv} = 0, \quad \eta^w_w = \frac{1}{2}C_3 v + 3C_7 - C4, \tag{1.122}$$

and additional conditions on the exact symmetry components (1.87) yield: $C_1 = C_2 = 0$. Hence, the exact symmetries X_0^1 and X_0^2 of the unperturbed ODE (1.86) are *unstable*. The infinitesimal component η^w is found from (1.122) and has the form

$$\eta^w = C_3\left(\frac{1}{4}x^3 + \frac{1}{2}vw\right) + (3C_7 - C_4)w + \frac{3}{2}C_6 + a_1 v + a_2 x + a_3. \tag{1.123}$$

Consequently, the system of ODEs (1.121) admits 9 exact symmetries (these are the approximate symmetries of the perturbed ODE (1.115)) given by the following categories

1. Exact symmetries inherited from the exact symmetries of the unperturbed ODE (1.86).

$$Z_1 = X_0^5(x,v) = \frac{\partial}{\partial x}, \quad Z_2 = X_0^8(x,v) = \frac{\partial}{\partial v}. \tag{1.124}$$

2. Genuine FS approximate symmetries:

$$\begin{gathered} Z_3 = x\frac{\partial}{\partial x} + 3w\frac{\partial}{\partial w}, \quad Z_4 = v\frac{\partial}{\partial x} + \frac{3}{2}x^2\frac{\partial}{\partial w}, \quad Z_5 = v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}, \\ Z_6 = \frac{1}{2}v^2\frac{\partial}{\partial x} + \frac{1}{2}xv\frac{\partial}{\partial v} + \left(\frac{1}{4}x^3 + \frac{1}{2}vw\right)\frac{\partial}{\partial w} \end{gathered} \tag{1.125}$$

   corresponding to the stable exact symmetries X_7^0, X_6^0, $X_4^0(x,v)$ and $X_3^0(x,v)$ of the unperturbed ODE (1.86), respectively. Note that X_3^0 is unstable in sense of BGI however it yields a new approximate FS symmetry Z_6 in (1.125). Also, in BGI case, there is only one new approximate symmetry (1.119b) corresponding to the stable exact symmetry X_6^0.

3. Trivial FS approximate symmetries:

$$Z_7 = \frac{\partial}{\partial w}, \quad Z_8 = x\frac{\partial}{\partial w}, \quad Z_9 = v\frac{\partial}{\partial w}. \tag{1.126}$$

1.3.6 Approximate invariant solutions

Approximate symmetries are useful in constructing approximate invariant solutions for differential equation with a small parameter [23].

Definition 1.3.7. An approximate function $J(x,u;\epsilon) = J_0(x,u) + \epsilon J_1(x,u) + o(\epsilon)$ is called an *approximate invariant* of a BGI approximate point transformation (1.90) if

$$J(x^*, u^*; \epsilon) = J(x,u;\epsilon) + o(\epsilon).$$

Theorem 1.3.3 ([76]). *An approximate function $J(x,u;\epsilon) = J_0(x,u) + \epsilon J_1(x,u) + o(\epsilon)$ is invariant under a one-parameter family of BGI approximate point transformations* (1.90) *with approximate symmetry generator* (1.91) *if and only if the identity*

$$XJ(x,u;\epsilon) = o(\epsilon) \tag{1.127}$$

holds.

The determining equation (1.127) for finding the approximate invariants of the Lie group of BGI approximate point transformations (1.90) splits into the system:

$$X^0 J_0 = 0, \quad X^0 J_1 + X^1 J_0 = 0.$$

Example 1.3.5. The perturbed wave equation

$$u_{tt} + \epsilon u_t = u u_{xx} \tag{1.128}$$

admits the approximate symmetry generator

$$X = u\frac{\partial}{\partial u} - \frac{t}{2}\frac{\partial}{\partial t} + \epsilon\left(\frac{tu}{5}\frac{\partial}{\partial u} - \frac{t^2}{20}\frac{\partial}{\partial t}\right). \tag{1.129}$$

Approximate invariants of (1.129) are given by

$$J(x,t,u;\epsilon) = J_0(x,t,u) + \epsilon J_1(x,t,u) + o(\epsilon).$$

These invariants are determined using

$$X^0 J_0 = 0, \quad X^0 J_1 = -X^1 J_0,$$

which leads to the following system of first-order PDEs

$$\begin{aligned} u\frac{\partial J_0}{\partial u} - \frac{t}{2}\frac{\partial J_0}{\partial t} &= 0, \\ u\frac{\partial J_1}{\partial u} - \frac{t}{2}\frac{\partial J_1}{\partial t} &= \frac{-tu}{5}\frac{\partial J_0}{\partial u} + \frac{t^2}{20}\frac{\partial J_0}{\partial t}. \end{aligned} \tag{1.130}$$

The solution of the above system yields two functionally independent approximate invariants for the operator (1.129) given by

$$J^1 = x + \epsilon\theta(x, t^2u), \quad J^2 = t^2u + \epsilon\left(\frac{t^3u}{5} + \vartheta(x, t^2u)\right),$$

with arbitrary functions θ and ϑ. In the simple case when $\theta = \vartheta = 0$, an approximately invariant solution given by the equation $J^2 \approx \phi(J^1)$ has the form

$$u(x,t) = \frac{\phi(x)}{t^2 + \frac{\epsilon}{5}t^3} \approx \phi(x)\left(\frac{1}{t^2} - \frac{\epsilon}{5t}\right). \tag{1.131}$$

Substituting (1.131) into the PDE (1.128) yields the following ODE

$$\frac{d^2\phi}{dx^2} = 6, \tag{1.132}$$

which has a solution

$$\phi(x) = 3x^2 + C_1 x + C_2,$$

where C_1, C_2 are arbitrary constants. Consequently,

$$u(x,t) = (3x^2 + C_1 x + C_2)\left(\frac{1}{t^2} - \frac{\epsilon}{5t}\right) \tag{1.133}$$

is an approximate solution for the perturbed wave equation (1.128).

Remark 1.3.3. One can readily verify that the remaining terms after substituting the approximate solution (1.133) into the perturbed wave equation (1.128) are $O(\epsilon^2)$.

In Fushchich-Shtelen framework, an approximate solution for the perturbed equations (1.82) can be found by first finding the exact solution $(v(x), w(x))$ of the corresponding FS system (1.104) using the classical methods (see Section 1.2.7), then the approximate solution $u(x;\epsilon)$ of (1.82) has the form (1.103).

1.4 Conservation laws and Noether's theorem

Conservation laws have significant mathematical and physical applications including existence, uniqueness and stability analysis, in various areas of science. Here, we assume that the system of differential equations (1.23) is totally nondegenerate.

1.4.1 Local and global conservation laws

Definition 1.4.1. A local conservation law of PDE system (1.23) is a divergence expression

$$D_i\Phi^i[u] = D_1\Phi^1[u] + ... + D_n\Phi^n[u] \tag{1.134}$$

which vanishes on all solutions of PDE system (1.23). $\Phi^i[u] = \Phi^i(x, u, \partial u, ..., \partial^q u)$, $i = 1, ..., n$, are called the fluxes of the conservation laws, and the highest derivative q is called the differential order of a conservation law.

For a scalar PDE $F[u] = 0$ with two independent variables, the local conservation law (1.134) has the form

$$D_t\Phi[u] + D_x\Psi[u] = 0. \tag{1.135}$$

The corresponding *global conservation law* is given by

$$\frac{d}{dt}\Theta[u] = \frac{d}{dt}\int_a^b \Phi[u]dx = -\Psi[u]\Big|_a^b. \tag{1.136}$$

If the flux $\Psi[u]$ vanishes on the boundary or at infinity or in the periodic case, then Θ defines a *global conserved quantity* [78]. When the independent variables are the time and space variables, the local conservation law (1.134) becomes

$$D_t\Phi[u] + D_j\Psi^j[u] = 0, \quad j = 1, 2, 3.$$

The corresponding global form is given by

$$\frac{d}{dt}\int_{\mathcal{V}} \Phi[u]dV = -\oint_{\partial\mathcal{V}} \mathbf{\Psi[u]} \cdot d\,\mathbf{S},$$

where $\mathcal{V} \subseteq \mathbb{R}^3$ is a closed volume with smooth boundary surface $\partial\mathcal{V}$ and $d\,\mathbf{S}$ is the surface element. For multidimensional PDE systems, several types of local and global conservation laws can arise such as surface-flux and circulatory conservation laws [79].

Example 1.4.1. Consider the Korteweg-de Vries equation given by

$$u_t + uu_x + u_{xxx} = 0, \tag{1.137}$$

where $u(x,t)$ is the amplitude of long surface waves on shallow water. The KdV equation (1.137) has an infinite sequence of conservation laws of increasing order [80] that are found using the Lax pair. A Lax pair refers to a set of time-dependent operators that satisfy a corresponding differential equation called the Lax equation [81]. Finding the Lax par is an essential step in solving nonlinear partial differential equations using the inverse scattering transform [82]. The KdV equation (1.137) can be viewed as a completely integrable Hamiltonian system. It also provides resources for studying integrability of nonlinear differential equations. Moreover, various physical solutions to the KdV equation can be presented explicitly in a simple way such as solitons, rational solutions, positons and negatons [83].

In particular, the KdV equation (1.137) has a conservation laws for mass, momentum and energy, given, respectively, by

$$\begin{aligned}
&D_t(u) + D_x(\tfrac{1}{2}u^2 + u_{xx}) &&= 0,\\
&D_t(\tfrac{1}{2}u^2) + D_x(\tfrac{1}{3}u^3 + uu_{xx} - \tfrac{1}{2}u_x^2) &&= 0,\\
&D_t(\tfrac{1}{6}u^3 - \tfrac{1}{2}u_x^2) + D_x(\tfrac{1}{8}u^4 - uu_x^2 + \tfrac{1}{2}(u^2u_{xx} + u_{xx}^2) - u_xu_{xxx}) &&= 0.
\end{aligned} \tag{1.138}$$

The local conservation laws (1.138) yield respectively the conserved integrals

$$I_1 = \int_a^b u\,dx, \quad I_2 = \int_a^b \frac{1}{2}u^2 dx, \quad I_3 = \int_a^b \left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) dx. \tag{1.139}$$

Equivalent conservation laws

A local conservation law (1.134) of the PDE system (1.23) could *trivially* hold in two different ways. The first type, each of the fluxes of (1.7) vanishes on the solution of the system (1.23). The second type occurs when a conservation law (1.7) vanishes identically as a differential identity. Trivial conservation laws apply to any system of differential equations and provide no new information about the given system.

Example 1.4.2. Consider the PDE system

$$v_x = u, \quad v_t = K(u)u_x. \tag{1.140}$$

The conservation law

$$D_t(u(u - v_x) + D_x(2(v_t - K(u)u_x)=0$$

is a trivial conservation law of the first type, and

$$D_t(u_{xx}) - D_x(u_{tx}) = 0$$

is a trivial conservation law of the second type.

Definition 1.4.2. Two conservation laws $D_i\Phi^i[u] = 0$ and $D_i\Psi^i[u] = 0$ are *equivalent* if they differ by a trivial conservation law. An *equivalence class* of conservation laws consists of all conservation laws equivalent to some given nontrivial conservation law.

1.4.2 The multiplier method for construction of conservation laws

The direct method [5, 6] provides an algorithmic approach to find conservation laws for any system of differential equations. Nontrivial conservation laws for a PDE system (1.23) arise from linear combinations of the equations of the PDE system (1.2.5) with *multipliers*. A set of multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^{N}$ yields a divergence expression for the PDE system (1.23) if the identity

$$\Lambda_\sigma[U]F^\sigma[U] \equiv D_i\Phi^i[U] \tag{1.141}$$

holds for an arbitrary functions $U(x)$. Then on solutions $U(x) = u(x)$ of the PDE system (1.23), if $\Lambda_\sigma[U]$ is non-singular, one obtains a local conservation law

$$\Lambda_\sigma[u]F^\sigma[u] = D_i\Phi^i[u] = 0. \tag{1.142}$$

Remark 1.4.1. A multiplier $\Lambda_\sigma[U]$ is *singular* if it is a singular function when evaluated on solutions $U(x) = u(x)$ of the given PDE system (1.23). In practice, one is interested in non-singular sets of multipliers, since considering singular multipliers can lead to arbitrary divergence expressions that are not conservation laws of the given system. An example of a singular multiplier is $\Lambda_\sigma[U] = D_i\Phi^i[U]/F^\sigma[U]$ yields $\Lambda_\sigma[U]F^\sigma[U] = D_i(N\Phi^i[U])$, in terms of arbitrary functions $\Phi^1[U], ..., \Phi^n[U]$.

Definition 1.4.3. The *Euler operator* with respect to u^μ, $\mu = 1, ..., m$ is given by

$$E_{u^\mu} = \frac{\partial}{\partial u^\mu} - D_i\frac{\partial}{\partial u^\mu_i} + ... + (-1)^r D_{i_1}...D_{i_r}\frac{\partial}{\partial {u^\mu}_{i_1...i_r}} + ..., \quad r \geq 1,\ \mu = 1, ..., m. \tag{1.143}$$

Theorem 1.4.1. *The identities*

$$E_{u^\mu}\left(A\left(x, u, \partial u, ..., \partial^\ell u\right)\right) \equiv 0$$

hold if and only if

$$A\left(x, u, \partial u, ..., \partial^\ell u\right) = D_i A^i(x, u, \partial u, ..., \partial^{\ell-1} u)$$

for some functions $A_i, i = 1, ..., n$.

Proof. [4] □

The following theorem shows that a given PDE system has a local conservation law if and only if there exist local multipliers such that their linear combinations with the differential equations of the given PDE system are annihilated by the Euler operator (1.143) [37].

Theorem 1.4.2. *A set of nonsingular local multipliers* $\{\Lambda_\sigma(x, U, \partial U, ..., \partial^l U)\}_{\sigma=1}^N$ *yields a local conservation law for the PDE system* (1.23) *if and only if the set of identities*

$$E_{U^\mu}(\Lambda_\sigma(x, U, \partial U, ..., \partial^l U) F^\sigma(x, U, \partial U, ..., \partial^k U)) \equiv 0, \quad \mu = 1, ..., m, \tag{1.144}$$

holds for arbitrary function $U(x)$.

Theorem 1.4.2 leads to a systematic way for the construction of local conservation laws:

- For the PDE system (1.23), define a set of conservation law multipliers up to some specified order.
- Solve the determining equations (1.144) for arbitrary $U(x)$ to find all such sets of multipliers.
- Find the corresponding fluxes satisfying

$$\Lambda_\sigma(x, U, \partial U, ..., \partial^l U) F^\sigma(x, U, \partial U, ..., \partial^k U)) \equiv D_i \Phi^i(x, U, \partial U, ..., \partial^r U). \tag{1.145}$$

- Each set of fluxes and multipliers yields a local conservation law

$$D_i \Phi^i(x, u, \partial u, ..., \partial^r u) = 0,$$

holding for all solutions $u(x)$ of the PDE system (1.23).

Example 1.4.3. Consider the KdV equation

$$F[u] = u_t + uu_x + u_{xxx} = 0. \tag{1.146}$$

Consider zeroth order multipliers, $\Lambda = \Lambda(x, t, U)$. The determining equation (1.144) for the multiplier Λ reads

$$E_U(\Lambda(x, t, U))(U_t + UU_x + U_{xxx}) = 0. \tag{1.147}$$

It follows that

$$(\Lambda_t + U\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxU}U_x + 3\Lambda_{xUU}U_x^2 + \Lambda_{UUU}U_x^3 + 3\Lambda_{xU}U_{xx} + 3\Lambda_{UU}U_xU_{xx} = 0. \tag{1.148}$$

Equation (1.148) splits into three equations

$$\Lambda_t + U\Lambda_x + \Lambda_{xxx} = 0, \quad \Lambda_{xU} = 0, \quad \Lambda_{UU} = 0,$$

with solution provides three local multipliers

$$\Lambda_1 = 1, \quad \Lambda_2 = U, \quad \Lambda_3 = tU - x, \tag{1.149}$$

where Λ_1, Λ_2 yield the conservation law for mass, momentum (1.138). The third multiplier Λ_3 yields a conservation law for center of mass motion given by

$$D_t\left(\frac{1}{2}tu^2 - xu\right) + D_x\left(-\frac{1}{2}xu^2 + tuu_{xx} - \frac{1}{2}tu_x^2 - xu_xx + u_x\right) = 0.$$

1.4.3 Noether's theorem

Consider a functional

$$\mathcal{L}[u] = \int_\Omega L[u]dx \tag{1.150}$$

defined on some domain Ω. The function $L[u]$ is called a *Lagrangian* and the functional (1.150) is called an *action integral*. A *variational problem* consists of finding the extremum of the action integral (1.150). The following theorem holds [37].

Theorem 1.4.3. *If a smooth function $U(x) = u(x)$ is an extremum of an action integral $\mathcal{L}[u]$ with $L[U] = L(x, U, \partial U, ..., \partial^k U)$, then $u(x)$ satisfies the equations*

$$E_{u^\mu} L = \frac{\partial L}{\partial u^\mu} - D_i \frac{\partial L}{\partial u^\mu_i} + ... + (-1)^r D_{i_1}...D_{i_k} \frac{\partial L}{\partial {u^\mu}_{i_1...i_k}} = 0, \quad \mu = 1, ..., m, \tag{1.151}$$

where E_{u^μ} is the Euler operator (1.143)*.*

Definition 1.4.4. Equations (1.151) are called the *Euler-Lagrange equations.*

A PDE system admits a variational principle if the PDEs of the system are precisely given by the Euler-Lagrange equations (1.151). Noether [39] considered transformations of the form

$$\begin{aligned}(x^i)^* &= x^i + a\xi^i(x, u, \partial u, ..., \partial^\ell u) + O(a^2), \quad i = 1, ..., n,\\ (u^\mu)^* &= u^\mu + a\eta^\mu(x, u, \partial u, ..., \partial^\ell u) + O(a^2), \quad \mu = 1, ..., m\end{aligned}$$

that leaves the action integral (1.150) invariant and established a direct relationship between the symmetries of the action integral and the conservation laws.

Definition 1.4.5. A Lie group of point transformations

$$\begin{aligned}(x^i)^* &= x^i + a\xi^i(x, u) + O(a^2), \quad i = 1, 2, \dots, n,\\ (u^\mu)^* &= u^\mu + a\eta^\mu(x, u) + O(a^2), \quad \mu = 1, 2, \dots, m\end{aligned} \tag{1.152}$$

is a *variational symmetry group* of the action integral (1.150) if

$$\int_{\Omega^*} L[u^*]dx^* = \int_\Omega L[u]dx,$$

where Ω^* is the image of Ω under the point transformation (1.152).

Theorem 1.4.4. *A Lie group of point transformations* (1.152) *with infinitesimal generator*

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\mu(x, u)\frac{\partial}{\partial u^\mu},$$

is a variational symmetry for the functional $\mathcal{L}[u]$ (1.150) *if and only if*

$$X^{(k)} L + L D_i \xi^i = 0. \tag{1.153}$$

Proof. [4]. □

For higher-order transformations (1.47) and the evolutionary form of (1.152) with infinitesimal generator (1.48), the Noether identity (1.153) becomes

$$\hat{X}^{(k)} L = D_i A^i$$

for some differential function $A[u] = (A^1[u], ..., A^n[u])$. Variational symmetries are also called *Noether transformations*. The following result hold [4]

Theorem 1.4.5. *If the Lie group of point transformations* (1.152) *is a variational symmetry group of the functional $\mathcal{L}[u]$, then* (1.152) *is a symmetry group of the Euler-Lagrange equations* (1.151).

The converse of the above theorem is not true, that is, not every symmetry of the Euler-Lagrange equations is a variational symmetry of the original variational problem.

Example 1.4.4. Consider the two-dimensional linear wave equation

$$u_{tt} = u_{xx} + u_{yy}, \tag{1.154}$$

which is the Euler-Lagrange equation for the functional

$$\mathcal{L}[u] = \frac{1}{2} \iiint \left(u_t^2 - u_x^2 - u_y^2\right) dtdxdy.$$

The wave equation (1.154) admits rotation and inversion symmetries

$$X_1 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}, \quad X_2 = 2yt\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (t^2 - x^2 + y^2)\frac{\partial}{\partial y} - yu\frac{\partial}{\partial u}.$$

For the rotation symmetry generator X_1, one has $X_1^{(1)} = X_1 - u_x\partial/\partial u_t - u_t\partial/\partial u_x$. Hence, the identity (1.153) reads

$$X_1^{(1)}\left(\frac{u_t^2 - u_x^2 - u_y^2}{2}\right) + D_t(x) + D_x(t) = -u_x u_t + u_t u_x = 0.$$

It follows that, X_1 is a variational symmetry for the functional $\mathcal{L}$ and hence a variational symmetry for the wave equation (1.154). For X_2, the first prolongation has the form

$$X_2^{(1)} = X_2 - (3yu_t + 2tu_y)\frac{\partial}{\partial u_t} + (2xu_y - 3yu_x)\frac{\partial}{\partial u_x} - (u + 2tu_t + 2xu_x + 3yu_y)\frac{\partial}{\partial u_y}.$$

Applying $X_2^{(1)}$ to the Lagrangian $L = \left(u_t^2 - u_x^2 - u_y^2\right)/2$, one has

$$\begin{aligned} X_2^{(1)} L &= -u_t\left(3yu_t + 2tu_y\right) - u_x\left(2xu_y - 3yu_x\right) + u_y\left(u + 2tu_t + 2xu_x + 3yu_y\right) \\ &= uu_y - 6yL. \end{aligned}$$

Equation (1.153) becomes

$$X_2^{(1)} + LD_i\xi^i = uu_y - 6yL + \left(D_t(2ty) + D_x(2xy) + D_y(t^2 - x^2 + y^2)\right) L = uu_y.$$

Consequently, X_2 is not a variational symmetry for the functional $\mathcal{L}$.

For variational differential equations, local symmetries and local conservation laws are related using the Noether's first theorem [37].

Theorem 1.4.6. *Suppose a PDE system* (1.23) *arises from a variational principle. Suppose a one-parameter Lie group of point transformations* (1.152) *is a variational symmetry for the functional* $\mathcal{L}[u]$ (1.150)*. Then the functions*

$$\zeta^\mu = \eta^\mu - u_i^\mu \xi^i$$

form a set of multipliers for the PDE system (1.23).

Example 1.4.5. The linear wave equation $F[u] = u_{tt} - c^2 u_{xx} = 0$ admits the time translation symmetry $X = \partial/\partial t$ with evolutionary symmetry component $\zeta = -u_t$. Hence, one gets a multiplier $\Lambda = \zeta = -u_t$. The corresponding conservation law is a conservation law of energy:

$$\Lambda F[u] = -u_t(u_{tt} - c^2 u_{xx}) = -D_t \left(\frac{u_t^2 + u_x^2}{2} \right) + D_x \left(c^2 u_t u_x \right) = 0.$$

Remark 1.4.2. Noether's theorem to find conservation laws is restricted to variational systems. However, the direct method for finding conservation laws is applicable to any differential equation whether or not it is variational.

2 Exact and Approximate Symmetries of Algebraic Equations and ODEs with a Small Parameter

2.1 Introduction

In Chapter 1, we have seen that under a perturbation of an ODE model, some exact point symmetries of the original system may be unstable and totally disappear from the classification of approximate symmetries of the perturbed model.

In this chapter, we follow the BGI and FS approximate symmetry frameworks to study the approximate symmetry properties of perturbed algebraic equations and ODEs. In particular, we provide the relation between exact and approximate symmetries of the original and perturbed algebraic and first-order ordinary differential equations. In summary, to every exact Lie point symmetry of an unperturbed equation, there correspond:

- an infinite set of exact Lie point symmetries of the perturbed equation,
- an infinite set of BGI and FS approximate point symmetries of the perturbed first-order ODE,
- an infinite set of BGI approximate point symmetries of the perturbed algebraic equation.

It follows that all point symmetries of algebraic systems and first-order ODEs are stable in the BGI and FS approximate symmetry senses.

By analogy with ODE systems, for higher-order ODEs, it is natural to expect that the correct framework is provided by local (including higher-order) symmetries. Indeed, we show that to every point or local symmetry of an unperturbed ODE of second or higher order, there corresponds a local BGI approximate symmetry of the perturbed ODE. We show how these higher-order approximate symmetries can be used to construct approximate solution of a perturbed Boussinesq ODE and we validate this solution by comparing it to numerical solutions of the Boussinesq equation.

We develop two approaches to construct approximate solutions for a perturbed ODE. In the first approach, we use BGI approximate point symmetries to determine the approximate integrating factors and we derive the determining equations of approximate integrating factors. We apply the approximate integrating factor to find approximate solutions for perturbed Boussinesq and BBM ODEs. The second approach consists in the approximate reduction of order of perturbed higher-order ODEs using admitted approximate contact and higher-order symmetries.

BGI and FS frameworks are different approaches which provide different approximate symmetry structures. For a class of perturbed higher-order ODEs, we show that a BGI approximate point symmetry yields a FS approximate point symmetry for the same model. Also, we find a connection between BGI and FS approximate point symmetries for a perturbed first-order ODE.

2.2 Exact and approximate point symmetries of algebraic equations

First we analyze the relationship between exact and approximate point symmetries of algebraic equations. Let $x = (x^1, ..., x^n) \in \mathbb{R}^n$, $n \geq 2$. Let $F_0(x)$ be a sufficiently smooth scalar function. An algebraic equation

$$F_0(x) = \text{const} \tag{2.1}$$

defines a family of surfaces (curves) in $\mathbb{R}^n$. A family of perturbed surfaces (curves) is given by

$$F(x; \epsilon) = F_0(x) + \epsilon F_1(x) = \text{const.} \tag{2.2}$$

2.2.1 Exact symmetries of unperturbed and perturbed algebraic equations

The exact symmetry generator of the unperturbed equation $F_0 = \text{const}$ is given by

$$X^0 = \sum_{i=1}^{n} \xi^{0i}(x) \frac{\partial}{\partial x^i}. \tag{2.3}$$

To find the infinitesimals, we apply the determining equations specifying the condition that every solution curve of (2.1) is mapped into a solution curve of (2.1):

$$X^0 F_0(x) = \sum_{i=1}^{n} \xi^{0i}(x) \frac{\partial F_0}{\partial x^i} \equiv 0. \tag{2.4}$$

Assuming without loss of generality that $\partial F_0 \big/ \partial x^1 \neq 0$, one can solve for

$$\xi^{01} = -\sum_{i=2}^{n} \xi^{0i}(x) \frac{\partial F_0}{\partial x^i} \bigg/ \frac{\partial F_0}{\partial x^1}, \tag{2.5}$$

keeping $\xi^{02}(x), ..., \xi^{0n}(x)$ arbitrary functions that parameterise an infinite-parameter Lie algebra of point symmetries of the family of surfaces (2.1). In the same fashion, an exact symmetry generator of the family of perturbed equations (2.2) is given by

$$Y = \sum_{i=1}^{n} \eta^i(x; \epsilon) \frac{\partial}{\partial x^i}. \tag{2.6}$$

Applying the determining equations to find exact symmetries of the perturbed equations (2.2), one has

$$YF(x; \epsilon) = \sum_{i=1}^{n} \eta^i(x; \epsilon) \left(\frac{\partial F_0}{\partial x^i} + \epsilon \frac{\partial F_1}{\partial x^i} \right) \equiv 0. \tag{2.7}$$

If at least one of the functions $\partial F_0/\partial x^1$, $\partial F_1/\partial x^1$ is nonzero, one can write

$$\eta^1(x;\epsilon) = -\sum_{i=2}^{n} \eta^i \left(\frac{\partial F_0}{\partial x^i} + \epsilon \frac{\partial F_1}{\partial x^i}\right) \Big/ \left(\frac{\partial F_0}{\partial x^1} + \epsilon \frac{\partial F_1}{\partial x^1}\right) \tag{2.8}$$

in terms of arbitrary functions $\eta^2(x;\epsilon)$, …, $\eta^n(x;\epsilon)$ that define the infinite-parameter symmetry generator (2.6). From the comparison of (2.5) and (2.8), the following simple theorem is established.

Theorem 2.2.1. *Suppose that the unperturbed algebraic equation* (2.1) *admits a point symmetry with infinitesimal generator* (2.3). *Then there exists a point symmetry generator* (2.6) *of the perturbed equation* (2.2) *such that* $Y \equiv X^0$ *when* $\epsilon = 0$.

Indeed, one can take $\eta^i(x;\epsilon) = \xi^{0i}$, $i = 1, ..., n$; then $\eta^1(x;\epsilon)$ (2.8) matches ξ^{01} (2.5) when $\epsilon = 0$. It follows that all exact symmetries of the unperturbed equation (2.1) carry over to the perturbed family (2.2).

2.2.2 BGI approximate symmetries of a perturbed algebraic equation

Let

$$X = X^0 + \epsilon X^1 = \sum_{i=1}^{n} \xi^{0i}(x) \frac{\partial}{\partial x^i} + \epsilon \sum_{i=1}^{n} \xi^{1i}(x) \frac{\partial}{\partial x^i} \tag{2.9}$$

be an BGI approximate point symmetry generator admitted by the family of perturbed surfaces (2.2), where X^0 is the exact symmetry generator of the unperturbed equations (2.1). Applying the determining equation

$$\left(X^0 + \epsilon X^1\right)(F_0 + \epsilon F_1) = o(\epsilon),$$

we find that the infinitesimals ξ^{1i} satisfy

$$\sum_{i=1}^{n} \xi^{1i}(x) \frac{\partial F_0}{\partial x^i} = -\sum_{i=1}^{n} \xi^{0i}(x) \frac{\partial F_1}{\partial x^i}. \tag{2.10}$$

As in equation (2.5), if $\partial F_0/\partial x_1 \neq 0$, one can solve for

$$\xi^{11} = -\left(\sum_{i=2}^{n} \xi^{1i}(x) \frac{\partial F_0}{\partial x^i} + \sum_{i=1}^{n} \xi^{0i}(x) \frac{\partial F_1}{\partial x^i}\right) \Big/ \frac{\partial F_0}{\partial x^1}, \tag{2.11}$$

where the infinitesimals $\xi^{12}(x)$, …, $\xi^{1n}(x)$ are arbitrary functions. The family of perturbed equations (2.2) consequently admits an infinite-parameter approximate symmetry generator

$$\begin{aligned} X &= \xi^{01} \frac{\partial}{\partial x^1} + \sum_{i=2}^{n} \xi^{0i}(x) \frac{\partial}{\partial x^i} + \epsilon \left(\xi^{11} \frac{\partial}{\partial x^1} + \sum_{i=2}^{n} \xi^{1i}(x) \frac{\partial}{\partial x^i}\right) \\ &= \left(\frac{\epsilon \xi^{01} \frac{\partial F_1}{\partial x^1} + \sum_{i=2}^{n} \xi^{0i} \left(\frac{\partial F_0}{\partial x^i} + \epsilon \frac{\partial F_1}{\partial x^i}\right) + \epsilon \xi^{1i} \frac{\partial F_0}{\partial x^i}}{-\partial F_0/\partial x^1}\right) \frac{\partial}{\partial x^1} + \sum_{i=2}^{n} \left(\xi^{0i} + \epsilon \xi^{1i}\right) \frac{\partial}{\partial x^i}. \end{aligned} \tag{2.12}$$

The following theorem holds.

Theorem 2.2.2. *For each exact symmetry generator* (2.3) *of the unperturbed algebraic equations* (2.1)*, there is a corresponding first-order deformation* X^1 *such that* (2.9) *is an approximate BGI symmetry generator of the family of perturbed equations* (2.2)*.*

It follows that every exact point symmetry of the unperturbed algebraic equation (2.1) is stable, that is, its generator X_0 is the $O(1)$ part of some approximate symmetry generator (2.9) of the perturbed equation (2.2). Moreover, due to the presence of additional arbitrary functions ξ^{1i}, $i = 2, \ldots, n$, the approximate symmetry generator (2.12) of the family of perturbed equations (2.2) is more general than the exact symmetry generator (2.6) of the same. We now consider a simple example in detail.

Example 2.2.1. Consider a family of circles in polar coordinates

$$F_0(r, \theta) = r = \text{const}, \tag{2.13}$$

and a family of perturbed circles

$$F(r, \theta) = r + \epsilon e^{-k\theta} = C = \text{const}, \tag{2.14}$$

where $k > 0$ is a fixed constant. Let

$$X^0 = \xi^0(r, \theta)\frac{\partial}{\partial r} + \eta^0(r, \theta)\frac{\partial}{\partial \theta}$$

be the symmetry generator of the family of equations (2.13). Using the determining equations (2.4), one gets $\xi^0 \equiv 0$, $\eta^0 = \eta^0(r, \theta)$. Consequently, all symmetries of the family of circles (2.13) are given by

$$X^0 = \eta^0(r, \theta)\frac{\partial}{\partial \theta}. \tag{2.15}$$

For example, if $\eta^0 = r$, the corresponding global transformation is the one-parameter (a) Lie group

$$r^* = r, \quad \theta^* = \theta + ar. \tag{2.16}$$

The equations (2.16) transforms circles to circles and lines to spirals as shown in Figure 2.1.

For the perturbed circles (2.14), the exact symmetry generator can be sought in the form

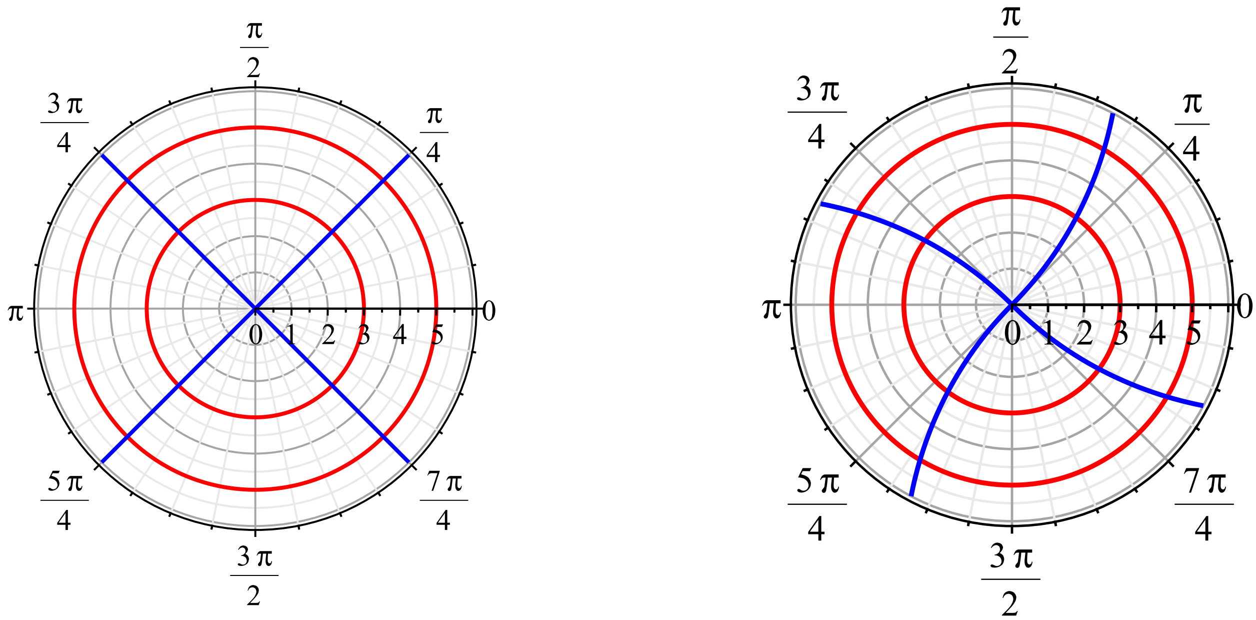
$$Y = \eta^1(r, \theta)\frac{\partial}{\partial r} + \eta^2(r, \theta)\frac{\partial}{\partial \theta}. \tag{2.17}$$

Using the formula (2.8), one gets $\eta^1 = \epsilon k e^{-k\theta}\eta^2$, where $\eta^2(r, \theta)$ is an arbitrary function. Take, for example, $\eta^2 = \eta^0 = r$. Then the perturbed equation (2.14) admits the Lie group of transformations:

$$\begin{aligned} r^* &= r^*(r, \theta; a, \epsilon) = r + a\epsilon k r e^{-k\theta} + o(a), \\ \theta^* &= \theta^*(r, \theta; a, \epsilon) = \theta + ar + o(a), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} \frac{dr^*}{da} &= \epsilon k r^* e^{-k\theta^*}, \quad r^*\Big|_{a=0} = r, \\ \frac{d\theta^*}{da} &= r^*, \quad \theta^*\Big|_{a=0} = \theta. \end{aligned} \tag{2.19}$$



(a) **(b)**

Figure 2.1: The family of circles (2.13) (a) and their shape under the transformations (2.16) for $a = 0.03$ (b). The radial lines are shown in blue for reference.

The above system is equivalent to

$$\frac{d^2\theta^*}{da^2} = \epsilon k \frac{d\theta^*}{da} e^{-k\theta^*}, \tag{2.20}$$

which has a solution

$$\theta^* = \frac{1}{k} \ln \left[\frac{kre^{k\theta + a[kr + \epsilon e^{-k\theta}]} + \epsilon}{kr + \epsilon e^{-k\theta}} \right]. \tag{2.21}$$

Using (2.19), r^* takes the form

$$r^* = \frac{[kr + \epsilon e^{-k\theta}] r e^{k\theta + a[kr + \epsilon e^{-k\theta}]}}{kre^{k\theta + a[kr + \epsilon e^{-k\theta}]} + \epsilon}. \tag{2.22}$$

Figure 2.2a shows the perturbation to the family of circles (2.13) caused by a small parameter (ϵ). Under the transformations (2.18), the perturbed circles (2.14) are rotated counter-clockwise as shown in Figure 2.2b. The action of the exact transformations (2.21) and (2.22) on the perturbed circles (2.14) is shown in Figure 2.2c.

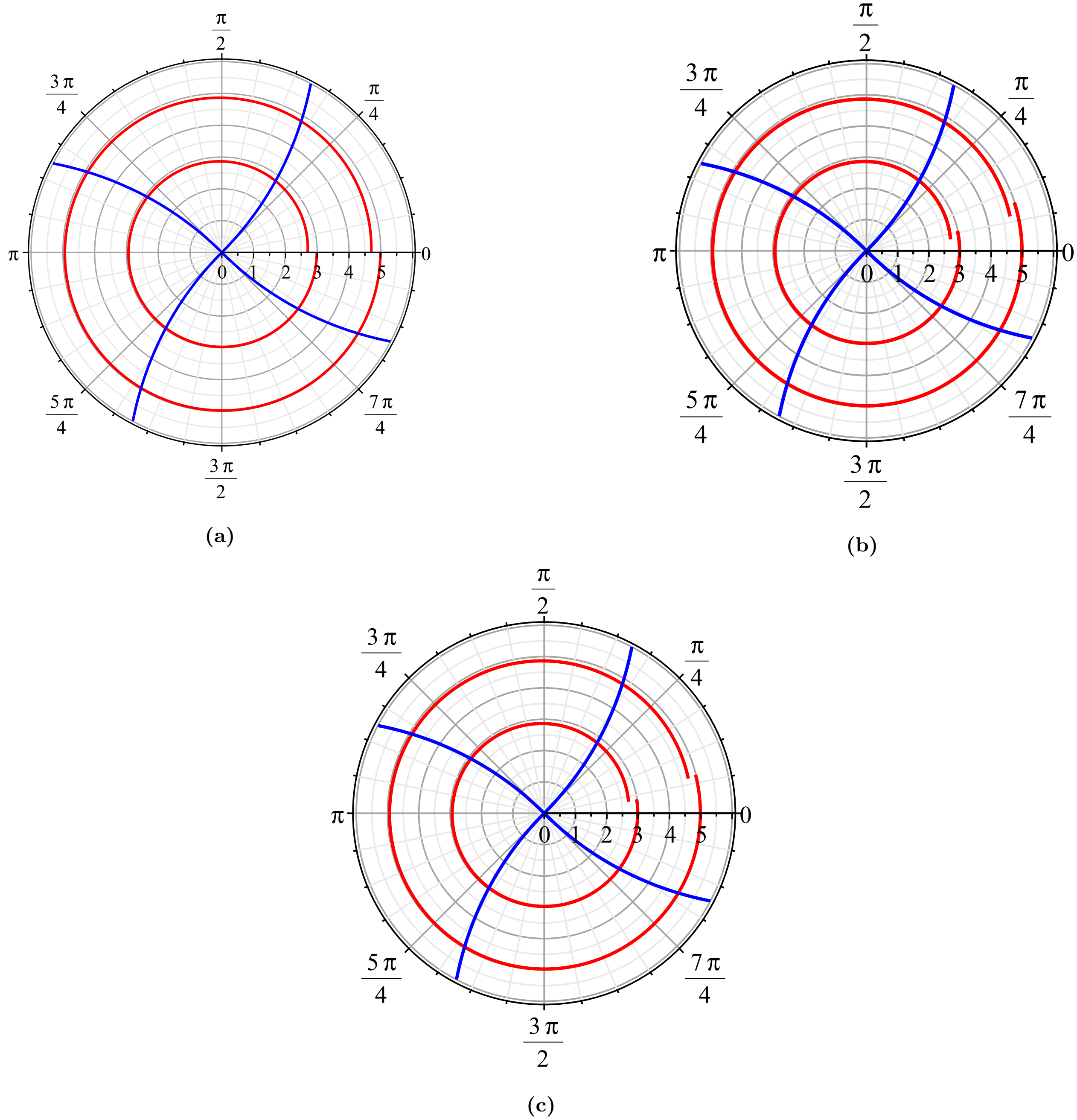


Figure 2.2: Family of perturbed circles (2.14) (a) and their graphs under the transformations (2.18)(b) and under the transformations (2.21), (2.22) for $a = 0.03$, $k = 0.5$, $\epsilon = 0.3$ (c).

The approximate symmetry generator for the perturbed equation (2.14) has the form

$$X = X^0 + \epsilon X^1 = \left(\xi^0(r,\theta) + \epsilon \xi^1(r,\theta)\right) \frac{\partial}{\partial r} + \left(\eta^0(r,\theta) + \epsilon \eta^1(r,\theta)\right) \frac{\partial}{\partial \theta}, \tag{2.23}$$

where X^0 is the exact symmetry generator given by equation (2.15). From the equation (2.11), one finds

$\xi^1 = ke^{-k\theta}\eta^0$, $\eta^1 = \eta^1(r,\theta)$. Thus, the approximate symmetry generator (2.23) becomes

$$X = \eta^0 \frac{\partial}{\partial\theta} + \epsilon \left(ke^{-k\theta}\eta^0 \frac{\partial}{\partial r} + \eta^1(r,\theta)\frac{\partial}{\partial\theta} \right). \tag{2.24}$$

The term $\eta^0\partial/\partial\theta$ in (2.24) corresponds to an exact symmetry of the unperturbed equation (2.13). It follows that the exact symmetry generator (2.15) of (2.13) is *stable*. By taking $\eta^0 = r$, the perturbed circles (2.14) admit Lie group of approximate transformations given by

$$r^* = r + a\epsilon kre^{-k\theta} + o(a), \quad \theta^* = \theta + ar + a\epsilon\eta^1 + o(a), \tag{2.25}$$

which coincides with (2.18) when $\eta^1 = 0$.

In Figure 2.3, action of (2.25) on the perturbed circles (2.14) is shown when $\eta^1 = r \neq 0$. If $\eta^1 = 0$, Figure 2.3 would coincide with Figure 2.2b.

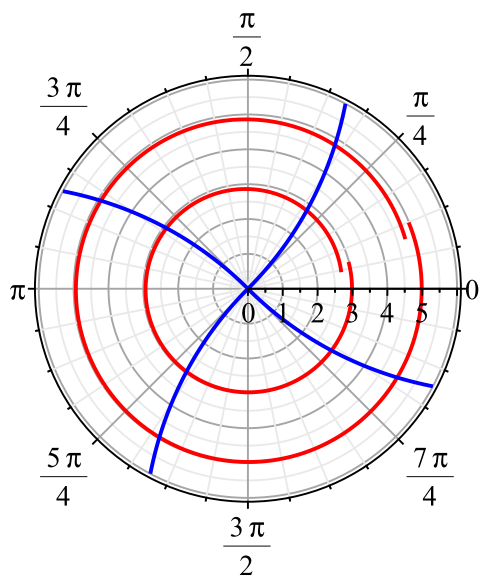


Figure 2.3: Perturbed circles under the transformation (2.25) for $a = 0.03$, $k = 0.5, \epsilon = 0.3$ and $\eta^1 = r$.

Summarizing the above results, the following statement has been established.

Proposition 2.2.1. *For any point symmetry* X^0 (2.3) *of an algebraic equation* (2.1)*, there exists a corresponding point symmetry* Y (2.6) *of the perturbed equation* (2.2)*. Moreover, in the BGI framework, any point symmetry* X^0 (2.3) *of* (2.1) *is stable; there always exists an approximate symmetry* X (2.9) *of the perturbed equation* (2.2) *corresponding to* X^0*.*

2.3 Exact and approximate point symmetries of first-order ODEs

We now analyze and compare the structures of exact point symmetries of perturbed and unperturbed first-order ODEs, and approximate point symmetries of perturbed ODE models. Let

$$y' = f_0(x, y) \tag{2.26}$$

denote a first-order ODE, and let

$$y' = f_0(x, y) + \epsilon f_1(x, y) + o(\epsilon) \tag{2.27}$$

be its perturbation.

2.3.1 Exact symmetries of an unperturbed first-order ODE

Let X^0 be an exact symmetry generator admitted by (2.26):

$$X^0 = \xi^0(x, y)\frac{\partial}{\partial x} + \eta^0(x, y)\frac{\partial}{\partial y}. \tag{2.28}$$

To find exact point symmetries of (2.26), one prolongs the exact symmetry generator X^0 to the first order:

$$X^{0^{(1)}} = \xi^0(x, y)\frac{\partial}{\partial x} + \eta^0(x, y)\frac{\partial}{\partial y} + \eta^{0^{(1)}}(x, y, y')\frac{\partial}{\partial y'}, \tag{2.29}$$

where $\eta^{0^{(1)}}(x, y, y')$ is given by

$$\eta^{0^{(1)}} = \eta^0_x + (\eta^0_y - \xi^0_x)y' - \xi^0_y y'^2. \tag{2.30}$$

Applying the determining equation (1.30) to find the exact symmetries of (2.26)

$$X^{0^{(1)}}(y' - f_0(x, y))\Big|_{y'=f_0(x,y)} = 0,$$

one obtains the following linear homogeneous first-order PDE:

$$\eta^0_x + \eta^0_y f_0 - \eta^0 f_{0_y} - \xi^0 f_{0_x} - \xi^0_x f_0 - \xi^0_y f_0^2 = 0 \tag{2.31}$$

for two unknown functions $\xi^0(x, y)$ and $\eta^0(x, y)$. Taking, for example, $\xi^0(x, y)$ as an arbitrary function, one can find $\eta^0(x, y)$ from the characteristic system

$$\frac{dx}{1} = \frac{dy}{f_0} = -\frac{d\eta^0}{\eta^0 f_{0_y} + \xi^0 f_{0_x} + \xi^0_x f_0 + \xi^0_y f_0^2}. \tag{2.32}$$

Note that the solution of the first characteristic equation is the solution of the differential equation (2.26) itself. It follows that for any ξ^0, one can find multiple η^0 so that (2.28) is a symmetry of (2.26). In particular, for an arbitrary $\xi^0 = \xi^0(x, y)$, it is well known that the choice $\eta^0(x, y) = \xi^0(x, y)f_0(x, y)$ yields a point symmetry of (2.26).

Example 2.3.1. Consider a first order ODE (its perturbed version will be used below)

$$y' = x. \tag{2.33}$$

For example, for $\xi^0(x, y) = y$, the PDE (2.31) becomes

$$\eta^0_x + x\eta^0_y = y + x^2.$$

First, take $\eta^0 = \xi^0 f_0 = xy$. Then

$$X^0_1 = x\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{2.34}$$

is a point symmetry for the ODE (2.33). More generally, using the method of characteristics, we have

$$\frac{dx}{1} = \frac{dy}{x} = \frac{d\eta^0}{y + x^2}, \tag{2.35}$$

which has a solution

$$\eta^0(x, y) = xy + A\left(y - \frac{x^2}{2}\right), \tag{2.36}$$

where A is an arbitrary function of its argument. The symmetry generator corresponding to the choice $\xi^0 = y$ is given by

$$X^0_2 = y\frac{\partial}{\partial x} + \left(xy + A\left(y - \frac{x^2}{2}\right)\right)\frac{\partial}{\partial y}. \tag{2.37}$$

2.3.2 Exact symmetries of a perturbed first-order ODE

Let

$$Y = \xi(x, y; \epsilon)\frac{\partial}{\partial x} + \eta(x, y; \epsilon)\frac{\partial}{\partial y} \tag{2.38}$$

be an exact symmetry generator of the perturbed equation (2.27). The prolongation of the symmetry generator Y is given by

$$Y^{(1)} = Y + \eta^{(1)}(x, y, y'; \epsilon)\frac{\partial}{\partial y'},$$

where

$$\eta^{(1)} = \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2.$$

The determining equation (1.30)

$$Y^{(1)}(y' - f_0 - \epsilon f_1)\Big|_{y'=f_0+\epsilon f_1} = 0$$

yields the following PDE

$$\eta_x + (f_0 + \epsilon f_1)\eta_y - \eta(f_{0_y} + \epsilon f_{1_y}) - (f_{0_x} + \epsilon f_{1_x})\xi - (f_0 + \epsilon f_1)\xi_x - (f_0 + \epsilon f_1)^2\xi_y = 0. \tag{2.39}$$

Again, for an arbitrary $\xi(x, y)$, one can obtain the following characteristic system to solve for $\eta(x, y; \epsilon)$:

$$\frac{dx}{1} = \frac{dy}{f_0 + \epsilon f_1} = -\frac{d\eta}{\eta(f_{0_y} + \epsilon f_{1_y}) + (f_{0_x} + \epsilon f_{1_x})\xi + (f_0 + \epsilon f_1)\xi_x + (f_0 + \epsilon f_1)^2\xi_y}. \tag{2.40}$$

If $\xi = \xi(x, y; \epsilon)$ depends on ϵ analytically, then so does $\eta = \eta(x, y; \epsilon)$; when $\epsilon = 0$, the symmetry (2.38) of the perturbed ODE (2.27) reduces to the exact point symmetry (2.28) of the unperturbed ODE (2.26). In particular, note that $\eta(x, y; \epsilon) = \xi(x, y; \epsilon)(f_0(x, y) + \epsilon f_1(x, y))$ solves the PDE (2.39), where ξ is an arbitrary function. Hence, by taking an arbitrary ξ with $\xi(x, y; 0) = \xi^0$, one obtains η with $\eta(x, y; 0) = \eta^0$.

Example 2.3.2. Consider a perturbed version of the ODE (2.33):

$$y' = x + \epsilon y. \tag{2.41}$$

From Example 2.3.1,

$$X^0 = y\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{2.42}$$

is an exact symmetry generator for the unperturbed ODE (2.33). By taking $\xi(x, y; \epsilon) = y + \epsilon x$, one gets $\eta(x, y; \epsilon) = xy + \epsilon(x^2 + y^2) + \epsilon^2 xy$. Therefore

$$Y = (y + \epsilon x)\frac{\partial}{\partial x} + (xy + \epsilon(x^2 + y^2) + \epsilon^2 xy)\frac{\partial}{\partial y} \tag{2.43}$$

is an exact symmetry generator for the perturbed ODE (2.41). When $\epsilon = 0$, the symmetry (2.43) of the perturbed ODE (2.41) reduces to the point symmetry (2.42) of the unperturbed ODE (2.33).

2.3.3 BGI approximate symmetries of a perturbed first-order ODE

The BGI approximate symmetry generator for the perturbed ODE (2.27) has the form

$$X = X^0 + \epsilon X^1 = \left(\xi^0(x, y) + \epsilon\xi^1(x, y)\right)\frac{\partial}{\partial x} + \left(\eta^0(x, y) + \epsilon\eta^1(x, y)\right)\frac{\partial}{\partial y}, \tag{2.44}$$

and its prolongation is given by

$$X^{(1)} = X^{0^{(1)}} + \epsilon X^{1^{(1)}} = X^{0^{(1)}} + \epsilon(X^1 + \eta^{1^{(1)}}(x, y, y')\frac{\partial}{\partial y'}), \tag{2.45}$$

where

$$\eta^{1^{(1)}} = \eta^1_x + (\eta^1_y - \xi^1_x)y' - \xi^1_y {y'}^2.$$

Applying the determining equation (1.96) for the approximate symmetries of (2.27), one obtains

$$\eta^1_x + (\eta^1_y - \xi^1_x)f_0 - \xi^1_y {f_0}^2 - \xi^1 f_{0_x} - \eta^1 f_{0_y} = (\xi^0_x - \eta^0_y)f_1 + 2\xi^0_y f_0 f_1 + \xi^0 f_{1_x} + \eta^0 f_{1_y}. \tag{2.46}$$

The above equation (2.46) is a linear nonhomogeneous PDE in two unknowns ξ^1 and η^1. To solve the PDE (2.46), one can pick an arbitrary function ξ^1, and solve the resulting PDE for η^1. In particular, (ξ^1, η^1) with $\eta^1 = \xi^0 f_1 + \xi^1 f_0$ and ξ^1 an arbitrary function are approximate symmetry components corresponding to the exact symmetry generator (2.28) with $\eta^0 = \xi^0 f_0$. The following theorem holds.

Theorem 2.3.1. *All exact symmetries of the unperturbed ODE* (2.26) *are stable in BGI sense.*

Remark 2.3.1. If the components (ξ, η) of the exact symmetry generator Y (2.38) are analytic in ϵ, then the approximate symmetry generator X (2.44) is contained as the first two terms of the Taylor expansion of Y in ϵ.

Example 2.3.3. Consider the perturbed ODE (2.41). For the exact symmetry generator (2.42) of the unperturbed ODE (2.33), one can find the approximate symmetry components by first taking an arbitrary value for ξ^1, say $\xi^1(x, y) = x$. Then $\eta^1(x, y)$ has the form $\eta^1 = \xi^0 f_1 + \xi^1 f_0 = x^2 + y^2$. Hence

$$X = (y + \epsilon x)\frac{\partial}{\partial x} + \left(xy + \epsilon(x^2 + y^2)\right)\frac{\partial}{\partial y} \tag{2.47}$$

is an approximate symmetry for (2.41). Note that by this choice of ξ^1, X (2.47) is contained in the exact symmetry generator (2.43).

The following statement summarizes the above results.

Proposition 2.3.1. *In the BGI framework, all point symmetries of the unperturbed ODE* (2.26) *are stable. If the components of the exact symmetry generator Y* (2.38) *of the perturbed first-order ODE* (2.27) *are analytic in ϵ, then the approximate symmetry generator X* (2.44) *is contained in the Taylor expansion of Y in terms of ϵ. Furthermore, when $\epsilon = 0$, Y reduces to the exact symmetry generator X^0* (2.28) *of the unperturbed first-order ODE* (2.26).

2.3.4 FS approximate symmetries of a perturbed first-order ODE

Substituting $y(x) = v(x) + \epsilon w(x)$ into the perturbed first-order ODE (2.27), then equating to zero the coefficients of zeroth and first order of ϵ, one obtains a FS system of ODEs:

$$\begin{aligned} v' &= f_0(x, v), \\ w' &= f_1(x, v) + w f_{0_v}. \end{aligned} \tag{2.48}$$

Let

$$Z = \lambda(x, v, w)\frac{\partial}{\partial x} + \phi^1(x, v, w)\frac{\partial}{\partial v} + \phi^2(x, v, w)\frac{\partial}{\partial w}.$$

be the exact symmetry generator admitted by the system (2.48). The first prolongation of Z is given by

$$Z^{(1)} = Z + \phi^{1(1)}(x, v, w, v', w')\frac{\partial}{\partial v'} + \phi^{2(1)}(x, v, w, v', w')\frac{\partial}{\partial w'},$$

where the extended infinitesimals are computed using (1.28):

$$\phi^{1(1)} = \phi^1_x + v'\phi^1_v + w'\phi^1_w - v'\left(\lambda_x + v'\lambda_v + w'\lambda_w\right),$$

$$\phi^{2(1)} = \phi^2_x + v'\phi^2_v + w'\phi^2_w - w'\left(\lambda_x + v'\lambda_v + w'\lambda_w\right).$$

Using the determining equations (1.30), the system of symmetry determining equations for the system of ODEs (2.48) is given by

$$\phi^1_x + \phi^1_v f_0 - \phi^1 f_{0_v} + (f_1 + w f_{0_v})\phi^1_w - \lambda f_{0_x} - \lambda_x f_0 - \lambda_v f_0^2 - f_0(f_1 + w f_{0_v})\lambda_w = 0, \tag{2.49a}$$

$$\phi^2_x+\phi^2_v f_0+(f_1+wf_{0_v})\phi^2_w-\phi^2 f_{0_v} = (f_1+wf_{0_v})\lambda_x+f_0(f_1+wf_{0_v})\lambda_v+\lambda f_{1_x}+\phi^1 f_{1_v}+w\lambda f_{0_{xv}}+w\phi^1 f_{0_{vv}}. \quad (2.49b)$$

The first determining equation (2.49a) is a linear homogeneous PDE in λ and ϕ^1. By taking, for example, an arbitrary value for λ, one can solve the resulting PDE for ϕ^1. After substituting these values into the second determining equation (2.49b), one can find ϕ^2 by solving the obtained PDE.

Remark 2.3.2. Since the first equation of the system (2.48) is equivalent to the unperturbed equation (2.26), one can take $\lambda = \xi^0(x,v)$, $\phi^1 = \eta^0(x,v)$, where ξ^0, η^0 are the unperturbed symmetry components for the unperturbed first-order ODE (2.26). The second determining equation (2.49b) is a first-order PDE in ϕ^2 with no restrictions on λ and ϕ^1. The following statement is established.

Theorem 2.3.2. *All point symmetries of the unperturbed ODE* (2.26) *are stable in FS sense.*

The *stability* of the exact symmetries of the unperturbed first-order ODE (2.26) in sense of both proposed methods allows us to find a relation between the BGI and FS approximate symmetries of the perturbed first-order ODE.

Theorem 2.3.3. *If*

$$X = \left(\xi^0(x,y)+\epsilon\xi^1(x,y)\right)\frac{\partial}{\partial x}+\left(\eta^0(x,y)+\epsilon\eta^1(x,y)\right)\frac{\partial}{\partial y}$$

is a BGI approximate symmetry for the perturbed first-order ODE (2.27)*. Then*

$$Z = \xi^0(x,v)\frac{\partial}{\partial x}+\eta^0(x,v)\frac{\partial}{\partial v}+(\eta^1(x,v)+R(x,v,w))\frac{\partial}{\partial w}.$$

is an exact symmetry for the system (2.48) *and hence a FS approximate symmetry for* (2.27)*, where R is a solution for the first-order PDE*

$$\begin{aligned} R_x + f_0 R_v + (f_1+wf_{0_v})R_w - f_{0_v}R = {} & w(f_{0_v}\xi^0_x + f_0 f_{0_v}\xi^0_v + \xi^0 f_{0_{xv}} + \eta^0 f_{0_{vv}}) \\ & + f_1\eta^0_v - f_0 f_1 \xi^0_v - \xi^1_x f_0 - \xi^1_v {f_0}^2 - \xi^1 f_{0_x}. \end{aligned} \quad (2.50)$$

Indeed, as noted in Remark (2.3.2), the infinitesimals ξ^0, η^0 satisfy the first determining equation (2.49a). By substituting $\phi^2 = \eta^1(x,v)+R(x,v,w)$ into the second determining equation (2.49b), one gets the PDE (2.50).

Example 2.3.4. Consider the first-order ODE

$$y' = y+\epsilon x. \quad (2.51)$$

Substituting $y(x) = v(x)+\epsilon w(x)$ into the ODE (2.51) leads to the FS system

$$\begin{aligned} v' &= v, \\ w' &= x+w. \end{aligned} \quad (2.52)$$

The ODE (2.51) admits a BGI approximate symmetry given by

$$X = (y + \epsilon\,(x+1))\,\frac{\partial}{\partial y}. \tag{2.53}$$

With $f_0 = y$, $f_1 = x$, $\xi^0 = \xi^1 = 0$, $\eta^0 = y$ and $\eta^1 = x+1$, the PDE (2.50) reduces to

$$R_x + vR_v + (x+w)R_w - R = x,$$

which has a particular solution $R(x, v, w) = w$. Hence

$$Z = v\frac{\partial}{\partial v} + (w + x + 1)\frac{\partial}{\partial w} \tag{2.54}$$

is an exact symmetry for the system (2.52) and hence an approximate FS symmetry for (2.51). An approximate particular solution for the perturbed ODE (2.51) under (2.53) is given by

$$y(x;\epsilon) = e^x + \epsilon\,(e^x - x - 1)\,. \tag{2.55}$$

Now, to find a solution for the system (2.52) under (2.54), one uses the characteristic system

$$\frac{dx}{0} = \frac{dv}{v} = \frac{dw}{w + x + 1}. \tag{2.56}$$

Hence, one gets an invariant $\alpha = x$, and the second characteristic equation can be written as an ODE for $w(v)$

$$\frac{dw}{dv} = \frac{w + x + 1}{v},$$

which has a solution

$$w = v - x - 1. \tag{2.57}$$

$v(x) = e^x$ solves the first equation of the system (2.52). And hence $w = e^x - x - 1$ is a solution for the system (2.52) under the exact symmetry (2.54). Using these values for v and w, the approximate solution of the ODE (2.51) is given by

$$y(x;\epsilon) = v + \epsilon w = e^x + \epsilon(e^x - x - 1),$$

which it is the same as the approximate solution (2.55) under the BGI approximate symmetry (2.53).

2.4 Exact and approximate point symmetries of higher-order ODEs

Here we discuss the BGI and FS approximate symmetries of a perturbed higher-order ODE, and the stability of the exact point symmetries of the unperturbed model. Consider the unperturbed higher-order ODE

$$y^{(n)} = f_0(x, y, y', ..., y^{(n-1)}), \quad n \geq 2, \tag{2.58}$$

and its perturbed version

$$y^{(n)} = f_0(x, y, y', ..., y^{(n-1)}) + \epsilon f_1(x, y, y', ..., y^{(n-1)}) + o(\epsilon). \tag{2.59}$$

2.4.1 Exact point symmetries of an unperturbed higher-order ODE

The exact symmetry generator for the unperturbed ODE (2.58) has the form

$$X^0 = \xi^0(x,y)\frac{\partial}{\partial x} + \eta^0(x,y)\frac{\partial}{\partial y}, \tag{2.60}$$

and the n^{th} prolongation of this operator is given by

$$X^{0^{(n)}} = \xi^0(x,y)\frac{\partial}{\partial x} + \eta^0(x,y)\frac{\partial}{\partial y} + \eta^{0^{(1)}}(x,y,y')\frac{\partial}{\partial y'} + ... + \eta^{0^{(n)}}(x,y,y',...,y^{(n)})\frac{\partial}{\partial y^{(n)}},$$

where the extended infinitesimals $\eta^{0^{(k)}}$ satisfy the recursion formula

$$\eta^{0^{(k)}} = D\eta^{0^{(k-1)}} - y^{(k)}D\xi^0,\ k \geq 1, \tag{2.61}$$

where $\eta^{0^{(0)}} = \eta^0$, and D is the total derivative operator (1.22) given by

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + ... + y^{(n+1)}\frac{\partial}{\partial y^{(n)}}. \tag{2.62}$$

The determining equations for exact symmetries of the unperturbed ODE (2.58)

$$X^{0^{(n)}}(y^{(n)} - f_0)\Big|_{y^{(n)}=f_0} = 0 \tag{2.63}$$

yield

$$\eta^{0^{(n)}}\Big|_{y^{(n)}=f_0} - \sum_{k=1}^{n-1}\eta^{0^{(k)}}\frac{\partial f_0}{\partial y^{(k)}} - \xi^0 f_{0_x} - \eta^0 f_{0_y} = 0, \tag{2.64}$$

which is a linear PDE system on (ξ^0, η^0). A sample point symmetry computation for $y'' = 0$ (1.86) is presented in Example 1.3.1.

2.4.2 BGI approximate point symmetries of a perturbed higher-order ODE

A perturbed ODE (2.59) generally has fewer exact point and local symmetries than the unperturbed ODE (2.58). Example 1.3.1 for the ODE $y'' = \epsilon(y')^{-1}$ illustrates this trend.

The BGI approximate symmetry generator of the perturbed ODE (2.59) is given by

$$X = X^0 + \epsilon X^1 = \left(\xi^0(x,y) + \epsilon\xi^1(x,y)\right)\frac{\partial}{\partial x} + \left(\eta^0(x,y) + \epsilon\eta^1(x,y)\right)\frac{\partial}{\partial y}, \tag{2.65}$$

and its prolongation is given by

$$X^{(n)} = \left(\xi^0 + \epsilon\xi^1\right)\frac{\partial}{\partial x} + \left(\eta^0 + \epsilon\eta^1\right)\frac{\partial}{\partial y} + \left(\eta^{0^{(1)}} + \epsilon\eta^{1^{(1)}}\right)\frac{\partial}{\partial y'} + ... + \left(\eta^{0^{(n)}} + \epsilon\eta^{1^{(n)}}\right)\frac{\partial}{\partial y^{(n)}}. \tag{2.66}$$

To find the approximate symmetries of the perturbed ODE (2.59), we apply the approximate invariance condition (1.96). In the zeroth order in ϵ, they are the same as (2.64) for exact point symmetries of the unperturbed ODE (2.58). At the first order in ϵ, one has

$$X^{1^{(n)}}(y^{(n)} - f_0)\Big|_{y^{(n)}=f_0} = -\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\left[X^{0^{(n)}}(y^{(n)} - f_0 - \epsilon f_1)\Big|_{y^{(n)}=f_0+\epsilon f_1}\right], \tag{2.67}$$

equivalent to

$$\left(\eta^{1^{(n)}} - \sum_{k=1}^{n-1} \eta^{1^{(k)}} \frac{\partial f_0}{\partial y^{(k)}}\right)\Big|_{y^{(n)}=f_0} - \xi^1 f_{0_x} - \eta^1 f_{0_y} =$$
$$-\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \left[\left(\eta^{0^{(n)}}\Big|_{y^{(n)}=f_0+\epsilon f_1} - \sum_{k=1}^{n-1} \eta^{0^{(k)}} \left(\frac{\partial f_0}{\partial y^{(k)}} + \epsilon \frac{\partial f_1}{\partial y^{(k)}}\right) - \xi^0(f_{0_x} + \epsilon f_{1_x}) - \eta^0(f_{0_y} + \epsilon f_{1_y})\right)\right]. \quad (2.68)$$

To find the derivative in the right-hand side of (2.68), we need first to find the terms of $\eta^{0^{(n)}}$ that contains $y^{(n)}$. Note that $\eta^{0^{(n)}}$ is linear in $y^{(n)}$, and satisfies the equation

$$\eta^{0^{(n)}} = D^n \eta^0 - \sum_{j=0}^{n-1} \binom{n}{j} D^j y' D^{n-j} \xi^0. \quad (2.69)$$

The first-term of (2.69) reads

$$D^n \eta^0 = D^{n-1}\left(\eta^0_x + y'\eta^0_y\right) = D^{n-2}\left(\eta^0_{xx} + 2y'\eta^0_{xy} + y'^2\eta^0_{yy} + y''\eta^0_y\right)$$
$$= \ldots = y^{(n)}\eta^0_y + R(x, y, y', ..., y^{(n-1)}).$$

The only terms of the sum in the equation (2.69) that yield $y^{(n)}$ are the terms corresponding to $j = 0$ and $j = n-1$. When $j = 0$, the term is $y'y^{(n)}\xi^0_y$. For $j = n-1$, the term is $n(\xi^0_x + y'\xi^0_y)y^{(n)}$. Hence the following statement is established.

Theorem 2.4.1. *The determining equation for the BGI approximate symmetries of the perturbed ODE* (2.59) *is given by the formula*

$$\left(\eta^{1^{(n)}} - \sum_{k=1}^{n-1} \eta^{1^{(k)}} \frac{\partial f_0}{\partial y^{(k)}}\right)\Big|_{y^{(n)}=f_0} - \xi^1 f_{0_x} - \eta^1 f_{0_y} =$$
$$(n\xi^0_x - \eta^0_y) f_1 + (n+1) y'\xi^0_y f_1 + \sum_{k=1}^{n-1} \eta^{0^{(k)}} \frac{\partial f_1}{\partial y^{(k)}} + \xi^0 f_{1_x} + \eta^0 f_{1_y}. \quad (2.70)$$

After replacing $y^{(n)}$ by $f_0(x, y, y', ..., y^{(n-1)})$, equation (2.70) constitutes of differential functions in $y', y'', \ldots, y^{(n-1)}$, whose coefficients are the unknown functions ξ^1, η^1, the unperturbed symmetry components ξ^0, η^0, and their partial derivatives up to n^{th} order. Hence equation (2.70) splits into overdetermined system of PDEs in ξ^1, η^1, with or without additional conditions on the unperturbed symmetry components ξ^0, η^0. When such additional conditions are present, an exact symmetry of the unperturbed ODE (2.58) may disappear from the approximate symmetry classification of the perturbed ODE (2.59), thus becoming unstable (see Example 1.3.4). The following example illustrates the case when there are no restrictions on the unperturbed symmetry components which leads to the stability of all point symmetries of the unperturbed equation.

Example 2.4.1. Consider a second order ODE

$$y'' = \epsilon y' \quad (2.71)$$

that is also a perturbed version of (1.86). Equation (2.70) for approximate symmetries of (2.71) reads

$$\eta^1_{xx} + (2\eta^1_{xy} - \xi^1_{xx})y' + (\eta^1_{yy} - 2\xi^1_{xy})(y')^2 - \xi^1_{yy}(y')^3 = \eta^0_x + \xi^0_x y' + 2\xi^0_y y'^2, \tag{2.72}$$

where ξ^0, η^0 are the unperturbed symmetry components (1.87). Obviously, equation (2.72) splits into the following system of PDEs in ξ^1, η^1

$$\eta^1_{xx} = C_1 y + C_2, \quad 2\eta^1_{xy} - \xi^1_{xx} = 2C_1 x + \frac{1}{2}C_3 y + C_7, \quad \eta^1_{yy} - 2\xi^1_{xy} = C_3 x + 2C_6, \quad \xi^1_{yy} = 0, \tag{2.73}$$

with no change on ξ^0, η^0. Solving the above system of PDEs yields the following values of ξ^1, and η^1

$$\xi^1 = -\frac{C_3}{4}x^2 y - C_6 xy - \frac{C_7}{2}x^2 + a_1 x^2 + \frac{a_2}{2}xy + a_3 x + a_4 y + a_5, \tag{2.74}$$

$$\eta^1 = \frac{C_1}{2}x^2 y + \frac{C_2}{2}x^2 + a_1 xy + \frac{a_2}{2}y^2 + a_6 x + a_7 y + a_8, \tag{2.75}$$

where a_i are arbitrary constants. Consequently, the perturbed ODE (2.71) admits 16 approximate symmetries given by

$$\begin{aligned}
X_1 &= x^2\frac{\partial}{\partial x} + \left(xy + \epsilon\frac{x^2 y}{2}\right)\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y} + \epsilon\frac{x^2}{2}\frac{\partial}{\partial y}, \quad X_3 = \left(\frac{xy}{2} - \epsilon\frac{x^2 y}{4}\right)\frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial y},\\
X_4 &= (y - \epsilon xy)\frac{\partial}{\partial x}, \quad X_5 = \left(x - \epsilon\frac{x^2}{2}\right)\frac{\partial}{\partial x}, \quad X_6 = y\frac{\partial}{\partial y}, \quad X_7 = \frac{\partial}{\partial y}, \quad X_8 = \frac{\partial}{\partial x},\\
X_9 &= \epsilon\left(xy\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial x}\right), \quad X_{10} = \epsilon x\frac{\partial}{\partial y}, \quad X_{11} = \epsilon\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right), \quad X_{12} = \epsilon y\frac{\partial}{\partial y},\\
X_{13} &= \epsilon\frac{\partial}{\partial y}, \quad X_{14} = \epsilon y\frac{\partial}{\partial x}, \quad X_{15} = \epsilon x\frac{\partial}{\partial x}, \quad X_{16} = \epsilon\frac{\partial}{\partial x}.
\end{aligned} \tag{2.76}$$

All exact symmetries (1.88) of the unperturbed ODE (1.86) are inherited by the approximate symmetries (2.76), and thus are stable by definition. Note that the symmetries X_9, X_{10},…, X_{16} are trivial symmetries arising from the point symmetries (1.88) of the unperturbed ODE (1.86). X_6, X_7, X_8 are exact symmetries directly carrying over from the unperturbed ODE (1.86), while X_1, X_2, ..., X_5 are genuine approximate symmetries with $O(\epsilon^0)$ parts inherited from the exact point symmetries of the unperturbed ODE (1.86).

2.4.3 FS approximate point symmetries of a perturbed higher-order ODE

For the perturbed ODE with $y(x) = v(x) + \epsilon w(x)$, the corresponding FS system (1.104) is given by

$$\begin{aligned}
v^{(n)} &= f_0(x, v, v', ..., v^{(n-1)}),\\
w^{(n)} &= f_1(x, v, v', ..., v^{(n-1)}) + w^{(k)}\frac{\partial f_0(x, v, v', ..., v^{(n-1)})}{\partial v^{(k)}},
\end{aligned} \tag{2.77}$$

where $w^{(0)} = w$, and $v^{(0)} = v$ and summation in repeated indices (as well as below) is assumed for $k = 0, 1, ..., n-1$. The first equation of the above system is equivalent to the unperturbed ODE (2.58). The first-order approximate symmetries of (2.59) are the exact symmetries of (2.77). The exact symmetry generator admitted by the system (2.77) is given by

$$Z = \xi(x, v, w)\frac{\partial}{\partial x} + \eta^v(x, v, w)\frac{\partial}{\partial v} + \eta^w(x, v, w)\frac{\partial}{\partial w}. \tag{2.78}$$

The system of symmetry determining equations for the system of ODEs (2.77) is given by

$$\eta^{v(n)} = \xi f_{0_x} + \eta^{v(k)} \frac{\partial f_0(x, v, v', ..., v^{(n-1)})}{\partial v^{(k)}}, \tag{2.79a}$$

$$\begin{aligned}\eta^{w(n)} = \xi f_{1_x} + \eta^{v(k)} \frac{\partial f_1}{\partial v^{(k)}} + \eta^{w(k)} \frac{\partial f_0}{\partial v^{(k)}} + w \left(\xi \frac{\partial^2 f_0}{\partial v \partial x} + \eta^{v(k)} \frac{\partial^2 f_0}{\partial v \partial v^{(k)}} \right) \\ + w' \left(\xi \frac{\partial^2 f_0}{\partial v' \partial x} + \eta^{v(k)} \frac{\partial^2 f_0}{\partial v' \partial v^{(k)}} \right) + \cdots + w^{(n-1)} \left(\xi \frac{\partial^2 f_0}{\partial v^{(n-1)} \partial x} + \eta^{v(k)} \frac{\partial^2 f_0}{\partial v^{(n-1)} \partial v^{(k)}} \right)\end{aligned} \tag{2.79b}$$

when (2.77) holds.

Remark 2.4.1. $\xi = \xi^0(x, v)$ and $\eta^v = \eta^0(x, v)$ satisfy the determining equation (2.79a), where ξ^0, η^0 are the infinitesimals of the exact symmetry generator of the unperturbed ODE (2.58). For $n > 1$, the second determining equation (2.79b) may have restrictions on the solutions of the first determining equation (2.79a) leads to some exact symmetries of the unperturbed equation (2.58) become unstable (see Example 1.3.4). If there is no conditions on ξ^0, η^0 in the determining equation (2.79b), then all exact symmetries of the unperturbed ODE (2.58) will survive, i.e., are stable in FS sense.

Example 2.4.2. Consider the perturbed second-order ODE

$$y'' = (y')^{-1} + \epsilon (y')^{-3}. \tag{2.80}$$

Let

$$X^0 = \xi^0(x, y) \frac{\partial}{\partial x} + \eta^0(x, y) \frac{\partial}{\partial y}$$

be the infinitesimal generator for the unperturbed ODE $y'' = (y')^{-1}$. Using the determining equation for exact symmetries (1.30), the unperturbed symmetry components have the form

$$\xi^0(x, y) = \frac{2C_1}{3} x + C_3, \tag{2.81a}$$

$$\eta^0(x, y) = C_1 y + C_2. \tag{2.81b}$$

Hence, the unperturbed ODE $y'' = (y')^{-1}$ admits three exact point symmetries

$$X_1^0 = y \frac{\partial}{\partial y} + \frac{2}{3} x \frac{\partial}{\partial x}, \quad X_2^0 = \frac{\partial}{\partial y}, \quad X_3^0 = \frac{\partial}{\partial x}. \tag{2.82}$$

Now, we find the FS approximate symmetries for the perturbed ODE (2.80). The FS system of differential equations resulting from the perturbed ODE (2.80) is given by

$$\begin{aligned} v'' &= (v')^{-1}, \\ w'' &= (v')^{-3} - w'(v')^{-2}. \end{aligned} \tag{2.83}$$

Let

$$Z = \lambda(x, v, w) \frac{\partial}{\partial x} + \phi^1(x, v, w) \frac{\partial}{\partial v} + \phi^2(x, v, w) \frac{\partial}{\partial w}$$

be the exact symmetry generator for the system (2.83). Then, the first determining equation for the exact symmetries of (2.83) reads

$$\phi^1_{vv} = 0, \quad \phi^1_x = 0, \quad 2\phi^1_v - 3\lambda_x = 0, \quad \lambda_v = 0, \quad \lambda_w = 0, \quad \phi^1_w = 0,$$

which has a general solution:

$$\phi^1 = \eta^0(x, v), \quad \lambda = \xi^0(x, v),$$

where ξ^0, η^0 are given by (2.81). The second determining equation for (2.83) splits to

$$\phi^2_x = 0, \quad \phi^2_v = 0, \quad \phi^2_w = \frac{C_1}{3}, \tag{2.84}$$

with no conditions on ξ^0, η^0. From (2.84), ϕ^2 has the form

$$\phi^2 = \frac{C_1}{3} w + C_4. \tag{2.85}$$

Hence, the system (2.83) admits 4 exact symmetries

$$Z_1 = X^0_1(x, v) + \frac{w}{3}\frac{\partial}{\partial w}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = \frac{\partial}{\partial v}, \quad Z_4 = \frac{\partial}{\partial w}. \tag{2.86}$$

The exact symmetries (2.86) of the system (2.83) are the first-order approximate symmetries of the perturbed ODE (2.80). In this example, all exact symmetries (2.82) of the unperturbed equation $y'' = (y')^{-1}$ are *stable* as FS approximate symmetries.

Remark 2.4.2. An important feature of the Fushchich-Shtelen approximate symmetry framework is the possibility of existence of approximate FS symmetries where the $O(1)$ components $\xi(x, v, w)$ and $\eta^v(x, v, w)$ of the generator (2.78) depend on $O(\epsilon)$ solution component w. For example, the second-order perturbed ODE $y'' = \epsilon y'^2$ admits a FS approximate point symmetry

$$Z = \left(4xw - 4xv^2\right)\frac{\partial}{\partial x} + \left(4vw - 2v^3\right)\frac{\partial}{\partial v} + \left(4w^2 - v^4\right)\frac{\partial}{\partial w}.$$

Such FS symmetries do not originate from stable point symmetries of the unperturbed ODE $y'' = 0$. Such symmetries cannot arise in the BGI framework.

2.4.4 Some connection between BGI and FS approximate point symmetries for a perturbed higher-order ODE

The determining equation (2.70) for BGI approximate symmetries of the perturbed higher-order ODE (2.59) is noticeably different than the determining equation (2.79) of FS approximate symmetries of (2.59) since the former has four components depend on x and y while the latter has three components depend on x, v and w.

By taking $\xi = \xi^0(x, v)$ and $\eta^v = \eta^0(x, v)$ in (2.79), where $\xi^0(x, y), \eta^0(x, y)$ are the infinitesimals of the exact symmetry generator of the unperturbed ODE (2.58), we can see that the restrictions on the infinitesimals

ξ^0, η^0 in both frameworks are also different. Hence, we may have *stable* symmetries in sense of BGI, which are *unstable* in FS approach, and vice versa.

As an example, consider the perturbed ODE

$$y'' = (y')^{-1} + \epsilon (y')^{-3}. \tag{2.87}$$

We found in Example 2.4.2 that all exact symmetries of the unperturbed ODE $y'' = (y')^{-1}$ are stable in sense of FS. To find the BGI approximate symmetries admitted by (2.87), one applies the determining equation (2.70) to get a split system of PDEs in ξ^1, η^1

$$2\eta^1_y - 3\xi^1_x = 0, \quad \eta^1_{yy} = 0, \quad \eta^1_x = 0, \quad \xi^1_y = 0,$$

and additional condition

$$4\eta^0_y - 5\xi^0_x = 0$$

on the unperturbed symmetry components (2.81). It follows that $C_1 = 0$ in (2.81) and hence the exact symmetry X^1_0 in (2.82) is *unstable* in BGI framework.

In the following theorem, we show that for a family of perturbed higher-order ODEs, a genuine BGI approximate point symmetry yields a genuine FS approximate symmetry.

Theorem 2.4.2. *If*

$$X = \left(\eta^0(x) + \epsilon \eta^1(x, y)\right) \frac{\partial}{\partial y} \tag{2.88}$$

is a genuine BGI approximate point symmetry for the perturbed ODE

$$y^{(n)} = f_0(x) + \epsilon f_1(x, y, y', ..., y^{(n-1)}). \tag{2.89}$$

Then

$$Z = \eta^0(x) \frac{\partial}{\partial v} + \eta^1(x, v) \frac{\partial}{\partial w} \tag{2.90}$$

is a FS approximate symmetry for (2.89) *corresponding to the stable exact point symmetry* $\eta^0(x)\partial/\partial v$.

Proof. The FS system (2.77) for the perturbed ODE (2.89) has the form

$$\begin{aligned} v^{(n)} &= f_0(x), \\ w^{(n)} &= f_1(x, v, v', ..., v^{(n-1)}). \end{aligned} \tag{2.91}$$

The first equation of the system (2.91) is equivalent to the unperturbed ODE $y^{(n)} = f_0(x)$. As noted in Remark 2.4.1, the first FS determining equation (2.79a) is satisfied with the solution $\xi = \xi^0 = 0$ and $\eta^v = \eta^0(x)$, where ξ^0, η^0 are the infinitesimals of the exact symmetry generator of the unperturbed ODE $y^{(n)} = f_0(x)$. The second determining equation (2.79b) with $\xi = 0$, $\eta^v = \eta^0(x)$, $\eta^w = \eta^1(x, v)$ and $f_0 = f_0(x)$ is independent of w and it simplifies to

$$\eta^{1(n)}\Big|_{v^(n) = f_0} = \eta^0 \frac{\partial f_1}{\partial v} + \sum_{k=1}^{n-1} \eta^{0(k)} \frac{\partial f_1}{\partial v^{(k)}}. \tag{2.92}$$

Now with the choice $\xi^0 = 0$ and $\eta^0 = \eta^0(x)$, the BGI determining equation (2.70) reduces to

$$\eta^{1^{(n)}}\Big|_{y^{(n)}=f_0} = \sum_{k=1}^{n-1} \eta^{0^{(k)}} \frac{\partial f_1}{\partial y^{(k)}} + \eta^0 f_{1_y}. \tag{2.93}$$

The latter equation is equivalent to (2.92), and this completes the proof. □

Example 2.4.3. Consider the perturbed second-order ODE (2.71):

$$y'' = \epsilon y', \tag{2.94}$$

and its equivalent FS system

$$\begin{aligned} v'' &= 0, \\ w'' &= v'. \end{aligned} \tag{2.95}$$

From Example 2.4.1, the ODE (2.94) admits a genuine BGI approximate point symmetry given by

$$X = x\frac{\partial}{\partial y} + \epsilon \frac{x^2}{2}\frac{\partial}{\partial y}.$$

The choice $\eta^v = \eta^0 = x$, $\eta^w = \eta^1 = x^2/2$ is a solution of the determining equations (2.79) for FS symmetries of (2.95). It follows that

$$Z = x\frac{\partial}{\partial v} + \frac{x^2}{2}\frac{\partial}{\partial w}$$

is a FS approximate point symmetry for the perturbed ODE (2.94).

2.5 Stability of local symmetries of unperturbed ODEs in terms of higher-order BGI approximate symmetries

We have seen that for algebraic equations and first-order ODEs, every point symmetry of the unperturbed equation is stable. This, however, is not the case for point symmetries of higher-order ODEs, as some of those may or may not be stable.

It is natural to expect that under a perturbation of an ODE model, an exact local symmetry of the original system would become an approximate local symmetry of a perturbed ODE system. In the current section, we show that a point symmetry of an ODE of any order corresponds to point or higher-order BGI approximate symmetry of the perturbed model.

2.5.1 Exact local symmetries of the unperturbed ODE

The local infinitesimal generator (1.48) for an unperturbed ODE (2.58) has the evolutionary form

$$\hat{X}^0 = \zeta^0(x, y, y', y'', ..., y^{(s)})\frac{\partial}{\partial y}, \quad 1 \le s \le n-1. \tag{2.96}$$

When $s = 1$, the local symmetry generator (2.96) corresponds to the point symmetry generator (2.60) of ODE (2.58) provided that $\zeta^0(x, y, y') = \eta^0(x, y) - y'\xi^0(x, y)$. If $\zeta^0(x, y, y')$ is not linear in y', then (2.96) corresponds to a contact symmetry generator of ODE (2.58). When $s \geq 2$, the local symmetry generator (2.96) corresponds to a higher-order symmetry generator of ODE (2.58). The n^{th} prolongation of (2.96) is given by

$$\hat{X}^{0^{(n)}} = \zeta^0 \frac{\partial}{\partial y} + \zeta^{0^{(1)}} \frac{\partial}{\partial y'} + ... + \zeta^{0^{(n)}} \frac{\partial}{\partial y^{(n)}},$$

with

$$\zeta^{0^{(j)}} = D^j \zeta^0, \quad j = 1, 2, ..., n, \tag{2.97}$$

where D is given by (2.62). The determining equation for the exact symmetries of the unperturbed equation is

$$\hat{X}^{0^{(n)}}(y^{(n)} - f_0)\Big|_{y^{(n)}=f_0} = 0, \tag{2.98}$$

or in detail,

$$\zeta^{0^{(n)}}\Big|_{y^{(n)}=f_0} - \sum_{k=1}^{n-1} \left(\zeta^{0^{(k)}} \frac{\partial f_0}{\partial y^{(k)}} \right)\Big|_{y^{(n)}=f_0} - \zeta^0 f_{0_y} = 0. \tag{2.99}$$

The latter is equivalent to

$$D^n \zeta^0\Big|_{y^{(n)}=f_0} = \sum_{k=1}^{n-1} \left(D^k \zeta^0 \frac{\partial f_0}{\partial y^{(k)}} \right)\Big|_{y^{(n)}=f_0} + \zeta^0 f_{0_y}. \tag{2.100}$$

If $s = n-1$, equation (2.100) is a linear homogeneous PDE for ζ^0 with independent variables $x, y, y', \ldots, y^{(n-1)}$. This PDE can be written in a solved form

$$\frac{\partial^n \zeta^0}{\partial x^n} = R(x, y, y', ..., y^{(n-1)}, \zeta^0, \partial\zeta^0, ..., \partial^n \zeta^0) \tag{2.101}$$

for the highest derivative of ζ^0 with respect to the independent variable x, where all derivatives with respect to x appearing in the right-hand side of (2.101) are of lower order than those appearing on the left-hand side. Hence, it is in Cauchy-Kovalevskaya form with respect to x. It follows that the PDE (2.100) is solvable when $s = n - 1$. When $s < n - 1$, equation (2.100) splits into an overdetermined system of linear homogeneous PDEs which has at most a finite number of linearly independent solutions (e.g., [3]).

2.5.2 BGI approximate local symmetries of the perturbed ODE

The higher-order approximate symmetry generator for the ODE (2.59) is given by

$$\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = \left(\zeta^0(x, y, y', ..., y^{(s)}) + \epsilon \zeta^1(x, y, y', ..., y^{(\ell)}) \right) \frac{\partial}{\partial y}. \quad s, \ell \leq n - 1. \tag{2.102}$$

The prolongation of this generator has the form

$$\hat{X}^{(n)} = \hat{X}^{0^{(n)}} + \epsilon \hat{X}^{1^{(n)}} = \hat{X}^{0^{(n)}} + \epsilon \big(X^1 + \zeta^{1^{(1)}} \frac{\partial}{\partial y'} + ... + \zeta^{1^{(n)}} \frac{\partial}{\partial y^{(n)}} \big), \tag{2.103}$$

with $\zeta^{1^{(j)}} = D^j \zeta^1$, $j = 1, 2, ..., n$. To find the approximate symmetries of the perturbed ODE (2.59), we apply the determining equations for approximate symmetries

$$\hat{X}^{(n)}(y^{(n)} - f_0 - \epsilon f_1)\Big|_{y^{(n)}=f_0+\epsilon f_1} = o(\epsilon). \tag{2.104}$$

First, one computes an exact local symmetry generator (2.96) of the unperturbed ODE (2.58). Then, the first-order deformation $\hat{X}^1$ can be found from the equation

$$\hat{X}^{1^{(n)}}(y^{(n)} - f_0)\Big|_{y^{(n)}=f_0} = G(x, y, y', ..., y^{(n-1)}),$$

where G is the coefficient of ϵ in

$$-\left(\hat{X}^{0^{(n)}}(y^{(n)} - f_0 - \epsilon f_1)\right)\Big|_{y^{(n)}=f_0+\epsilon f_1}. \tag{2.105}$$

The determining equation (2.104) becomes

$$D^n\zeta^1\Big|_{y^{(n)}=f_0} - \sum_{k=1}^{n-1}\left(D^k\zeta^1\frac{\partial f_0}{\partial y^{(k)}}\right)\Big|_{y^{(n)}=f_0} - \zeta^1 f_{0_y} = G. \tag{2.106}$$

Remark 2.5.1. When $\ell = n - 1$, equation (2.106) is a linear nonhomogeneous PDE in ζ^1, and it is in Cauchy-Kovalevskaya form with respect to the independent variable x, so it has solutions obtainable (at least implicitly) by the method of characteristics. If $\ell = n - 1$, any solution of the PDE (2.106) has no conditions on the unperturbed symmetry components ζ^0. The following theorem holds.

Theorem 2.5.1. *For each exact point or local symmetry* (2.96) *of an unperturbed ODE* (2.58)*, there is an approximate symmetry* (2.102) *of the perturbed ODE* (2.59)*, with the symmetry component ζ^1 being of order at most $n - 1$.*

We now consider two examples in detail.

2.5.3 First detailed example

For the second-order ODE (1.115) with a small parameter,

$$y'' = \epsilon(y')^{-1} \tag{2.107}$$

we apply Theorem 2.5.1 to find approximate symmetries of order $n - 1 = 1$ corresponding to unstable point symmetries of (1.115) (see Example 1.3.4). This ODE is a perturbed version of $y'' = 0$. In total it admits 12 approximate point symmetries; this set does not include the following *unstable* point symmetries of $y'' = 0$:

$$X_1^0 = xy\frac{\partial}{\partial y} + x^2\frac{\partial}{\partial x}, \quad X_2^0 = x\frac{\partial}{\partial y}, \quad X_3^0 = \frac{y^2}{2}\frac{\partial}{\partial y} + \frac{xy}{2}\frac{\partial}{\partial x}, \quad X_u^0 = X_4^0 - \frac{3}{2}X_7^0. \tag{2.108}$$

Let

$$\hat{X}^0 = \zeta^0(x, y, y')\frac{\partial}{\partial y} = \left(\eta^0(x, y) - y'\xi^0(x, y)\right)\frac{\partial}{\partial y} \tag{2.109}$$

be the symmetry generator of the ODE $y'' = 0$ in evolutionary form. Therefore, ζ^0 has the form

$$\zeta^0(x, y, y') = \alpha_1 xy + \alpha_2 x + \alpha_3\frac{y^2}{2} + \alpha_4 y + \alpha_5 - (\alpha_1 x^2 + \alpha_3\frac{xy}{2} + \alpha_6 y + \alpha_7 x + \alpha_8)y'. \tag{2.110}$$

In evolutionary form, the eight point symmetries (1.88) of $y'' = 0$ are given by

$$\begin{aligned} \hat{X}_1^0 &= \left(xy - x^2y'\right)\frac{\partial}{\partial y}, \quad \hat{X}_2^0 = x\frac{\partial}{\partial y}, \quad \hat{X}_3^0 = \left(y^2 - xyy'\right)\frac{\partial}{\partial y} \\ \hat{X}_4^0 &= y\frac{\partial}{\partial y}, \quad \hat{X}_5^0 = \frac{\partial}{\partial y}, \quad \hat{X}_6^0 = yy'\frac{\partial}{\partial y}, \quad \hat{X}_7^0 = xy'\frac{\partial}{\partial y}, \quad \hat{X}_8^0 = y'\frac{\partial}{\partial y}. \end{aligned} \tag{2.111}$$

Let

$$\hat{X} = \left(\zeta^0(x,y,y') + \epsilon\zeta^1(x,y,y')\right)\frac{\partial}{\partial y} \tag{2.112}$$

be the local approximate symmetry generator admitted by the perturbed ODE (2.107) where ζ^0 is given by equation (2.110). The determining equation (2.106) requires

$$\zeta^1_{xx} + 2y'\zeta^1_{xy} + {y'}^2\zeta^1_{yy} = (-\alpha_1 y - \alpha_2)(y')^{-2} + \left(4\alpha_1 x - \frac{\alpha_3}{2}y - 2\alpha_4 + 3\alpha_7\right)(y')^{-1} + 2\alpha_3 x + 4\alpha_6. \tag{2.113}$$

By change of variable $t = y - xy'$, the homogeneous PDE

$$\zeta^1_{xx} + 2y'\zeta^1_{xy} + {y'}^2\zeta^1_{yy} = 0 \tag{2.114}$$

in $\zeta^1(x,y,y') = u(x,t)$ becomes a PDE

$$u_{xx} = 0,$$

which has a solution $u(x,t) = R_1(t) + xR_2(t)$, where R_1, R_2 are arbitrary functions of their arguments. Hence, the PDE (2.114) has the solution $\zeta^1_c = R_1(y - xy') + xR_2(y - xy')$. Now, let

$$\zeta^1_p = P(x,y)(y')^{-2} + Q(x,y)(y')^{-1} + R(x,y)$$

be a particular solution for the nonhomogeneous PDE (2.113). Substituting this particular solution into the equation (2.113) yields the following system of PDEs

$$\begin{aligned} &P_{xx} = -\alpha_1 y - \alpha_2, \quad Q_{xx} + 2P_{xy} = 4\alpha_1 x - \frac{\alpha_3}{2}y - 2\alpha_4 + 3\alpha_7, \\ &R_{xx} + 2Q_{xy} + P_{yy} = 2\alpha_3 x + 4\alpha_6, \quad 2R_{xy} + Q_{yy} = 0, \quad R_{yy} = 0. \end{aligned} \tag{2.115}$$

Solving the above system gives the general solution of (2.113)

$$\begin{aligned} \zeta^1(x,y,y') = R_1(y - xy') + xR_2(y - xy') - \left(\frac{\alpha_1}{2}x^2y + \frac{\alpha_2}{2}x^2\right)(y')^{-2} \\ + \left(\alpha_1 x^3 - \frac{\alpha_3}{4}x^2y - \alpha_4 x^2 + \frac{3\alpha_7}{2}x^2\right)(y')^{-1} + \frac{\alpha_3}{2}x^3 + 2\alpha_6 x^2. \end{aligned} \tag{2.116}$$

For simplest solution, take $R_1 = R_2 = 0$. Then, ζ^1 becomes

$$\zeta^1(x,y,y') = \left(-\frac{\alpha_1}{2}x^2y - \frac{\alpha_2}{2}x^2\right)(y')^{-2} + \left(\alpha_1 x^3 - \frac{\alpha_3}{4}x^2y - \alpha_4 x^2 + \frac{3\alpha_7}{2}x^2\right)(y')^{-1} + \frac{\alpha_3}{2}x^3 + 2\alpha_6 x^2. \tag{2.117}$$

Now, we find, one by one, all approximate symmetry components ζ^1 corresponding to each symmetry in (2.111).

For $\hat{X}_1^0$, substituting $\alpha_1 = 1$, and $\alpha_i = 0$, $i = 2, ..., 8$ into equations (2.110) and (2.117), we obtain $\zeta^0 = xy - x^2y'$ and the corresponding ζ^1 is $\zeta^1(x, y, y') = -\frac{1}{2}x^2y(y')^{-2} + x^3(y')^{-1}$. Hence the corresponding first-order approximate symmetry to $\hat{X}_1^0$ is given by

$$\hat{X}_1 = \left(xy - x^2y' + \epsilon\left(\frac{1}{2}x^2y(y')^{-2} + x^3(y')^{-1}\right)\right)\frac{\partial}{\partial y}. \tag{2.118}$$

It was unstable as a point symmetry of the ODE (1.86), but corresponds to a first-order approximate symmetry (2.118).

For $\hat{X}_2^0$, we have $\zeta^0 = x$ and the corresponding ζ^1 is $\zeta^1(x, y, y') = -\frac{1}{2}x^2(y')^{-2}$. So, $\hat{X}_2^0$ used to be an unstable point symmetry, but in fact corresponds to a first-order approximate symmetry of the perturbed ODE (2.107) given by

$$\hat{X}_2 = \left(x - \epsilon\left(\frac{1}{2}x^2(y')^{-2}\right)\right)\frac{\partial}{\partial y}.$$

Similarly, the unstable point symmetry $\hat{X}_3^0$ of (1.86) becomes a local first-order approximate symmetry of (2.107) given by

$$\hat{X}_3 = \left(y^2 - xyy' + \epsilon\left(x^3 - \frac{1}{4}x^2y(y')^{-1}\right)\right)\frac{\partial}{\partial y}.$$

$\hat{X}_4^0$ and $\hat{X}_7^0$ are not approximate point symmetries of the perturbed ODE (2.107), while a combination $\hat{X}_4^0 - \frac{2}{3}\hat{X}_7^0$ is an evolutionary form of the approximate point symmetry X_9 in (1.119a). By substituting $\alpha_4 = 1$, $\alpha_7 = 2/3$, and other $\alpha_i = 0$, one gets $\zeta^0 = y - \frac{2}{3}xy'$ and $\zeta_1 = 0$. A transverse linear combination $\hat{X}_4^0 + \frac{3}{2}\hat{X}_7^0$ is the evolutionary form for the unstable point symmetry X_u^0 (1.120). Substituting $\alpha_4 = 1$, $\alpha_7 = -3/2$, and other $\alpha_i = 0$ into equations (2.110) and (2.117), one gets $\zeta^0 = y + \frac{3}{2}xy'$ and $\zeta^1 = -\frac{13}{4}x^2(y')^{-1}$. The first-order approximate symmetry of the perturbed ODE (2.107) corresponding to the transverse direction of $(\hat{X}_4^0, \hat{X}_7^0)$-space in (2.111) is given by

$$\hat{X}_u = \left(y + \frac{3}{2}xy' + \epsilon\left(-\frac{13}{4}x^2(y')^{-1}\right)\right)\frac{\partial}{\partial y}.$$

$\hat{X}_5^0$ is a stable symmetry as it is, with $\zeta^1 = 0$. This easily can be seen by substituting $\alpha_5 = 1$, and other $\alpha_i = 0$ into equation (2.117). Similarly, one obtains $\zeta^1 = 0$ corresponding to the stable symmetries $\hat{X}_7^0$ and $\hat{X}_8^0$.

Finally, $\hat{X}_6^0$ is an evolutionary form of X_6^0 in (1.88), it should be a genuine approximate symmetry coming from stable symmetries (1.119b), here $\zeta^1 \neq 0$. Substituting $\alpha_6 = 1$ and other $\alpha_i = 0$ into equations (2.110) and (2.117) gives $\zeta^0 = -yy'$ and $\zeta^1 = 2x^2$. The corresponding approximate symmetry of the perturbed ODE (2.107) is given by

$$\hat{X}_6 = \left(-yy' + 2\epsilon x^2\right)\frac{\partial}{\partial y}.$$

This is exactly the evolutionary form of the approximate point symmetry X_{11} in (1.119b).

Remark 2.5.2. We note that in the current example, one would obtain an infinite set of first-order approximate symmetries corresponding to each unstable point symmetry (2.108) of $y'' = 0$, if a more general form

(2.116) of $\zeta^1(x, y, y')$ was used instead of the simplified ansatz (2.117). This, however, does not make such first-order approximate symmetries trivial; they can be used, for example, for construction of approximate solutions of the perturbed ODE (2.107) through mappings or approximate reduction of order (see Section 2.6 below).

In the following example, we compute exact point and local symmetries of the fourth-order Boussinesq differential equation [84, 85] and discuss their stability.

2.5.4 Second detailed example

Consider a linear ODE

$$y^{(4)} + y'' = 0 \tag{2.119}$$

and its perturbed version

$$y^{(4)} + y'' - \epsilon\left(2yy'' + 2y'^2\right) = 0. \tag{2.120}$$

The latter ODE can be obtained as a time-independent or a traveling wave reduction of the Bousinesq PDE

$$u_{tt} - u_{xx} + \epsilon(u^2)_{xx} - u_{xxxx} = 0, \quad u = u(x,t), \tag{2.121}$$

that was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water [86]. In this example, some point and local symmetries of the unperturbed ODE (2.119) are shown to correspond to third-order local approximate BGI symmetries of the perturbed ODE (2.120), as guaranteed by Theorem (2.5.1). The calculated approximate symmetries are used in the next section to illustrate the construction of an approximate solution of the perturbed Boussinesq equation (2.120).

First, we seek exact point symmetries for (2.119) and approximate point symmetries for (2.120). Let

$$X^0 = \xi^0(x,y)\frac{\partial}{\partial x} + \eta^0(x,y)\frac{\partial}{\partial y} \tag{2.122}$$

be an exact point symmetry generator of the ODE (2.119). After the prolongation of X^0 to the fourth-order and applying the determining equations (1.30), one finds

$$\xi^0 = C_6, \quad \eta^0 = C_1 y + C_2 + C_3 x + C_4 \sin x + C_5 \cos x. \tag{2.123}$$

Consequently, the ODE (2.119) admits the following point symmetries:

$$X_1^0 = y\frac{\partial}{\partial y}, \quad X_2^0 = \frac{\partial}{\partial y}, \quad X_3^0 = x\frac{\partial}{\partial y}, \quad X_4^0 = \sin x\frac{\partial}{\partial y}, \quad X_5^0 = \cos x\frac{\partial}{\partial y}, \quad X_6^0 = \frac{\partial}{\partial x}. \tag{2.124}$$

Now, we proceed to find approximate point symmetries of the perturbed ODE (2.120). Let

$$X = X^0 + \epsilon X^1 = \left(\xi^0(x,y) + \epsilon\xi^1(x,y)\right)\frac{\partial}{\partial x} + \left(\eta^0(x,y) + \epsilon\eta^1(x,y)\right)\frac{\partial}{\partial y} \tag{2.125}$$

be the approximate BGI symmetry generator admitted by the perturbed ODE (2.120), where X^0 is an exact symmetry generator (2.122) of the unperturbed ODE (2.119). The determining equation for approximate symmetries (2.70) yields

$$\eta^1_{xxxx} + \eta^1_{xx} = 0, \quad \eta^1_{xy} = 0, \quad \eta^1_{yy} = 0, \quad \xi^1_x = C_2, \quad \xi^1_y = 0, \quad C_1 = C_3 = C_4 = C_5 = 0. \tag{2.126}$$

The above system has the solution

$$\xi^1(x,y) = C_2 x + a_6, \quad \eta^1(x,y) = a_1 y + a_2 + a_3 x + a_4 \sin x + a_5 \cos x, \tag{2.127}$$

also involving six arbitrary constants. Specifically, the perturbed ODE (2.120) admits six trivial symmetries $X_j = \epsilon X_j^0$, $j = 1,2,...,6$, corresponding to the free constants $a_1 \ldots, a_6$, where X_j^0 are the exact point symmetries (2.124) of the unperturbed ODE (2.119), and two nontrivial approximate point symmetries

$$X_7 = X_2^0 + \epsilon x \frac{\partial}{\partial x}, \quad X_8 = X_6^0. \tag{2.128}$$

It follows that the only two stable point symmetries of (2.119) are X_2^0 and X_6^0, and the unstable point symmetries are X_1^0, X_3^0, X_4^0, and X_5^0.

We now extend the above analysis, seeking exact local symmetries admitted by (2.119) up to second-order, in the form

$$V^0 = \varphi^0(x,y,y',y'') \frac{\partial}{\partial y}.$$

Applying the determining equation (2.100), one gets

$$\left(D^4 \varphi^0 + D^2 \varphi^0\right)\Big|_{y^{(4)}=-y''} = 0. \tag{2.129}$$

The above equation splits into system of PDEs. Solving this system gives

$$\begin{aligned}
\varphi^0 &= k_1 y' + k_2 + k_3 x + k_4 y + k_5 \sin x + k_6 \cos x + k_7 y'' + k_8 \left(y' \sin x + y'' \cos x\right) \\
&+ k_9 \left(y'^2 + y''^2\right) + k_{10} \left(\left(y''^2 - y'^2\right) \cos x + 2y'y'' \sin x\right) + k_{11} \left(\left(y'^2 - y''^2\right) \sin x + 2y'y'' \cos x\right) \\
&+ k_{12} \left(y'\left(2y - x + 2y''\right) - xy''^2\right) + k_{13} \left(\left(2\sin x - x\cos x\right)y'' - \left(x \sin x + \cos x\right)y' + y \sin x\right) \\
&+ k_{14} \left(\left(x \sin x + 3\cos x\right)y'' + \left(2 \sin x - x \cos x\right)y' + y \cos x\right) + k_{15} \left(y'' \sin x - y' \cos x\right),
\end{aligned} \tag{2.130}$$

involving 15 arbitrary constants k_j. Hence, the ODE (2.119) admits the following local symmetries

$$\begin{aligned}
&V_1^0 = y' \frac{\partial}{\partial y}, \quad V_2^0 = \frac{\partial}{\partial y}, \quad V_3^0 = x \frac{\partial}{\partial y}, \quad V_4^0 = y \frac{\partial}{\partial y}, \quad V_5^0 = \sin x \frac{\partial}{\partial y}, \quad V_6^0 = \cos x \frac{\partial}{\partial y}, \\
&V_7^0 = y'' \frac{\partial}{\partial y}, \quad V_8^0 = \left(y' \sin x + y'' \cos x\right) \frac{\partial}{\partial y}, \quad V_9^0 = \left(y'^2 + y''^2\right) \frac{\partial}{\partial y}, \\
&V_{10}^0 = \left(\left(y''^2 - y'^2\right) \cos x + 2y'y'' \sin x\right) \frac{\partial}{\partial y}, \quad V_{11}^0 = \left(\left(y'^2 - y''^2\right) \sin x + 2y'y'' \cos x\right), \\
&V_{12}^0 = \left(2y'\left(y + y''\right) - x\left(y' + y''^2\right)\right) \frac{\partial}{\partial y}, \\
&V_{13}^0 = \left(\left(2\sin x - x\cos x\right)y'' - \left(x \sin x + \cos x\right)y' + y \sin x\right) \frac{\partial}{\partial y}, \\
&V_{14}^0 = \left(\left(x \sin x + 3\cos x\right)y'' + \left(2 \sin x - x \cos x\right)y' + y \cos x\right) \frac{\partial}{\partial y}, \\
&V_{15}^0 = \left(y'' \sin x - y' \cos x\right) \frac{\partial}{\partial y}.
\end{aligned} \tag{2.131}$$

These generators were numbered to match the point symmetry classification (2.124) of the unperturbed ODE (2.119). In particular, the generators $V_1, \ldots, V_6$ in (2.131) are evolutionary forms of the point symmetries (2.124).

Now, we will find the approximate local symmetries for the perturbed ODE (2.120). Let

$$V=\left(\varphi^0(x,y,y',y'')+\epsilon\varphi^1(x,y,y',y'')\right)\frac{\partial}{\partial y} \tag{2.132}$$

be the local approximate symmetry generator admitted by the perturbed ODE (2.120) where φ^0 is given by equation (2.130). Using the determining equation (2.106), one obtains

$$\begin{aligned}
\varphi^1 &= Q_1(y)+y''Q_2(y)+a_3x+a_4y'+a_5\sin x+a_6\cos x+a_7\left(y'\left(2y-x+2y''\right)-xy''^2\right)\\
&+a_8\left(y'^2+y''^2\right)+a_9\left(\left(y'^2+y''^2\right)\sin x+2y'y''\cos x\right)+a_{10}\left(\left(y''^2-y'^2\right)\cos x+2y'y''\sin x\right)\\
&+a_{11}\left(y''\sin x-y'\cos x\right)+a_{12}\left(\left(2\sin x-x\cos x\right)y''-\left(x\sin x+\cos x\right)y'+2y\sin x\right)\\
&+a_{13}\left(y'\sin x+y''\cos x\right)+a_{14}\left(\left(x\sin x+3\cos x\right)y''+\left(2\sin x-x\cos x\right)y'+y\cos x\right)\\
&-k_2xy'+k_3\left(2xy''-\frac{1}{2}x^2y'+\frac{5}{2}xy\right)+k_7\frac{4}{3}y''^2,
\end{aligned} \tag{2.133}$$

k_1 is free, and $k_i=0$ for $i=4,5,6,8,\ldots,15$. Consequently, the local symmetries V_i^0 (2.131) for $i=4,5,6,8,\ldots,15$ of the unperturbed ODE (2.119) are unstable, while V_1^0, V_2^0, V_3^0 and V_7^0 in (2.131) are parts of the approximate symmetries of (2.120) given by

$$\begin{aligned}
&V_1=V_1^0=y'\frac{\partial}{\partial y},\quad V_2=V_2^0-\epsilon xy'\frac{\partial}{\partial y}=(1-\epsilon xy')\frac{\partial}{\partial y},\\
&V_3=V_3^0+\epsilon\left(2xy''-\frac{1}{2}x^2y'+\frac{5}{2}xy\right)\frac{\partial}{\partial y}=\left(x+\epsilon\left(2xy''-\frac{1}{2}x^2y'+\frac{5}{2}xy\right)\right)\frac{\partial}{\partial y},\\
&V_7=V_7^0+\frac{4}{3}\epsilon y''^2\frac{\partial}{\partial y}=\left(y''+\frac{4}{3}\epsilon y''^2\right)\frac{\partial}{\partial y}.
\end{aligned} \tag{2.134}$$

This set includes the evolutionary forms of the approximate point generators X_1 and X_2 of (2.128) in their evolutionary forms V_1 and V_2. Moreover, V_3 is a second-order approximate symmetry of the perturbed ODE (2.120) corresponding to the unstable point symmetry X_0^3 in (2.124), and V_7 is an evolutionary form of the approximate point symmetry X_7 in (2.128).

Higher-order approximate symmetries corresponding to unstable point and local symmetries of (2.119)

Let

$$\hat{X}^0=\zeta^0\frac{\partial}{\partial y} \tag{2.135}$$

be the evolutionary form of the exact point or local symmetry generator of the unperturbed ODE (2.119). Here $\zeta^0=\zeta^0(x,y,y')$ for point symmetries (2.124), and $\zeta^0=\phi^0(x,y,y',y'')$ for second-order local symmetries (2.131) of the unperturbed ODE (2.119).Following Theorem 2.5.1, for each unstable local symmetry V_0^j, $j=$

$4,5,6,8,\ldots,15$ in (2.131) of the ODE (2.119), there is a corresponding higher-order approximate symmetry for the perturbed ODE (2.120) of the form

$$\hat{X}=\left(\zeta^0+\epsilon\zeta^1(x,y,y',y'',y''')\right)\frac{\partial}{\partial y}.$$

First, consider the unstable point symmetry X_0^4 in (2.124) (V_4^0 in (2.131)). Its evolutionary components is $\zeta^0 = y$. The corresponding ζ^1 is any solution of the linear nonhomogeneous PDE

$$\left(D^4\zeta^1 + D^2\zeta^1\right)\Big|_{y^{(4)}=-y''} = 2yy'' + 2(y')^2.$$

A simple particular solution is given by

$$\zeta^1(x, y, y', y'', y''') = \left(\frac{1}{2}x^2 + \frac{5}{6}\right)y'^2 + \frac{1}{2}(x^2y' + 3xy + 2y'')y'''.$$

One consequently obtains

$$\hat{X}^4 = \left(y + \epsilon\left(\left(\frac{1}{2}x^2 + \frac{5}{6}\right)y'^2 + \frac{1}{2}(x^2y' + 3xy + 2y'')y'''\right)\right)\frac{\partial}{\partial y}$$

as a third-order approximate symmetry for the perturbed ODE (2.120) corresponds to the unstable point symmetry X_0^4, V_0^4.

In the same way, one can find a third-order approximate symmetry corresponding to each unstable point symmetry of (2.119) in (2.124) or unstable local symmetry in (2.131). Let

$$\hat{V} = \left(\varphi^0(x, y, y', y'') + \epsilon\hat{\varphi}^1(x, y, y', y'', y''')\right)\frac{\partial}{\partial y} \tag{2.136}$$

be approximate symmetry generator for the perturbed ODE (2.120) where φ^0 is given by the equation (2.130) . From the determining equation (2.106), one can find $\hat{\varphi}^1$ corresponds to each local symmetry of (2.131). For example, consider the unstable local symmetry $V_9^0 = \left(y'^2 + y''^2\right)\partial/\partial y$. By substituting $\varphi^0 = y'^2 + y''^2$ into the determining equation (2.106), one obtains

$$\left(D^4\hat{\varphi}^1 + D^2\hat{\varphi}^1\right)\Big|_{y^{(4)}=-y''} = 12yy'''^2 + 56y'y''y''' + 10y''^3 - 12yy''^2 - 6y'^2y''. \tag{2.137}$$

The above equation has a particular solution given by

$$\hat{\varphi}^1 = -2xy''^2y''' + \frac{7}{6}y''^3 + (2y - 3xy')\,y''^2 + \frac{1}{2}y'^2y'' - xy'^3. \tag{2.138}$$

Hence

$$\hat{V_9} = \left(y'^2 + y''^2 + \epsilon\left(-2xy''^2y''' + \frac{7}{6}y''^3 + (2y - 3xy')\,y''^2 + \frac{1}{2}y'^2y'' - xy'^3\right)\right)\frac{\partial}{\partial y}$$

is a third-order local approximate symmetry of the Boussinesq ODE (2.120) corresponding to the exact local symmetry V_9^0 of the unperturbed equation (2.119), which used to be unstable in the class of second-order local symmetries.

2.6 Reduction of order and approximately invariant solutions of perturbed differential equations

In this section we discuss approximate reduction techniques, including approximate integrating factors and approximate first integrals of perturbed differential equations, and the use of the higher-order approximate symmetries to find approximate solutions of some perturbed ODEs.

2.6.1 Approximate integrating factors using approximate point symmetries

A differential function

$$\mu(x,y,y',...,y^{(n-1)};\epsilon) = \mu_0(x,y,y',...,y^{(n-1)}) + \epsilon\mu_1(x,y,y',...,y^{(n-1)}) \tag{2.139}$$

is an *approximate integrating factor* for the perturbed ODE (2.59) if there is a differential function

$$\phi(x,y,y',...,y^{(n-1)};\epsilon) = \phi_0(x,y,y',...,y^{(n-1)}) + \epsilon\phi_1(x,y,y',...,y^{(n-1)})$$

such that

$$\mu(y^{(n)} - f_0 - \epsilon f_1) = D(\phi) = o(\epsilon).$$

Finding the integrating factor allows an approximate reduction of the equation (2.59) to an $(n-1)-$order equation

$$\phi(x,y,y',...,y^{(n-1)};\epsilon) = \text{const} + o(\epsilon). \tag{2.140}$$

Remark 2.6.1. The integrating factor for the perturbed first-order ODE (2.27) with exact symmetry generator (2.38) has the form

$$\mu(x,y;\epsilon) = \frac{1}{\eta - \xi(f_0 + \epsilon f_1)}, \tag{2.141}$$

provided that $\eta \neq \xi(f_0 + \epsilon f_1)$. If (ξ, η) are analytic in ϵ, then

$$\mu(x,y;0) = \mu_0(x,y)$$

is an integrating factor for the unperturbed first-order ODE (2.26). Moreover, $\mu_{approx}(x,y;\epsilon) = \mu_0(x,y) + \epsilon\mu_1(x,y) + o(\epsilon)$ with

$$\mu_0(x,y) = \frac{1}{\eta^0 - \xi^0 f_0}, \tag{2.142a}$$

$$\mu_1(x,y) = \mu_0^2\left(\xi^0 f_1 + \xi^1 f_0 - \eta^1\right) \tag{2.142b}$$

is an approximate integrating factor for the ODE (2.27) with approximate symmetry generator (2.44).

This follows from taking $\xi(x,y;\epsilon) = \xi^0(x,y) + \epsilon\xi^1(x,y) + o(\epsilon)$ and $\eta(x,y;\epsilon) = \eta^0(x,y) + \epsilon\eta^1(x,y) + o(\epsilon)$, substituting these values into (2.141) and taking the Taylor expansion about $\epsilon = 0$. Conversely, we have the following theorem.

Theorem 2.6.1. *If $\mu(x,y;\epsilon) = \mu_0(x,y) + \epsilon\mu_1(x,y) + o(\epsilon)$ is an approximate multiplier for the perturbed first-order ODE* (2.27). *Then any functions ξ^j, η^j, $j = 0,1$ satisfying* (2.142) *define a BGI approximate symmetry*

$$X = \left(\xi^0(x,y) + \epsilon\xi^1(x,y)\right)\frac{\partial}{\partial x} + \left(\eta^0(x,y) + \epsilon\eta^1(x,y)\right)\frac{\partial}{\partial y}$$

for the ODE (2.27).

Proof. An integrating factor $\mu(x,y;\epsilon)$ for the perturbed first-order ODE satisfies

$$-\frac{\partial}{\partial y}\left(\mu\left(f_0+\epsilon f_1\right)\right)=\frac{\partial}{\partial y}\left(\mu\right). \tag{2.143}$$

Substitute $\mu=\mu_0(x,y)+\epsilon\mu_1(x,y)+o(\epsilon)$ into equation (2.143) to get

$$\begin{aligned}&\mu_{0x}+\mu_{0y}f_0+\mu_0 f_{0y}=0,\\&\mu_{1x}+\mu_{1y}f_0+\mu_1 f_{0y}+\mu_{0y}f_1+\mu_0 f_{1y}=0.\end{aligned} \tag{2.144}$$

Substituting the values of μ_0,μ_1 given by (2.142) into (2.144) yields

$$\begin{aligned}&\mu_0^2\left(\eta_x^0+\eta_y^0 f_0-\eta^0 f_{0_y}-\xi^0 f_{0_x}-\xi_x^0 f_0-\xi_y^0 f_0^2\right)=0,\\&\mu_0^3\left(\eta_x^1+(\eta_y^1-\xi_x^1)f_0-\xi_y^1 {f_0}^2-\xi^1 f_{0_x}-\eta^1 f_{0_y}-(\xi_x^0-\eta_y^0)f_1-2\xi_y^0 f_0 f_1-\xi^0 f_{1_x}-\eta^0 f_{1_y}\right)=0.\end{aligned} \tag{2.145}$$

Comparison with the determining equation (2.31) of exact symmetries of the unperturbed equation (2.26) and determining equation (2.46) for approximate symmetries of the perturbed ODE (2.27) completes the proof. □

Example 2.6.1. The first-order ODE

$$y'=y+\epsilon xy \tag{2.146}$$

admits the approximate symmetry generator

$$X=(1+\epsilon)\,y\frac{\partial}{\partial y}.$$

The approximate integrating factor for (2.146) has the form

$$\mu(x,y;\epsilon)=\frac{1}{y}\left(1-\epsilon\right).$$

Using this integrating factor, one gets

$$\begin{aligned}o(\epsilon)&=\left(\frac{1}{y}\left(1-\epsilon\right)\right)\left(y'-y-\epsilon xy\right)\\&=\frac{y'}{y}-1+\epsilon\left(1-x-\frac{y'}{y}\right)\\&=D\left(\ln y-x+\epsilon\left(x-\frac{x^2}{2}-\ln y\right)\right)\end{aligned} \tag{2.147}$$

Hence

$$\ln y-x+\epsilon\left(x-\frac{x^2}{2}-\ln y\right)=C+o(\epsilon) \tag{2.148}$$

is a family of approximate solution curves for the perturbed ODE (2.146). Note that the first two terms of the Taylor expansion in ϵ of (2.148) agree with the first two terms of the Taylor expansion in ϵ of the exact solution

$$y=C_1 e^{\frac{\epsilon x^2}{2}+x}$$

of the ODE (2.146).

2.6.2 Determining equations for approximate integrating factors

For one independent variable x and one dependent variable y, the *Euler operator* (1.143) is given by

$$\frac{\delta}{\delta y} = \frac{\partial}{\partial y} - D\frac{\partial}{\partial y'} + D^2\frac{\partial}{\partial y''} - D^3\frac{\partial}{\partial y'''} + \cdots . \tag{2.149}$$

Since the *Euler-Lagrange operator*(2.149) annihilates the total derivative for any differential function, then the integrating factors (2.139) for the perturbed ODE (2.59) can be found from the following equation:

$$\frac{\delta}{\delta y}\left(\mu(y^{(n)} - f_0 - \epsilon f_1)\right) = 0. \tag{2.150}$$

For the perturbed first-order ODE (2.27), equation (2.150) has the form

$$(\mu f_0)_y + \epsilon(\mu f_1)_y + \mu_x = 0.$$

Substituting $\mu = \mu(x, y; \epsilon) = \mu_0(x, y) + \epsilon\mu_1(x, y)$ into the above equation and setting to zero the coefficients of ϵ^0, ϵ, we arrive at the following determining equations for μ_0 and μ_1:

$$\mu_{0_x} + (\mu_0 f_0)_y = 0, \quad \mu_{1_x} + (\mu_1 f_0)_y + (\mu_0 f_1)_y = 0. \tag{2.151}$$

In particular, for the second-order perturbed ODE

$$y'' = f_0(x, y, y') + \epsilon f_1(x, y, y'), \tag{2.152}$$

the integrating factor $\mu(x, y, y'; \epsilon) = \mu_0(x, y, y') + \epsilon\mu_1(x, y, y')$ for the ODE (2.152) satisfies

$$\frac{\delta}{\delta y}\,(\mu(y'' - f_0 - \epsilon f_1) = 0.$$

The above equation is equivalent to

$$y''\mu_y - (\mu f_0)_y - \epsilon(\mu f_1)_y - D\left(y''\mu_{y'} - (\mu f_0)_{y'} - \epsilon(\mu f_1)_{y'}\right) + D^2(\mu) = 0. \tag{2.153}$$

Finding the total derivatives appearing in equation (2.153), one obtains

$$\begin{aligned} y'\mu_{yy'} + \mu_{xy'} + 2\mu_y + (\mu f_0)_{y'y'} + \epsilon(\mu f_1)_{y'y'} &= 0, \\ y'^2\mu_{yy} + 2y'\mu_{xy} + \mu_{xx} + y'(\mu f_0)_{yy'} + (\mu f_0)_{xy'} + \epsilon y'(\mu f_1)_{yy'} + \epsilon(\mu f_1)_{xy'} - (\mu f_0)_y - \epsilon(\mu f_1)_y &= 0. \end{aligned}$$

Substituting $\mu(x, y, y'; \epsilon) = \mu_0(x, y, y') + \epsilon\mu_1(x, y, y')$ into the above equations, we arrive the following theorem.

Theorem 2.6.2. *The components μ_0, μ_1, of the approximate integrating factor $\mu(x, y, y'; \epsilon) = \mu_0(x, y, y') + \epsilon\mu_1(x, y, y')$ for the perturbed second-order ODE* (2.152) *satisfy the following equations*

$$y'\mu_{0_{yy'}} + \mu_{0_{xy'}} + 2\mu_{0_y} + (\mu_0 f_0)_{y'y'} = 0, \tag{2.154a}$$

$$y'^2\mu_{0_{yy}} + 2y'\mu_{0_{xy}} + \mu_{0_{xx}} + y'(\mu_0 f_0)_{yy'} + (\mu_0 f_0)_{xy'} - (\mu_0 f_0)_y = 0, \tag{2.154b}$$

$$y'\mu_{1_{yy'}} + \mu_{1_{xy'}} + 2\mu_{1_y} + (\mu_1 f_0)_{y'y'} + (\mu_0 f_1)_{y'y'} = 0, \tag{2.154c}$$

$$\begin{aligned} &y'^2\mu_{1_{yy}} + 2y'\mu_{1_{xy}} + \mu_{1_{xx}} + y'(\mu_1 f_0)_{yy'} + (\mu_1 f_0)_{xy'} - (\mu_1 f_0)_y - (\mu_0 f_1)_y \\ &\quad + y'(\mu_0 f_1)_{yy'} + (\mu_0 f_1)_{xy'} = 0. \end{aligned} \tag{2.154d}$$

As an application of the above theorem, we consider the perturbed Boussinesq ODE (2.120), and the Benjamin-Bona-Mahony (BBM) ODE reduction.

Example 2.6.2. Consider the perturbed Boussinesq ODE

$$y^{(4)} + y'' - \epsilon\left(2yy'' + 2y'^2\right) = 0. \tag{2.155}$$

Equation (2.155) can be written in the form

$$D^2(y'' + y - \epsilon y^2) = 0.$$

Hence, the Boussinesq ODE (2.155) reduces to the second-order ODE

$$y'' + y - \epsilon y^2 = C_1 x + C_2. \tag{2.156}$$

The general solution of (2.156) is unknown. An approximate solution can be constructed in the assumption of $C_1, C_2 = O(\epsilon)$. Let $C_1 = \epsilon c_1, C_2 = \epsilon c_2$; then the ODE (2.156) becomes

$$y'' = -y + \epsilon(c_1 x + c_2 + y^2). \tag{2.157}$$

Using the determining equations (2.154), one can easily find that $\mu = y' + \epsilon(y' - c_1)$ is an approximate integrating factor for the ODE (2.157). Multiplying this integrating factor by (2.157) yields

$$y'y'' + yy' + \epsilon(y'y'' - c_1 y'' + yy' - c_1 y - (c_1 x + c_2 + y^2)y') = o(\epsilon),$$

and consequently, an approximate first integral:

$$D\left(y'^2 + y^2 + \epsilon\left(y'^2 - 2c_1 y' + y^2 - (2c_1 x + 2c_2)y - \frac{2y^3}{3}\right)\right) = o(\epsilon).$$

Hence the perturbed Boussinesq ODE (2.155) is reduced to the first-order ODE

$$y'^2 + y^2 + \epsilon\left(y'^2 - 2y' + y^2 - (2x + 2)y - \frac{2y^3}{3}\right) = 2c_3^2 + o(\epsilon), \tag{2.158}$$

where c_1, c_2, c_3 are arbitrary constants. A series ansatz $y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + o(\epsilon)$ into the ODE (2.158) leads to the system of ODEs

$$\begin{aligned} &(y_0')^2 + y_0^2 = 2c_3^2, \\ &2y_0'y_1' + 2y_0 y_1 + (y_0')^2 - 2c_1 y_0' + y_0^2 - (2c_1 x + 2c_2)y_0 - \frac{2y_0^3}{3} = 0, \end{aligned}$$

with solutions

$$y_0(x) = c_3 (\sin x + \cos x),$$
$$y_1(x) = c_1 x + c_2 + c_3^2 - \frac{c_3^2}{3}\sin 2x - \frac{c_3}{2}(\cos x + \sin x) + c_4 (\cos x - \sin x).$$

Finally, a general approximate solution for the Boussinesq ODE (2.155) involving four arbitrary constants is obtained:

$$\begin{aligned} y(x;\epsilon) &= c_3(\sin x + \cos x) \\ &+\epsilon\left(c_1 x + c_2 + c_3^2 - \frac{c_3^2}{3}\sin 2x - \frac{c_3}{2}(\cos x + \sin x) + c_4(\cos x - \sin x)\right). \end{aligned} \tag{2.159}$$

Example 2.6.3. The Benjamin-Bona-Mahony (BBM) equation is the partial differential equation

$$u_t + u_x - u_{xxt} + \frac{3}{2}\epsilon u u_x = 0, \quad u = u(x,t) \tag{2.160}$$

modeling long surface gravity waves of small amplitude (ϵ) propagating unidirectionally in $1+1$ dimensions [49]. Using a travelling wave solution $u(x,t) = y(z) = y(x-ct)$, equation (2.160) reduces to an ODE:

$$y''' + \frac{1-c}{c}y' + \frac{3}{2c}\epsilon y y' = 0. \tag{2.161}$$

The BBM ODE (2.161) can be simplified to

$$y'' + \frac{1-c}{c}y + \frac{3}{4c}\epsilon y^2 = k, \tag{2.162}$$

where k is a constant of integration. By taking $k = 3\epsilon/4c$, the ODE (2.162) becomes

$$y'' = \frac{c-1}{c}y - \frac{3}{4c}\epsilon(y^2-1). \tag{2.163}$$

A possible approximate integrating factor for the perturbed ODE (2.163) is $\mu = (1+\epsilon)y'$. Multiplying this integrating factor by (2.163) and then integrating the resulting equation, one finds an approximate first integral:

$$D\left(y'^2 + \frac{1-c}{c}y^2 + \epsilon\left(y'^2 + \frac{1-c}{c}y^2 - \frac{3y}{2c} + \frac{y^3}{2c}\right)\right) = O(\epsilon^2). \tag{2.164}$$

Consequently, the perturbed BBM ODE (2.163) is reduced to the first-order ODE

$$y'^2 + \frac{1-c}{c}y^2 + \epsilon\left(y'^2 + \frac{1-c}{c}y^2 - \frac{3y}{2c} + \frac{y^3}{2c}\right) = C_1 + O(\epsilon^2). \tag{2.165}$$

Substituting $y(z;\epsilon) = y_0(z) + \epsilon y_1(z) + O(\epsilon^2)$ into the ODE (2.165) leads to the system of ODEs

$$\begin{aligned} &(y_0')^2 + \frac{1-c}{c}y_0^2 = C_1, \\ &2y_0'y_1' + \frac{2(1-c)}{c}y_0 y_1 + (y_0')^2 + \frac{1-c}{c}y_0^2 - \frac{3y_0}{2c} + \frac{y_0^3}{2c} = 0. \end{aligned} \tag{2.166}$$

The solution of the above system is bounded if $0 < c < 1$. Hence when $0 < c < 1$, the solution of the system (2.166) is given by

$$\begin{aligned}
y_0(z) &= \left(C_1c^2+1\right)\sin\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right),\\
y_1(z) &= \frac{1}{16c(1-c)^2}\Bigg[4\sqrt{c(1-c)}\left(C_1c^{5/2}-C_1c^{3/2}+c^{1/2}-1\right)\sin\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)\\
&-(C_1c^2+1)\cos^2\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)+16C_3c(1-c)^2\cos\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)\\
&-C_1^2c^4-2C_1c^2-12c^2+12c-1\Bigg].
\end{aligned} \tag{2.167}$$

Consequently, a general approximate solution for the BBM ODE (2.161) is given by

$$\begin{aligned}
y(z;\epsilon) &= \left(C_1c^2+1\right)\sin\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)\\
&+\frac{\epsilon}{16c(1-c)^2}\Bigg[4\sqrt{c(1-c)}\left(C_1c^{5/2}-C_1c^{3/2}+c^{1/2}-1\right)\sin\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)\\
&-(C_1c^2+1)\cos^2\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)+16C_3c(1-c)^2\cos\left(\sqrt{\frac{1-c}{c}}(z-C_2)\right)\\
&-C_1^2c^4-2C_1c^2-12c^2+12c-1\Bigg],\ 0<c<1.
\end{aligned} \tag{2.168}$$

Hence the general approximate solution of the BBM PDE (2.160) is $u(x,t) = y(x-ct;\epsilon)$.

We could not find harmonic-type solutions of BBM PDE in literature. The exact solutions of different forms of BBM equation is given in terms of Jacobi elliptic functions $\mathrm{cn}(v,k), \mathrm{sn}(v,k), 0 \le k \le 1$. When $k \to 1$, one obtains the solitary wave solutions of BBM equation (see, e.g. [87–89]). For the BBM model (2.160), the explicit cnoidal wave solutions are given by

$$u(x,t) = a^2\mathrm{cn}^2(B_1(x-ct),k)+H_1, \tag{2.169}$$

$$u(x,t) = a^2\mathrm{sn}^2(B_2(x-ct),k)+H_2, \tag{2.170}$$

where B_1, B_2, H_1 and H_2 are given in terms of a, ϵ, c and k:

$$B_1 = \frac{a\sqrt{2\epsilon c}}{4ck}, \quad H_1 = \frac{2ck^2+\epsilon a^2-2\epsilon a^2k^2-2k^2}{3\epsilon k^2},$$
$$B_2 = \frac{a\sqrt{-2\epsilon c}}{4ck}, \quad H_2 = \frac{2ck^2-\epsilon a^2-\epsilon a^2k^2-2k^2}{3\epsilon k^2}.$$

When $k \to 1$, (2.169) reduces to a solitary wave solution:

$$u(x,t) = a^2\mathrm{sech}^2\left(\frac{a\sqrt{2\epsilon c}\,(x-ct)}{4c}\right)+\frac{2c-\epsilon a^2-2}{3\epsilon}. \tag{2.171}$$

When $c < 0$ and $k \to 1$, the solution (2.170) simplifies to a left-travelling kink wave solution:

$$u(x,t) = a^2\tanh^2\left(\frac{a\sqrt{-2\epsilon c}\,(x-ct)}{4c}\right)+\frac{2c-2\epsilon a^2-2}{3\epsilon}. \tag{2.172}$$

Note that harmonic-type solutions like (2.168) do not follow from (2.169) and (2.170) as $k \to 0^+$.

2.6.3 Reduction of order under contact and higher-order symmetries

The higher-order approximate symmetry generator for an n^{th}−order ODE (2.59)

$$y^{(n)} = f_0(x, y, y', ..., y^{(n-1)}) + \epsilon f_1(x, y, y', ..., y^{(n-1)}) \tag{2.173}$$

is given by

$$\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = \left(\zeta^0(x, y, y', ..., y^{(s)}) + \epsilon\zeta^1(x, y, y', ..., y^{(\ell)})\right)\frac{\partial}{\partial y}. \quad s, \ell \geq 1. \tag{2.174}$$

The differential functions

$$\omega_k(x, y, y', ..., y^{(k)}; \epsilon) = \omega_k^0(x, y, y', ..., y^{(k)}) + \epsilon\omega_k^1(x, y, y', ..., y^{(k)}) + o(\epsilon),\ k = 1, ..., n,$$

are *approximate differential invariants* for the ODE (2.173) if

$$\hat{X}^{(k)}\omega_k(x, y, y', ..., y^{(k)}; \epsilon) = o(\epsilon).$$

Note that ω_k^0 are exact differential invariants for the unperturbed ODE (2.58). They arise as constant of integrations of the characteristic equations

$$\frac{dy}{\zeta^0} = \frac{dy'}{\zeta^{0^{(1)}}} = \cdots = \frac{dy^{(k)}}{\zeta^{0^{(k)}}}. \tag{2.175}$$

Then the differential invariants ω_k^1 are determined from the following equation

$$H(\omega^1_{k_y}, \omega^1_{k_{y'}}, \cdots) = \hat{X}^{1^{(k)}}(\omega_k^0)\Big|_{y^{(n)}=f_0},$$

where H is a differential exoression in ω_k^1 resulting from the coefficients of ϵ in

$$-\left(\hat{X}^{0^{(k)}}\omega_k\right)\Big|_{y^{(n)}=f_0+\epsilon f_1}.$$

Example 2.6.4. The first example of using approximate differential invariants to reduce ODEs is rather elementary and is used here for illustration purposes. Consider the second-order ODE

$$y'' = \epsilon x(y')^2. \tag{2.176}$$

This ODE admits an approximate contact symmetry given by

$$\hat{X} = \hat{X}^0 + \epsilon\hat{X}^1 = \left(x + \epsilon\left(\frac{x^3y'}{3} + y'^2\right)\right)\frac{\partial}{\partial y}.$$

We determine the invariants $\omega(x, y, y'; \epsilon) = \omega^0(x, y, y') + \epsilon\omega^1(x, y, y') + o(\epsilon)$ satisfying $\hat{X}^{(1)}\omega = o(\epsilon)$. Clearly, one invariant is x. Other invariants are determined by first finding ω^0 satisfying

$$\hat{X^0}^{(1)}\omega^0 = x\omega^0_y + \omega^0_{y'} = 0,$$

which has a general solution $\omega^0(x, y, y') = F(xy' - y)$ based on the fundamental invariant $xy' - y$. Let $\omega^0(x, y, y') = xy' - y$. Then one finds that the first-order correction satisfies the inhomogeneous linear PDE

$$x\omega^1_y + \omega^1_{y'} = -\frac{2}{3}x^3y' - y'^2.$$

The simplest particular solution is given by $\omega^1(x,y,y') = -(x^3 y'^2 + y'^3)/3$. Consequently,

$$\omega = \omega^0(x,y,y') + \epsilon\omega^1 = xy' - y - \frac{\epsilon}{3}(x^3 y'^2 + y'^3) = C_1 + o(\epsilon) \tag{2.177}$$

is an approximate invariant for the ODE (2.176); here $C_1 = \text{const}$ is a constant of integration. Thus the ODE (2.176) approximately reduces to a first-order ODE. By substituting $y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + o(\epsilon)$ into the ODE (2.177) and setting to zero coefficients at ϵ^0 and ϵ^1, one gets a system of ODEs

$$xy_0' - y_0 = C_1, \quad xy_1' - y_1 - \frac{y_0'^3}{3} - \frac{x^3 y_0'^2}{3} = 0.$$

Its solution yields an approximate solution of the perturbed ODE (2.176)

$$y = C_2 x - C_1 + \frac{\epsilon}{6} C_2^2\, x^3 + O(\epsilon^2). \tag{2.178}$$

We note that the ODE (2.176) is solvable by separation of variables, which makes it easy to compare its general solution with the approximate solution (2.178). The general solution is

$$y = \sqrt{\frac{2C_2}{\epsilon}}\ \tanh^{-1}\left(\sqrt{\frac{C_2\,\epsilon}{2}}\, x\right) - C_1. \tag{2.179}$$

The first two terms of its Taylor series with respect to ϵ indeed coincide with the approximate solution (2.178).

Example 2.6.5. We find an approximate solution for the perturbed Boussinesq ODE (2.155) using third-order approximate symmetries admitted by (2.155). The fundamental solution of the unperturbed equation (2.119) is

$$y(x) = C_1 x + C_2 \sin x + C_3 \cos x + C_4. \tag{2.180}$$

The solution (2.180) is invariant under the group generated by

$$X_1^0 - C_1 X_3^0 - C_2 X_4^0 - C_3 X_5^0 - C_4 X_2^0 = (y - C_1 x - C_2 \sin x - C_3 \cos x - C_4)\,\frac{\partial}{\partial y}, \tag{2.181}$$

where X_j^0, $j = 1, ..., 5$ are the point symmetries (2.124) for the unperturbed ODE (2.119). X_2^0 is stable as a point symmetry, the corresponding approximate symmetry is $X^2 = (1 - \epsilon x y')\partial/\partial y$. At the same time, X_1^0, X_3^0, X_4^0 and X_5^0 are unstable as point symmetries. But using Theorem 2.5.1, they correspond to third-order approximate symmetries of (2.155) given by

$$X^1 = \left(y + \epsilon\left(\left(\frac{x^2}{2} + \frac{5}{6}\right) y'^2 + \left(\frac{x^2 y' + 3xy + 2y''}{2}\right) y'''\right)\right)\frac{\partial}{\partial y}, \tag{2.182}$$

$$X^3 = \left(x + \epsilon\left(\frac{xy + 3x^2 y'}{2} + 2x^2 y'''\right)\right)\frac{\partial}{\partial y}, \tag{2.183}$$

$$X^4 = \left(\sin x + \epsilon\frac{\left((3x^2 - 17)y' - 6xy + (3x^2 - 36)y'''\right)\cos x + (15y - 12xy' - 18xy''')\sin x}{6}\right)\frac{\partial}{\partial y}, \tag{2.184}$$

$$X^5 = \left(\cos x + \epsilon\frac{\left((17 - 3x^2)y' + 6xy + (36 - 3x^2)y'''\right)\sin x + (15y - 12xy' - 18xy''')\cos x}{6}\right)\frac{\partial}{\partial y}. \tag{2.185}$$

The approximately invariant solution under $X^1 - C_1X^3 - C_2X^4 - C_3X^5 - C_4X^2$ is given by

$$y - C_1x - C_2\sin x - C_3\cos x - C_4 + \epsilon h(x, y, y', y'', y''') = o(\epsilon), \tag{2.186}$$

where h is given by

$$\begin{aligned} h = & \left(\frac{x^2}{2} + \frac{5}{6}\right) y'^2 + \left(\frac{x^2y' + 3xy + 2y''}{2}\right) y''' - C_1 \left(\frac{xy + 3x^2y'}{2} + 2x^2y'''\right) \\ & - C_2 \left(\frac{\left((3x^2 - 17)y' - 6xy + (3x^2 - 36)y'''\right)\cos x + (15y - 12xy' - 18xy''')\sin x}{6}\right) \\ & - C_3 \left(\frac{\left((17 - 3x^2)y' + 6xy + (36 - 3x^2)y'''\right)\sin x + (15y - 12xy' - 18xy''')\cos x}{6}\right). \end{aligned} \tag{2.187}$$

Substitute $y(x;\epsilon) = y_0(x) + \epsilon y_1(x)$ into the equation (2.186), and equate the coefficients of ϵ^0, ϵ^1, we find $y_0 = C_1x + C_2\sin x + C_3\cos x + C_4$ and $y_1 = -h(x, y_0, y_0', y_0'', y_0''')$. Hence, the approximate solution of the Boussinesq ODE (2.155) is given by

$$\begin{aligned} y(x;\epsilon) = & C_1x + C_2\sin x + C_3\cos x + C_4 + \epsilon\Bigg[\left(\frac{7C_1C_3 + 5C_2C_4}{2}\right)\sin x + \left(\frac{C_1C_2 + 2C_3C_4}{2}\right) x\sin x \\ & + \frac{C_1C_3}{2}x^2\sin x + \left(\frac{15C_2^2 + 17C_3^2}{6}\right)\sin^2 x - \frac{C_2C_3}{3}\sin 2x + \left(\frac{5C_3C_4 - 7C_1C_2}{2}\right)\cos x \\ & + \left(\frac{C_1C_3 - 2C_2C_4}{2}\right) x\cos x - \frac{C_1C_2}{2}x^2\cos x + \left(\frac{17C_2^2 + 15C_3^2}{6}\right)\cos^2 x + C_1^2x^2 - C_1C_4x - \frac{C_1^2}{3}\Bigg]. \end{aligned} \tag{2.188}$$

With the initial conditions $y(0) = 1, y'(0) = 1, y''(0) = -1, y'''(0) = -1$, the unperturbed ODE (2.119) has a particular solution

$$y(x) = \sin x + \cos x. \tag{2.189}$$

Using this particular solution, and the following initial conditions

$$y(0) = 1 + \frac{16\epsilon}{3}, \quad y'(0) = 1 - \frac{2\epsilon}{3}, \quad y''(0) = -1, \quad y'''(0) = -1 + \frac{8\epsilon}{3}, \tag{2.190}$$

one finds $C_1 = 0$, $C_2 = 1$, $C_3 = 1$, and $C_4 = 0$. Thus the approximate solution (2.188) of the perturbed ODE (2.155) reduces to the following particular approximate solution

$$y(x;\epsilon) = \sin x + \cos x + \epsilon\left(\frac{16 - \sin 2x}{3}\right). \tag{2.191}$$

In order to test the accuracy of the approximate solution (2.191), we convert the perturbed fourth-order ODE (2.155) into a system of four first-order ODEs, and compute numerical solutions of the resulting system with the initial conditions (2.190) using the `Matlab` native ODE solver `ode45`. The solver employs an adaptive Dormand-Prince algorithm [90] based on the use of a fourth- and a fifth-order Runge-Kutta (RK) method pair [91]. At every discrete independent variable step $i \to i + 1$, the algorithm chooses the optimal Runge-Kutta coefficients to minimize the error of the fifth-order RK solution, and also find the optimal variable step h_i for efficient computation.

In particular, on each step, the difference between the fourth- and the fifth-order RK solution values is given by

$$e_{i+1} = \left\| u_{i+1}^{[5]} - u_{i+1}^{[4]} \right\|, \tag{2.192}$$

where each $u_k^{[j]}$ is a four-component vector providing a numerical approximation of the exact solution $u = [y(x_k), y'(x_k), y''(x_k), y'''(x_k)]$. The one-step difference (2.192) is controlled by user-defined relative and absolute tolerances `RelTol`, `AbsTol` according to

$$e_{i+1} \leq \max\{\texttt{RelTol} \cdot |u_i|, \texttt{AbsTol}\}. \tag{2.193}$$

If the ODE (2.155) is solved numerically for $x \in [0, L]$ using N numerical steps, the conservative estimate of the global numerical error at $x = L$, for the small parameter value ϵ, is given by

$$E_{\text{num}}(\epsilon) = \sum_{i=0}^{N} e_i. \tag{2.194}$$

The difference between the numerical and the approximate solution at a numerical grid node x_i is given by

$$d(x_i; \epsilon) = |y_{\text{num}}(x_i; \epsilon) - y_{\text{approx}}(x_i; \epsilon)|. \tag{2.195}$$

For a sample numerical-approximate solution computation, we use tolerance values

$$\texttt{RelTol} = 10^{-8}, \quad \texttt{AbsTol} = 10^{-9}. \tag{2.196}$$

For example, for $\epsilon = 0.1$, this choice yields $N = 381$ steps in x, with variable step sizes h ranging from 0.00616 to 0.031948.

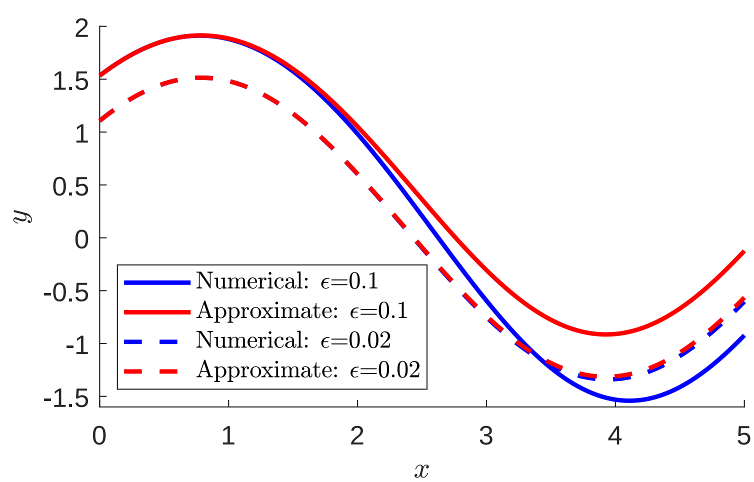


Figure 2.4: The approximate solution (2.191) of the perturbed Boussinesq equation (2.155) with initial conditions (2.190) vs. the numerical solution for the small parameter values $\epsilon = 0.02$ and $\epsilon = 0.1$.

Figure 2.4 shows numerical and approximate solution curves of $y(x)$ as functions of $x \in [0, L]$, $L = 5$, for $\epsilon = 0.02$ and $\epsilon = 0.1$. It is observed that for $\epsilon = 0.02$, the difference stays small for all x in the interval, while for a larger $\epsilon = 0.1$, the numerical and approximate solutions begin to differ significantly after $x \gtrsim 1$. (We note that for $\epsilon = 0$, the approximate solution (2.191) becomes exact, and the difference (2.195) is negligible).

To provide further details about the error and difference behaviour for the numerical and approximate solutions, Figure 2.5 shows the conservative estimate (2.194) of the total numerical error at $x = L$, the difference between the numerical and approximate solutions $d(l;\epsilon)$ (2.195) at $x = L$ as a function of ϵ, and also the typical behaviour of the difference (2.195) as a function of x for the specific small parameter value $\epsilon = 0.05$.

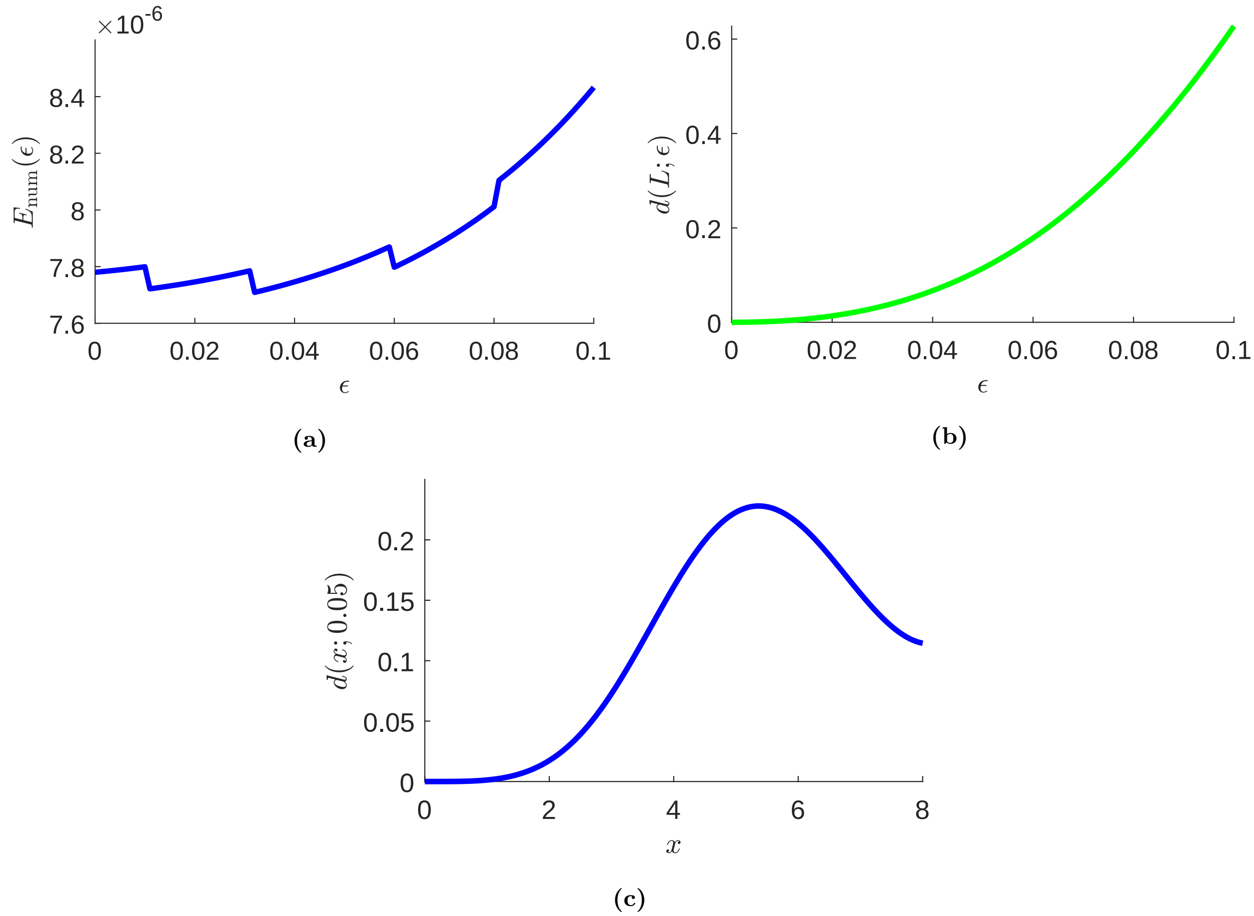


Figure 2.5: Numerical and approximate solution details for the Boussinesq ODE (2.155) with a small parameter ϵ. (a) The conservative estimate (2.194) of the total numerical error at $x = L = 5$ as a function of ϵ, for the tolerance values (2.196). (b) The difference between the numerical and approximate solutions $d(L;\epsilon)$ (2.195) at $x = L = 5$ as a function of ϵ. (c) The numerical-approximate solution difference $d(x;\epsilon)$ (2.195) as a function of x for the small parameter $\epsilon = 0.05$.

The above analysis indicates that for sufficiently small values of the parameter ϵ, the approximate solution (2.191) of the perturbed Boussinesq equation (2.155) indeed provides a precise approximation of the exact solution, with the error growing as the interval $x \in [0, L]$ lengthens and/or the parameter ϵ is increased.

2.7 Discussion

In the last three decades, BGI and FS approximate Lie symmetry frameworks [10–13] have been a subject of much discussion, including approximate symmetries and approximate solutions of perturbed models (e.g., [20,23,25,26,30]). However, the stability of exact local symmetries of unperturbed equations in terms of BGI and FS approximate symmetries of the perturbed models has not been addressed. In this chapter, we provided a complete answer to the question about stability of point and local symmetries of unperturbed algebraic equations and ODEs vs. their perturbed versions with a small parameter. In particular, we showed that all exact point symmetries of algebraic equations and first-order ODEs are stable (Theorem 2.2.2, Theorem 2.3.1 and Theorem 2.3.2). For higher-order ODEs, we showed that each point or local symmetry of an unperturbed ODE corresponds to a BGI approximate local symmetry of the perturbed model (Theorem 2.5.1).

The BGI and FS approaches are not equivalent. In the former, the approximate symmetry generator is expanded in a perturbation series in the small parameter with no change in the number of the independent and dependent variables. While in the latter, the dependent variables are expanded in a perturbation series. Therefore, the number of dependent variables of the original equations are doubled in the new equations with no small parameter. In Ref. [15], the FS approximate point symmetries of the Navier-Stokes equations were found and used to derive the approximate point symmetries in sense of BGI. For general perturbed diffusion equations, the BGI approximate point symmetries were obtained from the FS approximate point symmetries of the corresponding FS system [92]. For a perturbed first-order ODE and for a family of perturbed higher-order ODEs, we found a connection between BGI and FS approximate point symmetries (Theorem 2.3.3 and Theorem 2.4.2, respectively).

Contact and higher-order exact symmetries can be used to construct solutions for ordinary differential equations (e.g., [3]). In Ref. [41], it was shown how exact integrating factors for linear and nonlinear ordinary differential equations can be determined. A perturbation method based on integrating factors was developed for a system of regularly perturbed first-order ODEs [93]. In this chapter, we introduced approximate integrating factors using approximate point symmetries (Section 2.6.1) and we showed that the components of an approximate integrating factor for a perturbed first-order ODE yield a BGI approximate point symmetry for the same ODE (Theorem 2.6.1). We derived the determining equations for approximate integrating factors of perturbed second-order ODEs (Theorem 2.6.2). As an application, we found an approximate solution (2.159) of the perturbed Boussinesq ODE (2.155) and an approximate solution (2.168) of the perturbed BBM ODE (2.161). We presented a method consists of approximate reduction of order under contact and higher-order symmetries (Section 2.6.3). We applied this method to find an approximate solution of a perturbed second-order ODE (2.176) using admitted approximate contact symmetries (Example 2.6.4), and we used the approximate higher-order symmetries obtained using Theorem 2.5.1 to find another approximate solution for the fourth-order Boussinesq ODE (2.155) (Example 2.6.5). In the latter, the approximate solution was validated via a comparison to numerical solutions of the Boussinesq equation (2.155).

3 Exact and Approximate Symmetries of PDEs with a Small Parameter

3.1 Introduction

In this Chapter, we study exact and approximate (BGI and FS) symmetry properties for scalar PDEs with a small parameter.

In Chapter 2, we showed that all exact symmetries of a first-order ODE yield an infinite set of BGI and FS approximate point symmetries of the perturbed ODE. We proved that all exact symmetries of an unperturbed higher-order ODE are stable and they correspond to point or higher-order BGI approximate symmetries for the perturbed model. For PDEs, the situation is different: in general, unstable point symmetries of unperturbed PDEs do not yield higher-order approximate symmetries for perturbed PDEs. The reason for this instability and an illustration are given. As a detailed example, we investigate the stability in terms of BGI and FS frameworks of exact point symmetries for a family of one-dimensional wave equation involving an arbitrary function. This type of classification and comparison is important in sense that it helps to illustrate the difference between the both approaches and to check the stability of exact point symmetries and their corresponding approximate symmetry structures in both frameworks. For a general perturbed PDE, we prove that there is a relation between BGI and FS approximate local symmetries.

The exact symmetry classification problem for one-dimensional wave equations in different forms has been considered in many articles. In Ref. [94], the group properties of the nonlinear wave equation

$$u_{tt} = (f(u)u_x)_x \tag{3.1}$$

were discussed, (here and below subscripts denote partial derivatives). Bluman and Cheviakov [95] extended the group classification of the nonlinear wave equation (3.1) through a systematic construction of nonlocal symmetries. Point symmetry classifications for the generalized classes

$$u_{tt} = (f(x,u)u_x)_x, \quad u_{tt} = (f(u)u_x + g(x,u))_x \tag{3.2}$$

of (1.13) were considered in [96, 97]. The point symmetry classification for the PDE family

$$u_{tt} = f(x,u_x)u_{xx} + g(x,u_x) \tag{3.3}$$

was investigated in [98]. Further classifications of different classes of one-dimensional wave equation can be found, for example, in [99–102]. In this chapter, we classify exact point symmetries of the two-dimensional wave equation family (1.2) and exact and approximate point symmetries of the perturbed wave equations (1.3) and (1.4). We use new approximate symmetries to construct approximate solutions for a perturbed one-dimensional wave equation (1.3) with $T(u_x) = u_x^s$:

$$u_{tt} = (c^2 + \epsilon u_x^s)u_{xx}, \quad u = u(x,t). \tag{3.4}$$

As hyperbolic systems, wave equations have characteristic curves (or surfaces) along which the solution to the equation are simplified. If the characteristic curves intersect, the solution may become multi-valued. This is referred to as a shock or break in the wave and can have the physical meaning of a discontinuous solution [103]. A class of singularities that occurs in dynamical systems and reached in a finite time are called finite-time singularities [104]. Finite-time singularities have been found in many models, such as in the Euler equations of inviscid fluids [105], in the equations of general relativity coupled to a mass field leading to the formation of black holes [106]. In this Chapter, we use the numerically computed characteristic curves to approximate the finite-time singularity formation of (3.4) with $s = 2$. We also use an approximate solution of (3.4) to provide an alternative estimate of the finite-time singularities.

3.2 Exact local symmetries of unperturbed and perturbed PDEs

Let $x = (x^1, x^2, ..., x^n)$, $n > 1$, and $u = u(x)$ denote respectively the independent variables and the dependent variable of a given problem. We also denote partial derivatives by subscripts: $\partial u/\partial x_j \equiv u_j$, etc., and the set of all partial derivatives of u of order q by $\partial^k u$. A general k^{th}-order scalar PDE on u has the form

$$F_0[u] \equiv F_0(x, u, \partial u, \ldots, \partial^k u) = 0, \quad k \geq 1. \tag{3.5}$$

We assume that the PDE (3.5) as it stands, or after a point transformation, is of generalized Kovalevskaya type [4], that is, can be written in a *solved form* with respect to the highest pure derivative of u by one of the independent variables.

The *solution set* $\mathcal{S}$ of the PDE (3.5) in the jet space $J^s(x|u)$, $s \geq k$, is a hypersurface defined by the relations $F_0[u] = 0$ and its differential consequences $\partial F_0[u] = 0$, …, solved for the corresponding differential consequences of the leading derivative, up to the highest order s. Any differential function $f[u]$ can be evaluated *on the solution set* of (3.5) by substituting the expressions of the leading derivative and its differential consequences into $f[u]$, and the result is denoted by $f[u]|_{\mathcal{S}}$ or $f[u]|_{F_0[u]=0}$.

The exact symmetry generator for the unperturbed PDE (3.5) has the form

$$X^0 = \xi_0^i(x,u)\frac{\partial}{\partial x^i} + \eta_0(x,u)\frac{\partial}{\partial u}\,. \tag{3.6}$$

The determining equation (1.30) to find exact point symmetries of (3.5) reads

$$X^{0(k)} F_0[u]\big|_{F_0[u]=0} = 0 \tag{3.7}$$

in terms of the prolonged generator $X^{0(k)}$(1.29). The determining equation splits into an overdetermined PDE system on the unknown symmetry components ξ_0^i, η_0 (see, e.g., [4,37]). The evolutionary form of (3.6) is given by

$$\hat{X}^0 = \zeta_0[u]\frac{\partial}{\partial u}, \tag{3.8}$$

with the evolutionary component $\zeta_0[u] = \eta_0(x,u) - u_i\xi_0^i(x,u)$. Local (point, contact and higher-order) transformations and the related point, contact and higher-order local symmetries of the PDE (3.5) generalize (3.8) by allowing the evolutionary infinitesimal components $\zeta_0 = \zeta_0[u]$ to be general differential functions of u, depending on first and/or higher-order derivatives of u (see, e.g., Refs. [4,37] and references therein). The invariance condition (3.7) takes the form

$$\hat{X}^{0(k)}F_0[u]\big|_{F_0[u]=0} = 0, \tag{3.9}$$

where the prolongation of $\hat{X}^0$ is defined by

$$\hat{X}^{0(k)} = \zeta_0\frac{\partial}{\partial u} + (D_j\zeta_0)\frac{\partial}{\partial u_j} + \ldots . \tag{3.10}$$

The following elementary example will serve as a basis of further examples involving PDEs with a small parameter, and their exact and approximate symmetries.

Example 3.2.1. Consider a nonlinear wave-type equation [107]

$$u_{tt} = u_x u_{xx}, \quad u = u(x,t). \tag{3.11}$$

The exact symmetry generator for the PDE (3.11) is given by

$$X^0 = \xi_0^1(x,t,u)\frac{\partial}{\partial x} + \xi_0^2(x,t,u)\frac{\partial}{\partial t} + \eta_0(x,t,u)\frac{\partial}{\partial u}. \tag{3.12}$$

The determining equations (3.7) yield the solution

$$\xi_0^1 = C_4 + C_6 x, \quad \xi_0^2 = C_3 + \left(C_6 - \frac{C_5}{2}\right)t, \quad \eta_0 = C_1 + C_2 t + (C_5 + C_6)u. \tag{3.13}$$

Consequently, the PDE (3.11) admits six point symmetries given by

$$\begin{gathered} X_1^0 = \frac{\partial}{\partial u}, \quad X_2^0 = t\frac{\partial}{\partial u}, \quad X_3^0 = \frac{\partial}{\partial t}, \quad X_4^0 = \frac{\partial}{\partial x}, \\ X_5^0 = u\frac{\partial}{\partial u} - \frac{t}{2}\frac{\partial}{\partial t}, \quad X_6^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \end{gathered} \tag{3.14}$$

corresponding to three translations (X_1^0, X_3^0 and X_4^0), the Galilei group (X_2^0), and two scalings (X_5^0 and X_6^0).

A general first-order perturbation of a PDE (3.5) is a partial differential equation

$$F[u;\epsilon] = F_0[u] + \epsilon F_1[u] = o(\epsilon) \tag{3.15}$$

involving a small parameter ϵ. We assume that the perturbation $F_1[u]$ is *regular*, in the sense that the Kovalevskaya forms of the unperturbed PDE (3.5) and its perturbation (3.15) have the same leading derivatives. Exact point and local symmetry generators of the perturbed PDE (3.15) have the forms

$$Y = \alpha^i(x,u;\epsilon)\frac{\partial}{\partial x^i} + \beta(x,u;\epsilon)\frac{\partial}{\partial u}\,, \qquad \hat{Y} = \zeta[u;\epsilon]\frac{\partial}{\partial u}. \tag{3.16}$$

To find the exact symmetries of (3.15) holding for an arbitrary ϵ, one solves the determining equations (3.7) or (3.9).

It is commonly the case that due to the perturbation term, some or even all point and/or local symmetries of the unperturbed equations (3.5) disappear from the local symmetry classification of the perturbed model (3.15).

Example 3.2.2. We compute exact point symmetries of a perturbed version of the PDE (3.11),

$$u_{tt} + \epsilon u u_t = u_x u_{xx} \tag{3.17}$$

holding for an arbitrary ϵ. The leading derivative u_{tt} can be chosen for both (3.11) and (3.17). We obtain a Lie algebra of point symmetries spanned by

$$Y_1 = X_3^0 = \frac{\partial}{\partial t}, \quad Y_2 = X_4^0 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{4}{3}X_5^0 - \frac{1}{3}X_6^0 = -t\frac{\partial}{\partial t} - \frac{x}{3}\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\,, \tag{3.18}$$

a three-dimensional subalgebra of the six-dimensional Lie algebra of point symmetries (3.14) of the unperturbed wave equation (3.11).

3.3 BGI approximate point and local symmetries of a perturbed PDE

Approximate symmetries can be useful for finding additional symmetry-like structures for the perturbed equation (3.15) [23]. The BGI approximate point symmetry generator for the perturbed PDE (3.15) is given by

$$X = X^0 + \epsilon X^1 = \left(\xi_0^i(x,u) + \epsilon\xi_1^i(x,u)\right)\frac{\partial}{\partial x^i} + \left(\eta_0(x,u) + \epsilon\eta_1(x,u)\right)\frac{\partial}{\partial u}. \tag{3.19}$$

The $O(\epsilon)$ term of the determining equation (1.96) leads to the PDEs

$$X^{1\,(k)}F_0[u]\Big|_{F_0[u]=0} = H[u], \tag{3.20}$$

where H is obtained from the coefficients of ϵ in the expression

$$-X^{0\,(k)}(F_0[u] + \epsilon F_1[u])\Big|_{F_0[u]+\epsilon F_1[u]=o(\epsilon)}. \tag{3.21}$$

An alternative procedure for the calculation of BGI symmetries involves writing down exact symmetry determining equations for $F[u;\epsilon] = 0$ (cf. (3.15)), substituting $\zeta[u] = \zeta_0[u] + \epsilon\zeta_1[u]$, and collecting $O(1)$ and $O(\epsilon)$ coefficients of each split determining equation.

Similarly to exact local transformations with generators of the form (3.10), one can define more general *BGI approximate local transformations* with generators in evolutionary form given by

$$\hat{X} = \hat{X}^0 + \epsilon \hat{X}^1 = (\zeta_0[u] + \epsilon\zeta_1[u])\,\frac{\partial}{\partial u}\,. \tag{3.22}$$

BGI approximate local (including point, contact, and higher-order) symmetries of the perturbed PDE (3.15) can be found using the same procedure as described above for BGI approximate point symmetries. In particular, the analog of the first-order condition (3.20) takes the form

$$\left(\zeta_1 \frac{\partial}{\partial u} + (\zeta_1)_i \frac{\partial}{\partial u_i} + \ldots + (\zeta_1)_{i_1\ldots i_k} \frac{\partial}{\partial u_{i_1\ldots i_k}}\right) F_0[u]\bigg|_{F_0[u]=0} = H[u], \tag{3.23}$$

where the prolongations of the evolutionary components are

$$(\zeta_1)_{i_1\ldots i_k} = D_{i_1}\ldots D_{i_k}\,\zeta_1. \tag{3.24}$$

3.4 Fushchich-Shtelen approximate point and local symmetries of a perturbed PDE

Substituting the solution

$$u(x) = v(x) + \epsilon w(x) + o(\epsilon) \tag{3.25}$$

into the perturbed PDE (3.15) and equating to zero the coefficients $O(1)$ and $O(\epsilon)$, one obtains the Fushchich-Shtelen system given by

$$G_1[v,w] \equiv F_0[v] = 0, \tag{3.26a}$$

$$G_2[v,w] \equiv F_{0v}w + F_{0v_i}w_i + F_{0v_{ij}}w_{ij} + \ldots + F_{0v_{i_1 i_2\ldots i_k}}w_{i_1 i_2\ldots i_k} + F_1[v] = 0. \tag{3.26b}$$

Note that the FS system (3.26) is a Kovalevskaya system with respect to the same leading derivatives (for v and w) as the original PDE (3.15) (for u). The exact symmetry generator for the FS system (3.26) can be sought in the form

$$Z = \lambda^i(x,v,w)\frac{\partial}{\partial x^i} + \phi_1(x,v,w)\frac{\partial}{\partial v} + \phi_2(x,v,w)\frac{\partial}{\partial w} \tag{3.27}$$

The local (point or higher-order) symmetry generator for the FS system (3.26) in evolutionary form is given by

$$\hat{Z} = \psi_1[v,w]\frac{\partial}{\partial v} + \psi_2[v,w]\frac{\partial}{\partial w} \tag{3.28}$$

Remark 3.4.1. Similar to FS approximate symmetry properties for a perturbed ODE (Remark 2.4.2), there is also a possibility of existence of FS approximate symmetries for the perturbed PDE (3.15) where the component $\psi_1[v,w]$ of the generator (3.28) depends on w. Such FS symmetries do not correspond to stable local symmetries of the unperturbed PDE (3.5), and cannot arise in the BGI framework. An example is provided by the linear PDE $u_{tt} + \epsilon u_t = u_{xx}$ which admits a FS point symmetry (3.28) with

$$\psi_1[v,w] = tv + 2w + 2xtv_x + (x^2+t^2)v_t, \qquad \psi_2[v,w] = \frac{1}{2}x^2 v - tw + 2xtw_x + (x^2+t^2)w_t.$$

3.5 BGI and FS approximate symmetries: properties, connections, and examples

3.5.1 A computational example: exact and approximate point symmetry classification for a second-order nonlinear PDE with a small parameter

In this section, we compare exact point symmetries (3.14) of the (1+1)-dimensional wave equation $u_{tt} = u_x u_{xx}$ (3.11) with BGI and FS approximate local symmetry classifications for the family of perturbed wave equations

$$u_{tt} + \epsilon F_1(u, u_t) = u_x u_{xx}, \tag{3.29}$$

where $F_1(u, u_t)$ is an arbitrary function.

The PDE (3.11) admits six exact point symmetries given by (3.14). The BGI approximate point symmetries are computed and classified following the procedure described in Section 3.3, and the FS approximate local symmetries are obtained following Section 3.4. In particular, the Fushchich-Shtelen system (3.26) for (3.29) is given by

$$\begin{aligned} &v_{tt} - v_x v_{xx} = 0, \\ &w_{tt} + F_1(v, v_t) - v_x w_{xx} - w_x v_{xx} = 0, \end{aligned} \tag{3.30}$$

where $u(x,t) = v(x,t) + \epsilon w(x,t)$. The resulting classification is presented in Table 3.1. In the table, Q_i denote arbitrary functions of their arguments, and a_j arbitrary constants. The table is organized as follows. The first column lists evolutionary forms of the six point symmetry generators X_k^0 (3.14), $k = 1, \ldots, 6$, of the unperturbed PDE (3.11). The second column lists the forms of the arbitrary function $F_1(u, u_t)$ for which the corresponding X_k^0 is stable in the BGI sense, and the corresponding BGI approximate point symmetry of the perturbed wave equation (3.29). The third column contains the same information for the FS approximate local symmetries of the perturbed equation (3.29).

Table 3.1 illustrates differences between BGI and FS frameworks as there are stable symmetries in one framework and unstable in the other framework. Some specific examples are listed below.

- When $F_1 = u_t$ or $F_1 = \text{const}$, all point symmetries (3.14) of the unperturbed PDE (3.11) are stable as BGI and FS symmetries.

- When $F_1 = uu_t$, the u-translation symmetry $\hat{X}_1^0$ is stable as a BGI approximate point symmetry (generator $\hat{X}_1$) and stable as a FS approximate local symmetry ($\hat{Z}_1$). Similar argument holds for the point symmetry $\hat{X}_2^0$ when $F_1 = u_t^2$.

$\hat{X}_i^0$	BGI cases, approximate symmetry $\hat{X}_i$	FS cases, approximate symmetry $\hat{Z}_i$
$\hat{X}_1^0 = \frac{\partial}{\partial u}$	$F_1 = a_1 u u_t + a_2 u + Q_1(u_t)$, $\hat{X}_1 = \left(1 - \epsilon\left(\frac{2a_1}{5}tu + \frac{a_1}{10}t^2 u_t + \frac{a_2}{2}t^2\right)\right)\frac{\partial}{\partial u}$	$F_1 = e^{a_1 v}Q_4(v_t) + a_2 v_t + a_3 v v_t + a_4 v + a_5$, $\hat{Z}_1 = \frac{\partial}{\partial v} + \left(\frac{a_1 a_2}{10}t^2 v_t + \frac{2a_1 a_2}{5}tv + a_1 w - \frac{a_3}{10}t^2 v_t - \frac{a_3}{5}tv - \frac{a_4}{2}t^2\right)\frac{\partial}{\partial w}$
$\hat{X}_2^0 = t\frac{\partial}{\partial u}$	$F_1 = a_1 u_t + a_2 u + a_3 u_t^2 + a_4$, $\hat{X}_2 = \left(t - \epsilon\left(\frac{a_1}{2}t^2 + \frac{a_2}{6}t^3 + \frac{4a_3}{5}tu + \frac{a_3}{5}t^2 u_t\right)\right)\frac{\partial}{\partial u}$	• $F_1 = a_1 v_t + a_2 v + a_3 v_t^2 + a_4$, $\hat{Z}_2 = t\frac{\partial}{\partial v} - \left(\frac{a_1}{2}t^2 + \frac{a_2}{6}t^3 + \frac{4a_3}{5}tv + \frac{a_3}{5}t^2 v_t\right)\frac{\partial}{\partial w}$ • $F_1 = e^{a_1 v_t}, \hat{Z}_2 = t\frac{\partial}{\partial v} + a_1 w\frac{\partial}{\partial w}$
$\hat{X}_3^0 = u_t\frac{\partial}{\partial u}$	$F_1 = F_1(u, u_t),\ \hat{X}_3 = u_t \frac{\partial}{\partial u}$	$F_1 = F_1(v, v_t),\ \hat{Z}_3 = v_t \frac{\partial}{\partial v} + w_t \frac{\partial}{\partial w}$
$\hat{X}_4^0 = u_x\frac{\partial}{\partial u}$	$F_1 = F_1(u, u_t),\ \hat{X}_4 = u_x \frac{\partial}{\partial u}$	$F_1 = F_1(v, v_t),\ \hat{Z}_4 = v_x \frac{\partial}{\partial v} + w_x \frac{\partial}{\partial w}$
$\hat{X}_5^0 = \left(u + \frac{tu_t}{2}\right)\frac{\partial}{\partial u}$	$F_1 = a_1 + a_2 u_t + u^2 Q_2\left(u_t/u^{3/2}\right)$ $\hat{X}_5 = \left(u + \frac{tu_t}{2} + \epsilon\left(a_1 t^2 + a_2\left(\frac{tu}{5} + \frac{t^2 u_t}{20}\right)\right)\right)\frac{\partial}{\partial u}$	$F_1 = a_3 v^{a_1} v_t^{a_2} + a_4$, $\hat{Z}_5 = \left(v + \frac{tv_t}{2}\right)\frac{\partial}{\partial v} + \left(\left(\frac{a_1}{2} + \frac{3a_2}{4}\right)a_4 t^2 + \frac{tw_t}{2} + \left(a_1 + \frac{3a_2}{2} - 1\right)w\right)\frac{\partial}{\partial w}$
$\hat{X}_6^0 = (u - xu_x - tu_t)\frac{\partial}{\partial u}$	$F_1 = u^{-1}Q_3(u_t) + a_1 u_t + a_2$, $\hat{X}_6 = \left(u - xu_x - tu_t - \epsilon\left(\frac{a_1}{10}t^2 u_t + \frac{2a_1}{5}tu + \frac{a_2}{2}t^2\right)\right)\frac{\partial}{\partial u}$	• $F_1 = v^{a_1}Q_5(v_t) + a_2$, $\hat{Z}_6 = (v - xv_x - tv_t)\frac{\partial}{\partial v} + ((a_1 + 2)w + \frac{a_1 a_2}{2}t^2)\frac{\partial}{\partial w}$ • $F_1 = a_1 v_t v^{a_2} + a_3 v_t + a_4 v^{a_2} + a_5$, $\hat{Z}_6 = (v - xv_x - tv_t)\frac{\partial}{\partial v} + \left((a_2 + 2)w + \frac{a_2 a_3}{10}t^2 v_t + \frac{2a_2 a_3}{5}tv + \frac{a_2 a_5}{2}t^2 - xw_x - tw_t\right)\frac{\partial}{\partial w}$

Table 3.1: Stability of point symmetries of the wave equation (3.11) in terms of BGI and FS approximate local symmetries of the perturbed PDE (3.29), depending on the form of the arbitrary function F_1.

- For all forms $F_1(u, u_t)$, the PDE (3.29) and the FS system (3.30) are invariant under t- and x-translations. Consequently, the exact symmetries $\hat{X}_3^0$ and $\hat{X}_4^0$ reappear as BGI and FS approximate symmetries without change.
- When $F_1 = u^{-1}Q_3(u_t)$, the scaling symmetry $\hat{X}_6^0$ is stable in both the BGI and FS sense, with the corresponding generators $\hat{X}_6$ and $\hat{Z}_6$.
- When $F_1 = u^{a_1}Q_3(u_t)$, $a_1 \neq -1$, the symmetry $\hat{X}_6^0$ is stable in FS sense (generator $\hat{Z}_6$) but unstable as a BGI approximate point symmetry.

We also note that genuine BGI approximate symmetries are given by $\hat{X}_1$, $\hat{X}_2$, $\hat{X}_5$, and $\hat{X}_6$; genuine FS approximate symmetries are given by $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Z}_5$, and $\hat{Z}_6$.

3.5.2 Instability of local symmetries of unperturbed PDEs in terms of higher-order approximate symmetries: an example

For an ordinary differential equation (ODE), all local symmetries are stable in the BGI sense: each local symmetry of a given ODE corresponds to a BGI approximate local, often higher-order, symmetry of its perturbed version [57]. For a PDE, in general, this is not the case. For the BGI framework, differential functions $(\zeta_1)_{i_1\ldots i_p}$ (3.24) in the determining equation (3.23) contain derivatives of u of orders higher than those in the differential function ζ^1. It follows that the left-hand size of equation (3.23) splits into a system of linear PDEs in ζ_1. On the other side, the function H may contain derivatives of u with respect to other variables different than those in the left-hand side of equation (3.23). This can lead to some constraints on the unperturbed symmetry component ζ_0; in that case, an exact local symmetry of the unperturbed PDE (3.5) may not correspond to a local approximate BGI symmetry of the perturbed PDE (3.15). A similar argument holds for FS approximate symmetries. The main reason, as it can be seen in the example below, is the existence of multiple kinds of derivatives in PDEs, and thus more restrictive conditions that arise for ζ_0 when the determining equations are being split with respect to higher-order derivatives.

As an illustration, consider the PDE (3.29) with $F_1(u, u_t) = uu_t$:

$$u_{tt} + \epsilon u u_t = u_x u_{xx} \tag{3.31}$$

and the related Fushchich-Shtelen system (3.30)

$$v_{tt} - v_x v_{xx} = 0, \qquad w_{tt} + v v_t - v_x w_{xx} - w_x v_{xx} = 0. \tag{3.32}$$

From Table 3.1 one can see that $\hat{X}_2^0 = t\,\partial/\partial u$ is unstable as a point symmetry in both BGI and FS frameworks, that is, the point symmetry $\hat{X}_2^0$ admitted by the PDE (3.31) with $\epsilon = 0$ corresponds to no approximate point symmetry arising from BGI or FS approaches. First we examine whether or not it is possible to construct a local, possibly higher-order, BGI approximate symmetry of (3.31) that would correspond to $\hat{X}_2^0$. The generator of such a symmetry would have the form

$$\hat{X}_2 = C_2 \hat{X}_2^0 + \epsilon \hat{X}_2^1 = (C_2 t + \epsilon \zeta_1[u]) \frac{\partial}{\partial u}\,, \tag{3.33}$$

where $C_2 = \text{const} \neq 0$. The determining equation (3.23) for BGI local symmetries reads

$$\left(D_t^2 \zeta_1 - u_x D_x^2 \zeta_1 - u_{xx} D_x \zeta_1\right)\Big|_{u_{tt} = u_x u_{xx}} = H, \tag{3.34}$$

where one readily finds

$$H = C_2 (t u_t + u). \tag{3.35}$$

One can show by a direct computation that whatever the dependence of ζ_1 on partial derivatives of u is chosen to be, higher-order derivatives of u that arise in (3.34) lead to constraints on C_2 that result in $C_2 = 0$, which means that no nontrivial BGI point symmetry (3.33) corresponding to $\hat{X}_2^0$ exists.

Second, we seek a local, possibly higher-order, approximate FS symmetry of the PDE (3.31) corresponding to $\hat{X}_2^0$. Such a symmetry would arise as an exact local symmetry of Fushchich-Shtelen system (3.32). The corresponding evolutionary generator (3.28) has the form

$$Z = \psi_1[v,w]\frac{\partial}{\partial v} + \psi_2[v,w]\frac{\partial}{\partial w}. \tag{3.36}$$

As noted in Section 1.3.3, the determining equation for the first equation of the system (3.32) is satisfied when $\psi_1 = C_2 t$ as in $\hat{X}_2^0$. Now the determining equation for the second PDE of (3.32) leads to

$$\left(D_t^2\psi_2 - v_x D_x^2\psi_2 - v_{xx}D_x\psi_2\right)\Big|_{v_{tt}=v_x v_{xx},\, w_{tt}=-vv_t+v_x w_{xx}+w_x v_{xx}} = C_2(tv_t + v). \tag{3.37}$$

It can be shown that for any dependence $\psi_2[v,w]$, constraints on C_2 exist, leading to $C_2 = 0$. Consequently, there is no higher-order FS symmetry corresponding to the unstable point symmetry $\hat{X}_2^0$ admitted by the wave equation (3.31) with $\epsilon = 0$.

3.5.3 A relation between BGI and FS approximate symmetries

The computational example of Section 3.5.1 above illustrated the fact that BGI and FS frameworks can yield rather different approximate point symmetry classifications for the same PDE with a small parameter. However, in certain situations, the two approaches can lead to related results. We now show that for a specific class of (1+1)-dimensional PDEs, a stable BGI approximate point symmetry always correspond to a stable FS approximate local symmetry.

Consider the following class of PDEs on $u(x,t)$, written in the Kovalevskaya form with respect to an independent variable t:

$$\frac{\partial^n u}{\partial t^n} = F_0[u], \qquad F_0[u] \equiv F_0(x,t,u,\partial u,\partial^2 u,...,\partial^k u)\,, \tag{3.38}$$

and its perturbed version with a small parameter ϵ:

$$\frac{\partial^n u}{\partial t^n} = F_0[u] + \epsilon F_1[u], \qquad F_1[u] \equiv F_1(x,t,u,\partial u,\partial^2 u,...,\partial^\ell u). \tag{3.39}$$

A local BGI approximate symmetry of a PDE (3.39) has the form (3.22)

$$\hat{X} = \hat{X}^0 + \epsilon\hat{X}^1 = (\zeta_0[u] + \epsilon\zeta_1[u])\,\frac{\partial}{\partial u}\,. \tag{3.40}$$

As per Theorem 1.3.2, the $O(1)$ term in (3.40) corresponds to a local symmetry

$$\hat{X}^0 = \zeta_0[u]\,\frac{\partial}{\partial u} \tag{3.41}$$

of the unperturbed equation (3.38).

In order to compute FS approximate symmetries of a PDE (3.39), we substitute $u(x,t) = v(x,t) + \epsilon w(x,t) + o(\epsilon)$ into (3.39) and split the orders of ϵ to get the Fushchich-Shtelen system

$$\begin{aligned} \frac{\partial^n v}{\partial t^n} &= F_0[v], \\ \frac{\partial^n w}{\partial t^n} &= F_{0v}w + F_{0v_i}w_i + F_{0v_{ij}}w_{ij} + ... + F_{0v_{i_1 i_2\ldots i_k}}w_{i_1 i_2\ldots i_k} + F_1[v]. \end{aligned} \tag{3.42}$$

The evolutionary generator of a FS approximate local symmetry has the form (3.28). The determining equations (3.7) for exact local symmetries of (3.42) are

$$\hat{Z}^{(n)}\left(\frac{\partial^n v}{\partial t^n} - F_0\right) = 0\,, \tag{3.43a}$$

$$\hat{Z}^{(n)}\left(\frac{\partial^n w}{\partial t^n} - F_{0_v} w - F_{0_{v_i}} w_i - F_{0_{v_{ij}}} w_{ij} - \ldots - F_{0_{v_{i_1 i_2 \ldots i_k}}} w_{i_1 i_2 \ldots i_k} - F_1\right) = 0\,, \tag{3.43b}$$

holding on solutions of (3.42).

Theorem 3.5.1. *If* (3.40) *is a BGI approximate local symmetry generator of a PDE* (3.39) *having the specific form*

$$\hat{X} = (\zeta_0(x,t) + \epsilon \zeta_1(x,t,u,u_x,u_t))\frac{\partial}{\partial u} \tag{3.44}$$

and additionally, $F_0[u]$ *in* (3.39) *satisfies the following system of equations*

$$\begin{aligned}
&\zeta^0 F_{0_{uu}} + \zeta_i^{0^{(1)}} F_{0_{uu_i}} + \zeta_{i_1 i_2}^{0^{(2)}} F_{0_{uu_{i_1 i_2}}} + \ldots + \zeta_{i_1 i_2 \ldots i_k}^{0^{(k)}} F_{0_{uu_{i_1 i_2 \ldots i_k}}} = 0,\\
&\zeta^0 F_{0_{uu_i}} + \zeta_i^{0^{(1)}} F_{0_{u_i u_i}} + \zeta_{i_1 i_2}^{0^{(2)}} F_{0_{u_i u_{i_1 i_2}}} + \ldots + \zeta_{i_1 i_2 \ldots i_k}^{0^{(k)}} F_{0_{u_i u_{i_1 i_2 \ldots i_k}}} = 0,\\
&\vdots\\
&\zeta^0 F_{0_{uu_{i_1 i_2 \ldots i_k}}} + \zeta_i^{0^{(1)}} F_{0_{u_i u_{i_1 i_2 \ldots i_k}}} + \zeta_{i_1 i_2}^{0^{(2)}} F_{0_{u_{i_1 i_2} u_{i_1 i_2 \ldots i_k}}}\\
&\qquad + \ldots + \zeta_{i_1 i_2 \ldots i_k}^{0^{(k)}} F_{0_{u_{i_1 i_2 \ldots i_k} u_{i_1 i_2 \ldots i_k}}} = 0\,.
\end{aligned} \tag{3.45}$$

Then

$$\hat{Z} = \zeta^0(x,t)\frac{\partial}{\partial v} + \zeta^1(x,t,v,v_x,v_t)\frac{\partial}{\partial w} \tag{3.46}$$

is a FS approximate local symmetry of the perturbed PDE (3.39) *corresponding to the point symmetry generator* $\hat{X}^0 = \zeta^0\,\partial/\partial v$ *of the unperturbed PDE* (3.38).

Proof. We need to show that under the stated conditions, the determining equations (3.20) for BGI approximate symmetries of (3.39) are equivalent to the determining equations for FS approximate symmetries. Since the first PDE of the Fushchich-Shtelen system (3.42) is the same as the unperturbed equation (3.38), first FS determining equation (3.43a) is satisfied for any ζ^0 and ζ^1 as long as ζ^0 is an exact point symmetry component of (3.38).

The second FS determining equation (3.43b) with $\psi_1 = \zeta^0$, $\psi_2 = \zeta^1$ can be rewritten as

$$\left(\zeta_t^{1^{(n)}} - \zeta^1 F_{0_v} - \zeta_i^{1^{(1)}} F_{0_{v_i}} - \zeta_{i_1 i_2}^{1^{(2)}} F_{0_{v_{i_1 i_2}}} - \ldots - \zeta_{i_1 i_2 \ldots i_k}^{1^{(k)}} F_{0_{v_{i_1 i_2 \ldots i_k}}}\right)\Bigg|_{\partial^n v/\partial t^n = F_0} = G, \tag{3.47}$$

where

$$\begin{aligned}
G\ &= w(\zeta^0 F_{0_{vv}} + \zeta_i^{0^{(1)}} F_{0_{vv_i}} + \zeta_{i_1 i_2}^{0^{(2)}} F_{0_{vv_{i_1 i_2}}} + \ldots + \zeta_{i_1 i_2 \ldots i_k}^{0^{(k)}} F_{0_{vv_{i_1 i_2 \ldots i_k}}})\\
&\quad + w_i(\zeta^0 F_{0_{vv_i}} + \zeta_i^{0^{(1)}} F_{0_{v_i v_i}} + \zeta_{i_1 i_2}^{0^{(2)}} F_{0_{v_i v_{i_1 i_2}}} + \ldots + \zeta_{i_1 i_2 \ldots i_k}^{0^{(k)}} F_{0_{v_i v_{i_1 i_2 \ldots i_k}}}) + \ldots\\
&\quad + w_{i_1 i_2 \ldots i_k}(\zeta^0 F_{0_{vv_{i_1 i_2 \ldots i_k}}} + \zeta_i^{0^{(1)}} F_{0_{v_i v_{i_1 i_2 \ldots i_k}}} + \zeta_{i_1 i_2}^{0^{(2)}} F_{0_{v_{i_1 i_2} v_{i_1 i_2 \ldots i_k}}} + \ldots\\
&\quad + \zeta_{i_1 i_2 \ldots i_k}^{0^{(k)}} F_{0_{v_{i_1 i_2 \ldots i_k} v_{i_1 i_2 \ldots i_k}}}) + \zeta^0 F_{1_v} + \zeta_i^{0^{(1)}} F_{1_{v_i}} + \ldots + \zeta_{i_1 i_2 \ldots i_\ell}^{0^{(\ell)}} F_{1_{v_{i_1 i_2 \ldots i_\ell}}}.
\end{aligned}$$

As ζ^0 and $F_0[v]$ satisfy (3.45), G reduces to

$$G = \zeta^0 F_{1_v} + \zeta_i^{0^{(1)}} F_{1_{v_i}} + \ldots + \zeta_{i_1 i_2 \ldots i_\ell}^{0^{(\ell)}} F_{1_{v_{i_1 i_2 \ldots i_\ell}}}. \tag{3.48}$$

Now, we proceed to check the determining equation (3.20) of BGI approximate symmetries for (3.39). The left-hand side of (3.20) simplifies to

$$\left(\zeta_t^{1^{(n)}} - \zeta^1 F_{0u} - \zeta_i^{1^{(1)}} F_{0u_i} - \zeta_{i_1 i_2}^{1^{(2)}} F_{0u_{i_1 i_2}} - \ldots - \zeta_{i_1 i_2 \ldots i_k}^{1^{(k)}} F_{0u_{i_1 i_2 \ldots i_k}}\right)\Big|_{\partial^n u/\partial t^n = F_0}$$

which is equivalent to the left-hand side of (3.47). Now, the right-hand side of (3.20), the function H, is the coefficient of ϵ in

$$-\hat{X}^{0^{(n)}}\left(\frac{\partial^n u}{\partial t^n} - F_0 - \epsilon F_1\right)\Big|_{\partial^n u/\partial t^n = F_0 + \epsilon F_1}. \tag{3.49}$$

Since $\zeta^0 = \zeta^0(x,t)$, none of the terms in (3.49) contains $\partial^n u/\partial t^n$. Hence the coefficient of ϵ in (3.49) is

$$H = \hat{X}^{0^{(n)}} F_1 = \zeta^0 F_{1u} + \zeta_i^{0^{(1)}} F_{1u_i} + \ldots + \zeta_{i_1 i_2 \ldots i_\ell}^{0^{(\ell)}} F_{0u_{i_1 i_2 \ldots i_\ell}}. \tag{3.50}$$

The latter is equivalent to G (3.48). It follows that the determining equation (3.43) of FS symmetries for the system (3.42) and the determining equation (3.20) of BGI approximate symmetries for the PDE (3.39) are equivalent. Hence $\hat{Z}$ (3.46) is a FS approximate local symmetry of the system (3.42). □

The above theorem states that when a point symmetry of an unperturbed PDE yields a BGI approximate point symmetry but not an FS approximate point symmetry of the perturbed PDE, under the conditions of the theorem, there exists a corresponding *higher-order* FS approximate symmetry of the perturbed PDE instead.

Example 3.5.1. Consider again the PDE (3.31) $u_{tt} + \epsilon u u_t = u_x u_{xx}$. Using Table 3.1, we observe that $X_1^0 = \partial/\partial u$ is unstable as a FS approximate point symmetry but it is a stable point symmetry in the sense of BGI; the corresponding BGI generator is given by

$$\hat{X}_1 = \left(1 - \epsilon\left(\frac{2}{5}tu + \frac{1}{10}t^2 u_t\right)\right)\frac{\partial}{\partial u}.$$

Consider now the Fushchich-Shtelen system (3.32) for the PDE (3.31). Using determining equation (3.43) for exact local symmetries of (3.32), one can find that

$$\hat{Z}_1 = \frac{\partial}{\partial v} - \left(\frac{2}{5}tv + \frac{1}{10}t^2 v_t\right)\frac{\partial}{\partial w}$$

is a higher-order FS approximate symmetry generator of the PDE (3.31).

Remark 3.5.1. The conditions of Theorem 3.5.1 are not satisfied when $\zeta_u^0 \neq 0$, $\zeta_{u_x}^0 \neq 0$ or $\zeta_{u_t}^0 \neq 0$.

Example 3.5.2. The perturbed wave equation

$$u_{tt} + \epsilon u u_t = e^u u_{xx} \tag{3.51}$$

admits an approximate point symmetry with the evolutionary form

$$\hat{X} = \left(2 - x u_x - \epsilon\left(t^2 u_t + 4t\right)\right)\frac{\partial}{\partial u} \tag{3.52}$$

corresponding to the stable point symmetry $\hat{X}^0 = \zeta^0 \, \partial/\partial u = (2 - xu_x) \, \partial/\partial u$. Here $\zeta^0 = 2 - xu_x$ does not satisfy the conditions of Theorem 3.5.1 since it involves u_x. It turns out that $\hat{X}^0$ is unstable as a FS approximate point symmetry of (3.51). Indeed, it is easy to check that

$$\hat{Z} = (2 - xv_x) \frac{\partial}{\partial v} - \left(t^2 v_t + 4t\right) \frac{\partial}{\partial w}$$

is not a local symmetry of the Fushchich-Shtelen system of the PDE (3.51) given by

$$v_{tt} - v_x v_{xx} = 0, \qquad w_{tt} + v v_t - e^v w_{xx} - e^v w v_{xx} = 0.$$

3.6 Exact and approximate point symmetry classification of a one-dimensional perturbed wave model in a fiber-reinforced solid

One-dimensional nonlinear wave equations

$$u_{tt} = K(u_x) u_{xx} \tag{3.53}$$

on the unknown $u(x, t)$ and various forms of $K(u_x)$ arise in multiple physical contexts, in particular, in nonlinear mechanics [108]. The point symmetry classification of the PDE family (3.53) has been performed by Oron and Rosenau [107]. If $K(u_x) = c^2 = \text{const}$, the PDE (3.53) becomes linear:

$$u_{tt} = c^2 u_{xx}, \tag{3.54}$$

and consequently admits an infinite set of point symmetries described by the infinitesimal generator

$$X^0_\infty = (\alpha_1 + \alpha_2) \frac{\partial}{\partial t} + c(\alpha_1 - \alpha_2) \frac{\partial}{\partial x} + (C_1 u + \beta_1 + \beta_2) \frac{\partial}{\partial u}, \tag{3.55}$$

parameterized by an arbitrary constant C_1 and four arbitrary functions $\alpha_1(x + ct)$, $\beta_1(x + ct)$, $\alpha_2(x - ct)$, and $\beta_2(x - ct)$.

In the current section, we consider a special form of the arbitrary function $K(u_x) = c^2 + \epsilon Q(u_x)$ in (3.53), which yields a PDE family

$$u_{tt} = (c^2 + \epsilon Q(u_x)) u_{xx} \tag{3.56}$$

with a small parameter ϵ. It is assumed that $Q(u_x) \neq \text{const}$. Such models arise, for example, in the analysis of wave propagation in fiber-reinforced elastic solids [109, 110] with small fiber strengths. The PDEs (3.56) are nonlinear perturbed versions of the linear PDE (3.54), and therefore have a reduced set of symmetries compared to that of the linear wave equation. It is of interest to follow the algorithms presented in Sections 3.2-3.4 to compare the exact point symmetry classification of the PDE family (3.56) as it stands with approximate (BGI and FS) point symmetries of the PDEs (3.56) viewed as perturbations of the linear wave equation (3.54).

We classify exact and approximate (BGI and FS) point symmetries for (3.56). The classification is performed with respect to the forms of the arbitrary function $Q(u_x)$, with each classification case holding for an arbitrary ϵ. In the classifications, cases are simplified using the equivalence transformations of the perturbed equation (3.56), given by

$$t = C_1\tilde{t} + C_2, \quad x = C_3\tilde{x} + C_4, \quad u = C_5\tilde{u} + C_6\tilde{x} + C_7\tilde{t} + C_8,$$
$$c^2 = \frac{C_3^2}{C_1^2}\tilde{c}^2, \quad Q(u_x) = Q\left(\frac{C_5\tilde{u}_{\tilde{x}} + C_6}{C_3}\right) = \frac{C_3^2}{C_1^2}\tilde{Q}(\tilde{u}_{\tilde{x}}), \tag{3.57}$$

involving arbitrary constants C_i. It follows that by taking $C_1 = 1/c$, $C_3 = C_5 = 1$, and other constants zero, upon dropping tildes, one obtains the PDE (3.56) with $c^2 = 1$:

$$u_{tt} = (1 + \epsilon Q(u_x))u_{xx}, \tag{3.58}$$

which will be considered below.

The results below are presented modulo the equivalence transformations (3.57), usually without obvious trivial approximate symmetries (see Section 1.3.4); some trivial approximate symmetries will be pointed out.

3.6.1 Exact point symmetries of a one-dimensional perturbed wave equation

The exact symmetry generator for the PDE (3.56) has the form

$$Y = \xi^1(x, t, u; \epsilon)\frac{\partial}{\partial t} + \xi^2(x, t, u; \epsilon)\frac{\partial}{\partial x} + \eta(x, t, u; \epsilon)\frac{\partial}{\partial u}. \tag{3.59}$$

The following cases arise, holding for an arbitrary ϵ and non-constant Q.

1. In the general case of arbitrary $Q(u_x)$ and c, one has the five-dimensional Lie group of point symmetries generated by

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial u}, \quad Y_4 = t\frac{\partial}{\partial u}, \quad Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \tag{3.60}$$

   describing respectively translations in t, x, u, the Galilei transformation in the direction of the displacement u, and a homogeneous space-time scaling.

2. In the case when $Q(u_x) = u_x$, c arbitrary, the Lie algebra (3.60) is extended by a point symmetry generator

$$Y_6 = x\frac{\partial}{\partial u} - \epsilon\left(\frac{3t}{2c^2}\frac{\partial}{\partial t} + \frac{x}{c^2}\frac{\partial}{\partial x}\right). \tag{3.61}$$

3.6.2 BGI approximate point symmetries of a one-dimensional perturbed wave equation

The BGI approximate point symmetry generator for the PDE (3.56) has the form

$$X = X^0 + \epsilon X^1 = X^0 + \epsilon\left(\xi_1^1(x, t, u)\frac{\partial}{\partial t} + \xi_1^2(x, t, u)\frac{\partial}{\partial x} + \eta(x, t, u)\frac{\partial}{\partial u}\right), \tag{3.62}$$

where, according to Theorem 1.3.2, the freedom in X^0 does not exceed that in X^0_∞ (3.55). From the determining equations (3.20) for BGI approximate point symmetries, the following cases arise.

1. $Q(u_x)$ arbitrary: the $O(1)$ and $O(\epsilon)$ components of the generator (3.62) are given by

$$X^0 = (C_1 t + C_2)\frac{\partial}{\partial t} + (C_1 x + C_3)\frac{\partial}{\partial x} + (C_1 u + C_4 t + C_5)\frac{\partial}{\partial u}, \tag{3.63a}$$

$$X^1 = (\lambda_1 + \lambda_2)\frac{\partial}{\partial t} + c\,(\lambda_1 - \lambda_2)\,\frac{\partial}{\partial x} + (C_6 u + \lambda_3 + \lambda_4)\frac{\partial}{\partial u}, \tag{3.63b}$$

where $C_i = \text{const}$, λ_1 and λ_3 are arbitrary functions of $x+ct$, and λ_2 and λ_4 are arbitrary functions of $x-ct$. Consequently, the nonlinear wave equation (3.56) for an arbitrary $Q(u_x)$ admits the approximate symmetries

$$\begin{aligned} X_1 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial u}, \quad X_5 = \frac{\partial}{\partial u}, \\ X_\infty &= \epsilon\left[(\lambda_1 + \lambda_2)\,\frac{\partial}{\partial t} + c\,(\lambda_1 - \lambda_2)\,\frac{\partial}{\partial x} + (C_6 u + \lambda_3 + \lambda_4)\,\frac{\partial}{\partial u}\right], \end{aligned} \tag{3.64}$$

which are, respectively, the re-numbered exact point symmetries (3.60), and a trivial approximate symmetry X_∞ corresponding to the infinite symmetry set (3.55) of the linear wave equation (3.54). The difference between the freedom in (3.55) and (3.63a) corresponds to unstable point symmetries of the linear wave equation.

2. $Q(u_x) = u_x$: the exact symmetry generator (3.55) of the linear wave equation (3.54) reduces to

$$X^0 = (C_1 t + C_2)\frac{\partial}{\partial t} + (C_1 x + C_3)\frac{\partial}{\partial x} + (C_1 u + \beta_1 + \beta_2)\frac{\partial}{\partial u}\,, \tag{3.65}$$

and the $O(\epsilon)$ approximate symmetry components have the form

$$\begin{aligned} \xi_1^1 &= \lambda_1(x+ct) + \lambda_2(x-ct) - \frac{1}{4c^2}\int^t \left(\beta_2' + 2cz\beta_1''\,(c\,(t-2z)+x)\right)dz, \\ \xi_1^2 &= H(x,t), \qquad \eta = \left(C_4 + \frac{\beta_2' - \beta_1'}{4c^2}\right)u + \lambda_3(x+ct) + \lambda_4(x-ct), \end{aligned} \tag{3.66}$$

where $H(x,t)$ is an arbitrary solution of the PDEs: $H_t = c^2 \xi^1_{1_x}$, $H_x = \xi^1_{1_t} + \dfrac{1}{2c^2}\,(\beta_{1_x} + \beta_{2_x})$.

In this second case, the point symmetries of the unperturbed linear wave equation (3.54) with arbitrary $\beta_1(x{+}ct)$ and $\beta_2(x{-}ct)$ remain stable, and yield genuine approximate symmetries with $O(\epsilon)$ components given by the terms in (3.66) that contain β_1 and β_2.

3. $Q(u_x) = A\ln(u_x + B) + C$, where A, B and C are arbitrary constants: here the nonlinear wave equation (3.56) admits the approximate symmetries (3.64), and a genuine approximate symmetry given by

$$X_g = (u + Bx)\,\frac{\partial}{\partial u} - \epsilon\frac{At}{2c^2}\frac{\partial}{\partial t}. \tag{3.67}$$

3.6.3 FS approximate point symmetries of a one-dimensional perturbed wave equation

For the perturbed PDE (3.56) with $u(x,t) = v(x,t) + \epsilon w(x,t) + o(\epsilon)$, the Fuschich-Shtelen system (3.26) reads

$$v_{tt} = c^2 v_{xx}, \qquad w_{tt} = c^2 w_{xx} + Q(v_x)v_{xx}. \tag{3.68}$$

We find exact point symmetries of the system (3.68) that correspond to FS approximate point symmetries of the PDE (3.56). The infinitesimal generator of such symmetries has the form

$$Z = \lambda^1(x,t,v,w)\frac{\partial}{\partial x} + \lambda^2(x,t,v,w)\frac{\partial}{\partial t} + \phi_1(x,t,v,w)\frac{\partial}{\partial v} + \phi_2(x,t,v,w)\frac{\partial}{\partial w}. \tag{3.69}$$

The solution of the determining equations (3.7) leads to the following classification.

1. $Q(v_x)$ arbitrary:

$$\begin{aligned} &Z_1 = \frac{\partial}{\partial t}, \qquad Z_2 = \frac{\partial}{\partial x}, \qquad Z_3 = \frac{\partial}{\partial v}, \qquad Z_4 = t\frac{\partial}{\partial v}, \\ &Z_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, \qquad Z_6 = v\frac{\partial}{\partial w}, \\ &Z_\infty = (\beta_1(x+ct) + \beta_2(x-ct))\frac{\partial}{\partial w} \end{aligned} \tag{3.70}$$

   In this general case, no genuine FS approximate symmetries arise. Indeed, the generators $Z_1 \ldots, Z_5$ mimic the exact point symmetry generators (3.60), and Z_6, Z_∞ are trivial FS symmetries. Including the above symmetries, the system (3.68) admits additional point symmetries in the following case:

2. $Q(v_x) = v_x^s, s \neq 0$:

$$Z_7 = v\frac{\partial}{\partial v} + (s+1)w\frac{\partial}{\partial w}. \tag{3.71}$$

3. $Q(v_x) = e^{v_x}$:

$$Z_7' = x\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}. \tag{3.72}$$

The symmetries given by Z_7 and Z_7' are genuine FS approximate point symmetries of the perturbed PDE (3.58).

3.6.4 Summary

For an arbitrary Q, the perturbed one-dimensional wave equation (3.58) admits five exact symmetries given by (3.60) and it has these five symmetries and a trivial approximate symmetry as BGI approximate symmetries (3.64). For $Q = u_x$, the equation (3.58) admits (3.60) and an additional exact symmetry given by (3.61). For BGI classification with $Q(u_x) = u_x$, the PDE (3.58) has an infinite set of BGI approximate symmetries with approximate symmetry components given by (3.66). Note that the exact symmetry generator Y_6 in (3.61)

$$Y_6 = x\frac{\partial}{\partial u} - \epsilon\left(\frac{3t}{2}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right)$$

can be obtained from the BGI approximate components (3.66) by taking

$$\beta_1 = \frac{x+t}{2}, \quad \beta_2 = \frac{x-t}{2}, \quad \lambda_1 = -\frac{11}{16}(x+t), \quad \lambda_2 = \frac{11}{16}(x-t).$$

It follows that the BGI approximate symmetry classification of the wave equation (3.58) includes the exact symmetry classification of (3.58) but corresponds to a subset of exact point symmetries (3.55) of the unperturbed (linear) wave equation (3.54). An additional case appears in the BGI approximate symmetry classification when $Q = A\ln(u_x + B) + C$, with a corresponding additional approximate symmetry given by (3.67). This case does not arise in the FS symmetry classification.

For an arbitrary Q, the PDE (3.58) admits exact point symmetry generators (3.60) and trivial approximate symmetries given by (3.70). In comparison with the exact and BGI symmetry classifications of (3.58), two different cases appear in FS approximate symmetry classification: $Q = v_x^s$, $s \neq 0$ and $Q = e^{v_x}$. For $Q = v_x^s$, the PDE (3.58) admits an additional FS approximate symmetry given by (3.71). (By contrast, in the exact and BGI symmetry classifications of (3.58), this case appears only when $s = 1$.) For $Q = e^{v_x}$, a stable exact symmetry $x\,\partial/\partial u$ of the linear wave equation (3.54) yields a genuine FS approximate symmetry of (3.58) given by (3.72).

3.7 Approximate and numerical solutions modeling finite-time singularity formation in fiber-reinforced materials

The displacements in shear waves propagating in an incompressible hyperelastic material with a single family of fibers directed along the wave propagation are governed by a nonlinear one dimensional wave equation

$$u_{tt} = (\alpha + 3\beta u_x^2)u_{xx}, \quad u = u(x,t), \tag{3.73}$$

where the constants $\alpha, \beta > 0$ are the material parameters [110]. In this section, we consider wave equations (3.56) with $Q(u_x) = Bu_x^s$, $B > 0$, $s \neq 0$, which include the model (3.73). By a re-scaling of x, t and u, these PDEs can be brought into a simpler form

$$u_{tt} = (1 + \epsilon u_x^s)u_{xx}. \tag{3.74}$$

3.7.1 A general FS approximate solution of a perturbed wave model

Here we use Fuschich-Shtelen approximate symmetries to construct an approximate solution for the PDE (3.74) in the usual FS form

$$u(x,t) = v(x,t) + \epsilon w(x,t) + o(\epsilon). \tag{3.75}$$

In the first-order of precision in ϵ, the equation (3.74) is equivalent to the Fuschich-Shtelen system (3.68) with $Q(v_x) = v_x^s$:

$$v_{tt} = v_{xx}, \qquad w_{tt} = w_{xx} + v_x^s v_{xx}, \tag{3.76}$$

which admits the symmetry generator (3.71). The corresponding characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dv}{v} = \frac{dw}{(s+1)w}. \tag{3.77}$$

Consequently, if $v(x,t)$ is any solution for the first equation of the system (3.76), then the invariant solution following from the characteristic equations (3.77) is given by $w(x,t) = v^{s+1}\phi(x,t)$. Consider traveling wave solutions of the first equation in (3.76):

$$v = g\,(x \pm t)\,. \tag{3.78}$$

Substituting (3.78) and $w = g^{s+1}\phi$ into the second PDE of (3.76) one gets to the PDE in ϕ

$$g^{s+1}\left(\phi_{tt} - \phi_{xx}\right) + 2(s+1)g^s g'\left(\pm\phi_t - \phi_x\right) - (g')^s g'' = 0. \tag{3.79}$$

When $s \neq -1$, the PDE (3.79) has a general solution

$$\phi = g^{-s-1}\left[h \pm \frac{t(g')^{s+1} - \int^t (g')^{s+1}\left(\pm(2r-t)+x\right)dr}{2(s+1)}\right],$$

where $h = h(x,t)$ satisfies $h_{tt} = h_{xx}$. Similarly, when $s = -1$, the solution form changes to

$$\phi = h \pm \frac{1}{2}t\ln\left(g'\right).$$

In light of the above results, the higher-order solution part w has the form

$$w = \begin{cases} h \pm \dfrac{t(g')^{s+1} - \int^t (g')^{s+1}\left(\pm(2r-t)+x\right)dr}{2(s+1)}, & s \neq -1, \\ h \pm \dfrac{t\ln\left(g'\right)}{2}, & s = -1. \end{cases} \tag{3.80}$$

Finally, when $s \neq -1$, the perturbed equation (3.74) has the approximate solution (3.75) given by

$$u(x,t) = g(x \pm t) + \epsilon\left[h(x,t) \pm \frac{t(g')^{s+1} - \int^t (g')^{s+1}\left(\pm(2r-t)+x\right)dr}{2(s+1)}\right] + o(\epsilon). \tag{3.81a}$$

When $s = -1$, the approximate solution takes the form

$$u(x,t) = g(x \pm t) + \epsilon\left(h(x,t) \pm \frac{t\ln\left(g'\left(x \pm t\right)\right)}{2}\right) + o(\epsilon). \tag{3.81b}$$

Example 3.7.1. As a specific example that will be used below, we consider the PDE (3.73) describing shear waves in a fiber-reinforced solid, re-scaled to the form (3.74) with $s = 2$:

$$u_{tt} = \left(1 + \epsilon\left(u_x\right)^2\right)u_{xx}, \tag{3.82}$$

and assume in (3.73) that $\beta/\alpha \sim \epsilon \ll 1$, which corresponds to weak fiber effects. We also choose $v(x,t) = \exp(-(x-t)^2)$. Then the solution (3.81a) of the PDE (3.74) with $h = 0$ reduces to

$$\begin{aligned} u(x,t) \ &= e^{-(x-t)^2} + \frac{\epsilon}{6}\Big[8t\left(x-t\right)^3 e^{-3(x-t)^2} + \frac{1}{9}\Big(\left(12tx - 6t^2 - 6x^2 - 2\right)e^{-3(x-t)^2} \\ &+ \left(12tx + 6t^2 + 6x^2 + 2\right)e^{-3(x+t)^2}\Big)\Big] + o(\epsilon). \end{aligned} \tag{3.83}$$

Note that for any fixed t, the approximate solution (3.83) approaches zero as $x \to \infty$. Also, for any $x \in (-\infty, \infty)$, the solution (3.83) is bounded as $t \to \infty$. The solution (3.83) is not a purely right-traveling wave solution but describes an evolving wave form (see Section 3.7.2 below). In particular, the PDE (3.82) is known to have breaking wave-type solutions [110].

3.7.2 Numerical simulations of a perturbed wave model and finite-time singularity formation

We now compute numerical solutions of the wave equation (3.82) in Example 3.7.1 in order to model its finite-time singularity formation behaviour (see Section 3.7.3 below for details) and provide a reference for comparison with the approximate solutions developed in Section 3.7.1. Gaussian-type initial conditions corresponding to a right-traveling wave and periodic boundary conditions

$$u(x,0) = e^{-x^2}, \quad u_t(x,0) = 2xe^{-x^2}, \quad u(-L,t) = u(L,t), \quad L > 0. \tag{3.84}$$

are posed in the space-time domain $x \in [-L, L]$, $t \geq 0$, and the equation (3.82) is solved using an explicit finite difference cross-stencil scheme with constant spatial and temporal steps $\tilde{h}$, $\tilde{\tau}$. Following [78], we use a conservative finite-difference scheme developed for the PDE (3.82):

$$\begin{aligned} & U_{t\check{t}} - U_{x\bar{x}} - \epsilon \frac{U_x^3 - U_{\bar{x}}^3}{3\tilde{h}}, \qquad \tilde{h}, \tilde{\tau} = \text{const}, \\ & x_m = -L + m\tilde{h}, \quad m = 0, \ldots, M, \\ & t_n = 0 + n\tilde{\tau}, \quad n = 0, \ldots, N \end{aligned} \tag{3.85}$$

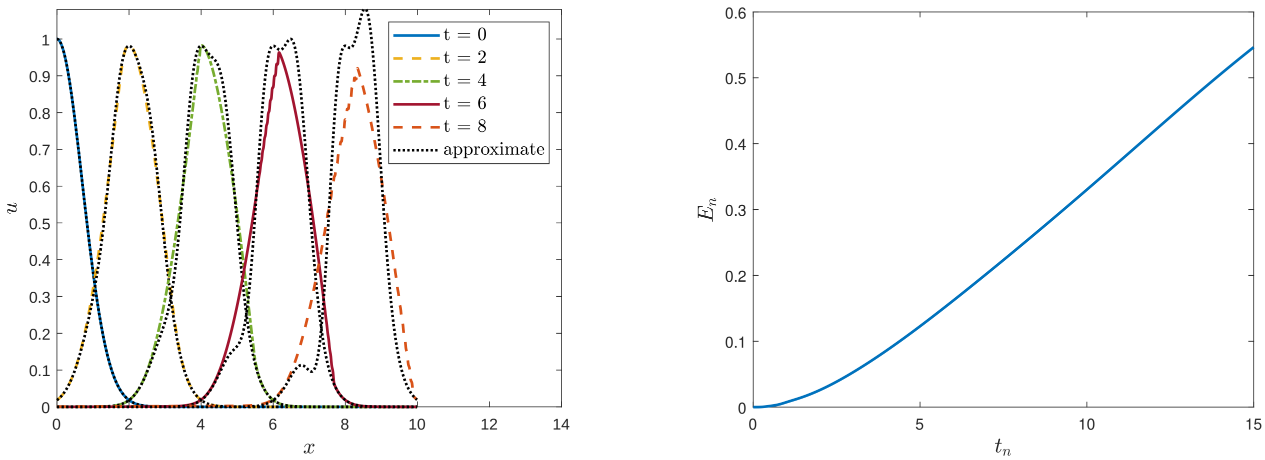
with $U = U_m^n$ approximating the value of $u(x,t)$ at the mesh node (x_m, t_n). Here $U_{t\check{t}}$ and $U_{x\bar{x}}$ represent the second-order central differences, U_x the first-order forward difference, and $U_{\bar{x}}$ the first-order backward difference:

$$\begin{aligned} & U_{t\check{t}} = \frac{U_m^{n+1} - 2U_m^n + U_m^{n-1}}{\tilde{\tau}^2}, \quad U_{x\bar{x}} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{\tilde{h}^2}, \\ & U_x = \frac{U_{m+1}^n - U_m^n}{\tilde{h}}, \quad U_{\bar{x}} = \frac{U_m^n - U_{m-1}^n}{\tilde{h}}. \end{aligned} \tag{3.86}$$

The numerical solutions provide a good agreement with approximate solutions (3.83), for a broad range of ϵ values, from the initial dimensionless time to the time close to the finite-time singularity. The time when singularity forms increases approximately as ϵ^{-1} as ϵ decreases (see Section 3.7.3 below). Here we present sample computation results for a relatively large value of the small parameter, $\epsilon = 0.5$. The computation is performed from $t = 0$ to $t = 4$ close to the finite-time singularity. A comparison of the numerical solution and the approximate solution (3.83) of PDE (3.82) with initial and boundary conditions (3.84) at several time snapshots is presented in Figure 3.1a. The relative difference at the time step t_n between the approximate and numerical solutions is calculated using 2-norms according to the formula

$$E_n = E(t_n) = \frac{||u_{approx} - u_{num}||_2}{||u_{approx}||_2} \tag{3.87}$$

and is shown in Figure 3.1b.



(a) **(b)**

Figure 3.1: (a) Numerical and approximate profiles of u according to the PDE (3.82) ($\epsilon = 0.5$) with initial conditions (3.84) for $L = 10$, $h = 0.01$, $\tau = h/8$, and $t = 0, 2, 4, 6, 8$. (b) Relative difference (3.87) between numerical and approximate solutions.

3.7.3 Estimates of finite-time singularity formation using approximate and numerical solutions

As discussed in Ref. [110], the variable nonlinearity $(1 + \epsilon\,(u_x)^2)$ leads to greater characteristic speed values at points where $|u_x|$ is larger. This can lead to the intersection of characteristic curves, which corresponds to the formation of a finite-time singularity. This behaviour can be studied using the method of characteristics. While for linear hyperbolic PDEs, such as the constant-coefficient wave equation $u_{tt} = c^2 u_{xx}$ in the simplest case, characteristic curves can be found in terms of explicit formulas such as $x = x_0 \pm ct$, and lead to explicit exact solutions, the situation is significantly more complex for nonlinear hyperbolic PDEs. Using the method described in Refs. [103, 111], one can show that (3.82) can be reduced to the first-order characteristic form

$$u_t = \pm\frac{1}{2\sqrt{\epsilon}}\left(\sqrt{\epsilon}\,u_x\sqrt{1+\epsilon\,(u_x)^2} + \ln\left(\sqrt{\epsilon}\,u_x + \sqrt{1+\epsilon\,(u_x)^2}\right)\right) \tag{3.88}$$

on the characteristic curves

$$\frac{dx}{dt} = \pm\sqrt{1+\epsilon\,(u_x)^2}\,. \tag{3.89}$$

In the physical terms, the part of the wave that has a time derivative given by (3.88) moves at a finite velocity given by (3.89). The integration of (3.89) yields a constant of integration x_0 that corresponds to the point on the characteristic curve where $t = t_0$ is some initial time. Thus $x = x(x_0, t)$, from which $u_t(x(x_0,t),t) = a(x_0,t)$ and $u_x(x(x_0,t),t) = b(x_0,t)$ in (3.88) and (3.89).

The formation of a shock where the solution becomes multi-valued takes place when characteristic curves intersect. Without explicit knowledge of $u_x(x,t)$, no explicit solution $x(x_0,t)$ of (3.89) is available. To estimate the time T_b when singularity forms, we use time-progressing linear approximations to characteristic curves, in conjunction with the finite-difference numerical solution of the PDE (3.82) described in Section 3.7.2. At each time layer t_n in (3.85), linearized characteristics are launched forward in time from each grid point (x_m, t_n). The smallest time of the intersection of such characteristics estimates the finite-time singularity.

For example, when the numerical computation has reached the time layer $t = t_n$, the linearized characteristics are launched from each spatial grid point x_m, $m = 1, \ldots, M-1$. In particular, the right-traveling characteristics are approximated by the lines

$$x = x_m + t\sqrt{1 + \epsilon\left(u_x(x_m, t_n)\right)^2}, \tag{3.90}$$

where $u_x = U_x$ is the first-order forward finite difference (3.86). To approximate the finite-time singularity formation numerically, we solve (3.90) for the time $t = \tau$ when two different characteristics intersect. Given two starting points, x_{m_1}, x_{m_2}, we get the system

$$x = x_{m_1} + \tau\sqrt{1 + \epsilon\left(u_x(x_{m_1}, t_n)\right)^2}, \quad x = x_{m_2} + \tau\sqrt{1 + \epsilon\left(u_x(x_{m_2}, t_n)\right)^2}. \tag{3.91}$$

Solving for τ yields

$$\tau = \frac{\xi_2 - \xi_1}{m_2 - m_1}, \tag{3.92}$$

where

$$m_i = \sqrt{1 + \epsilon\left(u_x(x_{m_i}, t_n)\right)^2}, \qquad i = 1, 2 \tag{3.93}$$

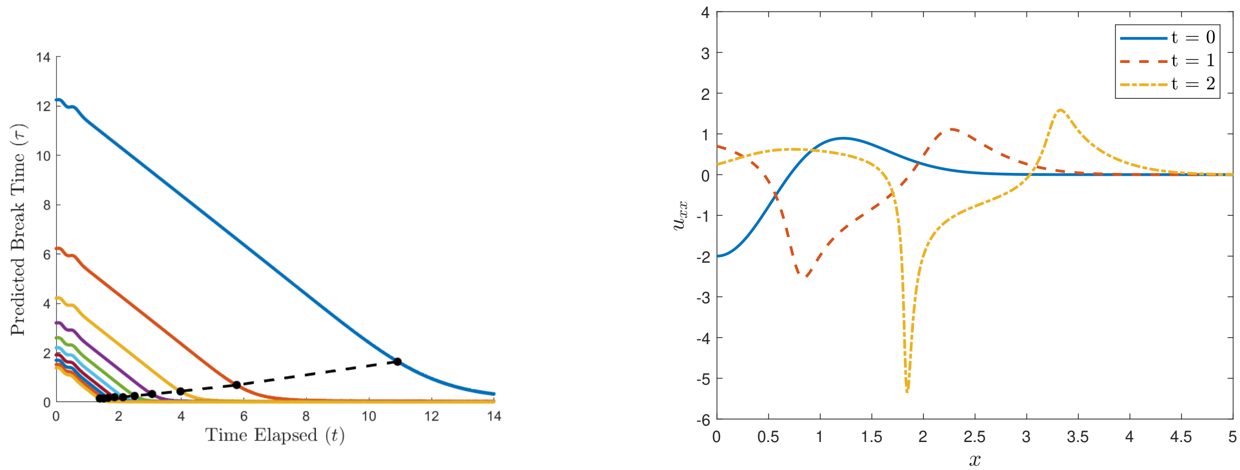
are the slopes of the characteristic lines. We choose x_{m_1} and x_{m_2} to be adjacent grid points, $x_{m_1} + \tilde{h} = x_{m_2}$. The numerator of (3.92) is constant, so the approximate finite-time singularity corresponds to the largest denominator of (3.92). We determine x_{m_1} corresponding to the largest difference between the slopes m_1 and m_2, then solve for τ.

The meaning of τ is thus the estimated time from t_n to the finite-time singularity T_b; one consequently has an estimate

$$T_b \sim t_n + \tau. \tag{3.94}$$

As the wave evolves, the slopes of the linearized characteristics will change and therefore so will τ. To account for this, at each time step, we repeat the calculation for τ. We use the first-order forward finite difference approximation to compute u_x in (3.93) at each time step. Figure 3.2a shows a plot of the value of the time to the singularity formation τ versus the time at which it was calculated, for several values of ϵ.

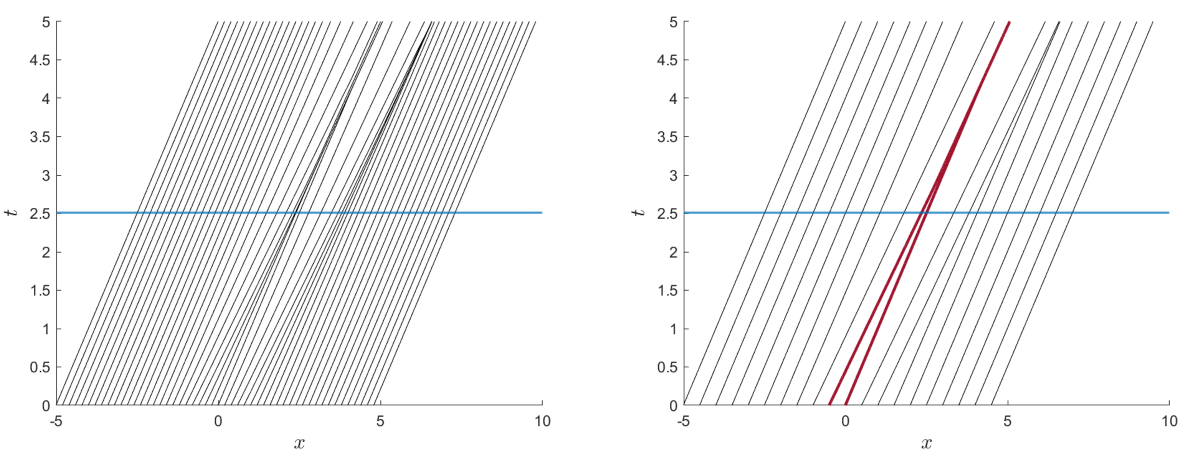
Alternatively, one can numerically estimate the finite-time singularity formation by defining it as the time when $\min(u_{xx}) \leq \delta$ for some negative number δ. Choosing for example $\delta = -5$, and using the second-order central difference approximation to the derivative for u_{xx} in the numerical solution, we calculate the numerical finite-time singularity for each ϵ. Using Richardson extrapolation of the finite-time singularities found with spatial step sizes $\tilde{h} = 0.01$ and $\tilde{h} = 0.005$ and temporal step size $\tilde{\tau} = 0.00125$, we found the finite-time singularities for each ϵ in the limit as $\tilde{h} \to 0$ as shown in Figure 3.2a. This approach uses the fact that as the wave approaches the finite-time singularity, the second spatial derivative u_{xx} tends to negative infinity (see Figure 3.2b).



(a) **(b)**

Figure 3.2: (a) Estimates (solid colour) and numerical (black dashed) values for the time-to singularity formation τ (3.92) for $\epsilon = 0.1, 0.2, \ldots, 1$ (right to left). (b) Numerical wave profiles of u_{xx} ($\epsilon = 0.5$) for $t = 0, 1, 2$.

We determined the actual, non-linearized characteristic curves by numerically integrating (3.89). The curves are shown in Figure 3.3.



(a) **(b)**

Figure 3.3: (a) Characteristic curves found by numerical integration of (3.89) with $\epsilon = 0.5$. The blue horizontal line is the finite-time singularity as determined by Richardson extrapolation. (b) The same plot as (a) with fewer characteristic lines shown. The thick red characteristic lines correspond to the earliest intersection.

It is interesting that one can also approximately determine the finite-time singularity from the approximate solution (3.83) by finding the time when the second spatial derivative u_{xx} of the approximate solution develops an additional root, as shown in Figure 3.4. This corresponds to an additional inflection point in the wave itself, which can be observed in Figure 3.1a closer to the finite-time singularity formation.

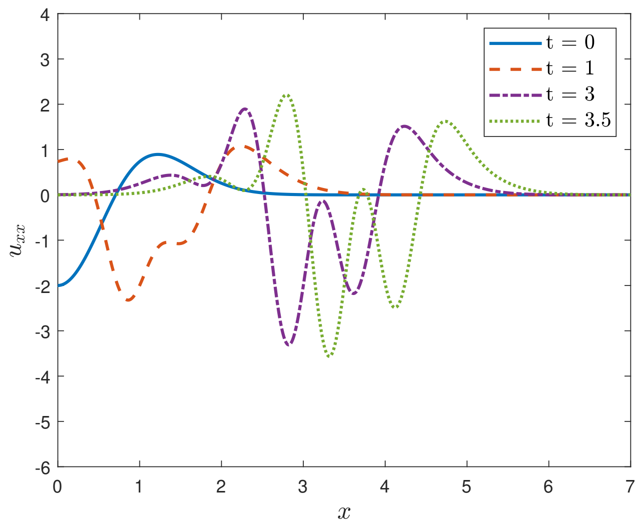


Figure 3.4: Wave profiles of the approximate solution u_{xx} ($\epsilon = 0.5$) for $t = 0, 1, 3, 3.5$. Note the development of extra roots as time increases.

The numerically determined finite-time singularities (τ_{num}) and the approximate-determined finite-time singularities (τ_{approx}) are given in Table 3.2.

ϵ	τ_{num}	τ_{approx}
1	1.3888	1.6050
0.9	1.1510	1.7875
0.8	1.6713	2.0175
0.7	1.8712	2.3263
0.6	2.1425	2.7363
0.5	2.5138	3.3238
0.4	2.3300	4.1150
0.3	3.0575	5.4988
0.2	5.7625	8.3162
0.1	10.9125	16.6737

Table 3.2: Numerical and approximate finite-time singularity formation estimates for the PDE (3.82) vs. the small parameter values (ϵ).

Both sets of data are also plotted in Figure 3.5. We observe a qualitative agreement in behaviour which suggests that the finite-time singularity of (3.82) with the initial value problem (3.84) goes as $\tau \sim \epsilon^{-1}$.

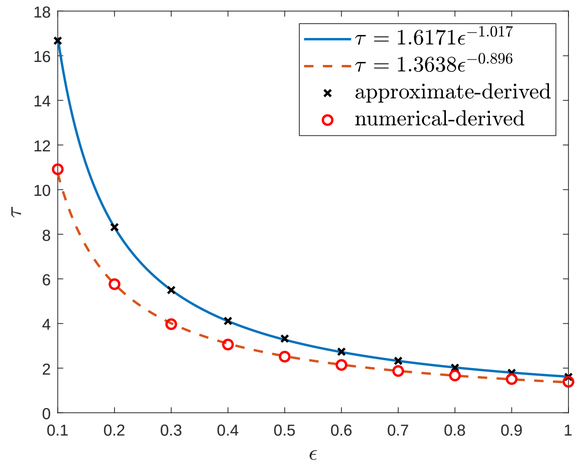


Figure 3.5: Numerical and approximate-derived finite-time singularities τ as a function of ϵ.

3.8 Exact and approximate point symmetry classification of a two-dimensional wave equation

Here, we classify exact point symmetries of a two-dimensional wave equation

$$u_{tt} = \left(u_x K(u_x^2 + u_y^2)\right)_x + \left(u_y K(u_x^2 + u_y^2)\right)_y, \quad u = u(t, x, y). \tag{3.95}$$

To find additional symmetries for (3.95), we consider $K(u_x) = c^2 + \epsilon Q(u_x)$ where ϵ is a small positive parameter. And then we classify exact and approximate point symmetries of a two-dimensional perturbed wave equation

$$u_{tt} = \left(u_x[c^2 + \epsilon Q(u_x^2 + u_y^2)]\right)_x + \left(u_y[c^2 + \epsilon Q(u_x^2 + u_y^2)]\right)_y, \quad u = u(t, x, y). \tag{3.96}$$

3.8.1 Exact point symmetries of an unperturbed wave model

The set of equivalence transformations for the family (3.95) is given by the symmetry generators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial u}, \quad X_6 = u\frac{\partial}{\partial u}, \\ X_7 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_8 = 2K\frac{\partial}{\partial K} - t\frac{\partial}{\partial t}, \quad X_9 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned} \tag{3.97}$$

The point transformations

$$\begin{aligned} &\tilde{t} = C_1 t + C_2, \quad \tilde{x} = C_3 x + C_4, \quad \tilde{y} = C_3 y + C_5, \quad \tilde{u} = C_6 u + C_7 t + C_8, \\ &\tilde{K}\left(\tfrac{C_3^2}{C_6^2}\left(\tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2\right)\right) = \frac{C_1^2}{C_3^2} K \end{aligned} \tag{3.98}$$

maps the nonlinear wave equation (3.95) to another PDE from the same family. Using the determining equations (3.7), the point symmetry classification of the nonlinear wave equation (3.95) modulo the equivalence transformations (3.97) is given by

1. K arbitrary: equation (3.95) admits seven point symmetries given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial u}, \\ X_6 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_7 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}. \end{aligned} \tag{3.99}$$

   Including the above symmetries, equation (3.95) admits additional point symmetries in the following cases:

2. $K(u_x^2 + u_y^2) = (u_x^2 + u_y^2)^q, \ q \neq 0, -2$:

$$X_8 = u\frac{\partial}{\partial u} - qt\frac{\partial}{\partial t}. \tag{3.100}$$

3. $K(u_x^2 + u_y^2) = (u_x^2 + u_y^2)^{-2}$:

$$X_8 = u\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial t}, \quad X_9 = tu\frac{\partial}{\partial u} + t^2\frac{\partial}{\partial t}. \tag{3.101}$$

   Note that the one-dimensional wave equation (3.53) with $K(u_x) = u_x^s$ admits the point symmetry (3.100) with $q = s/2$. When $s = -4$, it admits also the symmetry generator X_9 (3.101) (see, [107]).

When $K(u_x^2 + u_y^2) = c^2$, equation (3.95) reduces to a linear PDE

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right). \tag{3.102}$$

It admits the following exact symmetries [4]

$$\begin{aligned} X_1^0 &= \left(\frac{c^2t^2 + x^2 + y^2}{c^2}\right)\frac{\partial}{\partial t} + 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} - tu\frac{\partial}{\partial u}, \quad X_2^0 = \frac{y}{c^2}\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \\ X_3^0 &= 2yt\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (c^2t^2 - x^2 + y^2)\frac{\partial}{\partial y} - yu\frac{\partial}{\partial u}, \quad X_4^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ X_5^0 &= 2xt\frac{\partial}{\partial t} + (c^2t^2 + x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} - xu\frac{\partial}{\partial u}, \quad X_6^0 = \frac{x}{c^2}\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \quad X_7^0 = \frac{\partial}{\partial t} \\ X_8^0 &= u\frac{\partial}{\partial u}, \quad X_9^0 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad X_{10}^0 = \frac{\partial}{\partial x}, \quad X_{11}^0 = \frac{\partial}{\partial y}, \quad X_\infty^0 = \alpha(t,x,y)\frac{\partial}{\partial u}, \end{aligned} \tag{3.103}$$

where α satisfy the linear wave equation $\alpha_{tt} = c^2\left(\alpha_{xx} + \alpha_{yy}\right)$.

3.8.2 Exact point symmetries of a two-dimensional perturbed wave equation

The set of equivalence transformations for the family (3.96) is given by the symmetry generators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = \frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial u}, \quad X_6 = u\frac{\partial}{\partial u}, \\ X_7 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad X_8 = \epsilon\frac{\partial}{\partial \epsilon} - K\frac{\partial}{\partial K}, \quad X_9 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ X_{10} &= -t\frac{\partial}{\partial t} + 2K\frac{\partial}{\partial K} + c\frac{\partial}{\partial c}. \end{aligned} \tag{3.104}$$

The corresponding group of equivalence transformations are given by:

$$\begin{aligned}\tilde{t} &= C_1 t + C_2, \quad \tilde{x} = C_3 x + C_4, \quad \tilde{y} = C_3 y + C_5, \quad \tilde{u} = C_6 u + C_7 t + C_8, \\ \tilde{c} &= \frac{C_3^2}{C_1^2} c, \quad \tilde{\epsilon} = \frac{C_3^2}{C_1^2}\epsilon, \quad \tilde{Q}\left(\tfrac{C_3^2}{C_6^2}\left(\tilde{u}_{\tilde{x}}^2 + \tilde{u}_{\tilde{y}}^2\right)\right) = \frac{C_3^2}{C_1^2} Q.\end{aligned} \tag{3.105}$$

Applying the determining equation (3.7), we find that when $Q(u_x^2 + u_y^2) \neq 0$ is an arbitrary function, the PDE (3.96) admits the following symmetry generators

$$\begin{aligned}Y_1 &= \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial y}, \quad Y_4 = \frac{\partial}{\partial u}, \quad Y_5 = t\frac{\partial}{\partial u}, \\ Y_6 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \quad Y_7 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.\end{aligned} \tag{3.106}$$

Note that the symmetries (3.106) are equivalent to the exact symmetries (3.99) of the PDE (3.95). Hence, the classification of exact symmetries of the PDE (3.96) does not yield any new cases or new symmetries for (3.95). Therefore, we proceed now to classify the approximate symmetries of (3.96).

3.8.3 BGI approximate symmetries of a two-dimensional perturbed wave equation

The BGI approximate symmetry generator for the perturbed wave equation (3.96) has the form

$$\begin{aligned}X &= X^0 + \epsilon X^1 \\ &= \left(\xi_0^1(t,x,y,u) + \epsilon\xi_1^1(t,x,y,u)\right)\frac{\partial}{\partial t} + \left(\xi_0^2(t,x,y,u) + \epsilon\xi_1^2(t,x,y,u)\right)\frac{\partial}{\partial x} \\ &+ \left(\xi_0^3(t,x,y,u) + \epsilon\xi_1^3(t,x,y,u)\right)\frac{\partial}{\partial y} + \left(\eta_0(t,x,y,u) + \epsilon\eta_1(t,x,y,u)\right)\frac{\partial}{\partial u},\end{aligned} \tag{3.107}$$

where X^0 is an exact symmetry generator for the unperturbed equation (3.102) with components:

$$\begin{aligned}\xi_0^1 &= C_1\left(\tfrac{c^2t^2+x^2+y^2}{c^2}\right) + C_2\frac{y}{c^2} + 2C_3 yt + C_4 t + 2C_5 xt + C_6\frac{x}{c^2} + C_7, \\ \xi_0^2 &= 2C_1 xt + 2C_3 xy + C_4 x + C_5\left(c^2t^2 + x^2 - y^2\right) + C_6 t + C_9 y + C_{10}, \\ \xi_0^3 &= 2C_1 yt + C_2 t + C_3\left(\left(c^2t^2 - x^2 + y^2\right)\right) + C_4 y + 2C_5 xy - C_9 x + C_{11}, \\ \eta_0 &= -C_1 tu - C_3 yu - C_5 xu + C_8 u + \alpha(t,x,y).\end{aligned} \tag{3.108}$$

The solution of the determining equation (3.20) yields the following classification of BGI approximate symmetries for the PDE (3.96):

1. $Q(u_x^2 + u_y^2)$ arbitrary: the determining equation (3.20) provide some restrictions on the unperturbed symmetry components (3.108): $C_1 = C_2 = C_3 = C_5 = C_6 = 0$, $C_4 = C_8$ and $\alpha = k_1 t + k_2$. Therefore, the components (3.108) reduce to

$$\begin{aligned}\xi_0^1 &= C_4 t + C_7, \quad \xi_0^2 = C_9 y + C_4 x + C_{10}, \\ \xi_0^3 &= C_4 y - C_9 x + C_{11}, \quad \eta_0 = C_4 u + k_1 t + k_2.\end{aligned} \tag{3.109}$$

It follows that the exact symmetries: $X_j^0, j = 1, 2, 3, 5, 6$ and the linear combination $X_4^0 - X_8^0$ of the linear wave equation (3.102) are unstable. Consequently, the PDE (3.96) admits the following approximate symmetries divided to: approximate symmetries inherited from the exact symmetries (3.103):

$$X_{12} = X_7^0, \quad X_{13} = X_9^0, \quad X_{14} = X_{10}^0, \quad X_{15} = X_{11}^0, \quad X_{16} = t\frac{\partial}{\partial u}, \quad X_{17} = \frac{\partial}{\partial u} \tag{3.110}$$

and the linear combination of X_4^0 and X_8^0 ($C_4 = C_8$):

$$X_s = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \tag{3.111}$$

where X_j^0 are the exact symmetries of the unperturbed PDE (3.102) given by (3.103). Trivial approximate symmetries including $X_j = \epsilon X_j^0$, $j = 1 - 11$ and the infinite approximate symmetry

$$X_\infty = \epsilon\beta(t, x, y)\frac{\partial}{\partial u}, \tag{3.112}$$

where β satisfies the linear wave equation (3.102).

2. $Q(u_x^2 + u_y^2) = b\ln(u_x^2 + u_y^2) + d$, b, d are constants: the conditions $C_1 = C_2 = C_3 = C_5 = C_6 = 0$ on the exact symmetry components (3.108) reduce them to

$$\begin{aligned} \xi_0^1 &= C_4 t + C_7, \quad \xi_0^2 = C_9 y + C_4 x + C_{10}, \\ \xi_0^3 &= C_4 y - C_9 x + C_{11}, \quad \eta_0 = C_8 u + k_1 t + k_2. \end{aligned} \tag{3.113}$$

In addition to the approximate symmetries X_j, $j = 1, 2, ..., 17$, and X_∞, the PDE (3.96) admits a new approximate symmetry given by

$$X_g = u\frac{\partial}{\partial u} - \epsilon\frac{bt}{c^2}\frac{\partial}{\partial t}. \tag{3.114}$$

Note that the one dimensional wave equation (3.56) with $Q(u_x) = A\ln(u_x) + C$ admits the approximate symmetry (3.114) with $b = A/2$.

3.8.4 FS approximate symmetries of a two-dimensional perturbed wave equation

Substituting $u(t, x, y) = v(t, x, y) + \epsilon w(t, x, y) + o(\epsilon)$ transforms the perturbed wave equation (3.96) into the following system:

$$\begin{aligned} v_{tt} &= c^2 v_{xx} + c^2 v_{yy}, \\ w_{tt} &= \left(2v_x^2 v_{xx} + 2v_y^2 v_{yy} + 4v_x v_y v_{xy}\right) Q'(v_x^2 + v_y^2) + (v_{xx} + v_{yy})\, Q(v_x^2 + v_y^2) + c^2\left(w_{xx} + w_{yy}\right). \end{aligned} \tag{3.115}$$

The exact symmetry generator for the system (3.115) has the form

$$\begin{aligned} Y &= \xi^t(t, x, y, v, w)\frac{\partial}{\partial t} + \xi^x(t, x, y, v, w)\frac{\partial}{\partial x} + \xi^y(t, x, y, v, w)\frac{\partial}{\partial y} + \eta^v(t, x, y, v, w)\frac{\partial}{\partial v} \\ &+ \eta^w(t, x, y, v, w)\frac{\partial}{\partial w}. \end{aligned} \tag{3.116}$$

Using the determining equation (3.7), one finds $0 = \xi^t_w = \xi^x_w = \xi^y_w = \eta^v_w$, and hence

$$\xi^t = \xi^1_0(t,x,y,v), \quad \xi^x = \xi^2_0(t,x,y,v), \quad \xi^y = \xi^3_0(t,x,y,v), \quad \eta^v = \eta_0(t,x,y,v) \tag{3.117}$$

satisfy the determining equation of the first equation of the system (3.115), where $\xi^1_0, \xi^2_0, \xi^3_0$ and η^0 are the infinitesimals of the exact symmetry generator of the linear wave equation (3.102) given by (3.108). Applying the determining equation to the second equation of (3.115) leads to the following classification.

1. $Q(v_x^2 + v_y^2)$ arbitrary: the determining equation provides some conditions on the infinitesimals (3.117) which reduce them to:

$$\begin{aligned} \xi^t &= C_4 t + C_7, \quad \xi^x = C_4 x + C_9 y + C_{10}, \\ \xi^y &= C_4 y - C_9 x + C_{11}, \quad \eta^v = C_4 v + k_3 t + k_4. \end{aligned} \tag{3.118}$$

   Consequently, the system of equations (3.115) admits the following symmetries

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = \frac{\partial}{\partial y}, \quad Z_4 = \frac{\partial}{\partial v}, \quad Z_5 = t\frac{\partial}{\partial v}, \\ Z_6 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}, \quad Z_7 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \\ Z_8 &= v\frac{\partial}{\partial w}, \quad Z_\infty = \gamma(t,x,y)\frac{\partial}{\partial w}, \end{aligned} \tag{3.119}$$

   where γ satisfies the linear wave equation (3.102).

2. $Q = (v_x^2 + v_y^2)^n$, $n \neq 0$: the infinitesimals (3.117) reduce to:

$$\begin{aligned} \xi^t &= C_4 t + C_7, \quad \xi^x = C_4 x + C_9 y + C_{10}, \\ \xi^y &= C_4 y - C_9 x + C_{11}, \quad \eta^v = C_8 v + k_3 t + k_4. \end{aligned} \tag{3.120}$$

   The system of equations (3.115) admits the symmetries (3.119) and the following genuine point symmetry

$$Z_9 = v\frac{\partial}{\partial v} + (2n+1)w\frac{\partial}{\partial w}. \tag{3.121}$$

   This case also appears in the classification of Fushchich-Shtelen symmetries of the one dimensional wave equation (3.56) where the system of one-dimensional PDEs (3.68) with $Q(v_x) = v_x^s$ admits the symmetry generator (3.121) with $n = s/2$.

In this classification, the unstable point symmetries of the linear wave equation (3.102) are given by the difference between the freedom in (3.108) and (3.118), (3.120).

3.8.5 Summary

The classification of exact point symmetries of a one-dimensional wave equation (3.53) [107] has one more case, $K(u_x) = e^{u_x}$, than the classification of exact point symmetries of the two-dimensional wave equation (3.95) (Section 3.8.1).

Similarly, the classification of exact and BGI approximate point symmetries of the one-dimensional perturbed wave equation (3.56) (Section 3.6.1) has one more case, $Q(u_x) = u_x$, than the classification of exact and BGI approximate point symmetries of the two-dimensional perturbed wave equation (3.96). By comparing the exact symmetries (3.106) with the BGI approximate symmetries (3.110) and (3.111) of (3.96), one can see that the exact point symmetry classification of the wave equation (3.96) is included in the classification of BGI approximate symmetries of the same PDE with the latter has a new case when $Q(u_x^2+u_y^2) = b\ln(u_x^2+u_y^2)+d$ which yields a new approximate symmetry for (3.96) given by (3.114).

The classification of FS symmetries of the one-dimensional wave equation (3.56) (Section 3.6.3) has one more case than the FS symmetry classification of the two-dimensional perturbed wave equation (3.96). Finally, we note that the BGI and FS approximate symmetry classifications for the two-dimensional perturbed wave equation (3.96) are not equivalent.

3.9 Discussion

In chapter 2, we have seen that the determining equations (2.106) for higher-order BGI approximate symmetries of the perturbed ODE (2.59) is a linear PDE in the approximate symmetry component $\zeta^1(x, y, y', ..., y^{(n-1)})$ with no additional conditions on the unperturbed symmetry components. This led to stability of exact point or local symmetries of the unperturbed ODE (2.58) as higher-order BGI approximate symmetries. For a PDE, in general, the situation is different. Some exact/local symmetries of the unperturbed PDE (3.5) do not yield higher-order approximate symmetries of the perturbed PDE (3.15). This was clarified by noticing that the left-hand side of the determining equation (3.23) always splits into a system of linear PDEs in ζ^1. On the other side, the right-hand side of equation (3.23) may contain derivatives of u with respect to other variables different than those in the left-hand side of equation (3.23), this yields additional conditions on the unperturbed symmetry component ζ^0. A similar argument holds for FS approximate symmetries. As an example, we showed that there was no higher-order (BGI and FS) approximate symmetry for the perturbed PDE (3.31) corresponding to the unstable point symmetry $t\partial/\partial u$ of the unperturbed wave equation (3.11).

The knowledge of stability of exact point symmetries in BGI and FS senses helps in studying the symmetry properties of the perturbed models. The inheritance of all exact symmetries of an unperturbed PDE by its perturbed version can occur in some cases. As shown in [10], the perturbed evolution equations

$$u_t = h(u)u_x + \epsilon H[u],$$

where $h(u)$ is an arbitrary function of its argument and H is a differential function inherits the exact point symmetries of the unperturbed equation $u_t = h(u)u_x$. In this chapter, we found a classification of stable point symmetries for a nonlinear wave equation in BGI and FS frameworks (Table 3.1).

The classification of exact and BGI approximate local symmetries of perturbed ODEs and PDEs have been considered in many articles (see, e.g., [23] and references therein). FS point symmetry classification was performed for some PDEs with a small parameter [36,112]. The classification of exact and approximate (BGI and FS) symmetries for a perturbed PDE helps in illustrating the difference between these symmetries and providing different types of approximate symmetries than can be used to construct approximate solutions for the given PDE. In Section 3.6, we classified the exact and approximate (BGI and FS) point symmetries for the perturbed one-dimensional wave equation (3.56). We found general approximate solutions for a class of a perturbed one-dimensional wave equation (3.74) (Section 3.7.1). We found numerical solutions for the perturbed wave equation (3.82) and compared with its approximate solution (3.83) (Section 3.7.2). Using the approximate solution (3.83) of the wave model (3.82), we estimated the finite-time singularity formation of (3.82). We also estimated the finite-time singularity formation of (3.82) by a linear approximation of the characteristic curves using a finite difference scheme and compared the two sets of finite-time singularities (Section 3.7.3). We found a complete classification of exact point symmetries of the two dimensional wave equation (3.95), along with exact and (BGI and FS) approximate point symmetries of the perturbed two dimensional wave model (3.96) (Section 3.8).

The determining equation (2.79) for FS symmetries of the perturbed PDE (3.15) is different than the determining equation (3.20) for BGI approximate symmetries of (3.15) that yield different approximate symmetry structures. The relation between BGI and FS approximate point symmetries for Navier-Stokes equation and diffusion equations was discussed in [15, 92]. In this chapter, we showed some connection between the BGI and FS approximate symmetries for a family of perturbed PDEs, that each stable BGI point symmetry yields a higher-order FS approximate symmetry (Theorem 3.5.1).

4 Approximate Conservation Laws of PDEs with a Small Parameter

4.1 Introduction

In the previous chapters, we investigated the BGI and FS frameworks for approximate local symmetries of algebraic equations, ODEs and PDEs with a small parameter. We observed that new approximate symmetries can be obtained and we showed that how these approximate symmetries are useful in construction of new approximate solutions for perturbed models (ODEs and PDEs). Several examples were given.

The notion of approximate conservation laws was initiated in [23] with specific regard to approximate symmetries associated with approximate Lagrangian of the system of perturbed PDEs (approximate Noether symmetries [42]). For perturbed PDEs that do not admit variational principle, approximate conservation laws were constructed using known approximate symmetries of the given model [44,113] and using the direct method [46].

In this chapter, we apply the direct method [5,6] to obtain approximate conservation laws of a system of perturbed PDEs. We show, using examples of perturbed PDEs, that one can obtain additional approximate conservation laws for the given system that do not originate from the exact conservation laws of the same system. For a variational system of perturbed PDEs, we show that a set of approximate multipliers corresponds to a Noether approximate local symmetry of the PDE system. We show that if two systems of perturbed PDEs are approximately connected by an invertible approximate point transformation, then an approximate conservation law for one system is mapped using this transformation to a conservation law for the other system and a formula for the transformed conservation law is derived. Another formula is derived using the action of an approximate point symmetry of a system of perturbed PDEs on a given set of approximate multipliers of a known approximate conservation law for this system to obtain new set of approximate multipliers which could yield new approximate conservation law for the given system if the new set of approximate multipliers are independent of the given set of approximate multipliers. Using these formulas, we obtain new approximate conservation laws for perturbed wave equation and nonlinear telegraph system [37]. As an application for approximate conservation laws, we find the potential systems corresponding to approximate conservation laws of a nonlinear wave equation. We show that new approximate potential symmetries can be obtained and we provide a simple example to show that the approximate potential symmetries are useful in construction of new approximate solutions for perturbed PDEs.

4.2 Approximate local conservation laws

Consider a PDE system with a small parameter ϵ

$$F^\sigma[v;\epsilon] = F_0^\sigma(x, v, \partial v, \dots, \partial^k v) + \epsilon F_1^\sigma(x, v, \partial v, \dots, \partial^k v) = o(\epsilon), \tag{4.1}$$

$\sigma = 1, ..., N$.

Definition 4.2.1. An *approximate local conservation law* of (4.1) is a divergence expression

$$D_i \Phi^i[v;\epsilon] = o(\epsilon) \tag{4.2}$$

holds for all solutions of $F^\sigma[v;\epsilon] = o(\epsilon)$, where

$$\Phi^i[v;\epsilon] = \Phi_0^i[v] + \epsilon\Phi_1^i[v]$$

and D_i is the total derivative with respect to x^i.

Note that $D_i\Phi_0^i[v] = 0$ is a local conservation law of the unperturbed equations $F_0^\sigma[v] = 0$. For example, the nonlinear wave equation

$$u_{tt} + \epsilon u_t = (uu_x)_x \tag{4.3}$$

has an approximate conservation law

$$D_t\left[tu_t - u + \frac{\epsilon}{2}t^2u_t\right] - D_x\left[tuu_x + \frac{\epsilon}{2}t^2uu_x\right] = o(\epsilon).$$

When $\epsilon = 0$, one obtains an exact local conservation law $D_t(tu_t - u) - D_x(tuu_x) = 0$ for the unperturbed wave equation $u_{tt} = (uu_x)_x$.

4.2.1 Equivalent and trivial approximate conservation laws

An approximate conservation law (4.2) of the perturbed equations (4.1) is *trivial* when

1. its fluxes $\Phi^i[v;\epsilon]$ vanish identically or become $o(\epsilon)$ on the solutions of a PDE system (4.1),

2. the approximate conservation law itself vanishes to $o(\epsilon)$ as a differential identity.

   For example, consider the PDE system

$$v_x = u_t, \quad v_t = (c^2 + \epsilon u)u_x. \tag{4.4}$$

   The approximate conservation laws

$$D_t((v_x - u_t)) + D_x(v_t - (c^2 + \epsilon u)u_x) = o(\epsilon),$$

$$D_t((v_{xx})) - D_x(v_{tx}) = o(\epsilon)$$

   are trivial approximate conservation laws of the first and second type, respectively. The above cases are equivalent to the cases of trivial exact conservation laws [37]. Another kind of triviality appears in case of approximate conservation law when

3. an approximate conservation law (4.2) is of the form $\epsilon D_i\Phi_0^i[v] = o(\epsilon)$ where $\Phi_0^i[v]$ are fluxes of a local conservation law of the unperturbed equations $F_0^\sigma[v] = 0$. As an example, the divergence expression

$$\epsilon\left(D_t\left(u_t\right) - D_x\left(uu_x\right)\right) = o(\epsilon) \tag{4.5}$$

is a trivial approximate conservation law for the PDE (4.3) since on solutions of (4.3), the left-hand side of (4.5) reads $-\epsilon^2 u_t$.

Definition 4.2.2. Two approximate conservation laws are *equivalent* if their difference is a trivial approximate conservation law.

4.3 Approximate multipliers. The direct method

Definition 4.3.1. The *approximate multipliers*

$$\Lambda^\sigma[V;\epsilon] = \Lambda_0^\sigma[V] + \epsilon\Lambda_1^\sigma[V], \quad \sigma = 1, ..., N$$

yield a divergence expression for (4.1) if

$$\Lambda^\sigma F^\sigma \equiv (\Lambda_0^\sigma[V] + \epsilon\Lambda_1^\sigma[V])F^\sigma \equiv D_i\Phi^i[V;\epsilon] + o(\epsilon) \tag{4.6}$$

holds for arbitrary functions V. If $\Lambda^\sigma[V;\epsilon]$ is nonsingular, then on solutions $V(x) = v(x)$ of the PDE system (4.1) one has an approximate local conservation law

$$D_i\Phi^i[v;\epsilon] = o(\epsilon). \tag{4.7}$$

Remark 4.3.1. For perturbed ODEs, approximate conservation laws and approximate multipliers correspond respectively to approximate first integrals and approximate integrating factors for the perturbed ODEs. They have been discussed in detail in Chapter 2.

An approximate multiplier $\Lambda^\sigma[V;\epsilon] = \Lambda_0^\sigma[V]+\epsilon\Lambda_1^\sigma[V]$ is *singular* if it is a singular function as ϵ approaches to 0 when evaluated on $F^\sigma = o(\epsilon)$. In applications, one is only interested in nonsingular approximate multipliers since singular approximate multipliers can lead to divergence expressions that are not approximate conservation laws of a PDE system (4.1). For instance, for each nonsingular multiplier Λ_0^σ of the unperturbed PDE system $F_0^\sigma[V] = 0$, $\sigma = 1, \ldots, N$,

$$\Lambda^\sigma = \Lambda_0^\sigma[V] - \epsilon\frac{\Lambda_0^\sigma[V]F_1^\sigma[V]}{F_0^\sigma[V]}$$

satisfies (4.6). However, on solutions of (4.1), one has

$$\frac{\Lambda_0^\sigma[V]F_1^\sigma[V]}{F_0^\sigma[V]} = \frac{\Lambda_0^\sigma[v]F_1^\sigma[v]}{-\epsilon F_1^\sigma[v] + o(\epsilon)} \to \infty \quad \text{as} \quad \epsilon \to 0.$$

It follows that Λ^σ is a singular multiplier of the PDE system (4.1). In particular, one can show that Λ^σ yields a divergence expression that is not an approximate conservation laws of the PDE system (4.1). Take the nonlinear heat equation

$$u_t - u_{xx} - \epsilon u^3 = 0 \tag{4.8}$$

as an example. For a function $\Lambda = \Lambda_0 + \epsilon\Lambda_1 = 1 + \epsilon\frac{U^3}{U_t - U_{xx}}$, one has

$$\begin{aligned}\Lambda\left(U_t - U_{xx} - \epsilon U^3\right) &= \left(1 + \epsilon\frac{U^3}{U_t - U_{xx}}\right)\left(U_t - U_{xx} - \epsilon U^3\right) \\ &= U_t - U_{xx} - \varepsilon^2\frac{U^6}{U_t - U_{xx}} = U_t - U_{xx} + o(\epsilon)\end{aligned} \tag{4.9}$$

for arbitrary function U. However, on solutions of $u_t - u_{xx} - \epsilon u^3 = 0$,

$$\Lambda_1 = \frac{u^3}{u_t - u_{xx}} = \frac{u^3}{\epsilon u^3} \to \infty, \quad \text{as} \quad \epsilon \to 0.$$

It follows that $\Lambda = 1 + \epsilon\frac{U^3}{U_t - U_{xx}}$ is a singular multiplier. In fact, on solutions of (4.8), the expression in (4.9) becomes ϵu^3. Hence, $\Lambda = 1 + \epsilon U^3/(U_t - U_{xx})$ yields no approximate conservation law of the nonlinear heat equation (4.8).

Theorem 4.3.1. *The nonsingular approximate multipliers*

$$\Lambda^\sigma[V;\epsilon] = \Lambda_0^\sigma[V] + \epsilon\Lambda_1^\sigma[V], \quad \sigma = 1, ..., N$$

yield approximate conservation laws for the PDE system (4.1) *if and only if*

$$E_{V^\mu}((\Lambda_0^\sigma[V] + \epsilon\Lambda_1^\sigma[V])F^\sigma[V;\epsilon]) \equiv o(\epsilon), \quad \mu = 1, \ldots, m, \tag{4.10}$$

holds for arbitrary functions V, where E_{V^μ} is the Euler operator given by (1.143).

Proof. For the necessity, it is straightforward to obtain (4.10) by applying Euler operator E_V^μ to the identity (4.6). Conversely, if

$$E_{V^\mu}((\Lambda_0^\sigma[V] + \epsilon\Lambda_1^\sigma[V])F^\sigma[V;\epsilon]) \equiv o(\epsilon),$$

then, for each $\mu = 1, \ldots, m$, one has

$$\begin{aligned}o(\epsilon) &\equiv E_{V^\mu}((\Lambda_0^\sigma[V] + \epsilon\Lambda_1^\sigma[V])F^\sigma[V;\epsilon]) \\ &\equiv E_{V^\mu}(\Lambda_0^\sigma[V]F^\sigma[V;\epsilon]) + \epsilon E_{V^\mu}(\Lambda_1^\sigma[V]F^\sigma[V;\epsilon]) \\ &\equiv E_{V^\mu}(\Lambda_0^\sigma[V](F_0^\sigma[V] + \epsilon F_1^\sigma[V])) + \epsilon E_{V^\mu}(\Lambda_1^\sigma[V](F_0^\sigma[V] + \epsilon F_1^\sigma[V])) \\ &\equiv E_{V^\mu}(\Lambda_0^\sigma[V]F_0^\sigma[V]) + \epsilon E_{V^\mu}(\Lambda_0^\sigma[V]F_1^\sigma[V] + \Lambda_1^\sigma[V]F_0^\sigma[V]) + o(\epsilon)\end{aligned} \tag{4.11}$$

for arbitrary functions V. Setting to zero the coefficients of different powers of ϵ, one consequently has

$$\begin{aligned}&E_{V^\mu}(\Lambda_0^\sigma[V]F_0^\sigma[V]) \equiv 0, \\ &E_{V^\mu}(\Lambda_0^\sigma[V]F_1^\sigma[V] + \Lambda_1^\sigma[V]F_0^\sigma[V]) \equiv 0\end{aligned} \tag{4.12}$$

for arbitrary functions V and each $\mu = 1, \ldots, m$. According to Theorem 1.4.1, there exist differential functions $\Phi_0^i[V]$, $\Phi_1^i[V]$ such that

$$\Lambda_0^\sigma[V]F_0^\sigma[V] = D_i\Phi_0^i[V],$$

$$\Lambda_0^\sigma[V]F_1^\sigma[V] + \Lambda_1^\sigma[V]F_0^\sigma[V] = D_i\Phi_1^i[V]$$

for any arbitrary functions V. Let $\Phi^i[V;\epsilon] = \Phi^i_0[V] + \epsilon\Phi^i_1[V] + o(\epsilon)$, then it follows that

$$(\Lambda^\sigma_0[V] + \epsilon\Lambda^\sigma_1[V])F^\sigma[V;\epsilon] \equiv D_i\Phi^i + o(\epsilon).$$

Therefore, on solutions of (4.1), one obtains an approximate conservation law (4.7) for the PDE system (4.1) □

Remark 4.3.2. Equations (4.12) are the determining equations for approximate multipliers of the PDE system (4.1). From the first determining equation of (4.12), it follows that Λ^σ_0, $\sigma = 1, ..., N$ is a multiplier of the unperturbed PDE system $F^\sigma_0[V] = 0$.

Remark 4.3.3. It is clear from the determining equations (4.12), if $\Lambda^\sigma_0[V]$ is a multiplier of the unperturbed PDE system $F^\sigma_0[V] = 0$, then $\Lambda^\sigma_1[V] = \epsilon\Lambda^\sigma_0$ is an approximate multiplier of the PDE system (4.1). Such multipliers are called trivial approximate multipliers. In practice, one is only interested in approximate multipliers with $\Lambda^\sigma_0 \not\equiv 0$.

Remark 4.3.4. Suppose $\Lambda^\sigma[V;\epsilon]$ is an exact multiplier of the PDE system (4.1). It is straightforward to show that the first order expansion of $\Lambda^\sigma[V;\epsilon]$ about $\epsilon = 0$ is an approximate multiplier of (4.1).

4.3.1 Examples

Example 4.3.1. As a first example, consider the nonlinear wave equation

$$u_{tt} - (c^2(u)u_x)_x = 0, \tag{4.13}$$

and its perturbed version

$$u_{tt} - (c^2(u)u_x)_x + \epsilon u_t = 0, \tag{4.14}$$

with nonconstant wave speed $c(u)$. The second order Euler operator is given by

$$E_U = \frac{\partial}{\partial U} - D_x\frac{\partial}{\partial U_x} - D_t\frac{\partial}{\partial U_t} + D_{xx}\frac{\partial}{\partial U_{xx}} + D_{tt}\frac{\partial}{\partial U_{xx}} + D_{xt}\frac{\partial}{\partial U_{xt}}. \tag{4.15}$$

We seek all approximate multipliers of the perturbed nonlinear wave equation (4.14) of the form $\Lambda = \Lambda_0(x,t,U) + \epsilon\Lambda_1(x,t,U)$, where Λ_0 is an exact multiplier of the unperturbed equation (4.13). In terms of the second order Euler operator (4.15), the determining equations (4.12) become

$$\begin{aligned} &E_U(\Lambda_0(x,t,U)(U_{tt} - (c^2(U)U_x)_x)) \equiv 0, \\ &E_U(\Lambda_0(x,t,U)U_t + \Lambda_1(x,t,U)(U_{tt} - (c^2(U)U_x)_x)) \equiv 0. \end{aligned} \tag{4.16}$$

First, one find the exact multiplier Λ^0 using the first determining equation of (4.16). The split system of linear PDEs in Λ^0 is given by

$$\frac{\partial\Lambda_0}{\partial U} = 0, \quad \frac{\partial^2\Lambda_0}{\partial x^2} = 0, \quad \frac{\partial^2\Lambda_0}{\partial t^2} = 0 \tag{4.17}$$

for arbitrary nonconstant wave speed $c(u)$. One obtains the following general solution for Λ_0:

$$\Lambda_0(x,t,U) = c_1 + c_2x + c_3xt + c_4t. \tag{4.18}$$

Substituting Λ_0 into the second determining equation of (4.16) leads to the following splitting system of linear PDEs in Λ_1

$$\frac{\partial \Lambda_1}{\partial U} = 0, \quad \frac{\partial^2 \Lambda_1}{\partial x^2} = 0, \quad \frac{\partial^2 \Lambda_1}{\partial t^2} = c_3 x + c_4. \tag{4.19}$$

The general solution for the PDE system (4.19) has the form

$$\Lambda_1(x,t,U) = \frac{c_3}{2} x t^2 + \frac{c_4}{2} t^2 + c_5 x t + c_6 t + c_7 x + c_8, \tag{4.20}$$

where c_i, $i = 1, \ldots, 8$, are arbitrary constants. Hence there are eight nonsingular approximate multipliers for the perturbed nonlinear wave equation (4.14) given by

$$\begin{aligned}
&\Lambda_{(1)} = 1,\ \Lambda_{(2)} = x,\\
&\Lambda_{(3)} = xt + \frac{\epsilon}{2} x t^2,\ \Lambda_{(4)} = t + \frac{\epsilon}{2} t^2,\\
&\Lambda_{(5)} = \epsilon x t,\ \Lambda_{(6)} = \epsilon t,\ \Lambda_{(7)} = \epsilon x,\ \Lambda_{(8)} = \epsilon.
\end{aligned} \tag{4.21}$$

Note that $\Lambda_{(j)}, j = 5, ..., 8$ are trivial approximate multipliers that yield trivial approximate conservation law of the third type. For the multiplier $\Lambda_{(1)} = 1$, since the PDE (4.14) is in divergence form, one has

$$\Lambda_{(1)}(U_{tt} - (c^2(U)U_x)_x + \epsilon U_t) = D_t[U_t + \epsilon U] - D_x[c^2(U)U_x].$$

Consequently,

$$D_t[u_t + \epsilon u] - D_x[c^2(u)u_x] = o(\epsilon) \tag{4.22}$$

is an approximate conservation law for the PDE (4.14) corresponding to the approximate multiplier $\Lambda_{(1)} = 1$. For $\Lambda_{(3)} = xt + \frac{\epsilon}{2} x t^2$, one can use integration by parts to determine the flux and the density as follows

$$\begin{aligned}
\Lambda_{(3)}(U_{tt} - (c^2(U)U_x)_x + \epsilon U_t) \quad &= xtU_{tt} - xt(c^2(U)U_x)_x + \frac{\epsilon}{2} x t^2 U_{tt} - \frac{\epsilon}{2} x t^2 (c^2(U)U_x)_x \\
&+ \epsilon x t U_t + o(\epsilon) \\
&= D_t(xtU_t - xU) - D_x\left(xtc^2(U)U_x - t\int c^2(U)dU\right) \\
&+ D_t(\frac{\epsilon}{2} x t^2 u_t) - \epsilon x t U_t \\
&- D_x\left(\frac{\epsilon}{2}(xt^2 c^2(u)u_x - t^2 \int c^2(u)du)\right) + \epsilon x t U_t + o(\epsilon) \\
&= D_t(xtU_t - xU + \frac{\epsilon}{2} x t^2 u_t) \\
&- D_x\left(xtc^2(U)U_x - t\int c^2(U)dU - \frac{\epsilon}{2}(xt^2 c^2(u)u_x + t^2 \int c^2(u)du)\right).
\end{aligned}$$

Thus, the corresponding approximate conservation law is given by

$$\begin{aligned}
&D_t\left[xtu_t - xu + \tfrac{\epsilon}{2} x t^2 u_t\right] \\
&\qquad - D_x\left[xtc^2(u)u_x - t\int c^2(u)du + \frac{\epsilon}{2}\left(xt^2 c^2(u)u_x - t^2 \int c^2(u)du\right)\right] = o(\epsilon).
\end{aligned} \tag{4.23}$$

Similarly, one can find the approximate conservation laws of the perturbed nonlinear wave equation (4.14) corresponding to the approximate multipliers $\Lambda_{(2)}$ and $\Lambda_{(4)}$. They are given respectively by

$$D_t\left[xu_t + \epsilon xu\right] - D_x\left[xc^2(u)u_x - \int c^2(u)du\right] = o(\epsilon), \tag{4.24a}$$

$$D_t\left[tu_t - u + \frac{\epsilon}{2}t^2u_t\right] - D_x\left[tc^2(u)u_x + \frac{\epsilon}{2}t^2c^2(u)u_x\right] = o(\epsilon). \tag{4.24b}$$

The trivial approximate multipliers $\Lambda_{(j)}, j = 5, ..., 8$ yield trivial approximate conservation laws $\epsilon D_i\Phi_0^i = o(\epsilon)$, where Φ_0^i are fluxes of the exact conservation laws for the unperturbed equation (4.13). The trivial approximate conservation laws are given respectively by

$$D_t\left[\epsilon(xtu_t - xu)\right] - D_x\left[\epsilon\left(xtc^2(u)u_x - t\int c^2(u)du\right)\right] = o(\epsilon), \tag{4.25c}$$

$$D_t\left[\epsilon(tu_t - u)\right] - D_x\left[\epsilon tc^2(u)u_x\right] = o(\epsilon), \tag{4.25d}$$

$$D_t\left[\epsilon xu_t\right] - D_x\left[\epsilon\left(xc^2(u)u_x - \int c^2(u)du\right)\right] = o(\epsilon), \tag{4.25e}$$

$$D_t\left[\epsilon u_t\right] - D_x\left[\epsilon c^2(u)u_x\right] = o(\epsilon). \tag{4.25f}$$

If one considers the exact multipliers of the perturbed nonlinear wave equation (4.14), one can show that the perturbed nonlinear wave equation (4.14) has four exact multipliers given by

$$\begin{aligned} &A_{(1)} = 1,\ A_{(2)} = x, \\ &A_{(3)} = e^{\epsilon t} = 1 + \epsilon t + o(\epsilon), \\ &A_{(4)} = xe^{\epsilon t} = x + \epsilon xt + o(\epsilon). \end{aligned} \tag{4.26}$$

It follows that $A_{(1)} = \Lambda_{(1)}$, $A_{(2)} = \Lambda_{(2)}$, and the approximate multipliers $\Lambda_{(1)} + \Lambda_{(6)}$ and $\Lambda_{(2)} + \Lambda_{(5)}$ in (4.21) are the first two terms in the Taylor expansion in ϵ of the exact multipliers $A_{(3)}$ and $A_{(4)}$, respectively. Most importantly, the genuine approximate multipliers $\Lambda_{(3)}$ and $\Lambda_{(4)}$ in (4.21) of the perturbed equation (4.14) do not arise from the exact multipliers (4.26) of the same PDE.

Example 4.3.2. As a second example, consider the perturbed nonlinear diffusion equation

$$u_t - (u^{-2}u_x)_x - \epsilon(u - u^{-1})_x = 0. \tag{4.27}$$

We seek all approximate multipliers of the perturbed nonlinear diffusion equation (4.27) of the form $\Lambda = \Lambda_0(x,t,U) + \epsilon\Lambda_1(x,t,U)$. In terms of the second order Euler operator (4.15), the determining equations become

$$\begin{aligned} &E_U(\Lambda_0(x,t,U)(U_t - (U^{-2}U_x)_x) \equiv 0, \\ &E_U(\Lambda_0(x,t,U)(U - U^{-1})_x + \Lambda_1(x,t,U)(U_t - (U^{-2}U_x)_x) \equiv 0. \end{aligned} \tag{4.28}$$

Splitting the determining equations (4.28), one obtains the following explicit determining equations for the unknown functions $\Lambda_0(x,t,U)$ and $\Lambda_1(x,t,U)$:

$$\begin{aligned}
&-2U^{-2}\frac{\partial \Lambda_0}{\partial U}=0, \quad -2U^{-2}\frac{\partial^2 \Lambda_0}{\partial x \partial U}=0,\\
&-\frac{\partial \Lambda_0}{\partial t}-U^{-2}\frac{\partial^2 \Lambda_0}{\partial x^2}=0, \quad 2U^{-3}\frac{\partial \Lambda_0}{\partial U}-U^{-2}\frac{\partial^2 \Lambda_0}{\partial U^2}=0,\\
&-2U^{-2}\frac{\partial \Lambda_1}{\partial U}=0, \quad -2U^{-2}\frac{\partial^2 \Lambda_1}{\partial x \partial U}=0,\\
&-U^{-2}\frac{\partial^2 \Lambda_1}{\partial U^2}+2U^{-3}\frac{\partial \Lambda_1}{\partial U}=0, \quad -\frac{\partial \Lambda_1}{\partial t}+\frac{\partial \Lambda_0}{\partial x}+U^{-2}\frac{\partial \Lambda_0}{\partial x}-U^{-2}\frac{\partial^2 \Lambda_1}{\partial x^2}=0.
\end{aligned} \tag{4.29}$$

The general solutions of the determining equations (4.29) are given by

$$\begin{aligned}
&\Lambda_0(x,t,U)=c_2x+c_1,\\
&\Lambda_1(x,t,U)=c_2\left(t+\frac{x^2}{2}\right)+c_3x+c_4,
\end{aligned} \tag{4.30}$$

where c_i, $i=1,\ldots,4$, are arbitrary constants. Hence there are four approximate multipliers for the perturbed nonlinear diffusion equation (4.27) given by

$$\Lambda_{(1)}=1,\ \Lambda_{(2)}=x+\epsilon\left(t+\frac{x^2}{2}\right),\ \Lambda_{(3)}=\epsilon x,\ \Lambda_{(4)}=\epsilon. \tag{4.31}$$

Thus, on solutions of (4.27), one gets four corresponding approximate conservation laws of the perturbed nonlinear diffusion equation (4.27) given by

$$D_t[u]-D_x\left[u^{-2}u_x+\epsilon(u-u^{-1})\right]=o(\epsilon), \tag{4.32a}$$

$$D_t\left[xu+\epsilon\left(tu+\frac{x^2}{2}u\right)\right]-D_x\left[xu^{-2}u_x+u^{-1}+\epsilon\left(\left(t+\frac{x^2}{2}\right)u^{-2}u_x+xu\right)\right]=o(\epsilon), \tag{4.32b}$$

$$D_t[\epsilon xu]-D_x\left[\epsilon\left(xu^{-2}u_x+u^{-1}\right)\right]=o(\epsilon), \tag{4.32c}$$

$$D_t[\epsilon u]-D_x\left[\epsilon u^{-2}u_x\right]=o(\epsilon). \tag{4.32d}$$

One can show that the exact multipliers of the perturbed nonlinear diffusion equation (4.27) are given by

$$A_{(1)}=1,\ A_{(2)}=e^{\epsilon x}e^{\epsilon^2 t}=1+\epsilon x+o(\epsilon). \tag{4.33}$$

So, there are two exact multipliers vs. four approximate multipliers for the PDE (4.27). Note that the linear combination $\Lambda_{(1)}+\Lambda_{(3)}$ of the approximate multipliers $\Lambda_{(1)}$ and $\Lambda_{(3)}$ in (4.31) of the PDE (4.27) is contained in a Taylor expansion in ϵ of the exact multiplier $A_{(2)}$ (4.33) of (4.27). While, the multiplier $\Lambda_{(2)}$ is a new approximate multiplier for the PDE (4.27) that does not arise from the exact multipliers (4.33) of (4.27).

Remark 4.3.5. In Example 4.3.1 and Example 4.3.2, the exact multipliers of the unperturbed PDE carry over the approximate multipliers of the perturbed equation with some of them as genuine approximate multipliers. However this is not always the case for all PDEs since the determining equations (4.12) for approximate multipliers may contain additional constraints on the exact multipliers of the unperturbed equation.

As an illustration, consider the KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{4.34}$$

and its perturbed version

$$u_t + uu_x + u_{xxx} + \epsilon u_x^2 = 0. \tag{4.35}$$

These are Cauchy-Kovalevskaya PDEs with leading derivative u_t. As noted in [37], all nonsingular multipliers of the above equations have the form $\Lambda(x, t, U, U_x, U_{xx}, ...)$. The general second-order exact multiplier of the unperturbed equation (4.34) is found in [37] and has the form

$$\Lambda_0(x, t, U, U_x, U_{xx}) = c_1 + c_2 U + c_3(tU - x) + c_4 \left(U_{xx} + \frac{U^2}{2} \right) \tag{4.36}$$

Let

$$\Lambda = \Lambda_0(x, t, U, U_x, U_{xx}) + \epsilon \Lambda_1(x, t, U, U_x, U_{xx}, ..., U_{Nx})$$

for some finite number N be an approximate multiplier for the perturbed PDE (4.35). Take $N = 2$, then the determining equations (4.12) splits into linear PDEs in Λ_1 and additional conditions on the exact multiplier (4.36) leading to some constraints on the free constants in (4.36): $C_2 = C_3 = C_4 = 0$. Now increasing the dependance of Λ_1 on higher derivative of U yields additional terms with higher derivatives of U which does not help in removing the constraints on C_i in (4.36). Hence, the exact multipliers of the KdV equation (4.34) that correspond to the constants C_2, C_3 and C_4 do not yield approximate multipliers for the perturbed PDE (4.35).

4.4 Noether's theorem for approximate conservation laws

Consider an approximate Lagrangian

$$L[v; \epsilon] = L_0(x, v, \partial v, \ldots, \partial^k v) + \epsilon L_1(x, v, \partial v, \ldots, \partial^k v) = o(\epsilon), \tag{4.37}$$

and the action integral

$$\mathcal{L} = \int_\Omega L[v; \epsilon] dx \tag{4.38}$$

defined on some domain Ω. The approximate Euler-Lagrange equations are given by

$$E_{v^\mu}(L) = o(\epsilon), \quad \mu = 1, ..., m. \tag{4.39}$$

Definition 4.4.1. A one-parameter family of local approximate BGI transformations with infinitesimal generator (1.100) is an approximate variational symmetry of the action integral (4.38) if and only if

$$\hat{X}^{(k)} L = D_i A^i + o(\epsilon),$$

where $A^i[v; \epsilon]$ are differential functions of their arguments.

An approximate variational symmetry of the functional (4.38) yields an approximate local symmetry of the corresponding approximate Euler-Lagrange equations (4.39). Let the equations (4.39) be invariant under the one-parameter BGI approximate point transformations (1.90) with infinitesimal generator

$$\begin{aligned} X &= \xi^i(x, v; \epsilon) + \eta^\mu(x, v; \epsilon) \\ &= \left(\xi_0^i(x, v) + \epsilon \xi_1^i(x, v)\right) \frac{\partial}{\partial x^i} + \left(\eta_0^\mu(x, v) + \epsilon \eta_1^\mu(x, v)\right) \frac{\partial}{\partial v^\mu}. \end{aligned} \tag{4.40}$$

The one-parameter family of BGI approximate point transformations (1.90) with symmetry generator (4.40) is an approximate variational symmetry for (4.39) if

$$X^{(k)} L + L D_i \xi^i = D_i A^i + o(\epsilon). \tag{4.41}$$

For perturbed PDEs, Noether's theorem also provides a relation between approximate variational symmetries and corresponding approximate conservation laws [23].

Theorem 4.4.1. *Let X (4.40) be an approximate variational symmetry for the approximate Euler-Lagrange equations (4.39), then the differential functions*

$$\Phi^i = L\xi^i + \left(\eta^\mu - \xi^j v_j^\mu\right) \frac{\partial L}{\partial v_i^\mu} - A^i + o(\epsilon) \tag{4.42}$$

satisfy the approximate conservation law (4.2) for (4.39).

Though approximate Noether's theorem gives an explicit formula for the fluxes of approximate conservation laws, it is restricted to perturbed differential equations arising from a variational principle, i.e., the approximate Euler-Lagrange equations (4.39) that are approximate extremals of the action integral (4.38).

On the other hand, there are no restrictions on the direct method. It can be applied to any system of differential equations whether or not it arises from a variational principle. In the following theorem, we show that all approximate local conservation laws obtained by approximate Noether theorem are obtained by the direct method.

Theorem 4.4.2. *Let X (4.40) be an approximate variational symmetry for the approximate Euler-Lagrange equations (4.39), and let*

$$\zeta^\mu[v;\epsilon] = \zeta_0^\mu + \epsilon\zeta_1^\mu = \eta_0^\mu - v_i\xi_0^i + \epsilon\left(\eta_1^\mu - v_i\xi_1^i\right)$$

be the characteristic of X (4.40). Then $\zeta = (\zeta^1, \ldots, \zeta^m)$ is an approximate multiplier of an approximate conservation law for the approximate Euler-Lagrangian equations (4.39).

Proof. As an adaption of the relation between an exact point symmetry and its evolutionary form [4], it can be easily verified that the approximate point symmetry (4.40) and its evolutionary form satisfy

$$X^{(k)} - \tilde{X}^{(k)} = \xi^i D_i + o(\epsilon). \tag{4.43}$$

Substitute (4.43) into the Noether formula (4.41), one gets

$$\begin{aligned} o(\epsilon) &= \tilde{X}^{(k)}L + \xi^i D_i L + L D_i \xi^i - D_i A^i \\ &= \tilde{X}^{(k)}L + D_i(\xi^i L) - D_i A^i \\ &= \tilde{X}^{(k)}L + D_i\left(\xi^i L - A^i\right). \end{aligned}$$

The integration by parts for the first term of this equation leads to

$$\begin{aligned} \tilde{X}^{(k)}L &= \zeta^\mu \frac{\partial L}{\partial v^\mu} + \sum_{j,\mu} D_j \zeta^\mu \frac{\partial L}{\partial v^\mu_j} \\ &= \zeta^\mu \frac{\partial L}{\partial v^\mu} + \sum_{j,\mu} \zeta^\mu \left(-D\right)_j \frac{\partial L}{\partial v^\mu_j} + D_i B^i \\ &= \sum_{\mu=1}^{m} \zeta^\mu E_{v^\mu}(L) + D_i B^i, \end{aligned}$$

where $j = (j_1, ..., j_r), 1 \leq j_r \leq n$ and $B_i[v;\epsilon]$ are some functions depending on ζ^μ, L and their derivatives. Hence we proved that

$$\sum_{\mu=1}^{m} \zeta^\mu E_{v^\mu}(L) = D_i(A^i - B^i - \xi^i L) + o(\epsilon).$$

Therefore, $\zeta = (\zeta^1, \ldots, \zeta^m)$ is an approximate multiplier for the perturbed equations (4.39). □

Example 4.4.1. Consider the perturbed linear wave equation with damping

$$u_{tt} - u_{xx} + \epsilon u_t = 0. \tag{4.44}$$

This PDE has an approximate Lagrangian [113]

$$L_{approx} = \frac{1}{2}\left(u_x^2 - u_t^2\right) + \frac{\epsilon}{2} t \left(u_x^2 - u_t^2\right).$$

Applying the Euler operator (1.143) on L_{approx}, one gets (4.44) approximately:

$$E_u\left(L_{approx}\right) = u_{tt} - u_{xx} + \epsilon u_t - \epsilon^2 t u_t.$$

In fact, L_{approx} is contained in the Taylor expansion in ϵ of the exact Lagrangian

$$L = \frac{1}{2} e^{\epsilon t}\left(u_x^2 - u_t^2\right)$$

of the PDE (4.44). Note that $L_0 = \frac{1}{2}\left(u_x^2 - u_t^2\right)$ is a Lagrangian for the unperturbed equation ($\epsilon = 0$). The PDE (4.44) admits an approximate symmetry given by

$$X = \frac{\partial}{\partial t} - \epsilon\left(x\frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u}\right). \tag{4.45}$$

The first-order prolongation of (4.45) reads

$$X^{(1)} = X + \epsilon\left(\frac{1}{2}u_x\frac{\partial}{\partial u_x} - \frac{1}{2}u_t\frac{\partial}{\partial u_t}\right).$$

By direct computation, one can find that X satisfies the Noether formula (4.41) with $A_i[u;\epsilon] = 0$, which implies that X is an approximate Noether symmetry of (4.44). The evolutionary form of X has the form

$$\tilde{X} = \left(u_t + \epsilon\left(-xu_x + \frac{1}{2}u\right)\right)\frac{\partial}{\partial u}.$$

Let

$$\zeta = U_t + \epsilon\left(-xU_x + \frac{1}{2}U\right).$$

Then

$$\begin{aligned}
\zeta\left(U_{tt} - U_{xx} + \epsilon U_t\right) &= U_t\left(U_{tt} - U_{xx}\right) + \epsilon\left(\frac{UU_{tt}}{2} - \frac{UU_{xx}}{2} + xU_x\left(U_{xx} - U_{tt}\right) + U_t^2\right) \\
&= D_t\left(\frac{1}{2}U_t^2 + \frac{1}{2}U_x^2\right) - D_x\left(U_tU_x\right) + \epsilon\Bigg[D_t\left(\frac{1}{2}UU_t\right) - D_x\left(\frac{1}{2}UU_x\right) \\
&+ \frac{U_t^2}{2} - \frac{U_x^2}{2} + D_x\left(\frac{1}{2}\left(xU_x^2 + xU_t^2\right)\right) + \frac{U_x^2}{2} + \frac{U_t^2}{2} + xU_tU_{tx} \\
&- D_t\left(xU_tU_x\right) - xU_tU_{xt}\Bigg] + o(\epsilon) \\
&= D_t\left(\frac{1}{2}U_t^2 + \frac{1}{2}U_x^2 + \epsilon\left(\frac{1}{2}UU_t - xU_tU_x\right)\right) \\
&\quad - D_x\left(U_tU_x + \epsilon\left(\frac{1}{2}UU_x - \frac{1}{2}\left(xU_x^2 + xU_t^2\right)\right)\right) + o(\epsilon).
\end{aligned}$$

Hence, ζ is a nontrivial approximate multiplier that yields the following approximate conservation law for (4.44)

$$\begin{aligned}
&D_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \epsilon\left(\frac{1}{2}uu_t - xu_tu_x\right)\right) \\
&\qquad - D_x\left(u_tu_x + \epsilon\left(\frac{1}{2}uu_x - \frac{1}{2}\left(xu_x^2 + xu_t^2\right)\right)\right) = o(\epsilon). \quad (4.46)
\end{aligned}$$

4.5 Other connections between approximate symmetries and approximate conservation laws

For any unperturbed PDE system $F^\sigma[u]$, it was shown that an invertible transformation (point or contact transformation) that maps $F^\sigma[u]$ to another PDE system $R^\sigma[v]$, it maps each conservation law of $F^\sigma[u]$ to a corresponding conservation law of $R^\sigma[v]$. When the invertible transformation is a symmetry of $F^\sigma[u]$, then a known conservation law of $F^\sigma[u]$ is mapped to another conservation law of $F^\sigma[u]$. A formula related to construction of new set of exact multipliers from a known set of exact multipliers was derived which leads to obtaining new conservation law provided that the two sets of multipliers are independent [40].

We extend these results in case of system of perturbed PDEs and their approximate point transformations and approximate conservation laws. Consider an invertible approximate point transformation

$$\begin{aligned}
x^i &= f^i(\tilde{x}, \tilde{V}; \epsilon) = f_0^i(\tilde{x}, \tilde{V}) + \epsilon f_1^i(\tilde{x}, \tilde{V}), \quad i = 1, ..., n, \\
V^\mu &= g^\mu(\tilde{x}, \tilde{V}; \epsilon) = g_0^\mu(\tilde{x}, \tilde{V}) + \epsilon g_1^\mu(\tilde{x}, \tilde{V}), \quad \mu = 1, ..., m.
\end{aligned} \quad (4.47)$$

Under the transformation (4.47), a function $F^\sigma[V;\epsilon]$ with $V(x;\epsilon) = v(x;\epsilon)$ solves the system of PDEs $F^\sigma[v;\epsilon]$ (4.1) is mapped to some function $R^\sigma[\tilde{V};\epsilon]$ where the coordinates of $F^\sigma[V;\epsilon]$ are expressed in terms of the coordinates of $R^\sigma[\tilde{V};\epsilon]$ using (4.47). If $V(x;\epsilon) = v(x;\epsilon)$ is an approximate solution of the system of PDEs (4.1), then $\tilde{V}(\tilde{x}) = \tilde{v}(\tilde{x})$ is an approximate solution of PDE system $R^\sigma[\tilde{v};\epsilon]$ given by

$$R^\sigma[\tilde{v};\epsilon] = R_0^\sigma(x,\tilde{v},\partial\tilde{v},\ldots,\partial^k\tilde{v}) + \epsilon R_1^\sigma(x,\tilde{v},\partial\tilde{v},\ldots,\partial^k\tilde{v}) = o(\epsilon), \tag{4.48}$$

$\sigma = 1,...,N$, with n independent variables $\tilde{x} = (\tilde{x}^1,...,\tilde{x}^n)$ and m dependent variables $\tilde{v} = (\tilde{v}^1,...,\tilde{v}^m)$.

In the following theorem, we show how the invertible transformation (4.47) can be used to construct an approximate conservation law of $R^\sigma[\tilde{v};\epsilon]$ from a known approximate conservation law of $F^\sigma[v;\epsilon]$.

Theorem 4.5.1. *The invertible approximate point transformation* (4.47) *transforms an approximate conservation law* $D_i\Phi^i[v;\epsilon] = o(\epsilon)$ *of PDE system* $F^\sigma[v;\epsilon]$ *to the approximate conservation law*

$$D_i\Psi^i[\tilde{v};\epsilon] = o(\epsilon) \tag{4.49}$$

of PDE system $R^\sigma[\tilde{v};\epsilon]$(4.48) *with* $\Psi^i[\tilde{v};\epsilon]$ *is given in terms of the determinant obtained through replacing the ith column of the Jacobian determinant*

$$J[\tilde{v};\epsilon] = \frac{D\left(f^1,...,f^n\right)}{D\left(\tilde{x}^1,...,\tilde{x}^n\right)} \tag{4.50}$$

by $\begin{bmatrix}\Phi^1[v;\epsilon] & \ldots & \Phi^n[v;\epsilon]\end{bmatrix}^t$.

Proof. Let $D_i\Phi^i[v;\epsilon] = o(\epsilon)$ be an approximate conservation law for the PDE system $F^\sigma[v;\epsilon]$ (4.1). We prove that under the invertible transformation (4.47), the following statement holds

$$\tilde{D}_i\Psi^i[\tilde{V};\epsilon] = J[\tilde{V};\epsilon]D_i\Phi^i[V;\epsilon] + o(\epsilon), \tag{4.51}$$

where $\tilde{D}_i$ is the total derivative operator with respect to $\tilde{x}^i$ given by

$$\tilde{D}_i = \frac{\partial}{\partial\tilde{x}^i} + \tilde{V}_i^\mu\frac{\partial}{\partial\tilde{V}^\mu} + \tilde{V}_{ij}^\mu\frac{\partial}{\partial\tilde{V}_j^\mu} + \ldots + \tilde{V}_{ii_1i_2\ldots i_n}^\mu\frac{\partial}{\partial\tilde{V}_{i_1i_2\ldots i_n}^\mu} + \ldots.$$

Indeed, consider the determinants

$$\begin{gathered}\Psi^1[\tilde{V};\epsilon] = \begin{vmatrix}\Phi^1[V;\epsilon] & \tilde{D}_2f^1 & \cdots & \tilde{D}_nf^1\\ \Phi^2[V;\epsilon] & \tilde{D}_2f^2 & \cdots & \tilde{D}_nf^2\\ \vdots & \vdots & & \vdots\\ \Phi^n[V;\epsilon] & \tilde{D}_2f^n & \cdots & \tilde{D}_nf^n\end{vmatrix}, \quad \Psi^2[\tilde{V};\epsilon] = \begin{vmatrix}\tilde{D}_1f^1 & \Phi^1[V;\epsilon] & \cdots & \tilde{D}_nf^1\\ \tilde{D}_1f^2 & \Phi^2[V;\epsilon] & \cdots & \tilde{D}_nf^2\\ \vdots & \vdots & & \vdots\\ \tilde{D}_1f^n & \Phi^n[V;\epsilon] & \cdots & \tilde{D}_nf^n\end{vmatrix},\\ \cdots, \quad \Psi^n[\tilde{V};\epsilon] = \begin{vmatrix}\tilde{D}_1f^1 & \cdots & \tilde{D}_{n-1}f^1 & \Phi^1[V;\epsilon]\\ \tilde{D}_1f^2 & \cdots & \tilde{D}_{n-1}f^2 & \Phi^2[V;\epsilon]\\ \vdots & & \vdots & \vdots\\ \tilde{D}_1f^n & \cdots & \tilde{D}_{n-1}f^n & \Phi^n[V;\epsilon]\end{vmatrix}.\end{gathered} \tag{4.52}$$

Let $Q_{ik},\ k = 1, ..., n$ and $(i \neq k)$ be the determinant obtained by applying $\tilde{D}_i$ to the kth column of the determinant $\Psi^i[\tilde{V};\epsilon]$(4.52) and let S_i denote the determinant obtained by applying $\tilde{D}_i$ to the ith column of the determinant $\Psi^i[\tilde{V};\epsilon]$. In particular,

$$Q_{1k} = \begin{vmatrix} \Phi^1[V;\epsilon] & \tilde{D}_2 f^1 & \cdots & \tilde{D}_1\tilde{D}_k f^1 & \cdots & \tilde{D}_n f^1 \\ \Phi^2[V;\epsilon] & \tilde{D}_2 f^2 & \cdots & \tilde{D}_1\tilde{D}_k f^2 & \cdots & \tilde{D}_n f^2 \\ \vdots & \vdots & & \vdots & & \\ \Phi^n[V;\epsilon] & \tilde{D}_2 f^n & \cdots & \tilde{D}_1\tilde{D}_k f^n & \cdots & \tilde{D}_n f^n \end{vmatrix},$$

$$S_1 = \begin{vmatrix} \tilde{D}_1\Phi^1[V;\epsilon] & \tilde{D}_2 f^1 & \cdots & \tilde{D}_n f^1 \\ \tilde{D}_1\Phi^2[V;\epsilon] & \tilde{D}_2 f^2 & \cdots & \tilde{D}_n f^2 \\ \vdots & \vdots & & \vdots \\ \tilde{D}_1\Phi^n[V;\epsilon] & \tilde{D}_2 f^n & \cdots & \tilde{D}_n f^n \end{vmatrix}.$$

One consequently has

$$\tilde{D}_i\Psi^i[\tilde{V};\epsilon] = \sum_{j=1}^{n} S_j + \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}\left(Q_{ik} + Q_{ki}\right). \tag{4.53}$$

The second summation in (4.53) equals zero since the respective columns of the determinants Q_{ik} and Q_{ki} are odd permutations of each other. Thus, equation (4.53) simplifies to

$$\tilde{D}_i\Psi^i[\tilde{V};\epsilon] = \sum_{i=1}^{n} S_i. \tag{4.54}$$

Let γ_i^j be the cofactor of $\tilde{D}_j f^i$ for the Jacobian matrix given by

$$\begin{bmatrix} \tilde{D}_1 f^1 & \cdots & \tilde{D}_n f^1 \\ \vdots & & \vdots \\ \tilde{D}_1 f^n & \cdots & \tilde{D}_n f^n \end{bmatrix}.$$

Then (4.54) becomes

$$\tilde{D}_i\Psi^i[\tilde{V};\epsilon] = \left(\tilde{D}_j\Phi^i[V;\epsilon]\right)\gamma_i^j. \tag{4.55}$$

Using the chain rule, the right-hand side of (4.55) reads

$$\left(\tilde{D}_j\Phi^i[V;\epsilon]\right)\gamma_i^j = \left(D_\ell\Phi^i[V;\epsilon]\right)\left(\tilde{D}_j f^\ell\right)\gamma_i^j.$$

Now $\left(\tilde{D}_j f^\ell\right)\gamma_i^j = \delta_i^\ell J[\tilde{V};\epsilon]$, where δ_i^ℓ is the Kronecker symbol. Therefore, equation (4.55) leads to the equation (4.51). Hence for any solution $\tilde{V}(\tilde{x};\epsilon) = \tilde{v}(\tilde{x};\epsilon)$ of the PDE system $R[\tilde{v};\epsilon]$ (4.48), the approximate conservation law $D_i\Phi^i[v;\epsilon] = o(\epsilon)$ of PDE system $F^\sigma[v;\epsilon]$(4.1) is transformed to the approximate conservation law $D_i\Psi^i[\tilde{v};\epsilon] = o(\epsilon)$ of the PDE system (4.48). □

Example 4.5.1. The approximate point transformation [11]

$$s = x - \frac{\epsilon}{6}x^2, \quad y = t - \frac{\epsilon}{2}t^2, \quad v(s,y) = e^{u(x,t)}\left(1 + 2\epsilon\left(t - \frac{x}{3}\right)\right) \tag{4.56}$$

maps the perturbed PDE

$$v_{ss} + \epsilon v_s = \left(\frac{v_y}{v}\right)_y \tag{4.57}$$

to the nonlinear wave equation

$$u_{tt} + \epsilon u_t + o(\epsilon) = (e^u u_x)_x \, . \tag{4.58}$$

Using Theorem 4.5.1, we show that an approximate conservation law of the PDE (4.57) is transformed to an approximate conservation law of the PDE (4.58). A simple computation shows that the PDE (4.57) has approximate conservation law

$$D_s \Phi^1[v;\epsilon] + D_y \Phi^2[v;\epsilon] = o(\epsilon),$$

with fluxes given by

$$\begin{aligned} \Phi^1 &= s\frac{v_y}{v} + y v_s + \epsilon y v, \\ \Phi^2 &= -\frac{v_s}{v} - y\frac{v_y}{v}. \end{aligned} \tag{4.59}$$

Applying the approximate point transformation (4.56) to the fluxes (4.59) leads to

$$\begin{aligned} \Phi^1[u;\epsilon] &= x u_t + t e^u u_x + \epsilon\left(\frac{6xt - x^2}{6} u_t + \frac{9t^2 - 2xt}{6} e^u u_x + \frac{t e^u}{3} + 2x\right), \\ \Phi^2[u;\epsilon] &= -x u_x - t u_t + \epsilon\left(\frac{2x}{3} - 2t - \frac{x^2}{6} u_x - \frac{t^2}{2} u_t\right). \end{aligned} \tag{4.60}$$

The approximate conservation law of the PDE (4.57) with fluxes (4.59) is mapped to an approximate conservation law $D_t \Psi^1[u;\epsilon] + D_x \Psi^2[u;\epsilon] = o(\epsilon)$ for the PDE (4.58) with fluxes are found using the determinants (4.52):

$$\begin{aligned} \Psi^1 &= \begin{vmatrix} \Phi^1[u;\epsilon] & D_x s \\ \Phi^2[u;\epsilon] & D_x y \end{vmatrix} \\ &= \begin{vmatrix} x u_t + t e^u u_x + \epsilon\left(\frac{6xt - x^2}{6} u_t + \frac{9t^2 - 2xt}{6} e^u u_x + \frac{t e^u}{3} + 2x\right) & 1 - \frac{\epsilon}{3} x \\ -x u_x - t u_t + \epsilon\left(\frac{2x}{3} - 2t - \frac{x^2}{6} u_x - \frac{t^2}{2} u_t\right) & 0 \end{vmatrix} \\ &= t u_t + x u_x + \epsilon\left(\frac{3t^2 - 2xt}{6} u_t - \frac{x^2}{6} u_x\right), \end{aligned}$$

$$\begin{aligned} \Psi^2 &= \begin{vmatrix} D_t s & \Phi^1[u;\epsilon] \\ D_t y & \Phi^2[u;\epsilon] \end{vmatrix} \\ &= \begin{vmatrix} 0 & x u_t + t e^u u_x + \epsilon\left(\frac{6xt - x^2}{6} u_t + \frac{9t^2 - 2xt}{6} e^u u_x + \frac{t e^u}{3} + 2x\right) \\ 1 - \epsilon t & -x u_x - t u_t + \epsilon\left(\frac{2x}{3} - 2t - \frac{x^2}{6} u_x - \frac{t^2}{2} u_t\right) \end{vmatrix} \\ &= -x u_t - t e^u u_x + \epsilon\left(\frac{x^2}{6} u_t - \frac{t e^u}{3} + \frac{2xt - 3t^2}{6} e^u u_x\right). \end{aligned}$$

Consequently, one obtains an approximate conservation law

$$D_t\left(tu_t + xu_x + \epsilon\left(\frac{3t^2-2xt}{6}u_t - \frac{x^2}{6}u_x\right)\right) + D_x\left(-xu_t - te^u u_x + \epsilon\left(\frac{x^2}{6}u_t - \frac{te^u}{3} + \frac{2xt-3t^2}{6}e^u u_x\right)\right) = o(\epsilon) \quad (4.61)$$

for the wave equation (4.58) which is equivalent to the approximate conservation law (4.24b) given in Example 4.3.1 with $c^2(u) = e^u$.

We now consider the most important case where the invertible approximate transformation (4.47) is an approximate symmetry of the PDE system $F^\sigma[v;\epsilon]$ (4.1). We show that the action of an approximate symmetry on a known conservation law of $F^\sigma[v;\epsilon]$ can lead to a new approximate conservation law of (4.1). If the transformation (4.47) is an approximate symmetry of the PDE system $F^\sigma[v;\epsilon]$(4.1), then it leaves invariant the solution manifold of $F^\sigma[v;\epsilon]$. Hence there exist some functions $P^\sigma_\nu[\tilde{V};\epsilon]$ such that

$$F^\sigma[V;\epsilon] = R^\sigma[\tilde{V};\epsilon] + o(\epsilon) = P^\sigma_\nu[\tilde{V};\epsilon]F^\nu[\tilde{V};\epsilon] + o(\epsilon). \quad (4.62)$$

Using the formulas (4.51) and (4.52), we arrive at the following important result.

Corollary 4.5.1. *Suppose the invertible approximate point transformation* (4.47) *is an approximate symmetry of the PDE system* $F^\sigma[v;\epsilon]$ (4.1). *Then an approximate conservation law* $D_i\Phi^i[v;\epsilon] = o(\epsilon)$ *of system* (4.1) *is mapped to the approximate conservation law*

$$D_i\Psi^i[v;\epsilon] = o(\epsilon) \quad (4.63)$$

of the system of PDEs (4.1) *with conserved densities given by*

$$\Psi^1[v;\epsilon] = \begin{vmatrix} \Phi^1[\tilde{v};\epsilon] & D_2\tilde{f}^1 & \cdots & D_n\tilde{f}^1 \\ \Phi^2[\tilde{v};\epsilon] & D_2\tilde{f}^2 & \cdots & D_n\tilde{f}^2 \\ \vdots & \vdots & & \vdots \\ \Phi^n[\tilde{v};\epsilon] & D_2\tilde{f}^n & \cdots & D_n\tilde{f}^n \end{vmatrix}, \cdots, \Psi^n[v;\epsilon] = \begin{vmatrix} D_1\tilde{f}^1 & \cdots & D_{n-1}\tilde{f}^1 & \Phi^1[\tilde{v};\epsilon] \\ D_1\tilde{f}^2 & \cdots & D_{n-1}\tilde{f}^2 & \Phi^2[\tilde{v};\epsilon] \\ \vdots & & \vdots & \vdots \\ D_1\tilde{f}^n & \cdots & D_{n-1}\tilde{f}^n & \Phi^n[\tilde{v};\epsilon] \end{vmatrix}. \quad (4.64)$$

Proof. Since (4.47) is admitted by the PDE system (4.1), then (4.62) holds. Thus, $R^\sigma[V;\epsilon] = P^\sigma_\nu[V;\epsilon]F^\nu[V;\epsilon] + o(\epsilon)$ for arbitrary functions $V(x;\epsilon)$. It follows that $R^\sigma[v;\epsilon] = o(\epsilon)$ for any approximate solution $V(x;\epsilon) = v(x;\epsilon)$ of the PDE system $F^\sigma[v;\epsilon]$. Using Theorem 4.5.1, the approximate conservation law (4.63) is obtained where its fluxes $\Psi^i[v;\epsilon]$ given by formula (4.52) after replacing $\tilde{x}^i$ by x^i, $\tilde{V}^\mu$ by v^μ, etc. □

Corollary 4.5.1 shows that one can use the action of an approximate symmetry of the PDE system $F^\sigma[v;\epsilon]$(4.1) on a known approximate conservation law of (4.1) to construct an approximate conservation law (4.63) of (4.1) through the formula (4.51). Another interesting situation is using the action of an approximate symmetry of $F^\sigma[v;\epsilon]$ (4.1) on the approximate multipliers $\Lambda^\sigma[v;\epsilon]$ of known approximate conservation laws to construct approximate multipliers $\hat{\Lambda}^\sigma[v;\epsilon]$ for approximate conservation laws of (4.1). A new approximate conservation law for $F^\sigma[v;\epsilon]$ is obtained if the approximate multipliers $\hat{\Lambda}^\sigma[v;\epsilon]$ are independent of the approximate multipliers $\Lambda^\sigma[v;\epsilon]$.

Theorem 4.5.2. *If $\Lambda^\sigma[v;\epsilon]$ are approximate multipliers for an approximate conservation law $D_i\Phi^i[v;\epsilon]=o(\epsilon)$ of the PDE system $F^\sigma[v;\epsilon]$(4.1) and the approximate point transformation* (4.47) *is an approximate symmetry of the PDE system* (4.1)*, then*

$$\hat{\Lambda}^\nu[\tilde{V};\epsilon]F^\nu[\tilde{V};\epsilon]=\tilde{D}_i\Psi^i[\tilde{V};\epsilon]+o(\epsilon), \tag{4.65}$$

where

$$\hat{\Lambda}^\nu[\tilde{V};\epsilon]=J[\tilde{V};\epsilon]P^\sigma_\nu[\tilde{V};\epsilon]\Lambda^\sigma[V;\epsilon]+o(\epsilon), \quad \nu=1,...,N \tag{4.66}$$

with the coordinates of $\Lambda^\sigma[V;\epsilon]$ are expressed in terms of the transformation (4.47) *and its natural extensions. The fluxes $\tilde{\Psi}^i[\tilde{V};\epsilon]$ are given by* (4.52)*. In* (4.66)*, $J[\tilde{V};\epsilon]$ and $P^\sigma_\nu[\tilde{V};\epsilon]$ are given by* (4.50) *and* (4.62)*, respectively.*

Proof. Since the approximate point transformation (4.47) is an approximate symmetry of the PDE system (4.1), then equation (4.62) holds for arbitrary functions $\tilde{V}(\tilde{x};\epsilon)$. Since $\Lambda^\sigma[v;\epsilon]$ are approximate multipliers for an approximate conservation law $D_i\Phi^i[v;\epsilon]=o(\epsilon)$ of the PDE system $F^\sigma[v;\epsilon]$(4.1), it follows that the identity

$$\Lambda^\sigma[V;\epsilon]F^\sigma[V;\epsilon]=D_i\Phi^i[V;\epsilon]+o(\epsilon) \tag{4.67}$$

is satisfied for any arbitrary function $V(x;\epsilon)$. Substituting (4.62) into (4.67) leads to

$$D_i\Phi^i[V;\epsilon]=\Lambda^\sigma[V;\epsilon]F^\sigma[V;\epsilon]+o(\epsilon)=\Lambda^\sigma[V;\epsilon]P^\sigma_\nu[\tilde{V};\epsilon]F^\nu[\tilde{V};\epsilon]+o(\epsilon). \tag{4.68}$$

Multiplying (4.68) by $J[\tilde{V};\epsilon]$ and then using formula (4.51) yields

$$J[\tilde{V};\epsilon]D_i\Phi^i[V;\epsilon]=J[\tilde{V};\epsilon]\Lambda^\sigma[V;\epsilon]P^\sigma_\nu[\tilde{V};\epsilon]F^\nu[\tilde{V};\epsilon]=\tilde{D}_i\Psi[\tilde{V};\epsilon]+o(\epsilon).$$

Consequently, one has

$$\hat{\Lambda}^\nu[\tilde{V};\epsilon]F^\nu[\tilde{V};\epsilon]=\tilde{D}_i\Psi[\tilde{V};\epsilon]+o(\epsilon),$$

where $\hat{\Lambda}^\nu$ are given by (4.66). □

The following important corollary follows immediately from Theorem 4.5.2.

Corollary 4.5.2. *Suppose the approximate point transformation* (4.47) *is an approximate symmetry of the PDE system $F^\sigma[v;\epsilon]$* (4.1)*. If $\Lambda^\sigma[V;\epsilon]$ are approximate multipliers for an approximate conservation law of* (4.1)*, then $\hat{\Lambda}^\nu[V;\epsilon]$ are approximate multipliers of the PDE system $F^\sigma[v;\epsilon]$ where $\hat{\Lambda}^\nu[V;\epsilon]$ are given by* (4.66) *after replacing $\tilde{x}^i$ by x^i, $\tilde{V}^\mu(\tilde{x};\epsilon)$ by $V^\mu(x;\epsilon)$, $\tilde{V}^\mu_i$ by V^μ_i,etc..*

To illustrate the above formulas, we consider a nonlinear telegraph system of the form [114]

$$\begin{aligned} F^1[u,v]&=v_t-f(u)u_x-h(u)=0,\\ F^2[u,v]&=u_t-v_x=0. \end{aligned} \tag{4.69}$$

Example 4.5.2. Consider the nonlinear telegraph system (4.69) with $f(u) = -e^{2u} + \epsilon + 1$ and $h(u) = e^u$, where ϵ is a small parameter:

$$\begin{aligned} F^1[u,v;\epsilon] &= v_t - \left(\epsilon + 1 - e^{2u}\right) u_x - e^u = 0, \\ F^2[u,v;\epsilon] &= u_t - v_x = 0. \end{aligned} \tag{4.70}$$

Using the determining equation (4.12), we find the following pair of approximate multipliers for the system (4.70)

$$\begin{aligned} \Lambda^1[U,V;\epsilon] &= e^{\frac{V-U}{2}} \sin\left(\frac{t-x+e^U}{2}\right) + \epsilon\, e^{\frac{V-U}{2}} \left[\frac{x-e^U}{4} \cos\left(\frac{t-x+e^U}{2}\right) \right. \\ &\left. - \frac{V \sin\left(\frac{t-x+e^U}{2}\right)}{4}\right], \\ \Lambda^2[U,V;\epsilon] &= e^{\frac{V-U}{2}} \left(e^U \cos\left(\frac{t-x+e^U}{2}\right) - \sin\left(\frac{t-x+e^U}{2}\right)\right) \\ &- \frac{\epsilon}{4} e^{\frac{V-U}{2}} \left[\left(x + (V-1)\, e^U\right) \cos\left(\frac{t-x+e^U}{2}\right) \right. \\ &\left. + \left(xe^U - V - e^{2U} + 2\right) \sin\left(\frac{t-x+e^U}{2}\right)\right]. \end{aligned} \tag{4.71}$$

The fluxes of the approximate conservation law

$$D_t \Phi^1[u,v;\epsilon] + D_x \Phi^2[u,v;\epsilon] = o(\epsilon)$$

for the system (4.70) resulting from the approximate multipliers (4.71) have the form

$$\begin{aligned} \Phi^1 &= 2\, e^{\frac{v-u}{2}} \sin\left(\frac{t-x+e^u}{2}\right) + \epsilon\, e^{\frac{v-u}{2}} \left[\left(\frac{x-e^u}{2}\right) \cos\left(\frac{t-x+e^u}{2}\right) \right. \\ &\left. + \frac{(2-v)}{2} \sin\left(\frac{t-x+e^u}{2}\right)\right], \\ \Phi^2 &= 2\, e^{\frac{v-u}{2}} \left(\sin\left(\frac{t-x+e^u}{2}\right) - e^u \cos\left(\frac{t-x+e^u}{2}\right)\right) \\ &+ \frac{\epsilon}{2} e^{\frac{v-u}{2}} \left[\left(x + (v-3)\, e^u\right) \cos\left(\frac{t-x+e^u}{2}\right) \right. \\ &\left. + \left(xe^u - v - e^{2u} + 4\right) \sin\left(\frac{t-x+e^u}{2}\right)\right]. \end{aligned} \tag{4.72}$$

The system (4.70) has the approximate translation point symmetry

$$t = \tilde{t} + \epsilon a + a, \quad x = \tilde{x}, \quad u = \tilde{u}, \quad v = \tilde{v}. \tag{4.73}$$

We show that the action of the approximate symmetry (4.73) on the approximate conservation law with fluxes (4.72) and their corresponding approximate multipliers (4.71) yields a new set of approximate multipliers which leads to a new approximate conservation law for the nonlinear telegraph system (4.70). Indeed, one has

$$J[\tilde{U},\tilde{V};\epsilon] = \begin{vmatrix} \frac{\partial t}{\partial \tilde{t}} & \frac{\partial x}{\partial \tilde{t}} \\ \frac{\partial t}{\partial \tilde{x}} & \frac{\partial x}{\partial \tilde{x}} \end{vmatrix} = 1, \quad R^1[\tilde{U},\tilde{V};\epsilon] = F^1[U,V;\epsilon] + o(\epsilon) = F^1[\tilde{U},\tilde{V};\epsilon] + o(\epsilon),$$

$$R^2[\tilde{U},\tilde{V};\epsilon] = F^2[U,V;\epsilon] + o(\epsilon) = F^2[\tilde{U},\tilde{V};\epsilon] + o(\epsilon).$$

Hence, in equation (4.62), one gets

$$P_1^1 = P_2^2 = 1, \quad P_1^2 = P_2^1 = 0.$$

By applying the approximate translation symmetry (4.73) to the approximate multipliers (4.71) and then using the formula (4.66) and Corollary 4.5.2, one obtains new approximate multipliers given by

$$\begin{aligned}
\hat{\Lambda}^1[U,V;\epsilon] &= \frac{1}{2}e^{\frac{V-U}{2}}\cos\left(\frac{t-x+e^U}{2}\right) + \frac{\epsilon}{8}\,e^{\frac{V-U}{2}}\left[\left(e^U - x\right)\sin\left(\frac{t-x+e^U}{2}\right)\right.\\
&\left.+ (4-V)\cos\left(\frac{t-x+e^U}{2}\right)\right],\\
\hat{\Lambda}^2[U,V;\epsilon] &= -\frac{1}{2}e^{\frac{V-U}{2}}\left(e^U\sin\left(\frac{t-x+e^U}{2}\right) + \cos\left(\frac{t-x+e^U}{2}\right)\right)\\
&+\frac{\epsilon}{8}\,e^{\frac{V-U}{2}}\left[\left(x+(V-5)\,e^U\right)\sin\left(\frac{t-x+e^U}{2}\right)\right.\\
&\left.-\left(xe^U - V - e^{2U} + 6\right)\cos\left(\frac{t-x+e^U}{2}\right)\right].
\end{aligned} \tag{4.74}$$

Using the formula (4.64), we obtain an approximate conservation law

$$D_t\Psi^1[u,v;\epsilon] + D_x\Psi^2[u,v;\epsilon] = o(\epsilon)$$

for the system (4.70) with fluxes given by

$$\begin{aligned}
\Psi^1 &= e^{\frac{v-u}{2}}\cos\left(\frac{t-x+e^u}{2}\right) + \frac{\epsilon}{4}\,e^{\frac{v-u}{2}}\left[(e^u - x)\sin\left(\frac{t-x+e^u}{2}\right)\right.\\
&\left.+(6-v)\cos\left(\frac{t-x+e^u}{2}\right)\right],\\
\Psi^2 &= e^{\frac{v-u}{2}}\left(e^u\sin\left(\frac{t-x+e^u}{2}\right) + \cos\left(\frac{t-x+e^u}{2}\right)\right)\\
&+\frac{\epsilon}{4}\,e^{\frac{v-u}{2}}\left[((7-v)\,e^u - x)\sin\left(\frac{t-x+e^u}{2}\right)\right.\\
&\left.+\left(xe^u - v - e^{2u} + 8\right)\cos\left(\frac{t-x+e^u}{2}\right)\right].
\end{aligned} \tag{4.75}$$

It remains to check that an approximate conservation law with fluxes (4.75) is not equivalent to an approximate conservation law with fluxes (4.72). Indeed, we find the difference between both approximate conservation laws:

$$\begin{aligned}
\Phi^1 - \Psi^1 &= e^{\frac{v-u}{2}}\left(2\sin\left(\frac{t-x+e^u}{2}\right) - \cos\left(\frac{t-x+e^u}{2}\right)\right) + \frac{\epsilon}{4}\,e^{\frac{v-u}{2}}\left[(2x - e^u - 6 + v)\cos\left(\frac{t-x+e^u}{2}\right)\right.\\
&\left.+(x - e^u + 4 - 2v)\sin\left(\frac{t-x+e^u}{2}\right)\right],\\
\Phi^2 - \Psi^2 &= e^{\frac{v-u}{2}}\left((2-e^u)\sin\left(\frac{t-x+e^u}{2}\right) - (1+2e^u)\cos\left(\frac{t-x+e^u}{2}\right)\right)\\
&+\frac{\epsilon}{4}\,e^{\frac{v-u}{2}}\left[\left(e^{2u} - (x+6)e^u + (2e^u+1)v + 2x - 8\right)\cos\left(\frac{t-x+e^u}{2}\right)\right.\\
&\left.+\left(-2e^{2u} + (2x-7)e^u + (e^u-2)v + x + 8\right)\sin\left(\frac{t-x+e^u}{2}\right)\right].
\end{aligned} \tag{4.76}$$

It is clear that an approximate conservation law with fluxes (4.76) is nontrivial approximate conservation law. It follows that $D_t\Psi^1[u,v;\epsilon] + D_x\Psi^2[u,v;\epsilon] = o(\epsilon)$ with fluxes given by (4.75) is a new approximate conservation law for the perturbed telegraph system (4.70).

4.6 An application of approximate conservation laws

If at least one PDE of the system of PDEs $F^\sigma[u]$ can be written in a conserved form with respect to some choice of its variables. Then a conserved form leads to auxiliary dependent variables v which are *potentials* to an auxiliary system of PDEs $R^\tau[u,v]$. This leads to nonloclally related systems with the property that any solution $(u(x), v(x))$ of $R^\tau[u,v]$ will define a solution $u(x)$ of $F^\sigma[u]$ and to any solution $u(x)$ of $F^\sigma[u]$, there corresponds a function $v(x)$ such that $(u(x), v(x))$ is a solution of $R^\tau[u,v]$. Since a symmetry maps any solution of a PDE system to another solution of the same system, local symmetries of $R^\tau[u,v]$ induces symmetries admitted by $F^\sigma[u]$. A local symmetry of $R^\tau[u,v]$ that depends explicitly on the potential variables v induces a *nonlocal symmetry* of $F^\sigma[u]$. Such nonlocal symmetries are called *potential symmetries.* The natural way to find nonlocal related PDE systems is through using the local conservation laws of $F^\sigma[u]$. Consequently, each local conservation law of $F^\sigma[u]$ yields a potential variable that could yield a nonlocally related PDE system called a *potential system* [37, 60].

In [115], potential approximate symmetries are found for some perturbed PDEs written in a conserved form. In this section, we find potential systems and approximate potential symmetries for perturbed PDEs through the use of approximate local conservation laws.

Consider a scalar PDE with two independent variables (x,t) and one dependent variable u given by

$$F[u;\epsilon] = F_0[u] + \epsilon F_1[u] = o(\epsilon). \tag{4.77}$$

Suppose the perturbed PDE (4.77) has a nontrivial approximate conservation law

$$D_t\Psi[u;\epsilon] + D_x\Phi[u;\epsilon] = o(\epsilon). \tag{4.78}$$

arising from a nontrivial approximate multiplier $\Lambda = \Lambda_0 + \epsilon\Lambda_1$. The approximate conservation law (4.78) yields a PDE system consists of two *potential equations* given by

$$\begin{aligned} v_x &= \Psi[u;\epsilon] + o(\epsilon), \\ v_t &= -\Phi[u;\epsilon] + o(\epsilon). \end{aligned} \tag{4.79}$$

Definition 4.6.1. A system of perturbed PDEs (4.79) is a *potential system* with a *potential variable* v for the perturbed PDE (4.77) related to the approximate conservation law (4.78).

The potential variable v in (4.79) is a *nonlocal variable* in sense that it cannot be expressed as a local function of the variables x, t, u and partial derivatives of u. The PDE (4.77) and the potential system (4.79) are equivalent:

[1] For any approximate solution $u = f(x,t)$ of (4.77), there exists a function $v = g(x,t)$, unique to within a constant, such that $(u,v) = (f(x,t), g(x,t))$ is an approximate solution of (4.79).

[2] For any approximate solution $(u,v) = (f(x,t), g(x,t))$ of (4.79), by projection, $u = f(x,t)$ is an approximate solution of (4.77).

Definition 4.6.2. Suppose a potential system (4.79) of a given PDE (4.77) admits an approximate point symmetry

$$X = X^0 + \epsilon X^1 = \xi_0 \frac{\partial}{\partial x} + \tau_0 \frac{\partial}{\partial t} + \eta_0 \frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial v} + \epsilon \left(\xi_1 \frac{\partial}{\partial x} + \tau_1 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial v} \right), \tag{4.80}$$

where ξ_i, τ_i, η_i and ζ_i, $i = 1, 2$ are functions of x, t, u and v. Then X is called an approximate potential symmetry of (4.77) if and only if

$$[(\xi_0 + \epsilon\xi_1)_v]^2 + [(\tau_0 + \epsilon\tau_1)_v]^2 + [(\eta_0 + \epsilon\eta_1)_v]^2 \not\equiv 0. \tag{4.81}$$

Example 4.6.1. Consider the perturbed nonlinear wave equation (4.14) with $c^2(u) = u^2$

$$u_{tt} - (u^2 u_x)_x + \epsilon u_t = 0. \tag{4.82}$$

Based on the nontrivial approximate conservation laws of (4.82) found in Example 4.3.1:

$$D_t \left[u_t + \epsilon u\right] - D_x \left[u^2 u_x\right] = o(\epsilon), \tag{4.83a}$$

$$D_t \left[x u_t + \epsilon x u\right] - D_x \left[x u^2 u_x - \frac{1}{3} u^3\right] = o(\epsilon), \tag{4.83a}$$

$$D_t \left[t u_t - u + \frac{\epsilon}{2} t^2 u_t\right] - D_x \left[t u^2 u_x + \frac{\epsilon}{2} t^2 u^2 u_x\right] = o(\epsilon), \tag{4.83b}$$

$$D_t \left[x t u_t - x u + \frac{\epsilon}{2} x t^2 u_t\right] - D_x \left[x t u^2 u_x - \frac{1}{3} t u^3 + \frac{\epsilon}{2}\left(x t^2 u^2 u_x - \frac{1}{3} t^2 u^3\right)\right] = o(\epsilon), \tag{4.83c}$$

one can construct four systems of potential equations given respectively by

$$\begin{aligned} v_x &= u_t + \epsilon u, \\ v_t &= u^2 u_x; \end{aligned} \tag{4.84}$$

$$\begin{aligned} w_x &= x u_t + \epsilon x u, \\ w_t &= x u^2 u_x - \frac{1}{3} u^3; \end{aligned} \tag{4.85}$$

$$\begin{aligned} p_x &= t u_t - u + \frac{\epsilon}{2} t^2 u_t, \\ p_t &= t u^2 u_x + \frac{\epsilon}{2} t^2 u^2 u_x; \end{aligned} \tag{4.86}$$

$$\begin{aligned} q_x &= x t u_t - x u + \frac{\epsilon}{2} x t^2 u_t, \\ q_t &= x t u^2 u_x - \frac{1}{3} t u^3 + \frac{\epsilon}{2}\left(x t^2 u^2 u_x - \frac{1}{3} t^2 u^3\right) \end{aligned} \tag{4.87}$$

with potential variables $v(x,t)$, $w(x,t)$, $p(x,t)$ and $q(x,t)$.

Note that the conservation law (4.83a) is an exact conservation law for the PDE (4.82) which yields the potential system (4.84). We show that considering (4.84) as a system with a small parameter leads to new approximate potential symmetries for the perturbed PDE (4.82) that do not arise from the exact potential symmetries of (4.82). Indeed, let (4.80) be an approximate point symmetry for the potential system (4.84). The unperturbed system

$$v_x = u_t, \quad v_t = u^2 u_x \tag{4.88}$$

is mapped by a Hodograph transformation [60]

$$r = u, \quad s = v, \quad z = t, \quad w = x$$

to the linear system

$$w_s = z_r, \quad w_r = r^2 z_s. \tag{4.89}$$

It follows that the system (4.88) admits an infinite dimensional Lie algebra of point symmetries given by

$$\begin{aligned}
&X_1^0 = \frac{\partial}{\partial t},\ X_2^0 = \frac{\partial}{\partial x},\ X_3^0 = \frac{\partial}{\partial v},\ X_4^0 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\\
&X_5^0 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v},\\
&X_6^0 = (6vt + 2xu)\frac{\partial}{\partial t} + (2tu^3 + 2xv)\frac{\partial}{\partial x} - 4uv\frac{\partial}{\partial u} - \left(u^4 + 4v^2\right)\frac{\partial}{\partial v},\\
&X_\infty^0 = f(u,v)\frac{\partial}{\partial t} + g(u,v)\frac{\partial}{\partial x},
\end{aligned}$$

where $f(u,v)$ and $g(u,v)$ are arbitrary functions satisfy the following linear PDE system

$$\begin{aligned}
g_v &= f_u,\\
g_u &= u^2 f_v.
\end{aligned} \tag{4.90}$$

The potential symmetries of the perturbed wave equation (4.82) are given by X_6^0 and X_∞^0. Applying the determining equation (1.96), one can show that the PDE system (4.84) has an infinite number of approximate point symmetries given by

$$\begin{aligned}
&X_1^1 = \frac{\partial}{\partial t},\ X_2^1 = \frac{\partial}{\partial x},\ X_3^1 = \frac{\partial}{\partial v},\\
&X_4^1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v},\\
&X_5^1 = v\frac{\partial}{\partial t} + \frac{1}{3}u^3\frac{\partial}{\partial x} - \epsilon\left(xu\frac{\partial}{\partial t} + xv\frac{\partial}{\partial x} + uv\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v}\right),\\
&X_6^1 = \left(6v^2 + u^4\right)\frac{\partial}{\partial t} + 4vu^3\frac{\partial}{\partial x} - \epsilon\Bigg(\left(5tu^4 + 30tv^2 + 32xuv\right)\frac{\partial}{\partial t}\\
&\qquad + \left(16xv^2 + 8xu^4 + 20tvu^3\right)\frac{\partial}{\partial x} - \left(u^5 + 14uv^2\right)\frac{\partial}{\partial u} - \left(10vu^4 + \tfrac{28}{3}v^3\right)\frac{\partial}{\partial v}\Bigg),\\
&X_7^1 = \epsilon X_1^1,\ X_8^1 = \epsilon X_2^1,\ X_9^1 = \epsilon X_3^1,\ X_{10}^1 = \epsilon X_4^1,\ X_{11}^1 = \epsilon X_4^0,\ X_{12}^1 = \epsilon X_6^0,\ X_\infty^1 = \epsilon X_\infty^0.
\end{aligned} \tag{4.91}$$

X_5^1, X_6^1 and X_∞^1 yield approximate potential symmetries of the perturbed nonlinear wave equation (4.82). Moreover, X_5^1 and X_6^1 contain stable parts, it could be useful in application.

Similarly, one can find genuine approximate potential symmetries for the perturbed wave equation (4.82) using the potential systems (4.85-4.87). In particular, the PDE (4.82) has the approximate potential symmetries

$$\begin{aligned} X_1 &= u\frac{\partial}{\partial t} + \frac{w}{x}\frac{\partial}{\partial x} - \epsilon u^2 \frac{\partial}{\partial u}, \\ X_2 &= -5tp\frac{\partial}{\partial t} + t^2u^3\frac{\partial}{\partial x} + \epsilon\left(\frac{5}{2}tup\frac{\partial}{\partial u} + t^3u^4\frac{\partial}{\partial p}\right), \\ X_3 &= t\frac{\partial}{\partial t} + q\frac{\partial}{\partial u} - \epsilon\left(\frac{1}{2}tu + tq\right)\frac{\partial}{\partial u} \end{aligned} \tag{4.92}$$

arise respectively from the approximate potential systems (4.85-4.87). These approximate potential symmetries are useful in construction of approximate solutions for the perturbed PDEs.

Example 4.6.2. We find an approximate solution of (4.82) using X_1 in (4.92). Approximate invariants obtained from the characteristic system

$$\frac{dt}{u} = \frac{dx}{w/x} = \frac{du}{-\epsilon u^2} = \frac{dw}{0}$$

for X_1 are given by

$$u(x,t) = f(x)(1-\epsilon t), \quad w(x,t) = g(x,t;\epsilon) = g_0(x,t) + \epsilon g_1(x,t). \tag{4.93}$$

The substitution of (4.93) into the potential system (4.85) yields the system

$$g_x = 0, \quad g_{0_t} = k_0, \quad g_{1_t} = k_1 t, \quad f^2(3xf_x - f) = k_2,$$

with solutions

$$g(x,t;\epsilon) = k_0 t + k_3 + \epsilon\left(\frac{1}{2}k_1 t^2 + k_4\right), \quad f(x) = (k_2 x + k_5)^{1/3},$$

where $k_i, i = 0, 1, ..., 5$ are arbitrary constants. Consequently, the potential system (4.85) has the approximate solution

$$u(x,t) = (1-\epsilon t)(k_2 x + k_5)^{1/3}, \quad v(x,t) = k_0 t + k_3 + \epsilon\left(\frac{1}{2}k_1 t^2 + k_4\right).$$

Therefore

$$u(x,t) = (1-\epsilon t)(k_2 x + k_5)^{1/3} \tag{4.94}$$

is an approximate solution for the wave equation resulting from the potential symmetry X_1 in (4.92).

4.7 Discussion

In this chapter, we have seen that the direct method [5] is useful in construction of new approximate conservation laws for perturbed PDEs. We proved a theorem that provides a connection between approximate multipliers and approximate local conservation laws (Theorem 4.10). We applied the direct method on perturbed wave and heat equations and found new genuine approximate conservation laws for these models (Example 4.3.1 and Example 4.3.2).

For perturbed PDEs that admit an approximate Lagrangian, Noeher's theorem was generalized to provide a relation between approximate variational symmetries and approximate conservation laws [23]. For non-variational perturbed PDEs, a formula to construct approximate conservation laws using approximate local symmetries was derived [44, 113]. The direct method does not require the knowledge of the approximate symmetries and can be applied to approximately variational and non-variational systems. We proved that the direct method yields all the approximate conservation laws given by Noether's theorem (Theorem 4.4.2).

The action of a local symmetry to yield additional conservation laws has been considered in [4] and [116]. An extension to the action of any symmetry (continuous or discrete) appears in [40]. We generalized the results [40] to construction of approximate conservation laws under the action of approximate point transformations. In Section 4.5, two formulas were introduced related to obtaining new approximate conservation laws from known approximate conservation laws using the action of invertible approximate point transformations. We showed that if a system of perturbed PDEs is mapped to another system of perturbed PDEs under the action of an invertible approximate point transformation, then each approximate conservation law of the first system is mapped to an approximate conservation law of the transformed system (Theorem 4.5.1). The first formula (4.52) yields the transformed approximate conservation law. If the approximate transformation is an approximate point symmetry, then a given approximate conservation law for a PDE system is transformed to an approximate conservation law for the same PDE system (Corollary 4.5.1). We introduced another formula (4.65) which uses the action of an approximate point symmetry on a set of approximate multipliers to yield new set of approximate multipliers which leads to new approximate conservation laws if the transformed and the given approximate multipliers are independent (Theorem 4.5.2 and Corollary 4.5.2). Two examples including a nonlinear perturbed wave equation (Example 4.5.1) and a nonlinear perturbed telegraph system (Example 4.5.2) were discussed.

We used the approximate conservation laws of the nonlinear wave equation (4.82) to construct potential systems for (4.82). We found new approximate potential symmetries (4.91),(4.92) for (4.82). We found an approximate solution (4.94) for (4.82) using one of its approximate potential symmetries (Example 4.6.2).

5 Conclusion

Lie symmetry method has been widely used to study symmetry properties and find exact solutions of differential equations. Many nonlinear differential equations that arise in Science and Engineering depend on a small parameter. It is commonly the case that a perturbed model has fewer point and local symmetries than the unperturbed system. This limits the applicability of exact Lie group methods to perturbed models. Approximate Lie symmetries are useful to deal with perturbed models as some additional approximate symmetries may arise for some models with a small parameter. Approximate symmetries allow to build approximate solutions for perturbed differential equations, and to construct approximate conservation laws.

In this thesis, local symmetries of algebraic equations, ordinary differential equations and partial differential equations involving a small parameter ϵ were considered in comparison to the symmetry structure of their unperturbed versions (small parameter equal to zero). Exact symmetries of the unperturbed equations, and exact and approximate symmetries (in the Baikov-Gazizov-Ibragimov [10–12] and Fushchich-Shtelen [13] frameworks) of the perturbed models were investigated. We also investigated approximate local conservation laws of a system of perturbed PDEs.

It was observed by the original authors of the BGI method that while new approximate symmetries that are useful in constructing new approximate solutions for perturbed models can be sometimes found, some point symmetries of the unperturbed model may not appear in any form in the approximate point symmetry classification of a perturbed model, being thereby *unstable*. We also observed that similar situation appears in FS framework. The first goal of the thesis was to address the question of stability of symmetries, and to find out the conditions under which a local symmetry becomes unstable, the form it can assume in the approximate point symmetry classification of a perturbed equation.

The second goal in this thesis was to compare and discuss BGI and FS frameworks for perturbed ODEs and PDEs, and applications of approximate symmetries to compute approximate solutions of the perturbed models.

The study of approximate symmetries for perturbed models extends naturally to study the approximate conservation laws for these models. Our goal here was to investigate the possibility of finding new approximate conservation laws using the direct method and using invertible approximate point transformations, and to discuss the relation between the direct method and Noether's theorem for approximate conservation laws.

In Chapter 2, we showed that for algebraic equations and first-order ODEs, every point symmetry of the unperturbed equation is stable: a corresponding approximate point symmetry of the perturbed equation always exists; moreover, approximate point symmetry generators of perturbed algebraic equations are more general than the exact symmetry generators of perturbed algebraic equations, and the approximate symmetry components arise as first-order Taylor terms in the expansion of exact symmetry components of the perturbed equation in the small parameter.

For second and higher-order ODEs and PDEs, the situation is more complex (Section 2.4.2): some original symmetries of the unperturbed model (2.58) can be unstable, in the sense of not being inherited as nontrivial approximate point symmetries of a perturbed ODE (2.59) (Example 1.3.4). At the same time, for some ODEs, all point symmetries of the unperturbed model might be stable (Example 2.4.1). This occurs because in the approximate point symmetry computation of an ODE with a small parameter ϵ, additional conditions on the $O(\epsilon^0)$ approximate symmetry components may or may not arise. The situation is clarified in Section 2.5, where symmetries (point or local, exact and approximate) were written in the evolutionary form. Theorem 2.5.1 was proven, showing that to every point or local symmetry of an exact ODE (2.58) of any order, there corresponds an approximate symmetry of the perturbed ODE (2.59), being possibly a *higher-order symmetry* of order at most $n-1$. Two examples were considered in detail: a nonlinearly perturbed second-order ODE (2.107) (Section 2.5.3), and a fourth-order Boussinesq reduction ODE (2.120) (Section 2.5.4). A relation between genuine BGI approximate point symmetries and FS approximate point symmetries for perturbed ODEs was constructed. In FS framework, we found that there are FS approximate point symmetries of perturbed ODEs that do not correspond to the stable point symmetries and also do not appear in the BGI framework (Remark 2.4.2). This type of FS approximate point symmetries also appear in the case of Perturbed PDEs (Remark 3.4.1).

One of the most important applications of the approximate symmetry framework is the construction of closed-form approximate solutions to nonlinear ODE models with a small parameter. In Section 2.6, two approaches to obtain such solutions were developed. The first approach is based on approximate integrating factors using approximate point symmetries (Section 2.6.1). Equations satisfied by approximate integrating factor components were derived (Theorem 2.6.2) and applied to obtain a four-parameter approximate solution family (2.159) of the fourth-order Boussinesq ODE (2.120) and a BBM ODE (2.161). Another technique, approximate reduction of order under contact and higher-order symmetries, is presented in Section 2.6.3 and illustrated on two examples: an ODE (2.176) with a small parameter for which the exact general solution is known (Example 2.6.4), and again the fourth-order Boussinesq ODE (2.120) (Example 2.6.5). In the latter, the approximate solution was validated via a comparison to numerical solutions of the Boussinesq equation (2.120).

In Chapter 3, we investigated exact and approximate symmetries of scalar PDEs involving a small parameter. We discussed the stability of exact point symmetries of an unperturbed PDE. We found that the point symmetry of the unperturbed PDE does not correspond, in general, to a higher-order approximate symmetry of the perturbed model. The main reason of the instability is that the determining equation (3.23) for BGI local symmetries of the perturbed PDE (3.15), whatever the dependence of the approximate symmetry components ζ^1 (3.22), always contains derivatives of u (3.24) higher than those in ζ^1 which leads to a split system of PDEs in ζ^1 with some restrictions on the unperturbed symmetry components ζ^0. A similar argument holds in FS framework. As an illustration, we showed that there was no higher-order (BGI and FS) approximate symmetry for the perturbed PDE (3.31) corresponding to the unstable point symmetry of the unperturbed wave equation (3.11).

We found a classification of stable point symmetries for a nonlinear wave equation (3.11) in the sense of BGI and FS frameworks. This helped us to illustrate that both methods are different as there were stable symmetries in one framework and unstable in the other framework and showed some connection between the BGI and FS approximate symmetries that each stable BGI point symmetry of the form $(\zeta^0(x,t) + \epsilon\zeta^1(x,t,u,u_x,u_t))\partial/\partial u$ yields a higher-order approximate FS symmetry in the form $\zeta^0(x,t)\,\partial/\partial v +$ $\zeta^1(x,t,v,v_x,v_t)\,\partial/\partial w$ (Theorem 3.5.1).

Exact and approximate point symmetries (BGI and Fushchich-Shtelen) for the perturbed one-dimensional wave equation (3.56) were classified. Genuine BGI and FS approximate point symmetries for (3.56) were obtained and used to construct an approximate solution for a class of a perturbed one-dimensional wave equation (3.74). Using this approximate solution, we estimated the wave breaking time of a perturbed one-dimensional model (3.82). We also estimated the finite-time singularity formation of (3.82) by a linear approximation of the characteristic curves using a finite difference scheme and compared the two sets of singularities. The two different methods of approximating the finite-time singularity yielded qualitatively similar results that suggest the finite-time singularity of (3.82) with the initial value problem (3.84) goes as $\tau \sim \epsilon^{-1}$.

The classification of exact and approximate symmetries of the one-dimensional wave equation (3.56) yielded new approximate symmetries that were useful in constructing new approximate solutions that in turn helped to study the wave breaking time of (3.56), in comparison with the numerical methods. This motivated us to consider the two-dimensional wave models in order to have a basis of new exact and approximate symmetry structures that may be useful for further study of these models in future. In particular, we classified the exact point symmetries of the two dimensional wave equation (3.95), and found a complete classification of exact and (BGI and FS) approximate point symmetries of the perturbed two dimensional wave model (3.96). New BGI and FS approximate point symmetries for (3.96) were obtained.

In Chapter 4, we considered approximate conservation laws for systems of perturbed PDEs. We showed that a perturbed PDE system admits an approximate local conservation law if and only if there exist approximate multipliers such that their linear combinations with the differential equations of the given PDE system are approximately annihilated by the Euler operator (1.143) (Theorem 4.10). This relation was illustrated using perturbed wave and heat equations where new genuine approximate conservation laws for these models were found. We proved that all approximate conservation laws obtained through Noether approach are obtainable through the direct method (Theorem 4.4.2). We used the action of an invertible approximate point transformation to develop two formulas that yield additional approximate conservation law for a system of perturbed PDEs. These formulas were applied to obtain approximate conservation laws for a nonlinear perturbed wave equation (Example 4.5.1) and a nonlinear perturbed telegraph system (Example 4.5.2). We constructed potential systems for the nonlinear wave equation (4.82) using the approximate conservation laws of (4.82) and new approximate potential symmetries were found. An approximate solution (4.94) for (4.82) was found using approximate potential symmetries.

The main value of this contribution lies in new detailed examples of computation and comparison of exact and approximate symmetry structures and approximate conservation laws of multiple ODEs and PDEs, and the use of point and higher-order approximate symmetries to calculate closed-form approximate solutions of such perturbed models. This thesis is aimed at mathematicians, scientists and engineers interested in applications of exact and approximate Lie symmetries on differential equations, including finding exact and approximate solutions of differential equation involving a small parameter. There are different examples involving physical models. The reader whose interest is in conservation laws of differential equations can move to Chapter 4, which contains the extension of the study of exact conservation laws to the study of approximate conservation laws of systems of perturbed PDEs.

5.1 Future research work

- For higher-order ODEs, we showed that each point or local symmetry for an unperturbed ODE is stable in BGI framework and corresponds to higher-order BGI approximate symmetry for the perturbed model and we found that these symmetries are useful in construction new approximate solutions of perturbed ODEs. So, it is meaningful to investigate if these results about stability hold in FS framework. One can usually start by considering higher-order FS generator of the form

$$\hat{Z} = \eta^v(x, v, v')\frac{\partial}{\partial v} + \eta^w(x, v, w, v', ..., v^{(n-1)}, w', ..., w^{(n-1)})\frac{\partial}{\partial w}$$

  where $\eta^v(x, v, v') = \xi^0(x, v) - v'\eta^0(x, v)$ corresponds to the point symmetries of an unperturbed ODE.

- In chapters 2 and 3, we investigated the exact and approximate symmetries for regularly perturbed scalar ODEs and PDEs. In the singularly perturbed problems [117, 118], perturbations are functioning over a very narrow region across which the dependent variable experiences very rapid change. A singular perturbation generally occurs when a small parameter multiplies the highest derivative of the given problem. Thus, taking the parameter to be zero changes the nature of the problem. In the case of differential equations, boundary conditions cannot be satisfied; in algebraic equations, the possible number of solutions is decreased. So, it is important to extend the understanding of relationships between symmetry structures of unperturbed and perturbed models in the cases of systems of ODEs and PDEs, including regularly and singularly perturbed equations such as Navier-Stokes equations [79]

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ &\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p = \mu \Delta \mathbf{u}, \end{aligned}$$

and the one-dimensional shallow water equations [119]

$$\begin{aligned} u_t + h_x + \epsilon u u_x &= 0, \\ h_t + (hu)_x &= 0. \end{aligned}$$

- Through classical computations, we were able to classify the stable point symmetries of the nonlinear PDE (3.11) in terms of BGI and FS approximate symmetries of the perturbed equation (3.29). This type of classification is important since for any family of perturbed PDEs, it provides a complete information about the types of approximate symmetries that can be obtained for each class of this family. It is of interest to be able to understand the stability of local symmetries of the general differential equation $F_0[u]+\epsilon F_1[u] = o(\epsilon)$ based only on the knowledge of F_0, F_1 and the exact symmetries of the unperturbed equation a priori without computations.

  We were able to find an example where an unstable point symmetry, in BGI and FS senses, of an unperturbed PDE corresponds to higher-order FS approximate symmetry of the perturbed PDE: in Table 3.1, when $F_1 = e^v Q(v_t) + v_t$, the unstable point symmetry $\partial/\partial v$ (in BGI and FS frameworks) yields a local FS symmetry given by

$$\hat{Z} = \frac{\partial}{\partial v} + \left(\frac{1}{10}t^2 v_t + \frac{2}{5}tv + w\right)\frac{\partial}{\partial w}.$$

  On the other hand, we could not find a BGI approximate local symmetry of a perturbed PDE corresponds to unstable point symmetry of the unperturbed PDE. It is of interest to prove, for a certain class of perturbed PDEs, that all unstable point/local symmetries of the unperturbed PDE yield BGI and FS approximate local symmetries for the perturbed model.

- Throughout this thesis, we considered first-order perturbations of differential equations as first terms of the Taylor series in the small parameter ϵ. If the perturbed differential equation (1.82) contains a higher-order expansion in ϵ, such as

$$F[u;\epsilon] = F_0[u] + \epsilon F_1[u] + \ldots + +\epsilon^\ell F_\ell[u] = o(\epsilon^\ell),$$

  both BGI and FS frameworks can be naturally extended to those situations, by using, respectively, a higher-order expansion of the approximate generator (1.100) [11] or the solution (1.103). Higher approximation orders $u(x) = u_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \ldots$ can be considered in the same manner, leading to the corresponding Fushchich-Shtelen system containing more PDEs [92]. It is interesting to study the exact and approximate symmetry properties of PDEs with multiple independent small parameters such as the viscoelastic one-dimensional wave equation [110]

$$\begin{aligned} u_{tt} &= \left(\alpha + 3\beta u_x^2\right) u_{xx} + \eta u_x \left[\left(8u_x^2 + 2\right) u_{xx} u_{tx} + \left(2u_x^2 + 1\right) u_x u_{txx}\right] \\ &\quad + \zeta u_x^3 \left[\left(24 u_x^2 + 4\right) u_{xx} u_{tx} + \left(4u_x^2 + 1\right) u_x u_{txx}\right], \end{aligned}$$

where $\beta, \eta, \zeta << \alpha$, and the classical Boussinesq equations [48]

$$v_t + ((1+\epsilon v)u)_x = 0,$$
$$v_x + u_t + \epsilon u u_x + \frac{1}{3}\delta^2 u_{xxt} = 0.$$

- Classifications of exact and approximate symmetries of one- and two-dimensional wave models yielded new approximate symmetries. One can use these symmetries to construct further physical solutions for these models.

- For perturbed PDEs, we have considered the computation of their local approximate symmetries and the application of these symmetries to construct approximate solutions. Also, we observed that one can seek new approximate multipliers and new approximate local conservation laws that do not arise from the exact multipliers and conservation laws of the perturbed models. It is important to extend these applications to include:

  – the problem of construction of approximate invertible mappings that maps a given PDE approximately to some target PDE:
  theorems on invertible mappings of an exact PDE to some PDE were discussed in details in the literature (see, e.g., [37] and references therein). In [11], examples on approximate transformations that approximately connect two perturbed PDEs were presented. So, it is essential to generalize these results to determine the conditions for the existence of an approximate invertible mapping and to develop an algorithm to find such a mapping when it exists.

  – potential systems and approximate nonlocal symmetries (approximate potential symmetries):
  in Section 4.6, we constructed potential systems using approximate conservation laws. In the case of exact conservation laws, a local conservation law of the potential system is a nonlocal conservation law for the original PDE if it is not equivalent to a linear combination of the local conservation laws of the original PDE. These nonlocal symmetries have applications in PDE problem analysis and they are useful in obtaining PDE systems nonlocally related to the given PDE [37]. It is worthy to discuss these results in case of approximate conservation laws and to define the notion of approximate nonlocal conservation laws and their applications for obtaining additional approximate conservation laws for the original PDE and in construction of new nonlocally related systems.

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Appendix A

Maple Code for Approximate Symmetries and Approximate Conservation Laws

A.1 Maple code for BGI approximate point symmetries of the perturbed Boussinesq ODE

Consider the linear ODE (2.119) and its perturbed version, the Boussinesq ODE (2.120). We use the `GeM` package [9] to write a code for BGI approximate point symmetries of (2.120).

1. Initialize the `GeM` package using the command.

```
read(t:/gem32_12.mpl);
```

2. Define independent and dependent variables, parameters and arbitrary functions of the problem. We use U instead of y, and e instead of ϵ in the perturbed ODE (2.120).

```
gem_decl_vars(indeps=[x], deps=[U(x)], freeconst =[e]);
```

3. Define the Boussinesq equation.

```
gem_decl_eqs([diff(U(x), x, x)+diff(U(x), x, x, x, x)+
 e*(-2*U(x)*(diff(U(x), x, x))-2*(diff(U(x), x))^2)],
solve_for=[diff(U(x),x,x,x,x)]);
```

   A given ODE is written in a solved form. The leading derivative is specified in the `solve_for` parameter.

4. Find the exact symmetry determining equation for (2.120) using the command.

```
det_eq:=gem_symm_det_eqs([x,U(x)], return_unsplit=true);
```

   The arguments of the command define the dependence of symmetry components. In this example, we seek point symmetries. We keep the determining equation without splitting using the parameter `return_unsplit=true` since we will use it to find the approximate symmetry components. The exact symmetry component will be used in approximate symmetry computations. It is initialized using the function

```
sym_components:=gem_symm_components();
```

   The output is

```
sym_components := [xi_x(x, U), eta_U(x, U)].
```

   These are the infinitesimals of the exact symmetry generator $Y = \xi\partial/\partial x + \eta\partial/\partial y$.

5. Write the exact symmetry component in terms of BGI approximate symmetry components.

```
approx_BGI_comp:={sym_components[1]=xi0_x(x, U) + e*xi1_x(x, U),
sym_components[2]=eta0_U(x, U) + e*eta1_U(x, U)};
```

   One obtains

```
approx_BGI_comp:={eta_U(x, U) = eta0_U(x, U)+e*eta1_U(x, U),
xi_x(x, U) = xi0_x(x, U)+e*xi1_x(x, U)};
```

   Note that `xi0_x(x, U)` and `eta0_U(x, U)` are the unperturbed symmetry components of the unperturbed ODE (2.119).

6. Substitute the BGI approximate components in the exact determining equation to obtain the BGI approximate symmetry determining equations.

```
BGI_det_eq:=expand(subs(approx_BGI_comp,det_eq[1]));
```

7. Collect $O(1)$ and $O(e)$ coefficients of the BGI determining equation `BGI_det_eq`.

```
det_eq_e0:=subs(e=0,BGI_det_eq);
det_eq_e1:=subs(e=0,diff(BGI_det_eq,e));
```

   The first determining equation `det_eq_e0` is the determining equation for the exact symmetry components `xi0_x(x, U), eta0_U(x, U)` of the unperturbed equation (2.119). The second determining equation `det_eq_e1` is the determining equation (3.20) for the BGI approximate symmetries of the perturbed equation (2.120).

8. Find the exact point symmetries of the unperturbed ODE (2.119).
   Define again the unperturbed symmetry components

```
sym_components_e0:=[xi0_x(x, U), eta0_U(x, U)];
```

   Generate the split system of linear determining equations using the function

```
split_sys_e0:={coeffs(lhs(det_eq_e0),[Ux,Uxx,Uxxx])};
```

   The resulting determining equations can be simplified by eliminating the redundant determining equations using the `Maple rifsimp` routine.

```
simplified_sys_e0:=DEtools[rifsimp](split_sys_e0, sym_components_e0,mindim=1);
```

   The `mindim=1` option forces `rifsimp` to output linearly independent solutions. The resulting determining equations is stored in `simplified_sys_e0[solved]`. The solution of the determining equations is performed using the command.

```
symm_e0_soln:= pdsolve(simplified_sys_e0[solved],sym_components_e0);
```

   The final solution is

```
symm_e0_soln:={eta0_U(x, U) = _C1*U+_C2+_C3*x+_C4*sin(x)+_C5*cos(x),
xi0_x(x, U) = _C6};
```

   These infinitesimals yield six exact point symmetries of (2.119) given by (2.124).

9. Find the approximate symmetry components.
   This step involves the substitution of the unperturbed symmetry component `symm_e0_soln` (The function H (1.99)) into the determining equation `det_eq_e1`. Therefore, some additional conditions on the free constants C_i in `symm_e0_soln` may appear. Now define the approximate symmetry components

```
sym_components_e1:=[xi1_x(x,U),eta1_U(x, U)];
```

   Find the split system of linear determining equations

```
split_sys_e1:=simplify(eval(subs(symm_e0
,[coeffs(lhs(det_eq_e1),[Ux,Uxx,Uxxx])])));
```

   Simplify the resulting determining equations

```
simplified_sys_e1:=DEtools[rifsimp](split_sys_e1, sym_components_e1 ,mindim=1);
```

The output is

```
table([dimension = 6, Solved = [eta1_Uxxxx = −eta1_Uxx,
eta1_UxU= 0, eta1_UUU = 0, xi1_xx= _C2,
xi1_xU= 0, _C1 = 0, _C3 = 0, _C4 = 0, _C5 = 0]]);
```

Note that the exact symmetries of the unperturbed ODE (2.119) corresponding to the constants C_1, C_3, C_4, C_5 are unstable. While C_2 corresponds to a genuine approximate symmetry of the perturbed equation (2.120). Now we need some constant redefinition to make orders e^0 and e^1 work together.

```
subs_Cs:={_C1=0,_C2=A2,_C3=0,_C4=0,_C5=0,_C6=A6};
```

The updated determining equation for the approximate symmetry components is found using the command

```
simplified_sys_e1_new:=subs(subs_Cs,simplified_sys_e1[Solved][1..-5]);
```

The solution of these determining equation is found using the command

```
symm_e1_soln:=pdsolve(simplified_sys_e1_new,sym_components_e1);
```

The output is given by

```
symm_e1_soln := {eta1_U(x, U) = _C1*U+_C2+_C3*x+_C4*sin(x)+_C5*cos(x),
 xi1_x(x, U) = A2*x+_C6};
```

10. Write the general solution of the approximate symmetry infinitesimals.
Combine the unperturbed components with the approximate components

```
symm_e0_e1_BGI:=subs(symm_e1_soln union subs(subs_Cs,symm_e0_soln),approx_BGI_comp);
```

The result is

```
symm_e0_e1_BGI:={eta_U(x, U) = A2+e*(_C1*U+_C2+_C3*x+_C4*sin(x)+_C5*cos(x)),
xi_x(x, U) = A6+e*(A2*x+_C6)}
```

Redefine the constants $A2, A6$

```
symm_e0_e1_BGI_C:=subs({A2=_C12,A6=_C16},symm_e0_e1_BGI);
```

One gets

```
symm_e0_e1_BGI_C:={eta_U(x, U) =_C12+e*(_C1*U+_C2+_C3*x+_C4*sin(x)+_C5*cos(x)),
xi_x(x, U) = _C16+e*(_C12*x+_C6)}
```

Note that the constants $C_1, ..., C_6$ corresponds to trivial approximate symmetries of (2.120). The constants C_{12}, C_{16} correspond to stable point symmetries of (2.119) where C_{12} yields the new approximate symmetry X_7 in (2.128) of (2.120).

A.2 Maple code for approximate symmetry classification in Table 3.1

Here are the maple codes for the BGI and FS symmetry classifications for a nonlinear wave equation (3.29) (Table 3.1).

A.2.1 Maple code for BGI approximate symmetry classification in Table 3.1

1. The package is initialized using the command.

```
read(t:/gem32_12.mpl);
```

2. Define independent and dependent variables, parameters and arbitrary functions of the problem. We use U instead of u, K instead of F_1 and e instead of ϵ in the wave equation (3.29).

```
gem_decl_vars(indeps=[x,t], deps=[U(x,t)], freeconst =[e],
 freefunc=[K(U(x,t), diff(U(x,t),t))]);
```

3. The wave equation (3.29) is defined using the command.

```
gem_decl_eqs([diff(U(x,t),t,t)+e*K(U(x,t),diff(U(x,t),t))=
 diff(U(x,t),x)*diff(U(x,t),x,x)],solve_for=[diff(U(x,t),t,t)]);
```

4. Generate the exact symmetry determining equation for (3.29) using the command.

```
det_eq:=gem_symm_det_eqs([x,t,U(x,t),diff(U(x,t),x),diff(U(x,t),t)],
in_evolutionary_form=true, return_unsplit=true );
```

   In this example, we seek point symmetries in evolutionary form by adding the parameter `in_evolutionary_form=true`. The exact symmetry component is initialized using the function

```
sym_components:=gem_symm_components();
```

   The output is

```
sym_components := [eta_U(x, t, U, Ux, Ut)].
```

5. Write the exact symmetry component in terms of BGI approximate symmetry components.
`zeta_approx_BGI:={sym_components[1]=zeta0_U(x,t, U,Ux,Ut) + e*zeta1_U(x,t,U,Ux,Ut)};`

```
zeta_approx_BGI:={eta_U(x, t, U, Ux, Ut) = zeta0_U(x, t, U, Ux, Ut)+
e*zeta1_U(x, t, U, Ux, Ut)}.
```

6. Substitute the BGI approximate components in the exact determining equation to obtain the BGI approximate symmetry determining equations.

```
BGI_det_eq:=expand(subs(zeta_approx_BGI,det_eq[1]));
```

7. Collect $O(1)$ and $O(e)$ coefficients of the BGI determining equation `BGI_det_eq`.

```
det_eq_e0:=subs(e=0,BGI_det_eq);
det_eq_e1:=subs(e=0,diff(BGI_det_eq,e));
```

   The first determining equation `det_eq_e0` is the determining equation for the exact symmetry component `zeta0_U(x,t, U,Ux,Ut)` of the unperturbed wave equation (3.11). The second determining equation `det_eq_e1` is the determining equation (3.20) for the BGI approximate symmetries of the perturbed equation (3.29).

8. Find the exact point symmetries of the unperturbed wave equation (3.11).
`sym_components_e0:=[zeta0_U(x,t, U,Ux, Ut)];`

```
sym_components_e0 := [zeta0_U(x, t, U, Ux, Ut)].
```

We require zeta0_U(x,t, U,Ux,Ut) to be linear in the first derivatives of U.

```
linear_cond:={diff(sym_components_e0[1], Ux, Ux)=0,
diff(sym_components_e0[1], Ut, Ux)=0,diff(sym_components_e0[1], Ut, Ut)=0}
```

The split system of the linear PDEs is generated using the function

```
split_sys_e0:={coeffs(lhs(det_eq_e0) ,[Utxx,Utt,Utx,Uxx,Uxxx])} union linear_cond;
```

The resulting determining equations can be simplified by eliminating the redundant determining equations using the Maple rifsimp routine.

```
simplified_sys_e0:=DEtools[rifsimp](split_sys_e0, sym_components_e0,mindim=1);
```

The resulting determining equations is stored in simplified_sys_e0[solved]. The solution of the determining equations is performed using the command.

```
symm_e0_soln:= pdsolve(simplified_sys_e0[solved],sym_components_e0);
```

The final solution is

```
symm_e0_soln:={zeta0_U(x, t, U, Ux, Ut) = _C1+_C2*t+_C3*Ut+_C4*Ux
+_C5*(U+(1/2)*t*Ut)+_C6*(U-x*Ux-t*Ut)};
```

The latter is the evolutionary form of the exact point symmetries (3.14) of the unperturbed wave equation (3.11). The resulting six point symmetries of (3.11) given in Table 3.1 are printed using the command.

```
gem_output_symm({sym_components[1]=subs(symm_e0_soln, zeta0_U(x,t, U,Ux, Ut))});
```

Now, we find the form of the arbitrary function K (F_1 in Table 3.1) where an exact point symmetry of (3.11) is stable in BGI framework. Then we find the corresponding BGI approximate point symmetry of the perturbed equation (3.29).

9. Find the split system of the determining equation det_eq_e1 of the BGI approximate symmetry components.

```
sym_components_e1_ev:=[zeta1_U(x,t, U,Ux, Ut)];
```

```
split_sys_e1:=simplify({(coeffs(lhs(subs(symm_e0_soln,det_eq_e1)),
 [Utx,Utt,Uxx]))}) union  {diff(sym_components_e1_ev[1], Ux, Ux)=0} union
{diff(sym_components_e1_ev[1], Ut, Ux)=0} union
{diff(sym_components_e1_ev[1], Ut, Ut)=0};
```

Here we also require zeta1_U(x,t, U,Ux, Ut) to be linear in the first derivatives of U. This step involves the substitution of the unperturbed symmetry component symm_e0_soln (The function H (3.21)) into the determining equation det_eq_e1. Therefore, some additional conditions on the free constants C_i in symm_e0_soln may appear depending on the form of the arbitrary function K. Hence to classify the stability of exact point symmetries of the unperturbed wave equation (3.11), we find the general solution of K that removes the constraints on each C_i. Then we substitute the obtained value of K to find the approximate point symmetry corresponding to each exact point symmetry of (3.11). We find the form K and the BGI approximate point symmetry corresponding to $\hat{X}_2^0 = t\partial/\partial u$. The other cases can be found in the same way.

10. Substitute $C_2 = 1$ and $C_i = 0, i = 1, 3, ..., 6$ into the split system split_sys_e1.

```
split_sys_e1_X2 := subs([_C1 = 0, _C2 = 1, _C3 = 0, _C4 = 0, _C5 = 0,
_C6 = 0], split_sys_e1);
```

Then simplify using the command.

```
cases2 := simplify(DEtools[rifsimp](split_sys_e1_X2, sym_components_e1,
casesplit, mindim = 1));
```

For the classification, we use the variable casesplit. The result of cases2 contains only one case with a set of linear PDEs in zeta1_U(x, t, U, Ux, Ut) and K. The obtained system of PDEs in K is given by

$$K_{U_t U_t U_t} = 0, \quad K_{UU} = 0, \quad K_{UU_t} = 0$$

with solution

$$K(U, U_t) = a_1 U_t + a_2 U + a_3 U_t^2 + a_4.$$

11. Substitute the value of K into the system cases2.

```
eval(cases2[Solved],{K(U, Ut) =a1Ut+a2U+a3Ut^2+a4}).
```

The result of this step is a set of linear PDEs in zeta1_U(x, t, U, Ux, Ut). The solution can be found using the command pdsolve(%). Consequently, the solution of zeta1_U(x, t, U, Ux, Ut) has the form

```
{zeta1_U(x, t, U, Ux, Ut) = _C4*Ux*x-(1/6)*a2*t^3-(1/2)*
(2*a3*Ut*(1/5)+a1)*t^2+(1/30)*(-24*a3*U
+(45*_C4+15*_C5)*Ut+30*_C8)*t+_C5*U+Ut*_C6+_C7*Ux+_C3}
```

Note that each constant of a1,a2,a3 corresponds to a special form of K and a genuine BGI approximate symmetry of the stable point symmetry $\hat{X}_2^0 = t\partial/\partial u$. The other constants _C3,...,_C8 correspond to BGI trivial approximate symmetries of the perturbed equation (3.29).

A.2.2 Maple code for FS approximate symmetry classification in Table 3.1

1. We initialize the GeM package using the command.

```
read(t:/gem32_12.mpl);
```

2. Declare the variables and arbitrary functions in the FS system (3.30). Define the system of equations (3.30).

```
gem_decl_vars(indeps=[x,t], deps=[V(x,t),W(x,t)],
 freefunc=[K(V(x,t), diff(V(x,t),t))])
```

```
gem_decl_eqs([diff(diff(V(x, t), t), t) = (diff(V(x, t), x))*
(diff(diff(V(x, t), x), x)), diff(diff(W(x, t), t), t)+
K(V(x, t), diff(V(x, t), t))= (diff(W(x, t), x))*
diff(V(x, t), x, x)+(diff(V(x, t), x))*diff(W(x, t), x, x)],
solve_for=[diff(V(x,t),t,t),diff(W(x,t),t,t)]);
```

3. Generate the FS determining equations and the FS local symmetry generator for (3.30).

```
det_eq_Fush:=gem_symm_det_eqs([x,t,V(x,t),W(x,t),
diff(V(x,t),x), diff(V(x,t),t), diff(W(x,t),x), diff(W(x,t),t)],
in_evolutionary_form=true, return_unsplit=true );
```

This determining equations det_eq_Fush contain two non-split determining equations. The first one det_eq_Fush[1] is the determining equation for exact local symmetries of the unperturbed PDE (3.11)

in terms of V. Now define the FS symmetry components
`sym_components:=gem_symm_components();`

```
sym_components := [eta_V(x, t, V, W, Vx, Vt, Wx, Wt),
 eta_W(x, t, V, W, Vx, Vt, Wx, Wt)];
```

4. Find the exact point symmetries of the unperturbed equation (3.11) in terms of FS. The symmetry component `eta_V(x, t, V, W, Vx, Vt, Wx, Wt)` is redefined so that it is linear in the first derivatives of V and does not depend on W.
 `etaV_exact:=[sym_components[1]=eta_v(x, t, V,Vx, Vt)];`

```
etaV_exact:=[eta_V(x, t, V, W, Vx, Vt, Wx, Wt) = eta_v(x, t, V, Vx, Vt)];
```

 The constraints on `eta_v(x, t, V, Vx, Vt)` are performed using the command.

```
linear_cond:=[diff(eta_v(x, t, V, Vx, Vt),Vt,Vt)=0,
diff(eta_v(x, t, V, Vx, Vt),Vt,Vx)=0, diff(eta_v(x, t, V, Vx, Vt),Vx,Vx)=0];
```

 Now we find the determining equations for exact point symmetries of (3.11) in terms of V.

```
det_eqs_exact_V:=eval(subs(etaV_exact, det_eq_Fush[1]));
```

 Then we split the determining equation `det_eqs_exact_V` with respect to higher derivative of V.

```
split_exact_V:=[coeffs(lhs(det_eqs_exact_V), [Vxx,Vtx,Vtt])];
```

 The solution of the split system can be found using the function

```
symm_e0_Fush:=pdsolve([split_exact_V[],linear_cond[]],eta_v(x, t, V, Vx, Vt));
```

 The result is the evolutionary form of the point symmetries of (3.11) in terms of FS

```
symm_e0_Fush:={eta_v(x, t, V, Vx, Vt)= _C1+_C2*t+_C3*Vt+_C4*Vx
+_C5*(V+(1/2)*t*Vt)+_C6*(V-x*Vx-t*Vt);
```

5. Find the split system of determining equations in `eta_W(x, t, V, W, Vx, Vt, Wx, Wt)`.

```
sym_components_e1_Fush:=[eta_w(x, t, V, W,Vx,Vt, Wx, Wt)];
```

```
split_sys_e1_Fush:={coeffs(lhs(subs(symm_e0_Fush,det_eq_Fush[2])),
[Vtx,Vtt,Vxx,Wtx,Wxx,Wtt,Vtxx,Wtxx,Vxxx,Wxxx])}
```

 We note here that we find FS local symmetries, so no additional constraints on `eta_W(x, t, V, W, Vx, Vt, Wx, Wt)` are required. The substitution of the unperturbed symmetry component `symm_e0_Fush` into the determining equation `det_eq_Fush[2]` leads to some additional conditions on the free constants C_i in `symm_e0_Fush` depending on the form of the arbitrary function K. The classification of the stability of exact point symmetries of the unperturbed wave equation (3.11) is performed one by one. For a point symmetry $\hat{X}^0_k$, one substitutes $C_k = 1$ and $C_i = 0, i \neq k$ into the determining equation `split_sys_e1_Fush`. Then one finds the general solution of K that corresponds to C_k (a stable point symmetry $\hat{X}^0_k$). To find the corresponding approximate symmetry $\hat{Z}_k$, one substitutes the value of K into `split_sys_e1_Fush` and solve for the approximate symmetry component `eta_w(x, t, V, W,Vx,Vt, Wx, Wt)`. Here, we find the form K and the FS approximate local symmetry corresponding to $\hat{X}^0_1 = \partial/\partial v$. The other cases can be found in the same way.

6. Substitute $C_1 = 1$ and $C_i = 0, i = 2, ..., 6$ into the split system of equations `split_sys_e1_Fush`.

```
split_sys_e1_Fush_Z1 := subs([_C1 = 0, _C2 = 0, _C3 = 0, _C4 = 0, _C5 = 0,
_C6 = 1], split_sys_e1_Fush);
```

   Simplify the result using the command.

```
cases_F1:=simplify(DEtools[rifsimp](split_sys_e1_Fush_Z1 ,
sym_components_e1_Fush,casesplit, mindim=1));
```

   One gets a system of PDEs in the symmetry component `eta_w(x, t, V, W,Vx,Vt, Wx, Wt)` and the function K. The system of PDEs in K has the form

$$K_{VVV} = \frac{(K_{VV})^2}{K_V}, \quad K_{VVV} = \frac{K_{VV}K_{VV_t}}{K_V}, \quad K_{VV_tV_t} = \frac{K_{VV}K_{V_tV_t}}{K_V},$$

   which has a solution

$$K(V, V_t) = e^{a_1 V} Q(V_t) + a_2 V_t + a_3 V V_t + a_4 V + a_5.$$

7. Find the approximate symmetry component corresponding to the stable point symmetry $\hat{X}_1^0 = \partial/\partial v$. First, we substitute K into `cases_F1[Solved]`.

```
Z1_case := simplify(eval(cases_F1[Solved], K(V, Vt)
= e^{a1V}Q(V_t)+a2V_t+a3VV_t+a4V+a_5))
```

   Then we solve the resulting determining equation using the command

```
pdsolve(Z1_case);
```

   After simplifications, one obtains the solution

```
eta_W(x, t, V, W, Vx, Vt, Wx, Wt) = (1/10)*a1*(Vt*a2+5*a3)*t^2
+2*t*V*a1*a2*(1/5)+a1*W-(1/10)*a3*t^2*Vt-2*t*V*a3*(1/5)-(1/2)*a4*t^2;
```

   Each constant of `a1,...,a4` corresponds to a special form of K and a genuine FS approximate symmetry of the stable point symmetry $\hat{X}_1^0 = \partial/\partial v$.

A.3 Maple code for approximate multipliers and approximate conservation laws

In this section, we provide a maple code to compute approximate multipliers and approximate conservation laws for the system of perturbed PDEs (4.1) in n independent variables $x = (x^1, ..., x^n)$ and m dependent variables $(u^1, ..., u^m)$. In the code, we replace u by U and ϵ by e.

1. Initialize the `Maple` packages:

```
read(t:/gem32_12.mpl);

with(PDEtools):
```

2. Define the independent and dependent variables, and parameters of the given problem.

```
gem_decl_vars(indeps=[x^1,x^2,...,x^n], deps=[U^1(x),...,U^m(x)],
 freeconst=[e])
```

3. Define the N equations of the system of perturbed PDEs (4.1).

```
pde_1:=F_0^1+e*F_1^1=0;..., pde_N:=F_0^N+e*F_1^N=0;
```

4. Define the set of exact multipliers of the system (4.1).

```
exact_multip_pert:=[Lambda1[U],...,LambdaN [U]];
```

5. Use the exact multipliers to define the approximate multipliers of the system (4.1).

```
approx_multip :={exact_multip_pert[1]=Lambda01[U]+e*Lambda11[U],...,
 exact_multip_pert[N]=Lambda0N[U]+e*Lambda1N[U]};
```

Note that the unperturbed equation can be obtained from the perturbed equations using the function `eval(pde_k,e=0)`, and the perturbation term can be found using the command `diff(pde_k,e)`, $k = 1, ..., N$.

6. Find the $2N$ determining equations for exact multipliers of the unperturbed equations (`e=0`) and the determining equations for the $O(e)$ part of the approximate multiplier `approx_multip`.
First, define the $O(1)$ and $O(e)$ parts of the approximate multiplier `approx_multip`

```
multip_unpert:=[Lambda01[U],...,Lambda0N [U]];
```

```
multip_approx_part:=[Lambda11[U],...,Lambda1N [U]];
```

Then, generate the determining equations for `multip_unpert` and `multip_approx_part` using the Euler operators with respect to the dependent variables $U = (U^1, ..., U^N)$.

```
det_eq_multip_unpert:=Euler(Lambda01*eval(pde_1,e=0)+...+
Lambda0N*eval(pde_N,e=0),U)=0;
```

```
det_eq_multip_approx_part:= Euler(Lambda01*diff(pde_1,e)+Lambda11
*eval(pde_1,e=0)+...+Lambda0N*diff(pde_N,e)+Lambda1N*eval(pde_N,e=0),U)=0;
```

The above determining equations are equivalent to the determining equations (4.12).

7. Solve the above determining equations using the command.

```
lambda0_lambda1_sol:=pdsolve({det_eq_multip_unpert[]} union
 {det_eq_multip_approx_part[]});
```

8. Write the general form of the approximate multipliers.

```
approx_mult_general_k:=lambda0_lambda1_sol[k]+e*lambda0_lambda1_sol[N+k];
```

Now we use the approximate multipliers to obtain the approximate conservation laws (4.7). In the `Maple` notations, Φ^i is replaced by $\_J[i]$.

9. Find the approximate conservation laws corresponding to the approximate multipliers.
Set the divergence expressions (4.6)

```
div_term_e0:=Lambda01*eval(pde_1,e=0)+...+
Lambda0N*eval(pde_N,e=0);
```

```
div_term_e1:= Lambda01*diff(pde_1,e)+Lambda11
*eval(pde_1,e=0)+...+Lambda0N*diff(pde_N,e)+Lambda1N*eval(pde_N,e=0);
```

Find the exact conservation law of the unperturbed equation corresponding to the exact multipliers using `ConservedCurrents Maple` routine.

```
Phi0:=ConservedCurrents(div_term_e0,split=false);
```

Then find the $O(e)$ part of the approximate conservation law (4.7) using the command.

```
Phi1:=ConservedCurrents(div_term_e1,split=false);
```

We do not split here to get the general solution and to avoid the cases of trivial approximate conservation laws.

Example A.3.1. We write the code to find approximate multipliers and approximate conservation laws for the nonlinear perturbed heat equation (4.27) (Example 4.3.2).

1. Initialize the `Maple` packages:

```
with(PDEtools):

read(t:/gem32_12.mpl);
```

2. Define the independent and dependent variables, and parameters of (4.27) .

```
gem_decl_vars(indeps=[x,t], deps=[U],freeconst=[e]);
```

3. Define the heat equation (4.27).

```
pde_1:=diff(U(x,t),t)-diff((U(x,t))^(-2)*diff(U(x,t),x),x)
+e*diff((U(x,t)-U(x,t)^(-1)),x)=0;
```

4. Define the set of exact multipliers of (4.27) and specify their dependence .

```
exact_multip_pert:=[Lambda1(x,t,U);
```

5. Use the exact multipliers to define the approximate multipliers of the system (4.1).

```
approx_multip :=exact_multip_pert[]=Lambda01(x,t,U)+e*Lambda11(x,t,U);
```

6. Find the determining equations for exact multipliers of the unperturbed equations (`e=0`) and the determining equations for the $O(e)$ part of the approximate multiplier `approx_multip`.

```
multip_unpert:=[Lambda01(x,t,U);

multip_approx_part:=[Lambda11(x,t,U)];

det_eq_multip_unpert:=Euler(Lambda01*eval(pde_1,e=0))=0;

det_eq_multip_approx_part:= Euler(Lambda01*diff(pde_1,e)+
Lambda11*eval(pde_1,e=0))=0;
```

One gets

`det_eq_multip_unpert` $:= [\Lambda 01_{xx} = 0, \Lambda 01_t = 0, \Lambda 01_U = 0];$

`det_eq_multip_approx_part` $:= [\Lambda 11_{xx} = -\Lambda 01_x, \Lambda 11_t = -\Lambda 01_x, \Lambda 11_U = 0]$

7. Find the general solution of $\Lambda 01$ and $\Lambda 11$.

```
lambda0_lambda1_sol:=pdsolve({det_eq_multip_unpert[]} union
{det_eq_multip_approx_part[]});
```

The solution is given by

`lambda0_lambda1_sol:=`$[\Lambda 01(x,t,U) =$ `_C1*x+_C2,`
$\Lambda 11(x,t,U) =$ `(1/2)*(-x^2-2*t)*_C1+_C3*x+_C4`];

The general form of the approximate multiplier of (4.27) can be found using the command

```
approx_mult_general:=lambda0_lambda1_sol[1]+e*lambda0_lambda1_sol[2];
```

The output is

`approx_mult_general` $:= \Lambda 01(x,t,U) + e * \Lambda 11(x,t,U)$
= `_C1*x+_C2+e*((1/2)*(-x^2-2*t)*_C1+_C3*x+_C4)`

Now we find the approximate conservation law corresponding to the new approximate multiplier $\Lambda = x + e((1/2)(-x^2 - 2t))$.

8. Set the divergence expressions
`div_term_e0:=Lambda01*eval(pde_1,e=0);`

```
div_term_e0:=x*(Ut+2*Ux^2/U^3-Uxx/U^2);
```

`div_term_e1:= Lambda01*diff(pde_1,e)+Lambda11*eval(pde_1,e=0);`

```
div_term_e1:=x*(Ux+Ux/U^2)+(1/2)*(-x^2-2*t)*(Ut+2*Ux^2/U^3-Uxx/U^2);
```

9. Find the exact conservation law of the unperturbed equation corresponding to the exact multipliers $\Lambda 01 = x$.
`Phi0:=ConservedCurrents(div_term_e0,split=false);`

```
Φ0 := [_J[x](x, t, U, Ux, Ut) = -_F1(x, t, U)*Ut-x*Ux/U^2-1/U,
J[t](x, t, U, Ux, Ut) = _F1(x, t, U)*Ux+x*U]
```

Setting `_F1=0` using the command `eval(Phi0,_F1(x,t,U)=0)`, one gets the exact conservation law $D_t [xu] - D_x \left[xu^{-2}u_x + u^{-1}\right] = 0$ for the unperturbed heat equation $u_t - (u^{-2}u_x)_x$.

10. Find the $O(e)$ part of the approximate conservation law
`Phi1:=ConservedCurrents(div_term_e1,split=false);`

```
Φ1 := [_J[x](x, t, U, Ux, Ut) = -_F1(x, t, U)*Ut-(x^2/2+t)*Ux/U^2-xU,
J[t](x, t, U, Ux, Ut) = _F1(x, t, U)*Ux+(x^2/2+t)*U]
```

Setting `_F1=0` using the command `eval(Phi1,_F1(x,t,U)=0)`, one obtains the $O(e)$ part of the approximate conservation law of (4.27) given by (4.32b).