Diffusion in Cauchy Elastic Solid

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Abstract

It is commonly accepted that a starting point of the science of diffusion is the phenomenological diffusion equation postulated by German physiologist Adolf Fick inspired by experiments on diffusion by Thomas Graham and Robert Brown. Fick's diffusion equation has been interpreted decades later by Albert Einstein and Marian Smoluchowski. Here we will show that the theory of diffusion has its elegant mathematical foundations formulated three decades before Fick by French mathematician Augustin Cauchy (~1822). The diffusion equation is straightforward consequence of his model of the elastic solid - the classical balance equations for isotropic, elastic crystal. Basing on the Cauchy model and using the quaternion algebra we present a rigorous derivation of the quaternion form of the diffusion equation. The fundamental consequences of all derived equations and relations for physics, chemistry and the future prospects are presented.

Keywords

diffusion equation, Cauchy elastic solid, Schrödinger equation, quaternions, Planck-Kleinert crystal

1. Introduction

The original fundamental arguments to implement the classical mechanics equations in the field of wave mechanics were given by Kleinert [1]. The building blocks of the Kleinert solid Planck particles, that obey the laws of mass, momentum and energy conservation. Each particle exerts a short range force at the Planck length. The Planck-Kleinert concept combined with the Cauchy model of an elastic solid has been analyzed with the arbitrary assumption of the complex potential field [2]. Recently the Cauchy theory was rigorously combined with the quaternion algebra. Quaternion representation of the Cauchy equation of motion produced the Klein-Gordon wave equation [3]. In the present work the fundamental new result, explicitly the diffusion equation, is presented.

The Cauchy classical theory of elasticity was finished in 1822 [4]. Six years later Poisson [5] realized that displacement is composed of two types of fundamental waves, the longitudinal and transverse waves [6]. He showed that 1) every sufficiently regular solution of the Cauchy equation of motion can be represented by the sum of the gradients of a scalar and vector potential functions and that 2) these longitudinal and transverse wave equations have different speeds. The solution using both a scalar and vector potentials was first given by Lame [7]. Thus through the efforts of Poisson and Lame it was shown that the general elastodynamic displacement field is represented as the sum of the gradient of a scalar potential and the rotation of a vector potential, each satisfying a wave equation. A rigorous completeness proof of the Cauchy theory was given by Duhem [8]. In 1885 Neumann [9] gave the proof of the uniqueness of the solutions of the three fundamental boundary-initial value problems for the finite elastic solid. The Cauchy theory is the well-founded, self-consistent starting point allowing the advanced analysis of the wave phenomena.

The Cauchy and majority of physical problems cannot be reduced to vectorials models. The vector product does not permit the formulation of algebra, e.g., the multiplication and division operation are not defined [10]. In 1843 Hamilton considered necessity of the 4-dimensional space. Hence the algebra of quaternions has emerged, denoted as $\mathcal{H}$. The algebra of quaternions owns all laws of the algebra with the unique properties: the multiplication of quaternions is not commutative and they allow quantifying twists and rotations. The beauty of quaternions was immediately recognized by J.C. Maxwell, in 1869 he wrote [11]: "The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus… are fitted to be of the greatest in all parts of science."

In the following sections we will present the essentials of quaternion algebra, Section 2, and consequently the rigorous derivation of the quaternion representation of the Cauchy deformation field, Section 3. The quaternion representation allows considering quasi-stationary and standing waves in ideal elastic solid, Section 4. Next we consider the energy functional of the quasi-stationary wave and more complex situation that is the wave in the time invariant potential field, Section 5. Final result is the diffusion equation in the quaternion and complex forms, Section 6.

2. Essentials of the quaternion algebra

The review of basic definitions and formulas of the quaternion numbers and functions [12] is incomplete and limited to those used in this paper. Let $\mathbb{R}^4$ be the four-dimensional Euclidean vector space with the orthonormal basis $\{e_0, e_1, e_2, e_3\}$, where $e_0=(1,0,0,0)$, $e_1=(0,1,0,0)$, $e_2=(0,0,1,0)$, $e_3=(0,0,0,1)$ and with the three-dimensional vector subspace $P=span\{e_1,e_2,e_3\}$. In practice the following algebraical notation is useful: $e_0=1, e_1=i, e_2=j, e_3=k$. Thus for an arbitrary quaternion a, i.e., $a\in\mathcal{H}:=\mathbb{R}\otimes P$, we can write shortly

$$a = a_0 + a_1 i + a_2 j + a_3 k = a_0 + \hat{a}\,. \tag{1}$$

The quaternion imaginary units obey the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \qquad ki = -ik = j. \tag{2}$$

The multiplication in the quaternion algebra is non commutative. A conjugate quaternion is defined as

$$a^* = a_0 - a_1 i - a_2 j - a_3 k = a_0 - \hat{a}\,. \tag{3}$$

It follows immediately from (1) - (3) that $a\cdot a^* = a^*\cdot a = \sum_{i=0}^{3} a_i^2$. We will use here the Cauchy-Riemann operator D acting on the quaternion valued functions

$$D\sigma = \left(-\operatorname{div}\hat{\sigma}\right)e_0 + \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\sigma}, \quad \text{where} \quad \sigma = \sigma_0 + \hat{\sigma}. \tag{4}$$

Note that $DD\sigma = \Delta\sigma$ and hence D corresponds physically, under the constraint (13), to the gradient in $\mathbb{R}^3$. The exponent function has its trigonometrical representation

$$e^a = e^{a_0}\left(\cos\left|\hat{a}\right| + \frac{\hat{a}}{\left|\hat{a}\right|}\sin\left|\hat{a}\right|\right). \tag{5}$$

3. The quaternion representation of the deformation field

The Cauchy model of the elastic solid is the mathematical idealization of the isotropic elastic material [13,14]. We consider here the Cauchy theory of elasticity in a case of FCC structure, where the Poisson number ν = 0.25, l_P denotes the dimension of the FCC elementary cell that consists of four components[1].

1. The solid is treated as a closed system of the constant volume $\Omega \subset \mathbb{R}^3$.
2. The solid density, ρ_P, is high and we consider the small deformation limit only, $l_P = const.$, thus the density changes are negligible and $\rho_P = 4m_P / l_P^3 = const.$
3. The transverse wave velocity in such a solid equals $c = \sqrt{0.4Y/\rho_P} = const.$, where Y is the Young modulus.
4. The small deformation limit implies amplitude independent duration, Δt, of the periodic cycles in stationary, quasi-stationary and other waves.
5. In agreement with the Helmholtz decomposition theorem [15], every lattice deformation can be expressed as a sum of compression and twist, $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$.
6. We will consider here 1) the long evolution times, $t \gg \Delta t$, and 2) the waves at the low translation velocity, $\upsilon \ll c$, where υ denotes the velocity of the wave mass center.

The Cauchy momentum equation relates the displacement, $\mathbf{u} \in \mathbb{R}^3$, with the compression $\left(\operatorname{div}\mathbf{u}\right)$, twist $\left(\operatorname{rot}\mathbf{u}\right)$, and the external potential V. The scalar potential will be considered in the next sections

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3\frac{0.4Y}{\rho_P}\operatorname{grad}\operatorname{div}\mathbf{u} - \frac{0.4Y}{\rho_P}\operatorname{rot}\operatorname{rot}\mathbf{u}. \tag{6}$$

From Eq. (6), the local energy per boson mass in the deformation field follows [16]

$$e = \frac{\rho_e}{\rho_P} = \frac{1}{2}\dot{\mathbf{u}} \circ \dot{\mathbf{u}} + \frac{3}{2}\frac{0.4Y}{\rho_P}\left(\operatorname{div}\mathbf{u}\right)^2 + \frac{1}{2}\frac{0.4Y}{\rho_P}\operatorname{rot}\mathbf{u} \circ \operatorname{rot}\mathbf{u}, \tag{7}$$

where $\dot{\mathbf{u}} = \partial\mathbf{u}/\partial t$ and $\circ$ denotes the standard scalar inner product in $\mathbb{R}^3$. Equivalently

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2 \operatorname{grad}\operatorname{div}\mathbf{u} - c^2 \operatorname{rot}\operatorname{rot}\mathbf{u}, \tag{8}$$

$$e = \frac{1}{2}\dot{\mathbf{u}} \circ \dot{\mathbf{u}} + \frac{3}{2}c^2\left(\operatorname{div}\mathbf{u}\right)^2 + \frac{1}{2}c^2 \operatorname{rot}\mathbf{u} \circ \operatorname{rot}\mathbf{u}. \tag{9}$$

By the Helmholtz decomposition theorem every deformation can be expressed by the compression and twist, i.e., can be divided into an irrotational and a solenoidal component. Thus, if $\mathbf{u}$ belongs to the C^3 class of functions then: $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$, $\operatorname{rot}\mathbf{u}_0 = 0$ and $\operatorname{div}\mathbf{u}_\phi = 0$. Upon acting on Eq. (8) by the

[1] To avoid inconsistencies in terminology we call them bosons, scalar bosons showing the m_P mass.

divergence and rotation operators respectively, we get the transverse and the longitudinal elementary wave equations

$$\left.\begin{aligned} \operatorname{div}\left(\frac{\partial^2\mathbf{u}}{\partial t^2} = 3c^2 \operatorname{grad}\operatorname{div}\mathbf{u} - c^2 \operatorname{rot}\operatorname{rot}\mathbf{u}\right) \;&\Rightarrow\; \frac{\partial^2\sigma_0}{\partial t^2} = 3c^2\Delta\sigma_0, \\ \operatorname{rot}\left(\frac{\partial^2\mathbf{u}}{\partial t^2} = 3c^2 \operatorname{grad}\operatorname{div}\mathbf{u} - c^2 \operatorname{rot}\operatorname{rot}\mathbf{u}\right) \;&\Rightarrow\; \frac{\partial^2\hat{\sigma}}{\partial t^2} = c^2\Delta\hat{\sigma}, \end{aligned}\right\} \Rightarrow \begin{cases} \dfrac{\partial^2\sigma_0}{\partial t^2} = 3c^2\Delta\sigma_0, \\ \dfrac{\partial^2\hat{\sigma}}{\partial t^2} = c^2\Delta\hat{\sigma}, \end{cases} \tag{10}$$

where $\sigma_0 = \operatorname{div}\mathbf{u}_0,\quad \hat{\sigma} = \operatorname{rot}\mathbf{u}_\phi$ and the relation (9) becomes

$$e = \frac{1}{2}\dot{\mathbf{u}}\circ\dot{\mathbf{u}} + \frac{3}{2}c^2\sigma_0^2 + \frac{1}{2}c^2\hat{\sigma}\circ\hat{\sigma}. \tag{11}$$

Equations (10) and (11), imply that there exists a deformation field σ, such that one can represent the solenoidal (vector) and irrotational (scalar) fields as the superposition of a real and an imaginary field parts at each point

$$\sigma = \sigma_0 + \hat{\sigma} \quad \in \mathcal{H} \quad \text{and} \quad \sigma^* = \sigma_0 - \hat{\sigma} \quad \in \mathcal{H}, \tag{12}$$

where $\hat{\sigma} = \sigma_1 i + \sigma_2 j + \sigma_3 k$ and $(\sigma_1, \sigma_2, \sigma_3) = \operatorname{rot}\mathbf{u}_\phi$.

Deformation σ obeys constrain that

$$\operatorname{div}\hat{\sigma} = \operatorname{div}\operatorname{rot}\mathbf{u}_\phi = 0. \tag{13}$$

Taking into account the above constraint the Cauchy-Riemann operator (4) becomes

$$D\sigma = \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\sigma}. \tag{14}$$

By adding Eqs. (10) the momentum balance is expressed again by a single partial differential equation

$$\frac{\partial^2\sigma}{\partial t^2} = c^2\Delta\sigma + 2c^2\Delta\sigma_0,, \tag{15}$$

where c^2 is proportional to the lattice twist energy per mass of the boson $c^2 = 0.4Y/\rho_P$, m^2s^{-2}.

Because $\dot{\mathbf{u}}\circ\dot{\mathbf{u}} = \hat{\dot{u}}\circ\hat{\dot{u}} = -\hat{\dot{u}}\cdot\hat{\dot{u}} = \hat{\dot{u}}\cdot\hat{\dot{u}}^*$, where $\hat{\dot{u}} = \dot{u}_1 i + \dot{u}_2 j + \dot{u}_3 k$ and $\dot{\mathbf{u}} = (\dot{u}_1, \dot{u}_2, \dot{u}_3)$, the overall energy of the deformation field per mass of the boson, the formula (11), becomes in the quaternion form

$$e = \frac{\rho_e}{\rho_P} = \frac{1}{2}\hat{\dot{u}}\cdot\hat{\dot{u}}^* + \frac{1}{2}c^2\sigma\cdot\sigma^* + c^2\sigma_0^2,. \tag{16}$$

The energy is conserved, so relation (16) leads to the nonlocal boundary condition for Eqs.(13) and (15) [3].

Remark 1 The equation (8) and the relation (9) satisfy the Euler–Lagrange differential equation $\frac{\partial e}{\partial\mathbf{u}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial e}{\partial\dot{\mathbf{u}}}\right) = 0$, i.e., satisfy the fundamental equation of the calculus of variations. The Cauchy theory of elastic solid combined with the Helmholtz decomposition theorem and quaternion algebra results in four second order scalar differential equations (15) and the constraint (13). We can conclude that Eq. (15) implies the transverse, longitudinal and various forms of waves in the Cauchy elastic solid. It was already shown that upon splitting Eq. (15) into the system of 1) the basic quaternion wave and 2) the Poisson equation, the nonlinear form of the wave equation follows [3].

4. Quasi-stationary wave in the Cauchy elastic solid

The quasi-stationary wave, QS, means here that the wave can be treated as a particle in an arbitrary volume Ω. Such a wave has: 1) the overall energy, $E = E^0 + Q$, 2) the equivalent mass interrelated to the wave overall energy, 3) the overall mass center and 4) the excess energy, Q, due to the kinetic energy, e.g., as a result of the wave translation velocity υ.

In the following the label kinetic will denote the wave translation in solid: the wave velocity, υ, the wave kinetic energy, Q, the kinetic energy density ρ_Q, etc. The description dynamic will denote the local movements within the elastic body itself: the lattice local velocity, $\hat{u}$, the local kinetic energy density, k, etc. From the relation (16) the overall wave energy can be expressed by the formula

$$\begin{aligned} E = E^0 + Q = \int_\Omega \rho_e \,\mathrm{d}x &= c^2 \int_\Omega \rho_P \left(\frac{1}{2c^2} \hat{u} \cdot \hat{u}^* + \frac{1}{2} \tilde{\sigma} \cdot \tilde{\sigma}^* \right) \mathrm{d}x \\ &= c^2 \int_\Omega \rho_P \left(\quad k \quad + \quad s \quad \right) \mathrm{d}x \qquad (17) \\ &= \qquad K \quad + \quad S, \end{aligned}$$

where $\tilde{\sigma} = \tilde{\sigma}_0 + \hat{\sigma}$, $\tilde{\sigma}_0 = \sqrt{3}\sigma_0$, Q and E^0 are the kinetic and ground wave energies.

The overall particle mass and the particle local density follow from (17),

$$m = \int_\Omega \rho_m \mathrm{d}x = \int_\Omega \rho_P \left(\frac{1}{2c^2} \hat{u} \cdot \hat{u}^* + \frac{1}{2} \tilde{\sigma} \cdot \tilde{\sigma}^* \right) \mathrm{d}x \,, \qquad (18)$$

where m denotes the overall particle mass.

We may conclude that, the *"the overall mass of the wave"* follows from the relations (17) and (18): $m = E/c^2$. Note that in general m differs from the mass at the ground state, Eq. (17), i.e., because $m_0 = E^0/c^2$ then $m \neq m_0$ when $Q > 0$.

In simple words, for instance when single m_P boson in the Cauchy ideal crystal due to the arbitrary cyclic process has the overall energy E^P, then the ratio $m = E^P/c^2$ we call the mass of the *"particle localized on"* this single boson.

Note that the energy conservation, relation (16), in the arrangement shown in (17) displays the wave nature. The considered here QS wave has to satisfy the relation (17) and at every position, the energy density is a sum of the dynamic and strain terms: $\rho_e = c^2 \rho_P (k + s) = 0.4Y(k + s)$. The waves in Ω may differ in shape and many essentials. We start with an elementary situation when velocity of wave mass center υ is low and constant, $\upsilon \ll c$. By using the extremum principle, namely the action concept, one can quantify elementary properties of such QS wave. At every position in Ω:

1. the existence assumption of the quasi-stationary wave implies an equal duration of the periodic cycles in the whole volume occupied by the wave, $\Delta t = const$. Consequently implies, that the s- and k-actions are equal

$$\int_t^{t+\Delta t} s(\tau, x)\,\mathrm{d}\tau = \int_t^{t+\Delta t} k(\tau, x)\,\mathrm{d}\tau = \gamma(x)\Delta t \,, \qquad (19)$$

2. the sum of the overall strain, S, and the kinetic energy, K, in relation (17) equals the overall wave energy $E = E^0 + Q$, and is time invariant,

3. spans of the strain and the kinetic energy terms are equal, $\left[0,\max\left\{k\left(t,x\right)\right\}\right]=\left[0,\max\left\{s\left(t,x\right)\right\}\right]$ $=\left[0,\frac{\rho_e\left(t,x\right)}{c^2\rho_P}\right]=\left[0,\frac{\rho_e\left(t,x\right)}{0.4Y}\right]$.

The relation (19) is valid for the whole Ω so, $\int_t^{t+\Delta t}\int_\Omega s\left(\tau,x\right)\mathrm{d}x\,\mathrm{d}\tau=\int_t^{t+\Delta t}\int_\Omega k\left(\tau,x\right)\mathrm{d}x\,\mathrm{d}\tau=\gamma_\Omega\,\Delta t$, and also for arbitrary number of cycles: $t=n\,\Delta t$. Thus, from the assumptions 2. and 3. above and the relation (17) it follows that both actions in Ω can be approximated by the discrete formula

$$\gamma_\Omega\,n\,\Delta t=\int_0^{n\Delta t}\int_\Omega s\,\mathrm{d}x\,\mathrm{d}\tau=\int_0^{n\Delta t}\int_\Omega k\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{20}$$

Taking into account that we consider time evolution in a case when $t\gg\Delta t$, the continuous expressions for both actions follow

$$\gamma_\Omega\,t=\int_0^t\int_\Omega s\,\mathrm{d}x\,\mathrm{d}\tau=\int_0^t\int_\Omega k\,\mathrm{d}x\,\mathrm{d}\tau\,. \tag{21}$$

Taking time derivative of the relations in (21) we get:

$$\int_\Omega s\,\mathrm{d}x=\int_\Omega k\,\mathrm{d}x\quad\text{for}\quad t\gg\Delta t\,. \tag{22}$$

Both terms, s and k, in (22) oscillate and depend on the time and position. It will be useful to normalize the displacement term s in (17) with respect to the overall particle mass, the relation (18). Note that because we restrict our analysis to the low velocities $\upsilon\ll c$, the translation energy has an minor impact on the overall wave mass and $m\cong m_0$. From the formulae (17), (18) and (22), the normalized strain energy density, s, equals

$$\int_\Omega\frac{\rho_P}{m}\tilde{\sigma}\cdot\tilde{\sigma}^*\mathrm{d}x=\int_\Omega\psi\cdot\psi^*\mathrm{d}x=1,\quad\text{where}\qquad\psi=\sqrt{\frac{\rho_P}{m}}\,\tilde{\sigma}\,. \tag{23}$$

Consequently from (17), (22) and (23)

$$0=\int_\Omega\left(\rho_P\,c^2\;\tilde{\sigma}\cdot\tilde{\sigma}^*-E\,\psi\cdot\psi^*\right)\mathrm{d}x\,, \tag{24}$$

and also

$$0=\int_\Omega\left[\rho_P\,\hat{\dot{u}}\cdot\hat{\dot{u}}^*-E\,\psi\cdot\psi^*\right]\mathrm{d}x\,. \tag{25}$$

Note that, the relations (17), (18) and (24) imply the relation between the overall energy of the wave, the overall wave mass $E=mc^2$, and the probability density because the relation (23) is satisfied. Obviously, the both terms, $\psi=\sqrt{\frac{\rho_P}{m}}\,\tilde{\sigma}$ and $\psi\cdot\psi^*$, vary in time.

Remark 2. The relation (24) links up frequency of the wave with its overall energy:

1) the kinetic, i.e., the excess energy Q, and ground, E^0, energies are entangled in (24) that does not allow their separating,
2) the overall wave energy is increased by translation term, $E=E^0+Q$, accordingly also all displacements and velocities are affected,

3) the wave periodicity implies that by solving the relation (25), one should expect only the discrete values if excess translation energy Q,
4) when the wave overall energy equals its ground energy, $Q = 0$, then relation (24) results in

$$0=\int_{\Omega}\left(\rho_P c^2\ \tilde{\sigma}^0\cdot\left(\tilde{\sigma}^0\right)^* - E^0\ \psi\cdot\psi^*\right)\mathrm{d}x\,, \tag{26}$$

where $\tilde{\sigma}^0$ denotes the displacement in the wave at the ground energy E^0.

5. Wave in the time invariant potential field

Let consider now the evolution of the wave as in the relation (17) in the time invariant potential field, e.g., in the field generated by other particles. The overall energy is now a sum of the ground and excess energy Q,

$$E=E^0+Q=\int_{\Omega}\left(\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+\frac{1}{2}\rho_P c^2\tilde{\sigma}\cdot\tilde{\sigma}^*+V\left(x\right)\psi\cdot\psi^*\right)\mathrm{d}x\,, \tag{27}$$

where $\sqrt{\tilde{\sigma}\cdot\tilde{\sigma}^*}=\left|\tilde{\sigma}^0+\tilde{\sigma}_Q\right|=\left|\tilde{\sigma}\right|$, where $\tilde{\sigma}=\tilde{\sigma}^0+\tilde{\sigma}_Q$.

We consider the low excess energies only, $\upsilon\ll c$. Consequently: 1) the influence of Q on the overall particle mass is marginal, $m\cong m_0$, and 2) the displacement $\tilde{\sigma}$ in (27) can be normalized using the formulae (23). Thus relation (27) becomes

$$\begin{aligned}E=E^0+Q&=\int_{\Omega}\left(\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+\frac{1}{2}mc^2\psi\cdot\psi^*+V\left(x\right)\psi\cdot\psi^*\right)\mathrm{d}x\\&=\frac{1}{2}mc^2+\int_{\Omega}\left(\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+V\left(x\right)\psi\cdot\psi^*\right)\mathrm{d}x.\end{aligned} \tag{28}$$

Both, the E^0 and m are constant, thus it is enough to minimize the relation

$$Q=\int_{\Omega}\left(\frac{1}{2}\rho_P\,\hat{u}\cdot\hat{u}^*+V\left(x\right)\psi\cdot\psi^*\right)\mathrm{d}x\,. \tag{29}$$

Above relation contains the unknown velocity due to the potential $V\left(x\right)$. The Cauchy–Riemann operator of the deformation, $D\tilde{\sigma}$, can be understood, by means of the relations (4) and (14), as an analogy of the gradient in $\mathbb{R}^3$. In the classical dynamics, the potential gradient results in acceleration. For the quaternion representation of the deformation field it is reasonable to guess that the local lattice momentum, $\hat{p}=m_p\hat{u}$, is related to the Cauchy–Riemann operator of the quaternion displacement, $D\tilde{\sigma}$. Namely, the local lattice velocity is proportional to 1) the force that is the normalized Cauchy–Riemann derivative of local displacement, $l_P D\tilde{\sigma}$, and 2) the transverse wave velocity c. Accordingly

$$\hat{p}=-m_P c\,l_P D\tilde{\sigma}=-\hbar D\tilde{\sigma}\,, \tag{30}$$

where we introduced constants $\hbar=m_P c^2 t_P$ and t_P is the time that transverse wave travels at the lattice distance: $l_P=ct_p$.

The overall and local translation velocities of the QS wave equal each other, $\hat{\upsilon}=\hat{u}$.Thus the momentum balance requires

$$\hat{u}=\frac{\hat{p}}{m}=-\frac{\hbar}{m}D\tilde{\sigma}\,. \tag{31}$$

By introducing (31), the relation (29) becomes

$$Q=\int_{\Omega}\left(\rho_P\frac{\hbar^2}{2m^2}\left(D\tilde{\sigma}\right)\cdot\left(D\tilde{\sigma}\right)^*+V\left(x\right)\psi\cdot\psi^*\right)\mathrm{d}x. \tag{32}$$

Normalization using (23) results in the functional

$$Q\left[\psi\right]=\int_{\Omega}\left(\frac{\hbar^2}{2m}\left(D\psi\right)\cdot\left(D\psi\right)^*+V\left(x\right)\psi\cdot\psi^*\right)\mathrm{d}x. \tag{33}$$

There are numerous methods solving above problem, e.g., the path integrals, the Hamilton Jacobi equation, etc. We minimize the functional $Q\left[\psi\right]$, that is the integral above, with respect to a quaternion function such that ψ satisfies the normalization introduced in Section 4, the relation (23). We look for a differential equation that has to be satisfied by the ψ function to extremize (here minimize) the energies allowed by (33). Subsequently we will show that the extremum problem leads to the quaternion analog of the time–independent diffusion equation.

Given the functional (33) and the constraint in (13), we seek for the conditional extreme and use the Lagrange coefficients method combined with a procedure presented in the Appendix. Then there exists a multiplayer $\lambda\neq 0$ such that ψ minimalizes the functional

$$\tilde{Q}\left[\psi\right]=\int_{\Omega}\left[\frac{\hbar^2}{2m}\left(D\psi\right)\cdot\left(D\psi\right)^*+V\left(x\right)\psi\cdot\psi^*+\lambda\left(\frac{1}{\left|\Omega\right|}-\psi\cdot\psi^*\right)\right]\mathrm{d}x. \tag{34}$$

It follows from the Appendix that ψ satisfies the differential equation

$$-\frac{\hbar^2}{2m}\Delta\psi+V\left(x\right)\psi=\lambda\psi. \tag{35}$$

A constant factor on the right–hand side can be considered as an extra energy of the particle in the presence of the field $V = V(x)$. For $E=\lambda$, Eq. (35) is clearly the time independent Schrödinger equation satisfied by the particle in the ground state of the energy E,

$$-\frac{\hbar^2}{2m}\Delta\psi+V\left(x\right)\psi=E\psi. \tag{36}$$

It has to be satisfied together with the condition

$$\operatorname{div}\hat{\psi}=0 \quad \text{where} \quad \psi=\psi_0+\hat{\psi}. \tag{37}$$

Upon using the NIST data [17] of Planck's natural units m_P, l_P, t_p and the light velocity c, the computed constant $\hbar$ equals the Planck constant.

6. Time dependent diffusion equation

By analogy to the complex time-dependent Schrödinger equation $i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi+V\left(x\right)\Psi$, we propose a quaternion form

$$\frac{1}{3}\left(i+j+k\right)\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi+V\left(x\right)\Psi, \tag{38}$$

or equivalently

$$\frac{1}{3}(i+j+k)\frac{\partial\Psi}{\partial t} = -\frac{\hbar}{2m}\,\Delta\Psi + \frac{1}{\hbar}V(x)\,\Psi,$$
$$\frac{\partial\Psi}{\partial t} = \Theta_P\,\Delta\Psi - \frac{i+j+k}{\hbar}V(x)\,\Psi, \quad \text{where } \Theta_P = (i+j+k)\frac{\hbar}{2m}. \tag{39}$$

When the external potential $V(x)$ is negligible, then it can be seen that we generated a quaternion form of the diffusion equation

$$\frac{\partial\Psi}{\partial t} = \Theta_P\,\Delta\Psi\,. \tag{40}$$

We will show now that by the substitution $\Psi(t,x) = e^{-(i+j+k)\frac{E}{\hbar}t}\,\psi(x)$, the equation (38) leads to the time–independent Schrödinger equation (36). Note that by the trigonometric formula (5), we have

$$\Psi(t,x) = \left[\cos\left(\sqrt{3}\frac{E}{\hbar}t\right) - \frac{1}{\sqrt{3}}(i+j+k)\sin\left(\sqrt{3}\frac{E}{\hbar}t\right)\right]\psi(x), \tag{41}$$

$$\begin{aligned}\frac{\partial\Psi}{\partial t}(t,x) &= \left[-\sqrt{3}\frac{E}{\hbar}\sin\left(\sqrt{3}\frac{E}{\hbar}t\right) - (i+j+k)\frac{E}{\hbar}\cos\left(\sqrt{3}\frac{E}{\hbar}t\right)\right]\psi(x)\\ &= -(i+j+k)\frac{E}{\hbar}\left[\cos\left(\sqrt{3}\frac{E}{\hbar}t\right) - \frac{1}{\sqrt{3}}(i+j+k)\sin\left(\sqrt{3}\frac{E}{\hbar}t\right)\right]\psi(x)\\ &= -(i+j+k)\frac{E}{\hbar}\cdot e^{-(i+j+k)\frac{E}{\hbar}t}\,\psi(x).\end{aligned} \tag{42}$$

Obviously

$$\Delta\Psi(t,x) = e^{-(i+j+k)\frac{E}{\hbar}t}\Delta\psi(x)\,. \tag{43}$$

Hence it is immediately seen that Eq. (38) implies Eq. (36).

Consider the special case $\Psi_1 = \Psi_2 = \Psi_3$ and put $\tilde{\Psi} := \Psi_1 = \Psi_2 = \Psi_3$. It follows from elementary calculations that $\Psi := \Psi_0 + \frac{1}{\sqrt{3}}(i+j+k)\tilde{\Psi}$ solves the quaternion Schrödinger equation (38) if and only if $\Psi := \Psi_0 + i\,\tilde{\Psi}$ solves the complex Schrödinger equation

$$\frac{1}{\sqrt{3}}i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\,\Delta\Psi + V(x)\,\Psi\,, \tag{44}$$

or equivalently

$$i\frac{\partial\Psi}{\partial t} = -\frac{\hbar}{2m_1}\,\Delta\Psi + \frac{1}{\hbar}V_1(x)\,\Psi,$$
$$\frac{\partial\Psi}{\partial t} = \Theta_P\,\Delta\Psi - \frac{i}{\hbar}V_1(x)\,\Psi\,, \tag{45}$$

where $D_P = i\frac{\hbar}{2m}$, $m_1 = m/\sqrt{3}$ and $V_1 = \sqrt{3}V$.

7. Conclusions

The aim of our work has been to show the ontology of the diffusion equation. We demonstrated that energy conservation in the elastic Navier–Cauchy continuum implies a quaternion form of the Schrödinger equation and that it can be regarded as the fundamental diffusion equation.

Upon combining the Navier–Cauchy model of the elastic solid with the quaternion algebra we presented the approach that allowed the self–consistent classical interpretation of the wave phenomena. The wave, i.e., the collective movement of the constituents forming the elastic Navier-Cauchy continuum, is considered as equivalent to the particle. Thus, the quantum space is regarded as an analog to the elastic solid. Our derivation provides a new evidence that there is a rigorously defined mathematical connection between classical and quantum mechanics. All the obtained results support the physical reality at the Planck scale and allow for the interpretation of quantum mechanics.

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Appendix

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with the smooth boundary $\partial\Omega$. Moreover, let $\alpha, \beta : \bar{\Omega} \to \mathbb{R}$ be given sufficiently regular functions, in a special case they can be constants. Define a real functional

$$F[\psi] = \int_\Omega \left[\alpha(x)\, \psi \cdot \psi^* + \beta(x)(D\psi) \cdot (D\psi)^* \right] \mathrm{d}x \tag{46}$$

acting on a set

$$S = \left\{ \psi : \bar{\Omega} \to \mathcal{H} \text{ of the } \mathrm{C}^2 \text{ class} \right\} \cap \left\{ \psi = g \text{ on } \partial\Omega \right\},$$

where g is a given function. The functional F can be written in the form

$$\begin{aligned} F[\psi] &= \int_\Omega \left[\alpha(x)\, |\psi|^2 + \beta(x)\left(|\nabla \psi_0|^2 + |\mathrm{rot}\,\hat{\psi}|^2 \right) + 2\beta(x)\,\mathrm{rot}\,\hat{\psi} \circ \nabla \psi_0 + 2\beta(x)\left(\mathrm{div}\,\hat{\psi}\right)^2 \right] \mathrm{d}x \\ &= \int_\Omega \Bigg\{ \alpha(x)\left(\psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 \right) \\ &\quad + \beta(x)\left[\left(\frac{\partial \psi_0}{\partial x} \right)^2 + \left(\frac{\partial \psi_0}{\partial y} \right)^2 + \left(\frac{\partial \psi_0}{\partial z} \right)^2 \right] \\ &\quad + \beta(x)\left[\left(\frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_3}{\partial y} \right)^2 + \left(\frac{\partial \psi_3}{\partial x} - \frac{\partial \psi_1}{\partial z} \right)^2 + \left(\frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x} \right)^2 \right] \\ &\quad + 2\beta(x)\left[\left(\frac{\partial \psi_2}{\partial z} - \frac{\partial \psi_3}{\partial y} \right)\frac{\partial \psi_0}{\partial x} + \left(\frac{\partial \psi_3}{\partial x} - \frac{\partial \psi_1}{\partial z} \right)\frac{\partial \psi_0}{\partial y} + \left(\frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x} \right)\frac{\partial \psi_0}{\partial z} \right] \\ &\quad + \beta(x)\left(\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} \right)^2 \Bigg\} \mathrm{d}x. \end{aligned}$$

Suppose that ψ minimizes F on S. We will show that ψ solves some differential equation. Let $\varphi : \bar{\Omega} \to \mathcal{H}$ be any smooth function and $\varphi = 0$ on $\partial\Omega$, i.e., $\varphi \in C_0^\infty\left(\bar{\Omega}, \mathcal{H}\right)$. Define a real function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(\tau) = F[\psi + \tau\varphi]. \tag{47}$$

Obviously $f'(0) = 0$, because $\psi + \tau\varphi \in S$ and f has minimum in $\tau = 0$. We make the calculations:

$$
\begin{aligned}
f(\tau) = \int_\Omega \Bigg\{ & \alpha(x)\left[\left(\psi_0 + \tau\varphi_0\right)^2 + \left(\psi_1 + \tau\varphi_1\right)^2 + \left(\psi_2 + \tau\varphi_2\right)^2 + \left(\psi_3 + \tau\varphi_3\right)^2 \right] \\
& + \beta(x)\left[\left(\frac{\partial \psi_0}{\partial x} + \tau \frac{\partial \varphi_0}{\partial x} \right)^2 + \left(\frac{\partial \psi_0}{\partial y} + \tau \frac{\partial \varphi_0}{\partial y} \right)^2 + \left(\frac{\partial \psi_0}{\partial z} + \tau \frac{\partial \varphi_0}{\partial z} \right)^2 \right] \\
& + \beta(x)\left[\left(\frac{\partial \psi_2}{\partial z} + \tau \frac{\partial \varphi_2}{\partial z} - \frac{\partial \psi_3}{\partial y} - \tau \frac{\partial \varphi_3}{\partial y} \right)^2 \right. \\
& \qquad + \left(\frac{\partial \psi_3}{\partial x} + \tau \frac{\partial \varphi_3}{\partial x} - \frac{\partial \psi_1}{\partial z} - \tau \frac{\partial \varphi_1}{\partial z} \right)^2 \\
& \qquad \left. + \left(\frac{\partial \psi_1}{\partial y} + \tau \frac{\partial \varphi_1}{\partial y} - \frac{\partial \psi_2}{\partial x} - \tau \frac{\partial \varphi_2}{\partial x} \right)^2 \right] \\
& + 2\beta(x)\left[\left(\frac{\partial \psi_2}{\partial z} + \tau \frac{\partial \varphi_2}{\partial z} - \frac{\partial \psi_3}{\partial y} - \tau \frac{\partial \varphi_3}{\partial y} \right)\left(\frac{\partial \psi_0}{\partial x} + \tau \frac{\partial \varphi_0}{\partial x} \right) \right. \\
& \qquad + \left(\frac{\partial \psi_3}{\partial x} + \tau \frac{\partial \varphi_3}{\partial x} - \frac{\partial \psi_1}{\partial z} - \tau \frac{\partial \varphi_1}{\partial z} \right)\left(\frac{\partial \psi_0}{\partial y} + \tau \frac{\partial \varphi_0}{\partial y} \right) \\
& \qquad \left. + \left(\frac{\partial \psi_1}{\partial y} + \tau \frac{\partial \varphi_1}{\partial y} - \frac{\partial \psi_2}{\partial x} - \tau \frac{\partial \varphi_2}{\partial x} \right)\left(\frac{\partial \psi_0}{\partial z} + \tau \frac{\partial \varphi_0}{\partial z} \right) \right] \\
& + \beta(x)\left[\frac{\partial \psi_1}{\partial x} + \tau \frac{\partial \varphi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} + \tau \frac{\partial \varphi_2}{\partial y} + \frac{\partial \psi_3}{\partial z} + \tau \frac{\partial \varphi_3}{\partial z} \right]^2 \Bigg\} \mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
f'(\tau) = 2\int_\Omega \Big\{ & \alpha(x)\Big[\big(\psi_0 + \tau\varphi_0\big)\varphi_0 + \big(\psi_1 + \tau\varphi_1\big)\varphi_1 + \big(\psi_2 + \tau\varphi_2\big)\varphi_2 + \big(\psi_3 + \tau\varphi_3\big)\varphi_3\Big] \\
& + \beta(x)\left[\left(\frac{\partial\psi_0}{\partial x} + \tau\frac{\partial\varphi_0}{\partial x}\right)\frac{\partial\varphi_0}{\partial x} + \left(\frac{\partial\psi_0}{\partial y} + \tau\frac{\partial\varphi_0}{\partial y}\right)\frac{\partial\varphi_0}{\partial y} + \left(\frac{\partial\psi_0}{\partial z} + \tau\frac{\partial\varphi_0}{\partial z}\right)\frac{\partial\varphi_0}{\partial z}\right] \\
& + \beta(x)\left[\left(\frac{\partial\psi_2}{\partial z} + \tau\frac{\partial\varphi_2}{\partial z} - \frac{\partial\psi_3}{\partial y} - \tau\frac{\partial\varphi_3}{\partial y}\right)\left(\frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi_3}{\partial y}\right)\right. \\
& \qquad + \left(\frac{\partial\psi_3}{\partial x} + \tau\frac{\partial\varphi_3}{\partial x} - \frac{\partial\psi_1}{\partial z} - \tau\frac{\partial\varphi_1}{\partial z}\right)\left(\frac{\partial\varphi_3}{\partial x} - \frac{\partial\varphi_1}{\partial z}\right) \\
& \qquad \left. + \left(\frac{\partial\psi_1}{\partial y} + \tau\frac{\partial\varphi_1}{\partial y} - \frac{\partial\psi_2}{\partial x} - \tau\frac{\partial\varphi_2}{\partial x}\right)\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right)\right] \\
& + \beta(x)\left[\left(\frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi_3}{\partial y}\right)\left(\frac{\partial\psi_0}{\partial x} + \tau\frac{\partial\varphi_0}{\partial x}\right) + \left(\frac{\partial\psi_2}{\partial z} + \tau\frac{\partial\varphi_2}{\partial z} - \frac{\partial\psi_3}{\partial y} - \tau\frac{\partial\varphi_3}{\partial y}\right)\frac{\partial\varphi_0}{\partial x}\right. \\
& \qquad + \left(\frac{\partial\varphi_3}{\partial x} - \frac{\partial\varphi_1}{\partial z}\right)\left(\frac{\partial\psi_0}{\partial y} + \tau\frac{\partial\varphi_0}{\partial y}\right) + \left(\frac{\partial\psi_3}{\partial x} + \tau\frac{\partial\varphi_3}{\partial x} - \frac{\partial\psi_1}{\partial z} - \tau\frac{\partial\varphi_1}{\partial z}\right)\frac{\partial\varphi_0}{\partial y} \\
& \qquad \left. + \left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right)\left(\frac{\partial\psi_0}{\partial z} + \tau\frac{\partial\varphi_0}{\partial z}\right) + \left(\frac{\partial\psi_1}{\partial y} + \tau\frac{\partial\varphi_1}{\partial y} - \frac{\partial\psi_2}{\partial x} - \tau\frac{\partial\varphi_2}{\partial x}\right)\frac{\partial\varphi_0}{\partial z}\right] \\
& + \beta(x)\left[\frac{\partial\psi_1}{\partial x} + \tau\frac{\partial\varphi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \tau\frac{\partial\varphi_2}{\partial y} + \frac{\partial\psi_3}{\partial z} + \tau\frac{\partial\varphi_3}{\partial z}\right]\left[\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}\right]\Big\}\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
0 = f'(0) = 2\int_\Omega \Big\{ & \alpha(x)\big(\psi_0\varphi_0 + \psi_1\varphi_1 + \psi_2\varphi_2 + \psi_3\varphi_3\big) \\
& + \beta(x)\left(\frac{\partial\psi_0}{\partial x}\frac{\partial\varphi_0}{\partial x} + \frac{\partial\psi_0}{\partial y}\frac{\partial\varphi_0}{\partial y} + \frac{\partial\psi_0}{\partial z}\frac{\partial\varphi_0}{\partial z}\right) \\
& + \beta(x)\left[\left(\frac{\partial\psi_2}{\partial z} - \frac{\partial\psi_3}{\partial y}\right)\left(\frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi_3}{\partial y}\right) + \left(\frac{\partial\psi_3}{\partial x} - \frac{\partial\psi_1}{\partial z}\right)\left(\frac{\partial\varphi_3}{\partial x} - \frac{\partial\varphi_1}{\partial z}\right)\right. \\
& \qquad \left. + \left(\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial x}\right)\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right)\right] \\
& + \beta(x)\left[\left(\frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi_3}{\partial y}\right)\frac{\partial\psi_0}{\partial x} + \left(\frac{\partial\psi_2}{\partial z} - \frac{\partial\psi_3}{\partial y}\right)\frac{\partial\varphi_0}{\partial x} + \left(\frac{\partial\varphi_3}{\partial x} - \frac{\partial\varphi_1}{\partial z}\right)\frac{\partial\psi_0}{\partial y}\right. \\
& \qquad \left. + \left(\frac{\partial\psi_3}{\partial x} - \frac{\partial\psi_1}{\partial z}\right)\frac{\partial\varphi_0}{\partial y} + \left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right)\frac{\partial\psi_0}{\partial z} + \left(\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial x}\right)\frac{\partial\varphi_0}{\partial z}\right] \\
& + \beta(x)\left(\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z}\right)\left(\frac{\partial\varphi_1}{\partial x} + \frac{\partial\varphi_2}{\partial y} + \frac{\partial\varphi_3}{\partial z}\right)\Big\}\,\mathrm{d}x.
\end{aligned}
$$

After differentiation by parts we have

$$
\begin{aligned}
&\int_\Omega \Big\{ \alpha(x)\big(\psi_0\varphi_0+\psi_1\varphi_1+\psi_2\varphi_2+\psi_3\varphi_3\big)-\beta(x)\big(\Delta\psi_0\big)\varphi_0 \\
&\quad+\beta(x)\Bigg\{\left[\frac{\partial}{\partial z}\left(\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)\right]\varphi_1 \\
&\qquad+\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y}\right)\right]\varphi_2 \\
&\qquad+\left[\frac{\partial}{\partial y}\left(\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z}\right)\right]\varphi_3\Bigg\} \\
&\quad+\beta(x)\Bigg\{\left[-\frac{\partial}{\partial x}\left(\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)\right]\varphi_0 \\
&\qquad+\left[\left(\frac{\partial^2\psi_0}{\partial z\partial y}-\frac{\partial^2\psi_0}{\partial y\partial z}\right)\varphi_1+\left(\frac{\partial^2\psi_0}{\partial x\partial z}-\frac{\partial^2\psi_0}{\partial z\partial x}\right)\varphi_2+\left(\frac{\partial^2\psi_0}{\partial y\partial x}-\frac{\partial^2\psi_0}{\partial x\partial y}\right)\varphi_3\right]\Bigg\} \\
&\quad+\beta(x)\left[-\left(\frac{\partial}{\partial x}\operatorname{div}\hat{\psi}\right)\varphi_1-\left(\frac{\partial}{\partial y}\operatorname{div}\hat{\psi}\right)\varphi_2-\left(\frac{\partial}{\partial z}\operatorname{div}\hat{\psi}\right)\varphi_3\right]\Big\}\mathrm{d}x=0.
\end{aligned}
$$

The last equation takes the form

$$
\begin{aligned}
&\int_\Omega \Big\{ \alpha(x)\big(\psi_0\varphi_0+\psi_1\varphi_1+\psi_2\varphi_2+\psi_3\varphi_3\big)-\beta(x)\big(\Delta\psi_0\big)\ \varphi_0 \\
&\quad+\beta(x)\Bigg(\left[\frac{\partial}{\partial z}\left(\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)-\frac{\partial}{\partial x}\operatorname{div}\hat{\psi}\right]\varphi_1 \\
&\qquad+\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_2}{\partial x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y}\right)-\frac{\partial}{\partial y}\operatorname{div}\hat{\psi}\right]\varphi_2 \\
&\qquad+\left[\frac{\partial}{\partial y}\left(\frac{\partial\psi_2}{\partial z}-\frac{\partial\psi_3}{\partial y}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\psi_3}{\partial x}-\frac{\partial\psi_1}{\partial z}\right)-\frac{\partial}{\partial z}\operatorname{div}\hat{\psi}\right]\varphi_3\Bigg)\Big\}\mathrm{d}x=0.
\end{aligned}
$$

Equivalently we can write

$$
\begin{aligned}
&\int_\Omega \Big\{\big[\alpha(x)\psi_0-\beta(x)\Delta\psi_0\big]\varphi_0+\big[\alpha(x)\psi_1-\beta(x)\Delta\psi_1\big]\varphi_1 \\
&\quad+\big[\alpha(x)\psi_2-\beta(x)\Delta\psi_2\big]\varphi_2+\big[\alpha(x)\psi_3-\beta(x)\Delta\psi_3\big]\varphi_3\Big\}\mathrm{d}x=0
\end{aligned}
\tag{48}
$$

for all $\varphi\in C_0^\infty\left(\overline{\Omega},\mathcal{H}\right)$. The Du Bois Reymond variational lemma [18] used for (48) implies

$$\alpha(x)\psi_i-\beta(x)\Delta\psi_i=0 \ \text{ for } \ i=0,1,2,3.$$

In consequence ψ must be a solution of the differential equation

$$\alpha(x)\psi-\beta(x)\Delta\psi=0\,. \tag{49}$$

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