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Masterarbeit zum Thema:

Belief Revision in light of Lindenbaum-Tarski Algebra

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1 Introduction

The theory of belief revision is commonly discussed in philosophy and computer science. It describes how new information may change what an agent believes. In this paper, I show how belief revision is representable in a Lindenbaum-Tarski algebra. Although this correspondence is mentioned sometimes, for example in [7] and [11], it seems like a side-note. To my knowledge, this connection was never shown rigorously, which is the goal of this paper. Advantages of an algebraic representation include visualization in a Hasse diagram and an intuitive demonstration of combinatorical calculations to enumerate the number of possible revision results. My approach uses tools from propositional logic as well as lattice and order theory. I introduce them in detail in their respective sections.

In the next section, I list some formal preliminaries. In chapter 2, I define and prove relevant aspects of propositional logic. The set of well-formed formulas as the syntactic basis is defined in section 2.1. Then the semantic basis follows. It consists of the concept of a model, which is an interpretation satisfying a formula or a set of formulas. I use models to define the central term of logical consequence. In section 2.3, the consequence operator corresponding to logical consequence is introduced and some of its properties are proven. The goal is to properly define deductively closed sets, since these are the main subjects of the theory of belief revision. The section after provides an axiomatic system, necessary to construct the Lindenbaum-Tarski algebra. I also introduce additional axioms to simplify this process. The chapter finishes with relevant properties of propositional logic: The conincidence lemma, the deduction theorem and the connection between logical equivalence and tautologies.

In chapter 3, I give a brief overview of the AGM framework. The three basic functions in this formalism are expansion, revision and contraction. They are used to define addition of unproblematic information, changing knowledge when faced with inconsistency and the removal of information, respectively. I only explain expansion and revision, since contraction is definable via the Harper identity. To obtain a specific function satisfying the AGM postulates, I afterwards discuss the ideas of Katsuno and Mendelzon. Instead of using sets of formulas, they work with knowledge bases. Knowledge bases are propositional formulas fulfilling the role of deductively closed sets. The AGM postulates are rephrased for these knowledge bases and I prove that the conditions are equivalent. The KM revision operator is a model based revision, where revision results are determined by an order on the set of interpretations. I introduce these components formally and exemplify them in two examples at the end of the chapter.

Chapter 4 presents the Lindenbaum-Tarski algebra and its properties. I introduce several basic terms of lattice theory in the first section. It ends with the accumulation of these properties to define a Boolean algebra. In the next section, the concept of filters is discussed. I introduce filters in general and specify two kinds of related filters. These are filters generated by a set and principal filters. I prove two of their properties: Cut of filters and how filters generated by unions of sets function. To obtain deductively closed filters, the Lindenbaum-Tarski algebra is needed. I motivate and properly define the construction of this algebra in the following section. The carrier set of the Boolean algebra is partitioned into equivalence classes based on the deductive system defined in section 2.4. By adequacy this is just the set of equivalence classes of logically equivalent formulas. The standard operations infimum, supremum and complement are constructed and shown to be well-defined. I show that it is possible to define a well-defined partial order, which is in fact the consequence relation. A Lindenbaum-Tarski algebra is then visualized in a Hasse diagramm. Following that, I list the first two representation results: The atoms of the order are models and generated filters are deductively closed sets.

In the fifth chapter, the main ideas are collected. The first section provides a reason for the algebraic representation in general. Afterwards I list several concepts of the theory of belief revision and their corresponding algebraic constructions. An example is then visualized using Hasse diagrams. In the last section of the chapter, I use combinatorical calculations to provide equations determining the number of revision results.

I conclude with a summary of the results and provide future directions and related work.

1.1 Formal Preliminaries

In this section, I list some formal preliminaries and notational conventions. Some of them are reintroduced in their respective chapters.

The uppercase greek letters Γ and Δ are used for arbitrary sets. The number of elements of any set is written $|\Gamma| = n$, meaning the cardinality of Γ is the natural number n. The power set of a set Γ is denoted by 2^{Γ}, indicating the relationship of the number of the elements in the power set construction. Other concepts of set theory follow the usual conventions. If sets are denoted by other letters in the greek alphabet, then they are named after the specific sets they describe, so Π as the symbol for a set indicates that the set is partially ordered. Special sets, like the set of well-formed formulas, are written with calligraphic letters; those are also chosen in analogy to or by the first letter of the terms they describe. Uppercase latin letters are sometimes used for some elements of these special sets, so for example $I \in \mathcal{I}$ is an interpretation in the set of interpretations.

Fracture font letters denote algebras, so for example $\mathfrak{B}$ is a Boolean algebra. Variables for algebras are the lowercase latin letters starting with x, specific elements of the carrier set are denoted by the letters a, b and c. Lowercase greek letters starting with φ are variables for formulas of propositional logic, ranging over atomic and complex formulas. Atomic formulas are denoted by lowercase latin letters starting with the letter p. The symbol $\circ$ is also used in the beginnings of this paper as a variable for logical connectives. However after the second chapter, it is only used for one specific operator in the theory of belief revision, so confusion is avoided. Some specific operators and functions are written in bold, so for example $\mathbf{C}$ denotes a closure operator. The latin letters g, l and r are used as functions for the sake of exposition or to provide a definition.

Any other symbols for relations and abbreviations, like the logical connectives or the consequence relation are defined in their respective chapters. Some constructs, like conjunction and infimum, have the same symbol, however this double usage is justified by similar behavior. Context provides the intended meaning. To differentiate elements of the same set they are sometimes given indices, starting with the number 1. An indexed pair of elements, for example x_1, x_2, is in general preferred over the option x', x'', however to avoid cluttering, the latter is sometimes used as well. The number 0 as an index is only used for one specific set: the set of atomic formulas $\mathcal{L}_0$. Occasionally, the meta theoretical conditions "if and only if" and its two directions "if ..., then ..." as well as "only if ..., then ..." are denoted by $\Longleftrightarrow$, $\Rightarrow$ and $\Leftarrow$, respectively.

2 Propositional Logic

In this chapter, a brief overview of relevant aspects of propositional logic is given.[1] The first section introduces the syntax of propositional logic by defining the set of well-formed formulas. The semantics section lists how those formulas are assigned truth values, while also introducing terms connected to the semantic aspects like logical consequence, tautologies and logical equivalence. In section 2.3 the concept of theories is characterized. The definition of theories is based on deductive closure, which is captured by a closure operator corresponding to logical consequences. The next paragraph lists axioms, which provide a full axiomatization of propositional logic. However, to simplify proofs for the construction of a Lindenbaum-Tarski algebra additional axioms are added. Lastly, some relevant meta properties of propositional logic are listed.

2.1 Syntax

The basis for the definition the propositional language is a denumerable set of atomic formulas, $\mathcal{L}_0 = \{p_1, ..., p_n\}$[2], which is the set containing factual statements or *propositions*. They are atomic in the sense that they don't contain connectives like *and* or *or*. To form more complex sentences, atomic sentences are combined via the logical connectives $\mathcal{C} = \{\wedge, \vee, \neg, \rightarrow, \leftrightarrow\}$. The propositional language Σ consist of two sets, a set of atomic formulas and a set of logical connectives: $\Sigma = \mathcal{L}_0 \cup \mathcal{C}$, with $\mathcal{L}_0 \cap \mathcal{C} = \emptyset$. For the sake

[1] This chapter is based on introductory works to mathematical logic, cf. [10], [12], [14] and [15].
[2] Sometimes small letters starting with p, $\{p, q, r, ...\}$, are also used to denote atomic formulas.

of readability, brackets are also used occasionally. Small greek letters are variables for formulas, and range over atomic as well as complex formulas.

The set of *well-formed formulas*, denoted by $\mathcal{L}$, is defined inductively as follows:

Definition 2.1.1. (*Well-formed formulas*)

(i) $\mathcal{L}_0 \subseteq \mathcal{L}$

(ii) If $\varphi \in \mathcal{L}$, then $\neg\varphi \in \mathcal{L}$.

(iii) If $\varphi, \psi \in \mathcal{L}$, then $\varphi \circ \psi \in \mathcal{L}$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

(iv) Nothing else is element of $\mathcal{L}$.

The set $\mathcal{L}$ contains all possible well-formed formulas of a given set of propositions. Note, that implicitly each arity of the logical connectives is given; the arity of $\neg$ is one and $\wedge, \vee, \rightarrow, \leftrightarrow$ have an arity of two.

With definition 2.1.1. the structure of propositional formulas are characterized, and in the next chapter the definitions of semantics will follow, that is, how those atomic and complex formulas are assigned truth values.

2.2 Semantics

Semantics is the theory of truth values. Similar to the syntactic characterization, semantics is defined inductively. We start with the definition of a function called *interpretation*:

Definition 2.2.1. (*Interpretation*)

Let φ be a formula and $p_1, ..., p_n$ be the atomic formulas occurring in φ. Then an *interpretation* of φ is a function $I : \{p_1, ..., p_n\} \mapsto \{1, 0\}$.

In other words, an interpretation of a formula assigns to each atomic formula occurring in it one truth value. If that truth value is 1, in symbols $I(p_i) = 1$, I is called a *model* of p_i and alternatively is written $I \vDash p_i$. Models of complex formulas are defined inductively:

Definition 2.2.2. (*Model*)

Let φ and ψ be formulas and I an interpretation. Then:

(i) $I \vDash p_i$ iff $I(p_i) = 1$

(ii) $I \vDash \neg\varphi$ iff $I \nvDash$ [3] φ

(iii) $I \vDash \varphi \wedge \psi$ iff $I \vDash \varphi$ and $I \vDash \psi$

(iv) $I \vDash \varphi \vee \psi$ iff $I \vDash \varphi$ or $I \vDash \psi$

(v) $I \vDash \varphi \rightarrow \psi$ iff $I \nvDash \varphi$ or $I \vDash \psi$

(vi) $I \vDash \varphi \leftrightarrow \psi$ iff $I \vDash \varphi \rightarrow \psi$ and $I \vDash \psi \rightarrow \varphi$

[3] The symbol $\nvDash$ means that it is not the case that $\vDash$ holds.

The definition of models can be extended to sets of formulas. An interpretation is a model for any set of sentences Γ, in symbols $I \vDash \Gamma$, if $I \vDash \varphi$ for all $\varphi \in \Gamma$. The set of all interpretations is denoted by $\mathcal{I}$. The function $\mathbf{Mod}(\Gamma)$ maps a set Γ to all of its models, defined as: $\mathbf{Mod}(\Gamma) = \{I \in \mathcal{I} \mid I \vDash \Gamma\}$. A formula or a set of formulas is called *satisfiable* if there exist at least one model for it. Consequently, a formula as well as a set of formulas is called *unsatisfiable* if there is no model for it.

The preceding definitions characterize the syntactic structure of well-formed formulas and their interpretations. The latter are models, if the assigned truth value is 1. Central to these is the relation $\vDash$, since it determines if an interpretation satisfies a formula, so if it is a model of the formula or not. With the following definition, we can extend this relation to the notion of logical consequences:

Definition 2.2.3. (*Logical consequence*)

Let Γ be a set of formulas and φ be a formula. The formula φ is a logical consequence of Γ, in symbols $\Gamma \vDash \varphi$, iff $\mathbf{Mod}(\Gamma) \subseteq \mathbf{Mod}(\varphi)$.

In other words, φ is a consequence of Γ iff every interpretation I that is a model of Γ is also a model of φ.

To list some notational abbreviations: $\Gamma \vDash \varphi, \psi$ means $\Gamma \vDash \varphi$ and $\Gamma \vDash \psi$. $\Gamma \vDash \Delta$ denotes $\Gamma \vDash \psi$, for all $\psi \in \Delta$. For singleton sets like $\{\varphi\}$, brackets in context of consequences will be omitted, that is, $\varphi \vDash \psi$, instead of $\{\varphi\} \vDash \psi$. Similarly $\Gamma, \varphi \vDash \psi$ is shorthand for $\Gamma \cup \{\varphi\} \vDash \psi$.

We can use the consequence relation to define additional central terms of propositional logic. The definitions of tautology and logical equivalence are as follows:

Definition 2.2.4

(i) A formula φ is called a *tautology*, in symbols $\vDash \varphi$, iff for all interpretations $I \in \mathcal{I}$: $I(\varphi) = 1$.

(ii) Two formulas φ and ψ are called *equivalent*, in symbols $\varphi \equiv \psi$, iff $\varphi \vDash \psi$ and $\psi \vDash \varphi$ iff $\mathbf{Mod}(\varphi) = \mathbf{Mod}(\psi)$.

As usual, $\vDash \varphi$ as a definition of a tautology is shorthand for $\emptyset \vDash \varphi$. This is justified because the empty set has no elements. Hence, every interpretation is trivially a model for it. In this case, logical consequence is reduced to the condition that every interpretation must be a model of φ, which is the definition above.

2.3 Theories

The notion of a consequence relation is central to logic. This relation can have different properties, depending on its defined conditions, the language used or the model characteristics. So the consequence relation has properties reflecting structural properties of the corresponding logic. The important properties in the case of propositional logic are the following:

Theorem 2.3.1.
Let Γ, Γ', Δ be sets of formulas and φ be a formula. Then:

(R) If $\varphi \in \Gamma$, then $\Gamma \vDash \varphi$ (*Reflexivity*)

(M) If $\Gamma \vDash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vDash \varphi$ (*Monotonicity*)

(T) If $\Gamma \vDash \Delta$ and $\Delta \vDash \varphi$, then $\Gamma \vDash \varphi$ (*Transitivity*)

Proof.

(R) Let $\varphi \in \Gamma$. An interpretation I is a model for Γ iff it is a model for each element of Γ. Thus, if $I \vDash \Gamma$, then $I \vDash \psi$ for all $\psi \in \Gamma$, so especially $I \vDash \varphi$. By definition of $\vDash$, $\Gamma \vDash \varphi$.

(M) Suppose $\Gamma \vDash \varphi$ and $\Gamma \subseteq \Gamma'$, but it is not the case that $\Gamma' \vDash \varphi$. The latter is true, if there is an interpretation I with $I \vDash \Gamma'$ and $I \nvDash \varphi$. Hence, $I \vDash \psi'$ for all $\psi' \in \Gamma'$ and since $\Gamma \subseteq \Gamma'$, $I \vDash \psi$ for all $\psi \in \Gamma$. Then however, due to $\Gamma \vDash \varphi$, it holds that $I \vDash \varphi$. Contradiction! Thus, $\Gamma' \vDash \varphi$.

(T) Suppose $\Gamma \vDash \Delta$ and $\Delta \vDash \varphi$. The former holds if $\mathbf{Mod}(\Gamma) \subseteq \mathbf{Mod}(\Delta)$, the latter if $\mathbf{Mod}(\Delta) \subseteq \mathbf{Mod}(\varphi)$. Since $\subseteq$ is transitive, $\mathbf{Mod}(\Gamma) \subseteq \mathbf{Mod}(\varphi)$, therefore: $\Gamma \vDash \varphi$.

□

Additionally, $\vDash$ is called *finitary* and *structural*, if the following conditions hold:

(F) If $\Gamma \vDash \varphi$, then $\Gamma' \vDash \varphi$, for a finite $\Gamma' \subseteq \Gamma$ (*Finiteness*)

(S) If $\Gamma \vDash \varphi$, then $\sigma(\Gamma) \vDash \sigma(\varphi)$, for every substitution σ (*Structurality*)

The consequence relation for propositional logic is a reflexive, monotone, transitive, finitary and structural[4] relation. A consequence relation fulfilling the conditions (R), (M), (T), (F) and (S) is usually called a *structural tarskian consequence relation.*

[4] A substitution is a mapping that, if applied on one or more formulas, substitutes each occurrence of a formula with another formula. So, for example, obviously $p \vDash p$, and if $\sigma(p) = p \vee q$ is fixed, then the consequence $p \vee q \vDash p \vee q$ still holds. The more nuanced aspects of substitution are omitted, since they are not important in the context of this paper.

We define an operator **Cn** corresponding to this consequence relation, the *consequence operator*, which maps sets of formulas to sets of formulas; $\mathbf{Cn} : 2^{\mathcal{L}} \mapsto 2^{\mathcal{L}}$. $\mathbf{Cn}(\Gamma)$ denotes the set of all consequences of Γ:

$$\mathbf{Cn}(\Gamma) = \{\varphi \in \mathcal{L} \mid \Gamma \vDash \varphi\}$$

An operator **C** is called a *closure*, if it satisfies the following conditions:

(In) $\Gamma \subseteq \mathbf{C}(\Gamma)$ (*Inclusion*)

(Mo) If $\Gamma \subseteq \Delta$, then $\mathbf{C}(\Gamma) \subseteq \mathbf{C}(\Delta)$ (*Monotonicity*)

(Id) $\mathbf{C}(\mathbf{C}(\Gamma)) = \mathbf{C}(\Gamma)$ (*Idempotence*)

Theorem 2.3.2.
The operator **Cn** corresponding to $\vDash$ satisfies the conditions (In), (Mo) and (Id).

Proof.

(In) Suppose $\varphi \in \Gamma$. By property (R), $\Gamma \vDash \varphi$. By definition of **Cn**, $\varphi \in \mathbf{Cn}(\Gamma)$, therefore $\Gamma \subseteq \mathbf{Cn}(\Gamma)$.

(Mo) Suppose $\Gamma \subseteq \Delta$. For any $\varphi \in \Gamma$ it holds that $\Gamma \vDash \varphi$, therefore $\varphi \in \mathbf{Cn}(\Gamma)$. By monotonicity of $\vDash$ and assumptions, $\Delta \vDash \varphi$. Then however, $\varphi \in \mathbf{Cn}(\Delta)$. Thus, if $\varphi \in \mathbf{Cn}(\Gamma)$, then $\varphi \in \mathbf{Cn}(\Delta)$, which is $\mathbf{Cn}(\Gamma) \subseteq \mathbf{Cn}(\Delta)$.

(Id) Two properties have to be shown:

($\supseteq$) Since $\mathbf{Cn}(\Gamma)$ is a set of formulas, the property (In) applies. Therefore: $\mathbf{Cn}(\Gamma) \subseteq \mathbf{Cn}(\mathbf{Cn}(\Gamma))$.

($\subseteq$) Suppose $\varphi \in \mathbf{Cn}(\mathbf{Cn}(\Gamma))$. By (R) it holds that $\mathbf{Cn}(\mathbf{Cn}(\Gamma)) \vDash \varphi$. To apply (T) we need $\Gamma \vDash \mathbf{Cn}(\mathbf{Cn}(\Gamma))$. For this, note that by (In): $\Gamma \subseteq \mathbf{Cn}(\Gamma)$, by the first half of (Id): $\mathbf{Cn}(\Gamma) \subseteq \mathbf{Cn}(\mathbf{Cn}(\Gamma))$. Therefore $\Gamma \subseteq \mathbf{Cn}(\mathbf{Cn}(\Gamma))$. Obviously $\Gamma \vDash \Gamma$, applying (M) yields: $\Gamma \vDash \mathbf{Cn}(\mathbf{Cn}(\Gamma))$. The transitivity of $\vDash$ ensures that $\Gamma \vDash \varphi$, therefore $\varphi \in \mathbf{Cn}(\Gamma)$.

□

The consequence operator **Cn** is called *tarskian* iff it satisfies the conditions of a closure operator. Note, that the properties of the operator reflect properties of the corresponding relation: Reflexivity of $\vDash$ is captured by idempotence of **Cn**, etc.

Definition 2.3.3. (*Theory*)
A subset $\Gamma \subseteq \mathcal{L}$ is called a *theory* iff it satisfies: If $\Gamma \vDash \varphi$, then $\varphi \in \Gamma$.

In other words a set of formulas Γ is a theory iff every logical consequence of Γ is also an element of Γ. By property (R) of $\vDash$, the condition can be combined to: $\Gamma \vDash \varphi$ iff $\varphi \in \Gamma$.

This means, a subset $\Gamma \subseteq \mathcal{L}$ is a theory iff $\Gamma = \mathbf{Cn}(\Gamma)$. A set satisfying the preceding condition is also called *deductively closed*.

The consequence operator can also be used to define the set of tautologies. Note that the latter are all well-formed formulas φ, which are by Definition 2.2.4. logical consequences of the empty set, more formally: $\{\varphi \in \mathcal{L} \mid \emptyset \vDash \varphi\}$. This coincides with the definition of the consequence operator, where $\Gamma = \emptyset$. Hence, $\mathbf{Cn}(\emptyset)$ is the set of all tautological formulas.

2.4 Axiomatization

So far, the concept of logical consequences was the semantic consequence, meaning logical consequences via truth values and aspects of model theory. Another dimension of logical consequences is syntactic consequence. This is the deduction of formulas via a proof system.

Let $\vdash$ denote the syntactic consequence. $\varphi \vdash \psi$ means ψ can be derived from φ syntactically. A syntactic derivation is the deduction of ψ in a calculus. There are many different calculi with different strengths and weaknesses, this paper is based on a Hilbert-style axiomatization. This calculus has two components, a set of axioms and a set of rules. In this case there is only one rule, the *Modus Ponens*: $\varphi, \varphi \rightarrow \psi \vdash \psi$.

The axioms are based on Frege's Axiomatization:

Definition 2.4.1. (*Axioms*)

(A1) $\varphi \rightarrow (\psi \rightarrow \varphi)$

(A2) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$

(A3) $(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$

(A4) $\varphi \leftrightarrow \neg\neg\varphi$

This axiomatization captures all theorems of propositional logic. It holds that if $\varphi \vdash \psi$, then $\varphi \vDash \psi$. This *soundness* property of a calculus ensures that whenever one formula can be derived from a set of formulas , then it is a logical consequence of the formulas in the set. If the set is the empty set, then any derived formula is a tautology.

Hilbert-style calculi have been shown to also be *complete*: If $\varphi \vDash \psi$, then $\varphi \vdash \psi$. Completeness means that if a formula is a logical consequence, this can also be shown in the calculus without referring to model theory.

Combining these two properties yields *adequacy*:

Theorem 2.4.2. (*Adequacy*)

For any formula φ and ψ, $\varphi \vdash \psi$ iff $\varphi \vDash \psi$.[5]

[5] A proof for this theorem can be found in [14], p. 6-8.

We can use this property to add axioms to the list above, which will be an advantage for some proofs in this paper. By adding tautologies to the axioms, there is no actual change of the axiomatic system. Let φ be a tautology, so $\vDash \varphi$. Then by adequacy, it holds that $\vdash \varphi$. Consequently, the formula φ is already provable in the axiomatic system, adding it to the list doesn't change the provable formulas.

The following tautologies are added:

Definition 2.4.3. (*Additional axioms*)

(A5) $(\varphi \to \psi) \to ((\varphi \wedge \chi) \to \psi))$

(A6) $(\varphi \to \psi) \to (\varphi \to (\psi \vee \chi))$

(A7) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$

(A8) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$

The axioms have to be tautologies, which has to be proven. All proofs rely on a contradiction. To simplify the proofs, note that each tautology is an implication. By assuming that this implication is false, it follows that in all cases the assumptions are that the antecedent is true, but the consequence is false. The contradiction in all cases will be a truth value for some formula being 1 and 0 simultaneously.

(A5) $I(\varphi \to \psi) = 1, I(\varphi \to (\psi \vee \chi)) = 0$ iff $I(\varphi \to \psi) = I(\varphi) = 1, I(\psi \vee \chi) = 0$ iff $I(\varphi \to \psi) = I(\varphi) = I(\psi) = 1$. Thus, $I(\psi \vee \chi) = 1 = 0$.

(A6) $I(\varphi \to \psi) = 1, I((\varphi \wedge \chi) \to \psi)) = 0$ iff $I(\varphi \to \psi) = I(\varphi \wedge \chi) = 1, I(\psi) = 0$ iff $I(\varphi \to \psi) = I(\varphi) = 1$. Thus, $I(\psi) = 1 = 0$.

(A7) $I((\varphi \to \psi) \wedge (\varphi \to \chi)) = 1, I(\varphi \to (\psi \wedge \chi)) = 0$ iff $I(\varphi) = 1, I(\psi \wedge \chi) = 0$ iff $I(\varphi) = 1, I(\psi) = I(\chi) = 0$ iff $I(\varphi \to \psi) = I(\varphi \to \chi) = 0$, contradicting the assumption that the antecedent is true.

(A8) $I((\varphi \to \chi) \wedge (\psi \to \chi)) = 1, I((\varphi \vee \psi) \to \chi) = 0$ iff $I(\varphi \to \chi) = I(\psi \to \chi) = I(\varphi \vee \psi) = 1, I(\chi) = 0$ iff $I(\varphi) = I(\psi) = 0$. Thus, $I(\varphi \vee \psi) = 1 = 0$.

2.5 Meta Properties

In this section, some meta-theoretical properties of propositional logic are listed. The first one is the *conincidence lemma.* To prove this lemma a proof by structural induction is necessary.

To do this, the definition of a rank of a formula is used:

Definition 2.5.1 (*Rank of a formula*)

Let φ and ψ be formulas. The rank of φ, in symbols $r(\varphi)$, is defined inductively as follows:

(i) $r(p_i) = 0$, for any atomic formula p_i.

(ii) $r(\neg\varphi) = r(\varphi) + 1$.

(iii) $r(\varphi \circ \psi) = r(\varphi) + r(\psi) + 1$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

The rank of a formula is the number of logical connectives occurring in it. Thus, it is a measure for the structural complexity of a formula. The conincidence lemma can now be stated and proven:

Lemma 2.5.2 (*Coincidence lemma*)

Let φ be a formula and I, I' be interpretations, such that $I(p_i) = I'(p_i)$ for all atomic formulas p_i occurring in φ. Then: $I \vDash \varphi$ iff $I' \vDash \varphi$.

Proof. By structural induction:

(IB) $\varphi \equiv p_i$ (φ is atomic)

$I \vDash \varphi$ iff $I \vDash p_i$ iff $I' \vDash p_i$ iff $I' \vDash \varphi$

(IS) Let (IB) be true for formulas of rank $\leq n$. Let φ be a formula of rank $n+1$. Then there are the following cases:

1.) $\varphi \equiv \neg\varphi'$

$I \vDash \varphi$ iff $I \vDash \neg\varphi'$ iff $I \nvDash \varphi' \overset{\text{(IB)}}{\text{iff}} I' \nvDash \varphi'$ iff $I' \vDash \neg\varphi'$ iff $I' \vDash \varphi$

2.) $\varphi \equiv \varphi' \circ \varphi''$ $\quad\quad$ $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$; φ' and φ'' of rank $\leq n$

Case $\wedge$:

$I \vDash \varphi$ iff $I \vDash \varphi' \wedge \varphi''$ iff $I \vDash \varphi'$ and $I \vDash \varphi'' \overset{\text{(IB)}}{\text{iff}} I' \vDash \varphi'$ and $I' \vDash \varphi''$ iff $I' \vDash \varphi' \wedge \varphi''$ iff $I' \vDash \varphi$

The other cases are similar.

□

The next property of propositional logic, the *deduction theorem*, captures the interrelation of logical consequence and implication.

Theorem 2.5.3 (*Deduction theorem*)

Let Γ be a set of formulas and φ, ψ be formulas. Then: $\Gamma, \varphi \vDash \psi$ iff $\Gamma \vDash \varphi \rightarrow \psi$.

Proof.

($\Rightarrow$) Suppose $\Gamma, \varphi \vDash \psi$ and I is a model of Γ. There are two cases:

1.) If $I \vDash \varphi$, then $I \vDash \Gamma, \varphi$ and by assumption, $I \vDash \psi$. Thus, $I \vDash \varphi \rightarrow \psi$ and since I is a model of Γ: $\Gamma \vDash \varphi \rightarrow \psi$.

2.) If $I \not\models \varphi$, then by definition of $\rightarrow$, $I \models \varphi \rightarrow \psi$ and, again, since I is a model of Γ: $\Gamma \models \varphi \rightarrow \psi$.

($\Leftarrow$) By contraposition: If $\Gamma, \varphi \not\models \psi$, then $\Gamma \not\models \varphi \rightarrow \psi$.

Suppose $\Gamma, \varphi \not\models \psi$. Then there is an interpretation I, such that $I \models \Gamma$ and $I \models \varphi$, but $I \not\models \psi$. Therefore also $I \not\models \varphi \rightarrow \psi$ and since I is a model of Γ, it holds that $\Gamma \not\models \varphi \rightarrow \psi$.

□

Another property of propositional logic is that equivalence and tautology have a direct connection in the following sense:

Theorem 2.5.4

Let φ and ψ be formulas. Then: $\varphi \equiv \psi$ iff $\models \varphi \leftrightarrow \psi$.

Proof.

Suppose $\varphi \equiv \psi$. By definition, $\mathbf{Mod}(\varphi) = \mathbf{Mod}(\psi)$. So, $\mathbf{Mod}(\varphi) \subseteq \mathbf{Mod}(\psi)$ and $\mathbf{Mod}(\psi) \subseteq \mathbf{Mod}(\varphi)$. Therefore $\varphi \models \psi$ and $\psi \models \varphi$, which by the deduction theorem is equivalent to $\models \varphi \rightarrow \psi$ and $\models \psi \rightarrow \varphi$. Then, by truth-conditions of $\leftrightarrow$: $\models \varphi \leftrightarrow \psi$.

□

3 Belief Revision

The theory of belief revision is a theory describing changes of beliefs when given new information. The crux is that new knowledge might be inconsistent with existing knowledge. Revision is concerned with adding new information while also avoiding (or removing) inconsistency if it occurs.

This chapter introduces one approach to belief revision, which uses set-theoretical and logical methodology.[6] The first section lists the AGM postulates. Within this framework, beliefs are represented by propositional formulas and the change of these beliefs is formalized with functions on formulas and sets of formulas, in order to fulfill certain rationality constrains. In the paragraph after, a closely related formalism is introduced. The KM revision differs insofar that instead of using sets to describe beliefs, it uses propositional formulas equivalent to the former called knowledge bases. Additionally, the revision function within this framework is based on a model-theoretic order, which determines revision by closeness to the original knowledge base.

[6] Another approach for example, is a Bayesian approach to belief revision where the underlying formalism is probability theory.

3.1 AGM Postulates

The AGM paradigm is based on the work of the authors Carlos Alchourrón, Peter Gärdenfors and David Makinson. They published "On the logic of theory change: Partial meet contraction and revision functions", laying the groundwork for a logic based approach to belief revision.[7]

The AGM framework represents beliefs by propositional formulas. The syntactic logical basis is $\mathcal{L}$, the set of well-formed formulas introduced in chapter 2.1. The collection of all beliefs are formalized via deductively closed sets, which are called belief sets or knowledge sets[8]. The other basic aspect taken from propositional logic is that of the tarskian consequence operator **Cn** to define theories as well as tautologies. This yields the following definitions and notations:

A set of formulas K is a knowledge set iff $\mathrm{K} = \mathbf{Cn}(\mathrm{K})$. An inconsistent knowledge set is denoted by $\mathrm{K}_\perp$, where $\mathrm{K}_\perp = \mathcal{L}$. The set of tautologies is defined with the consequence operator: If $\varphi \in \mathbf{Cn}(\emptyset)$, then $\vDash \varphi$.

Before going into the formal aspects of belief revision, we start with an ordinary language example to provide some basic ideas and intuitions.

Suppose an agent[9] knows the fact "The sun is shining". The representation for this simple sentence would be an atomic formula, say p. So the agent should have a knowledge set containing p. Intuitively, the requirement of a knowledge set to be deductively closed is not that the agent now should know *all* logical consequences of the belief, but should affirm if asked "Do you also know x", where x is a logical consequence of p. In other words, the agent ought to belief the logical consequences, but is not expected to be aware of all of them simultaneously.

Now suppose that the agent learns a new fact. The agent sees that whenever the sun is shining, people are outside. Concluding that "If the sun is shining, then people are outside", the new belief $p \to q$ is formed by the agent. This new information can just be added to the existing knowledge, it is unproblematic since it does not yield contradictions. Then the new knowledge set has two sentences: "The sun is shining" and "If the sun is shining, then people are outside", which would be formally $\{p, p \to q\}$. Moments after, the agent notices that the people are inside, while still knowing the aforementioned sentences. This is represented by the formula $\neg q$. Adding this information to the knowledge set is inconsistent. The easiest way to see this is that if p and $p \to q$ are known, then q should also be known. The formulas q and $\neg q$ cannot be known together without

7 For the groundwork, cf. [4]. General overviews can be found in [1], [6], [13] and [16]. The following definitions and theorems are mostly from [16].

8 Obviously, there are philosophical considerations, which differentiate the terms "belief" and "knowledge". In this paper however, those differences aren't discussed. Henceforth, the different beliefs a person might hold are captured by the term "knowledge set"

9 The term "agent" will be used for a carrier of a belief.

inconsistency. However, the new information is taken seriously, it has to be added to the existing knowledge.

Consequently, the agent should revise her knowledge set. Now the agent has to give up some knowledge, since all existing knowledge taken together yields an inconsistency. As already stated, $\neg q$ is definitely part of the revised knowledge. The targets of revision have to be the known facts p and $p \rightarrow q$. Note, that each of them alone with $\neg q$ is not inconsistent. The agent has to choose one of the existing facts over the other. If the agent dismisses "The sun is shining", then the new knowledge would be "If the sun is shining, then people are outside" and "People are not outside". Otherwise, the new knowledge set would be "The sun is shining" and "People are not outside".

Choosing which belief to dismiss is not of logical nature; both results are valid as new knowledge sets. Which facts are preferred over others is determined by the agent. For this reason, belief revision always has multiple solutions.

The formalization of belief revision will be introduced below. The three essential operators on knowledge sets are *expansion*, *revision* and *contraction*. Expansion is a function that simply adds new information to the existing knowledge set. Revising a knowledge set with a formula applies, if there is a conflict. The new information is added, but there are changes to the existing knowledge necessary to avoid inconsistency. The agent wants to remove as few facts as necessary, keeping as much information as possible. Contraction is related to this: It removes facts from a knowledge set. The definitions of expansion and revision are:

Definition 3.1.1. *Expansion*

Let K be a knowledge set and φ a formula. A function $+ : 2^{\mathcal{L}} + \mathcal{L} \mapsto 2^{\mathcal{L}}$ is an *expansion* iff $\mathrm{K} + \varphi = \mathbf{Cn}(\mathrm{K} \cup \{\varphi\})$.

Definition 3.1.2. *Revision*

Let K be a knowledge set and φ a formula. A function $* : 2^{\mathcal{L}} * \mathcal{L} \mapsto 2^{\mathcal{L}}$ is a *revision* iff it satisfies the following conditions:

(P1) $\mathrm{K} * \varphi = \mathbf{Cn}(\mathrm{K} * \varphi)$ (*Closure*)

(P2) $\varphi \in \mathrm{K} * \varphi$ (*Success*)

(P3) $\mathrm{K} * \varphi \subseteq \mathrm{K} + \varphi$ (*Inclusion*)

(P4) $\neg\varphi \notin \mathrm{K} \Rightarrow \mathrm{K} * \varphi = \mathrm{K} + \varphi$ (*Vacuity*)

(P5) $\mathrm{K} * \varphi = \mathrm{K}_{\perp} \Rightarrow \neg\varphi \in \mathbf{Cn}(\emptyset)$ (*Consistency*)

(P6) $(\varphi \leftrightarrow \psi) \in \mathbf{Cn}(\emptyset) \Rightarrow \mathrm{K} * \varphi = \mathrm{K} * \psi$ (*Extensionality*)

These conditions for the revision function are the basic AGM postulates. Supplementary conditions were added to define iterated revisions. However, they are not utilized in this paper. For completeness, they are noted beneath:

(P7) $\mathrm{K} * (\varphi \wedge \psi) \subseteq (\mathrm{K} * \varphi) + \psi$ (*Superexpansion*)

(P8) $\neg\psi \notin \mathbf{Cn}(\mathrm{K} * \varphi) \Rightarrow (\mathrm{K} * \varphi) + \psi \subseteq \mathrm{K} * (\varphi \wedge \psi)$ (*Subexpansion*)

The postulates for revision can be viewed as principles one should obey to revise rationally. Each basic AGM postulate is justified:

The first postulate, (P1), states that the revised set should also be a knowledge set; it is also a deductively closed set of formulas. Condition (P2) states that the new information φ should be part of the new set. The third postulate (P3) ensures that a revision by φ does not introduce more information than given by φ. (P4) ensures that if there is no conflict with a new information φ, revision is identical to expansion. Since the conflict is based on the containment of both φ and $\neg\varphi$ in the knowledge set, the antecedent of this postulate demands that $\neg\varphi \notin \mathrm{K}$. Condition (P5) ensures that neither the revision function itself nor the knowledge set are inconsistent. If the revision fails to produce a consistent result, that is, if revision of K by φ leads to the inconsistent theory, then $\neg\varphi$ is a tautology. Consequently, φ is then a contradiction. Since any contradiction itself is inconsistent, the revision by a contradiction is always inconsistent. The last postulate (P6) states that it is the semantic content of new information φ which matters for the revision, not its syntactic structure.

The supplementary revision postulates capture revision with multiple formulas. They are best understood together: If a knowledge set K revised by a formula φ is consistent with another formula ψ, then the iterated revision by φ and then ψ should have the same result as revision by the conjunction of both formulas.

The last of the essential operators on knowledge sets is called *contraction*. Contraction is a function $\div : 2^{\mathcal{L}} \div \mathcal{L} \mapsto 2^{\mathcal{L}}$. To contract φ from any knowledge set K means removing the known fact φ from the set. However, deductive closure may lead to φ reappearing in the contracted knowledge set. For this reason, the function has its own set of postulates. These are not discussed here, because of the following identities which define revision via contraction and vice versa. With these identities, only one of the operators revision and contraction is needed:

Theorem 3.1.3.

Let K be a knowledge set and φ a formula. Then:

(I1) $\mathrm{K} * \varphi = (\mathrm{K} \div \neg\varphi) + \varphi$ (*Levi identity*)

(I2) $\mathrm{K} \div \varphi = \mathrm{K} \cap (\mathrm{K} * \neg\varphi)$ (*Harper identity*)

3.2 Knowledge Base Revision

In this section, the AGM framework will be rephrased in the formalism introduced by Katsuno and Mendelzon. In their paper “Propositional knowledge base revision and minimal change"[10], they proposed a framework that has its focus on the semantics of revision by defining model based revision functions.

The knowledge sets are formalized slightly differently: They are transformed into knowledge bases. A knowledge base is a knowledge set K represented by a propositional formula. In the finite case, this formula is easy to obtain: Suppose $\psi_1, ..., \psi_n$ is the collection of all non-equivalent formulas $\psi_i \in \mathrm{K}$. Then $\kappa \equiv \psi_1 \wedge ... \wedge \psi_n$ is the desired formula, such that $\mathrm{K} = \{\psi \mid \kappa \vDash \psi\}$.

Within this framework, the expansion of a knowledge set by a formula functions as expected. By definition, we know that $\mathrm{K} + \varphi = \mathbf{Cn}(\mathrm{K} \cup \{\varphi\})$, therefore $\mathrm{K} \cup \{\varphi\}$ is the new knowledge set. Given that the knowledge set K is represented by the formula κ, the new knowledge base may also be written as a conjunction. Then $\mathrm{K} \cup \{\varphi\}$ is represented by $\kappa \wedge \varphi$. Since knowledge bases are also deductively closed, we define the expansion as $\mathrm{K} + \varphi = \{\psi \mid \kappa \wedge \varphi \vDash \psi\}$. The formula $\kappa \wedge \varphi$ represents the knowledge base κ expanded by the new information φ.

Katsuno and Mendelzon also provide a characterization of the AGM postulates[11] for knowledge bases:

Theorem 3.2.1. *Knowledge Base Revision*

Let $*$ be a revision operator on knowledge sets and $\circ$ be its corresponding operator on knowledge bases. Then $*$ satisfies conditions (P1) - (P6) iff $\circ$ satisfies the following conditions:

(R1) $\kappa \circ \varphi \vDash \varphi$.

(R2) If $\kappa \wedge \varphi$ is satisfiable, then $\kappa \circ \varphi \equiv \kappa \wedge \varphi$.

(R3) If φ is satisfiable, then $\kappa \circ \varphi$ is also satisfiable.

(R4) If $\kappa_1 \equiv \kappa_2$ and $\varphi_1 \equiv \varphi_2$, then $\kappa_1 \circ \varphi_1 \equiv \kappa_2 \circ \varphi_2$.

Proof.

The notational change proposed by Katsuno and Mendelzon provides the following form for a revised knowledge base: $\mathrm{K} * \varphi = \{\psi \mid \kappa \circ \varphi \vDash \psi\}$. Therefore (P1) is satisfied by construction.

(P2) $\Longleftrightarrow$ (R1):

$\varphi \in \mathrm{K} * \varphi$ iff $\varphi \in \{\psi \mid \kappa \circ \varphi \vDash \psi\}$ iff $\kappa \circ \varphi \vDash \varphi$

(P3) and (P4) $\Longleftrightarrow$ (R2):

(P3) and (P4) $\Rightarrow$ (R2):

10 For the original article, cf. [9]. Advanced aspects of the framework are discussed in [3], [8] and [11].
11 Cf. [9], p. 267.

Since $K + \varphi = \{\psi \mid \kappa \wedge \varphi \models \psi\}$, (P3) holds iff $\kappa \wedge \varphi \models \kappa \circ \varphi$:
$K * \varphi \subseteq K + \varphi$ iff $\{\psi \mid \kappa \circ \varphi \models \psi\} \subseteq \{\chi \mid \kappa \wedge \varphi \models \chi\}$ iff $\kappa \circ \varphi \in \{\chi \mid \kappa \wedge \varphi \models \chi\}$ iff $\kappa \wedge \varphi \models \kappa \circ \varphi$

It holds that if $\kappa \wedge \varphi$ is satisfiable, then $\neg\varphi \notin K$. Therefore (P4) is equivalent to the condition: If $\kappa \wedge \varphi$ is satisfiable, then $\kappa \circ \varphi \models \kappa \wedge \varphi$.

Thus, (P3) and (P4) imply (R2).

(P3) and (P4) $\Leftarrow$ (R2):

It is the case that if $\neg\varphi \in K$, then $K + \varphi = K_\perp$. Then however, $K * \varphi \subseteq K + \varphi$ is trivially true. Consequently, (P3) holds.

If $\kappa \wedge \varphi$ is satisfiable, then $\kappa \circ \varphi \equiv \kappa \wedge \varphi$ is equivalent to: If $\neg\varphi \notin K$, then $\{\psi \mid \kappa \circ \varphi \models \psi\} = \{\chi \mid \kappa \wedge \varphi \models \chi\}$. This is in terms of knowledge sets: If $\neg\varphi \notin K$, then $K * \varphi = K + \varphi$, which is (P4).

(P5) $\Longleftrightarrow$ (R3):

If $K * \varphi = K_\perp$, then $\neg\varphi \in \mathbf{Cn}(\emptyset)$ is in terms of knowledge bases: If $\kappa \circ \varphi$ is unsatisfiable, then φ is unsatisfiable. By contraposition: If φ is satisfiable, then $\kappa \circ \varphi$ is satisfiable.

(P6) $\Longleftrightarrow$ (R4):

Since $\kappa_1 \equiv \kappa_2$ iff $K_1 = K_2$, we can assume without loss of generality that both κ_1 and κ_2 are knowledge bases for the knowledge set K. Therefore, if $\kappa_1 \equiv \kappa_2$ and $\varphi_1 \equiv \varphi_2$, then $\kappa_1 \circ \varphi_1 \equiv \kappa_2 \circ \varphi_2$ is just: If $\varphi_1 \equiv \varphi_2$, then $K * \varphi_1 = K * \varphi_2$. Thus, by Corollary 2.2.5, if $\varphi_1 \leftrightarrow \varphi_2 \in \mathbf{Cn}(\emptyset)$, then $K * \varphi_1 = K * \varphi_2$.

□

There are also rephrased postulates for the conditions (P7) and (P8):

(R5) $(\kappa \circ \varphi) \wedge \psi \models \kappa \circ (\varphi \wedge \psi)$.

(R6) If $(\kappa \circ \varphi) \wedge \psi$ is satisfiable, then $\kappa \circ (\varphi \wedge \psi) \models (\kappa \circ \varphi) \wedge \psi$

The basic KM postulates[12] are equivalent to the basic AGM postulates. Before giving the revision function based on this approach, an order on interpretations has to be introduced, the *faithful assignment*.

Let $\mathcal{I}$ be the set of all interpretations of $\mathcal{L}$. The cardinality of $\mathcal{I}$ depends on the number of atomic formulas, it holds that $|\mathcal{I}| = 2^n$, for n atomic formulas. As an example, if $\mathcal{L}_0$ contains two atomic formulas p, q, then there are $2^2 = 4$ possible interpretations. This means $\mathcal{I} = \{I_1, I_2, I_3, I_4\}$, with the following interpretations:

[12] Analogous to the name AGM, the KM postulates are conditions (R1) - (R6), based on the names of the authors Katsuno and Mendelzon. The basic KM postulates are (R1) - (R4).

$I_1: I(p) = 1,\ I(q) = 1$ $\qquad$ $I_3: I(p) = 0,\ I(q) = 1$

$I_2: I(p) = 1,\ I(q) = 0$ $\qquad$ $I_4: I(p) = 0,\ I(q) = 0$

Because of the strict[13] truth conditions of conjunction and negation, we can represent each of these interpretations with a single formula: $p \wedge q, \neg p \wedge q, p \wedge \neg q$ and $\neg p \wedge \neg q$, respectively. Since $I_1 \vDash p \wedge q$, but $I_2, I_3, I_4 \nvDash p \wedge q$, we can think of the formula $p \wedge q$ as representing the interpretation, as it is the only model of said formula. Henceforth, the formulas representing interpretations are denoted by $(\neg)p \wedge (\neg)q$.

The relation $\leq$ over $\mathcal{I}$ is a *pre-order*, which is a reflexive and transitive relation. An Interpretation I_1 is strictly less than I_2, in symbols $I_1 < I_2$, iff $I_1 \leq I_2$ and $I_2 \nleq I_1$. The definition of a *faithful assignment*[14] is as follows:

Definition 3.2.2. *Faithful assignment*

Let φ be a propositional formula and $\mathcal{I}$ the set of interpretations of $\mathcal{L}$. A pre-order $\leq_\varphi$ over $\mathcal{I}$ with respect to φ is called faithful assignment iff the following conditions hold:

(F1) If $I, I' \in \mathbf{Mod}(\varphi)$, then $I <_\varphi I'$ does not hold.

(F2) If $I \in \mathbf{Mod}(\varphi)$ and $I' \notin \mathbf{Mod}(\varphi)$ then $I <_\varphi I'$ holds.

(F3) If $\varphi \equiv \psi$, then $\leq_\varphi = \leq_\psi$.

In other words, any model of φ cannot be strictly less than any other model of φ, but must be for any interpretation which is not a model of φ. The faithful assignment is defined for any formula φ, but the idea is to give an order for a knowledge base.

We can define the minimum for any order; The minimal elements of any ordered set Γ are the elements with: $x \in \Gamma$, but there is no $y \in \Gamma$, such that $y < x$. We therefore define: $min(\Gamma, \leq) = \{x \mid x \in \Gamma$ and there is no $y \in \Gamma$, such that $y < x\}$. Then $min(\Gamma, \leq)$ yields all minimal elements of Γ, according to an order $\leq$.

The definition of the KM revision operator based on an order of models is:

Theorem 3.2.3 *KM Revision*

Let κ be a knowledge base, φ be a formula and $\circ$ be an operator on knowledge bases. The revision operator $\circ$ satisfies conditions (R1) - (R4) iff there exists a faithful assignment with respect to κ, such that:

$$\mathbf{Mod}(\kappa \circ \varphi) = min(\mathbf{Mod}(\varphi), \leq_\kappa)$$

Katsuno and Mendelzon proved this theorem in their paper.[15]

[13] Strict in the sense that both have exactly one interpretation as a model.

[14] Cf. [9], p. 268.

[15] The definition can be found on page 269 of their paper, the proof is on pages 283-286, cf. [9].

In summary: The revision operator yields a model for the new, revised knowledge base. This model is a model of the new information φ closest to the original model of the knowledge base. Closeness is determined according to the faithful assignment of κ. The agent determines which models are preferred over others.

To clarify this approach, two examples are given:

Example 1

Let $\mathrm{K} = \{p, p \leftrightarrow q\}$ be a knowledge set. To transform it into a knowledge base, we construct the conjunction of all non-equivalent formulas in K, that is, $p \wedge p \leftrightarrow q$. The formula $p \wedge p \leftrightarrow q$ is equivalent to $p \wedge q$. Hence, $p \wedge q$ is the knowledge base for the knowledge set and consequently a representation of K.

Suppose the new information $\neg q$ is acquired. Expanding the knowledge base with this formula, that is, $p \wedge q \wedge \neg q$, yields an inconsistent formula. Therefore we need to revise the knowledge base.

Now a faithful assignment with respect to the knowledge base has to be defined. There are multiple orders fulfilling the criteria of a faithful assignments for the formula $p \wedge q$. Note that by (F2) I_1 is minimal, because it is the only model of $p \wedge q$. The remaining interpretations can be ordered as one wishes. Some distinct orders[16] for $\leq_{p \wedge q}$ are the following:

$O_1 : I_1 < I_2 < I_3 < I_4$

$O_2 : I_1 < I_2 < I_4 < I_3$

$O_3 : I_1 < I_3 < I_2 < I_4$

$O_4 : I_1 < I_3 < I_4 < I_2$

$O_5 : I_1 < I_4 < I_2 < I_3$

$O_6 : I_1 < I_4 < I_3 < I_2$

The revision operator is based on the models of the new information, so in this case, $I_2, I_4 \in \mathbf{Mod}(\neg q)$. This yields: $\mathbf{Mod}((p \wedge q) \circ \neg q) = min(\{I_2, I_4\}, \leq_{p \wedge q})$

The result according to the different orders is:

$$min(\{I_2, I_4\}, O_1) = min(\{I_2, I_4\}, O_2) = min(\{I_2, I_4\}, O_3) = \{I_2\}$$

In these cases, $\mathbf{Mod}((p \wedge q) \circ \neg q) = \{I_2\}$. The new knowledge base is $p \wedge \neg q$.

$$min(\{I_2, I_4\}, O_4) = min(\{I_2, I_4\}, O_5) = min(\{I_2, I_4\}, O_6) = \{I_4\}$$

For these orders, $\mathbf{Mod}((p \wedge q) \circ \neg q) = \{I_4\}$. The new knowledge base is $\neg p \wedge \neg q$.

In this example, the formulas representing models coincided with the knowledge bases. However, this is only true for simple languages. The next example will provide an overview of this process with an expanded set of atomic formulas.

[16] All faithful assignments in the examples are total pre-orders. The number of possible arrangements for all orders, including those that are not total, is far too big to list it. Since total orders are sufficient and more intuitive to demonstrate the revision, only those are considered here.

Example 2

Again, let K $= \{p, p \leftrightarrow q\}$ be a knowledge set with $p \wedge q$ as its knowledge base and $\neg q$ the new information. Suppose however, that the set of atomic formulas has been expanded: $\mathcal{L}_0 = \{p, q, r\}$. The set of interpretation $\mathcal{I}$ now has $2^3 = 8$ elements represented by the formulas:

$I_1 : p \wedge q \wedge r$

$I_2 : p \wedge q \wedge \neg r$

$I_3 : p \wedge \neg q \wedge r$

$I_4 : p \wedge \neg q \wedge \neg r$

$I_5 : \neg p \wedge q \wedge r$

$I_6 : \neg p \wedge q \wedge \neg r$

$I_7 : \neg p \wedge \neg q \wedge r$

$I_8 : \neg p \wedge \neg q \wedge \neg r$

Since the faithful assignment needed for revision depends on the knowledge base κ, we again need to define $\leq_\kappa$. The models of κ are $\mathbf{Mod}(p \wedge q) = \{I_1, I_2\}$. Because these are always minimal, both I_1 and I_2 are not relevant for the outcome of the revision, so we can set them together as the two least elements.

The remaining interpretations have to be ordered according to the conditions of faithful assignment. Since the number of distinct orders is quite large, consider the following way to reduce their number.

The revision function depends on the models of the new information $\neg q$, note that $\mathbf{Mod}(\neg q) = \{I_3, I_4, I_7, I_8\}$. Only the order of these interpretations matter for outcome of revision, since neither I_5 nor I_6 change the result. In the previous example, I_3 increased the number of orders similarly, but in the end the order of the models of $\neg q$ mattered, independent of the position of I_3.

However, even when removing all interpretations which are neither models of the knowledge base nor models of the new information, the faithful assignment yields a large number of possible arrangements. For the sake of exposition, suppose the following selected (total) orders for $\leq_{p \wedge q}$:

$O_1 : I_1 = I_2 < I_3 < I_4 < I_5 < I_6 < I_7 < I_8$

$O_2 : I_1 = I_2 < I_8 < I_7 < I_6 < I_5 < I_4 < I_3$

To get the result of revision we have to calculate $min(\{I_3, I_4, I_7, I_8\}, \leq_{p \wedge q})$ for the new model of the revised knowledge base $(p \wedge q) \circ \neg q$. Therefore:

$$\mathbf{Mod}((p \wedge q) \circ \neg q) = min(\{I_3, I_4, I_7, I_8\}, O_1) = \{I_3\}$$

$$\mathbf{Mod}((p \wedge q) \circ \neg q) = min(\{I_3, I_4, I_7, I_8\}, O_2) = \{I_8\}$$

There are two results of revision for these faithful assignments, in one case the model of the new knowledge base $(p \wedge q) \circ \neg q$ is the interpretation $p \wedge \neg q \wedge r$, in the other $\neg p \wedge \neg q \wedge \neg r$.[17]

One question arises: How is the atomic formula r relevant for the revision of a knowledge base, which does not contain it, by a formula not containing it either? This is an inevitable consequence of a model-theoretic approach: By characterizing revision via models, all atomic formulas are assigned a truth value, even if these formulas do not occur in the revision components.

In this sense, the revision results may be overgeneralized. Each result yields maximal information, in the sense that *all* atomic facts are interpreted as either true or false with every revision. This is counterintuitive not only because of an unsensible information increase by revision, but also for computability reasons.

There are two immediate responses. First of all, the faithful assignment can help to justify the information increase. The assignment has some restrictions concerning the models of the ordered knowledge base, but can be ordered in any way for interpretation which are not models. For the example above, an agent may order them, such that all interpretations, in which $I(r) = 1$, are minimal, because r is taken to be true. Then the result of revision has no additional information about r, since the information about it was given in the first place.

The second response is the conincidence lemma, listed in chapter 2.5. Seeing that belief revision is built upon propositional logic, this lemma holds true in the context of revision. To paraphrase this property: The interpretations which are models of any formula φ depend solely on the interpretations of atomic formulas occurring in this formula. If we think of the knowledge base $(p \wedge q) \circ \neg q$ as a formula φ, then the latter should only contain the atomic formulas p and q.[18] Then, by the conincidence lemma, all interpretations I which are models of these atomic formulas are models of φ independent of other atomic formulas. So, if we conclude $I_3 \in \mathbf{Mod}((p \wedge q) \circ \neg q)$, which means $I(p) = I(r) = 1$ and $I(q) = 0$, then $I_4 \in \mathbf{Mod}((p \wedge q) \circ \neg q)$, since $I(p) = 1$ and $I(q) = I(r) = 0$ fulfills the criteria of the conincidence lemma. This however is just another way of stating that there should not be an information increase with respect to atomic formulas not occurring in the revision components.

These responses taken together are in conflict. On the one hand the faithful assignment fixes the information given with atomic formulas, on the other hand, the interpretation

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17 The uniqueness of the revision results in the examples depends on the faithful assignment being totally ordered. It is possible that there are multiple results of this calculation, if the order is not total. As an example, consider a result $\{I_3, I_4\}$. A viable revision result would be each on of the interpretations or a formula which is satisfied by both of them. In this case, this could be the formula $p \wedge \neg q$ or even the revising formula $\neg q$.

18 This holds true for the knowledge base seen as the formula φ. Obviously, the knowledge set itself has the atomic formula r occurring in some of its formulas.

of atomic formulas not occurring in any formula φ should not influence the interpretation of φ. This problem of overgeneralization will be addressed further in chapter 5, where revision is characterized algebraically. There, the overgeneralization can be solved by explicating which knowledge bases fulfill the revision conditions without being formulas representing models.

4 Lindenbaum-Tarski Algebra

This chapter defines a Lindenbaum-Tarski algebra, henceforth called LT algebra. To do so, the first section introduces lattices as mathematical structures. Some properties a lattice can have, such as being complemented and distributive, are introduced. The section then ends in defining a specific kind of lattice called Boolean algebra.[19] Afterwards, the definition of a important kind of set called *filter* is given and some of its properties are listed and proven. The next paragraph shows how to construct a LT algebra and describes concepts of lattice theory in the context of the algebra. This includes filters, which have their own section in the last part.

4.1 Boolean Algebra

The basis of a lattice is a *partially ordered* set. The definition of a partial order is:

Definition 4.1.1. (*Partial Order*)

A relation $\leq$ on a set Π is a partial order, if it satisfies the following conditions for all $x, y, z \in \Pi$:

(i) $x \leq x$ (*Reflexivity*)

(ii) If $x \leq y$ and $y \leq z$, then $x \leq z$ (*Transitivity*)

(iii) If $x \leq y$ and $y \leq x$, then $x = y$ (*Anti-Symmetry*)

Note that two elements of a partially ordered set need not be comparable, that is, it need not be true that for all $x, y \in \Pi$ either $x \leq y$ or $y \leq x$. The case where two elements of $x, y \in \Pi$ are incomparable is denoted by $x \otimes y$.

To define a lattice, the notions *infimum* and *supremum* are needed. Again, we start with a partially ordered set Π. In a subset Γ of Π, any element $a \in \Pi$ is called *upper bound* of Γ, if for all elements $x \in \Gamma$: $a \geq x$. Similarly, $b \in \Pi$ is called *lower bound*, if for all $x \in \Gamma$: $b \leq x$. Thus, a is larger than or equal and b on the other hand smaller than or equal to every element of Γ.

[19] The definition in the following chapter are mostly based on [10], p. 55-124. Introductions to lattices can be found in [2]. Boolean algebras are discussed in [18], cf. [17] for their connection to propositional logic.

We also need the *least* element and the *greatest* element. Any element of $\Gamma \subseteq \Pi$ is the least element, in symbols $l(\Gamma)$, if $l(\Gamma) \in \Gamma$ and for all $x \in \Gamma$: $l(\Gamma) \leq x$. The least element of Γ is the smallest element of the subset, making it in fact a lower bound as well. The greatest element $g(\Gamma)$ is defined analogously: If $g(\Gamma) \in \Gamma$ and $l(\Gamma) \geq x$ for all $x \in \Gamma$, then $g(\Gamma)$ is the greatest element of Γ. Combining these two notions in a certain way yields the definitions of the infimum, the *greatest lower bound*, and the supremum, the *least upper bound*.

We can define a lattice in the following way:

Definition 4.1.2. (*Lattice*)

A lattice $\mathfrak{A} = \langle \Pi, \leq \rangle$ is a partially ordered set Π, in which each $a, b \in \Pi$ has a supremum $a \vee b$ and an infimum $a \wedge b$.

One may transform the relational characterization of a lattice into an algebraic one and vice versa, with the equivalence: $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$. An algebra is simply a set combined with operations on this set. An operation is a function where the domain and codomain are the same set.

The algebraic definition of a lattice is as follows:

Definition 4.1.3. (*Lattice Algebra*)

A lattice algebra $\mathfrak{A} = \langle \Pi, \wedge, \vee \rangle$ is an algebra with operations $\wedge$ and $\vee$, such that the following conditions hold for all $x, y, z \in \Pi$:

(L1) $x \wedge y = y \wedge x$ (L1′) $x \vee y = y \vee x$

(L2) $x \wedge x = x$ (L2′) $x \vee x = x$

(L3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (L3′) $x \vee (y \vee z) = (x \vee y) \vee z$

(L4) $x \wedge (x \vee y) = x$ (L4′) $x \vee (x \wedge y) = x$

The equations listed above hold for lattices in general. To acquire specific lattices, we add more equations for *bounded* as well as *distributive* lattices. For *complemented* lattices, a new operator $\neg$ has to be defined.

We start with the definitions for distributivity. Note that in the definition of a lattice there is no interactivity between both operations. There are conditions (L4) and (L4′), however, as the name *absorption* suggests, the operations are able to cancel each other out. To have an interaction between the two, we can require them to distribute over each other.

Definition 4.1.4. (*Distributivity*)

Let $\mathfrak{A} = \langle \Pi, \wedge, \vee \rangle$ be a lattice algebra. It is called *distributive*, if it satisfies:

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$

Since the notion of upper and lower bound has been defined for subsets of the carrier set of a lattice, the same conditions can be generalized for the whole set. A lattice is bounded, if it has an upper bound, that is, a greatest element, and a lower bound, which coincides with the least element. As usual, these special elements are denoted by 1 and 0 and are captured by the following definitions:

Definition 4.1.5. (*Bounded Lattice Algebra*)

Let $\mathfrak{A} = \langle \Pi, \wedge, \vee \rangle$ be a lattice algebra. It is called *bounded*, if it satisfies:

(Bo1) $1 \wedge x = x$ (Bo1′) $1 \vee x = 1$

(Bo2) $0 \wedge x = 0$ (Bo2′) $0 \vee x = x$

Obviously, any finite lattice, a lattice based on a finite set, is bounded in both directions. All lattices henceforth are finite and therefore bounded. With both bounds we can define the property of being *complemented*:

Definition 4.1.6. (*Complemented Lattice Algebra*)

Let $\mathfrak{A} = \langle \Pi, \wedge, \vee, 1, 0 \rangle$ be a lattice algebra. It is called *complemented*, if for each $x \in \Pi$, there exists a $y \in \Pi$, such that:

(C1) $x \vee y = 1$ (C2) $x \wedge y = 0$

The element y is called the complement of x. It is possible that more then one element fulfills the equations above, so complementation is not unique. The following theorem provides conditions for uniqueness:

Theorem 4.1.7.

In a distributive lattice, complementation is unique.

Proof.

Let $\mathfrak{A}$ be a distributive lattice algebra. Suppose y and y' are both complements for any element $x \in \Pi$. Then, by Definition 4.1.5., it follows that:

$x \vee y = 1$ $x \vee y' = 1$ $x \wedge y = 0$ $x \wedge y' = 0$

Therefore:

$$\begin{aligned} y = y \vee 0 = y \vee (x \wedge y') &= (y \vee x) \wedge (y \vee y') \\ &= 1 \wedge (y \vee y') = y \vee y' \end{aligned}$$

With $y = y \vee y'$, it follows, that $y' \leq y$. For the other direction consider:

$$\begin{aligned} y' = y' \vee 0 = y' \vee (x \wedge y) &= (y' \vee x) \wedge (y' \vee y) \\ &= 1 \wedge (y' \vee y) = y' \vee y \end{aligned}$$

This yields $y \leq y'$, by anti-symmetry: $y = y'$. Therefore complementation is unique.

□

Since complementation is unique, the complement of any element x of a lattice can be denoted by the operator: $\neg x$. Combining the properties above yields a *Boolean algebra*, which is a complemented distributive lattice algebra:

Definition 4.1.8. (*Boolean Algebra*)

Let $\mathfrak{B} = \langle \Pi, \wedge, \vee, \neg, 1, 0 \rangle$ be a lattice algebra with binary operations $\wedge$ and $\vee$, the unary operation $\neg$ and designated elements 1 and 0. Then it is a Boolean algebra iff the following conditions hold for all $x, y, z \in \Pi$:

(B1) $x \wedge y = y \wedge x$	(B1′) $x \vee y = y \vee x$
(B2) $x \wedge x = x$	(B2′) $x \vee x = x$
(B3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(B3′) $x \vee (y \vee z) = (x \vee y) \vee z$
(B4) $x \wedge (x \vee y) = x$	(B4′) $x \vee (x \wedge y) = x$
(B5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	(B5′) $x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$
(B6) $x \wedge \neg x = 0$	(B6′) $x \vee \neg x = 1$

The first three conditions state that both operations $\wedge$ and $\vee$ are commutative, idempotent and associative. Conditions (B4) and (B4′) define contexts in which one operation absorbs the other, conditions (B5) and (B5′) on the other hand demand that both operations distribute over each other.[20] Conditions (B6) and (B6′) give two bounds and define complementation.

As mentioned above, lattices and lattice algebras are interdefinable by the equivalence $x \leq y$ iff $x \wedge y = x$. Hence, the lattice algebra $\mathfrak{B} = \langle \Pi, \wedge, \vee, \neg, 1, 0 \rangle$ can also be defined as a lattice $\mathfrak{B} = \langle \Pi, \leq \rangle$. Since a Boolean algebra can be characterized as a lattice, we can define the concept of *atoms* in the order-theoretic sense:

Definition 4.1.9. (*Atoms*)

Let Π be a partially ordered set with least element $0 \in \Pi$. Then any element $x \in \Pi$ is called *atom* iff $0 < x$ and $\nexists y \in \Pi$, such that $0 < y < x$.

In other words, any element x of the carrier set is an atom iff it is not the least element, but there is no other element between x and 0.

There are a variety of Boolean algebras, but for the purposes of this paper only two are of interest: the *minimal* Boolean algebra and the LT algebra. Both of them have a deep connection to propositional logic. The former will be discussed below, the latter in the following chapter.

The minimal Boolean algebra is defined on the set $\Pi = \{1, 0\}$, where $0 \leq 1$. The importance of this structure lies in the fact that it captures the essential truth value

[20] Distribution needs at least three elements, note the similarity of absorption and distributivity.

properties of the various connectives defined in Chapter 2.

Take for example the connective $\wedge$: A formula $\varphi \wedge \psi$ is true iff both φ and ψ are true. The infimum $\wedge$ functions in a similar way: $1 \wedge 1 = 1$, $1 \wedge 0 = 0$ as well as $0 \wedge 1 = 0$ and of course $0 \wedge 0 = 0$. We can think of the minimal Boolean algebra as a truth value algebra, where the algebraic operations of infimum, supremum and complementation correspond to the truth conditions of the logical connectives $\wedge$, $\vee$ and $\neg$, respectively.

4.2 Filters

For the purposes of this paper, a specific subset of the carrier set of a Boolean algebra is of importance.

These sets are called filters and their definition is as follows:

Definition 4.2.1. (*Filter*)

Let $\mathfrak{B} = \langle \Pi, \leq \rangle$ be a Boolean algebra. A non-empty subset $\Phi \subseteq \Pi$ is called *Filter*, if it satisfies the following conditions:

(Fi1) $\forall x, y \in \Pi$: If $x, y \in \Phi$, then $x \wedge y \in \Phi$.

(Fi2) $\forall x \in \Phi : \forall y \in \Pi$: If $x \in \Phi$ and $x \leq y$, then $y \in \Phi$.

A filter Φ is a set closed under infima and contains all elements of Π greater than arbitrary elements already in the filter.

Obviously not every set of elements of the lattice is a filter. An easy example is a set $\{a, b\}$ of the lattice, where $a \otimes b$. Since both elements are incomparable, the equivalence $x \leq y$ iff $x \wedge y = x$ does not reduce one to the other when the infimum is applied. Then we know $a \wedge b \notin \{a, b\}$, which violates condition (Fi1).

By definition of lattices the infimum of the elements a and b does exist, although it may not be in the set. The following definition provides a way to construct a filter from any set $\Gamma \subseteq \Pi$, even if it is not yet a filter:

Definition 4.2.2. (*Filter generated by a set*)

Let $\mathfrak{B} = \langle \Pi, \leq \rangle$ be a Boolean algebra and $\Gamma \subseteq \Pi$. The filter generated by Γ, denoted by $\mathcal{F}(\Gamma)$, is the smallest filter containing Γ and has the following form: $\mathcal{F}(\Gamma) = \{y \in \Pi \mid x_1 \wedge ... \wedge x_n \leq y, \text{ for some } x_1, ..., x_n \in \Gamma\}$

Note that a generated filter is constructed by forming and adding every infimum of every element, even if any of those are not in the set Γ. For this reason it is possible to construct a filter from any set Γ.

If the set Γ is a singleton, i.e. $\Gamma = \{a\}$, then $\mathcal{F}(\{a\})$ is called principal, and can be characterized by the following equation:

Definition 4.2.3. (*Principal Filter*)

Let $\mathfrak{B} = \langle \Pi, \leq \rangle$ be a Boolean algebra and x be any element of Π. Then: $\mathcal{F}(\{x\}) = \{y \in \Pi \mid x \leq y\}$

The next theorem characterizes the cut of two filters Φ and Ψ:

Theorem 4.2.4.

Let $\mathfrak{B}$ be a Boolean algebra and Φ, Ψ be filters on $\mathfrak{B}$. Then $\Phi \cap \Psi$ is a filter on $\mathfrak{B}$, with $\Phi \cap \Psi = \{z \mid \exists x \in \Phi, \exists y \in \Psi$, such that $x \vee y \leq z\}$.

Proof.

$$\begin{aligned}
\Phi \cap \Psi &= \{z \mid a \leq z \text{ and } b \leq z\} && a \in \Phi, b \in \Psi \\
&= \{z \mid a \vee z = z \text{ and } b \vee z = z\} && \text{Definition} \leq \\
&= \{z \mid a \vee (b \vee z) = z\} && \text{Insertion of equation} \\
&= \{z \mid (a \vee b) \vee z = z\} && \text{Assiocativity} \vee \\
&= \{z \mid a \vee b \leq z\} && \text{Definition} \leq \\
&= \{z \mid \exists x \in \Phi, \exists y \in \Psi \text{ such that } x \vee y \leq z\} && \text{Generalization}
\end{aligned}$$

□

The filter generated[21] by a union of a filter and another filter or an arbitrary element has the following properties:

Theorem 4.2.5.

Let $\mathfrak{B}$ be a Boolean algebra, Φ, Ψ be filters on $\mathfrak{B}$ and $a \in \Pi$. Then:

(i) $\mathcal{F}(\Phi \cup \Psi) \quad = \{z \mid \exists x \in \Phi, \exists y \in \Psi, \text{ such that } x \wedge y \leq z\}$.

(ii) $\mathcal{F}(\Phi \cup \{a\}) = \{z \mid \exists x \in \Phi, \text{ such that } x \wedge a \leq z\}$.

Proof.[22]

By definition, any element $z \in \mathcal{F}(\Phi \cup \Psi)$ iff $\exists z_1, ..., z_n \in \Phi \cup \Psi$, such that $z_1 \wedge ... \wedge z_n \leq z$. This is true iff either

1.) $\exists x_1, ..., x_i \in \Phi$, $\exists y_1, ..., y_k \in \Psi$, such that $i + k = n$ and $x_1 \wedge ... \wedge x_i \wedge y_1 \wedge ... \wedge y_k \leq z$, or

2.) $\exists x_1, ..., x_n \in \Phi$, such that $x_1 \wedge ... \wedge x_n \leq z$, or

3.) $\exists y_1, ..., y_n \in \Psi$, such that $y_1 \wedge ... \wedge y_n \leq z$.

Since both Φ and Ψ are filters, the sets are closed under infima. Therefore:

1.) This case is true iff $\exists x \in \Phi$ and $\exists y \in \Psi$, such that $x \wedge y \leq z$, where $x = x_1 \wedge ... \wedge x_i$ and $y = y_1 \wedge ... \wedge y_k$.

[21] Occasionally the set-brackets for generated filters will be omitted, i.e. $\mathcal{F}(a)$ means $\mathcal{F}(\{a\})$.

[22] This proof follows the proof of Theorem 3.18.12, cf. reference [10], page 118.

2.) This case is true iff $\exists x \in \Phi$ and $\exists y \in \Psi$, such that $x \wedge y \leq z$, where $x = x_1 \wedge ... \wedge x_n$ and y is any element of Ψ.

3.) This case is true iff $\exists x \in \Phi$ and $\exists y \in \Psi$, such that $x \wedge y \leq z$, where $y = y_1 \wedge ... \wedge y_n$ and x is any element of Φ.

For equation (ii); because Φ is a filter, $\mathcal{F}(\Phi \cup \{a\}) = \mathcal{F}(\Phi \cup \mathcal{F}(a))$, since any $x \geq a$ must be in $\mathcal{F}(\Phi \cup \{a\})$. Then however, it suffices to show that the identity $\mathcal{F}(\Phi \cup \mathcal{F}(a)) = \{z \mid \exists x \in \Phi, \text{ such that } x \wedge a \leq z\}$ holds:

($\supseteq$) Since $\mathcal{F}(a)$ is itself a filter, equation (i) applies, therefore $\mathcal{F}(\Phi \cup \mathcal{F}(a)) = \{z \mid \exists x \in \Phi, \exists y \in \mathcal{F}(a), \text{ such that } x \wedge y \leq z\}$. Now the following subset relations holds: $\{z \mid \exists x \in \Phi, \text{ such that } x \wedge a \leq z\} \subseteq \{z \mid \exists x \in \Phi, \exists y \in \mathcal{F}(a), \text{ such that } x \wedge y \leq z\} \subseteq \mathcal{F}(\Phi \cup \mathcal{F}(a))$.

($\subseteq$) Suppose $z \in \mathcal{F}(\Phi \cup \mathcal{F}(a))$. Then $\exists x \in \Phi$, $\exists y \in \mathcal{F}(a)$, such that $x \wedge y \leq z$. Since for any $y \in \mathcal{F}(a)$, $y \geq a$, it holds, that $x \wedge a \leq z$, we conclude $z \in \{z \mid \exists x \in \Phi, \text{ such that } x \wedge a \leq z\}$. Therefore $\mathcal{F}(\Phi \cup \mathcal{F}(a)) \subseteq \{z \mid \exists x \in \Phi, \text{ such that } x \wedge a \leq z\}$, concluding the set identity.

□

4.3 Lindenbaum-Tarski Algebra

A LT algebra is a specific kind of Boolean algebra and for our purposes a handier representation of propositional logic.[23]

As already mentioned in the previous chapter, Boolean algebras are connected to logic. However, there are some limitations to this.

Take for example the formula $\varphi \wedge \varphi$. In the context of algebra, by idempotence of $\wedge$, it follows that $\varphi \wedge \varphi = \varphi$. This fits for the truth values, since it is true that $I(\varphi \wedge \varphi) = I(\varphi)$. However, syntactically the formulas $\varphi \wedge \varphi$ and φ are different. Similarly, the equivalence $\varphi \wedge \psi \equiv \neg(\neg\psi \vee \neg\varphi)$ is semantically identical, but the formulas therein are syntactically different.

The equivalence is the basis for the construction of a LT algebra. Instead of taking single formulas as elements of the carrier set, the elements are sets of equivalent formulas. The equivalence is defined via provability:

$$\varphi \sim \psi \text{ iff } \vdash \varphi \to \psi \text{ and } \vdash \psi \to \varphi$$
$$\text{iff } \vdash \varphi \leftrightarrow \psi$$

It is easy to see that $\sim$ is an equivalence relation. Reflexivity and symmetry follow immediately from the definition and transitivity follows by a deduction of the following general form using axioms (A1) and (A2) and modus ponens:

[23] The construction of this algebra is based on [10], p. 3-9 as well as [12], p. 76-80.

(1) $\vdash \varphi \to \psi$ (*Assumption*)

(2) $\vdash \psi \to \chi$ (*Assumption*)

(3) $\vdash (\psi \to \chi) \to (\varphi \to (\psi \to \chi))$ (*Axiom* 1)

(4) $\vdash \varphi \to (\psi \to \chi)$ (*MP* (2), (3))

(5) $\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (*Axiom* 2)

(6) $\vdash (\varphi \to \psi) \to (\varphi \to \chi)$ (*MP* (4), (5))

(7) $\vdash \varphi \to \chi$ (*MP* (1), (6))

□

The equivalence class of φ is denoted by $[\varphi]_\sim$. This set contains all formulas ψ which are equivalent to φ, that is, $[\varphi]_\sim = \{\psi \mid \varphi \sim \psi\}$.

Let $\mathcal{L}_{/\sim} = \{[\varphi]_\sim \mid \varphi \in \mathcal{L}\}$. Then $\mathcal{L}_{/\sim}$ is the set $\mathcal{L}$ *modulo* the equivalence $\sim$: It is the set of formulas partitioned into its equivalence classes, so the formulas $\varphi \wedge \psi$ and $\neg(\neg\psi \vee \neg\varphi)$ are in the same equivalence class.

Note that atomic formulas are contained in equivalence classes of complex formulas. For example, the atomic formula p is in the same class as the complex formula $p \wedge p$. Consequently, the representative of any equivalence class is one or more greek letters combined via the logical connectives.

An algebra on the set $\mathcal{L}_{/\sim}$ is formed by defining the operations of a lattice, infimum $\curlywedge$, supremum $\curlyvee$, complementation $-$, and the distinguished elements, the greatest element 1 and least element 0, with respect to the equivalence classes:

$$[\varphi]_\sim \curlywedge [\psi]_\sim := [\varphi \wedge \psi]_\sim$$
$$[\varphi]_\sim \curlyvee [\psi]_\sim := [\varphi \vee \psi]_\sim$$
$$-[\varphi]_\sim := [\neg\varphi]_\sim$$
$$0 := [\varphi \wedge \neg\varphi]_\sim$$
$$1 := [\varphi \vee \neg\varphi]_\sim$$

Definition 4.3.1. (*Lindenbaum-Tarski algebra*)

The algebra $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ is called a Lindenbaum-Tarski algebra.

To show that these operators[24] are well-defined on equivalence classes, we have to

[24] The same operators have different symbols in this algebra, whereas in Boolean algebras the infimum and the conjunction (for example) coincided. This has two reasons: The first is that the curved symbols indicate their dependency on the equivalence relation $\sim$ the algebra is based on, while also being recognizable as the represented operation. The second and more important reason is that in this algebra the difference between $\wedge$ and the infimum has more weight. In the one case two sentences are combined to form a new sentence, in the other the greatest lower bound of two equivalence-classes is constructed. The logical conjunction of two equivalence classes 'makes no sense' with respect to the basic concepts of the conjunction. However, the definition shows that there is still a connection between the two, albeit not as direct as in a Boolean algebra.

show for example: If $[\varphi]_\sim = [\varphi']_\sim$ and $[\psi]_\sim = [\psi']_\sim$, then $[\varphi \vee \psi]_\sim = [\varphi' \vee \psi']_\sim$.

Proof.

Suppose $[\varphi]_\sim = [\varphi']_\sim$ and $[\psi]_\sim = [\psi']_\sim$. That is, $\vdash \varphi \leftrightarrow \varphi'$ and $\vdash \psi \leftrightarrow \psi'$. Therefore $\vdash \varphi \rightarrow \varphi'$ and $\vdash \varphi' \rightarrow \varphi$ as well as $\vdash \psi \rightarrow \psi'$ and $\vdash \psi' \rightarrow \psi$.

Weakening the consequent by using axiom (A6) and the modus ponens yields $\vdash \varphi \rightarrow (\varphi' \vee \psi')$ and $\vdash \varphi' \rightarrow (\varphi \vee \psi)$ as well as $\vdash \psi \rightarrow (\psi' \vee \varphi')$ and $\vdash \psi' \rightarrow (\psi \vee \varphi)$. By combining $\vdash \varphi \rightarrow (\varphi' \vee \psi')$ and $\vdash \psi \rightarrow (\psi' \vee \varphi')$ via axiom (A8), we get $\vdash (\varphi \vee \psi) \rightarrow (\varphi' \vee \psi')$ and vice versa, which means $\vdash (\varphi \vee \psi) \leftrightarrow (\varphi' \vee \psi')$. Thus, $[\varphi \vee \psi]_\sim = [\varphi' \vee \psi']_\sim$.

The other cases are similar, using axiom (A3) for complementation $-$ and the additional axioms (A5) and (A7) for the infimum $\curlywedge$.

□

Since the LT algebra is a Boolean algebra for propositional logic, the properties listed in Definition 4.1.8 apply. We can also characterize $\mathfrak{L}$ via a partial order:

Definiton 4.3.2.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra. The partial order $\preceq$ is defined as: $[\varphi]_\sim \preceq [\psi]_\sim$ iff $\vdash \varphi \rightarrow \psi$

To show that this definition of a partial order is justified, this relation has to be well-defined on equivalence classes: For this, suppose $[\varphi]_\sim = [\varphi']_\sim$ and $[\psi]_\sim = [\psi']_\sim$. Then the following implication must hold: If $[\varphi]_\sim \preceq [\psi]_\sim$, then $[\varphi']_\sim \preceq [\psi']_\sim$. So by definition: If $\vdash \varphi \rightarrow \psi$, then $\vdash \varphi' \rightarrow \psi'$. Since $\varphi \sim \varphi'$ iff $\vdash \varphi \rightarrow \varphi'$ and $\vdash \varphi' \rightarrow \varphi$, and the same for $\psi \sim \psi'$, we have five assumptions with $\vdash \varphi \rightarrow \psi$.

Transitivity of $\rightarrow$ has already been shown, therefore we can immediately conclude $\vdash \varphi \rightarrow \psi'$. By using an instance of (A1), where φ is substituted with $\varphi \rightarrow \psi'$, and MP, we get $\vdash \varphi \rightarrow (\varphi' \rightarrow \psi')$. Applying the modus ponens on axiom (A2) and the last result as its antecedent yields $\vdash (\varphi' \rightarrow \varphi) \rightarrow (\varphi' \rightarrow \psi')$. By using modus ponens again, this can be reduced to $\vdash \varphi' \rightarrow \psi'$, which is the desired result.

□

Because the LT algebra for propositional logic is a Boolean algebra, the equivalence $x \leq y$ iff $x \wedge y = x$ is expected to be true: $[\varphi]_\sim \preceq [\psi]_\sim$ iff $[\varphi]_\sim \curlywedge [\psi]_\sim = [\varphi]_\sim$ should hold as well. By definition of the the equivalence relation, the partial order and the infimum in $\mathfrak{L}$, this reduces to showing that $\vdash \varphi \rightarrow \psi$ iff $\vdash \varphi \wedge \psi \leftrightarrow \psi$. This is proven by using (A5) to show that $\vdash \varphi \wedge \psi \rightarrow \varphi$ and (A7) to get $\vdash \varphi \rightarrow \varphi \wedge \psi$.

By defining the partial order as above, we get the following result:

Theorem 4.3.3.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra. Then: $[\varphi]_\sim \preceq [\psi]_\sim$ iff $\varphi \vDash \psi$.

Proof.

$[\varphi]_\sim \preceq [\psi]_\sim$ iff $\vdash \varphi \to \psi$, by definition. $\vdash \varphi \to \psi$ iff $\vDash \varphi \to \psi$, since propositional logic is adequate. $\vDash \varphi \to \psi$ iff $\varphi \vDash \psi$, by the deduction theorem.

□

This means we can use the consequence relation as an appropriate partial order for a LT algebra $\mathfrak{L}$. Any partially ordered set can be visualized in a Hasse diagram.

To give an example of a LT algebra, let $\mathcal{L}_0 = \{p, q\}$ be a set of atomic formulas.[25] The number of equivalence classes of a set of n atoms is 2^{2^n}. In our example, the number of elements in $\mathcal{L}_{/\sim}$ is $2^{2^2} = 16$. Again, since the equivalence class of any atomic proposition contains more than just atomic formulas, φ is the representative of p and ψ that of q. The Hasse diagram of $\mathfrak{L}$ is visualized in the following figure:

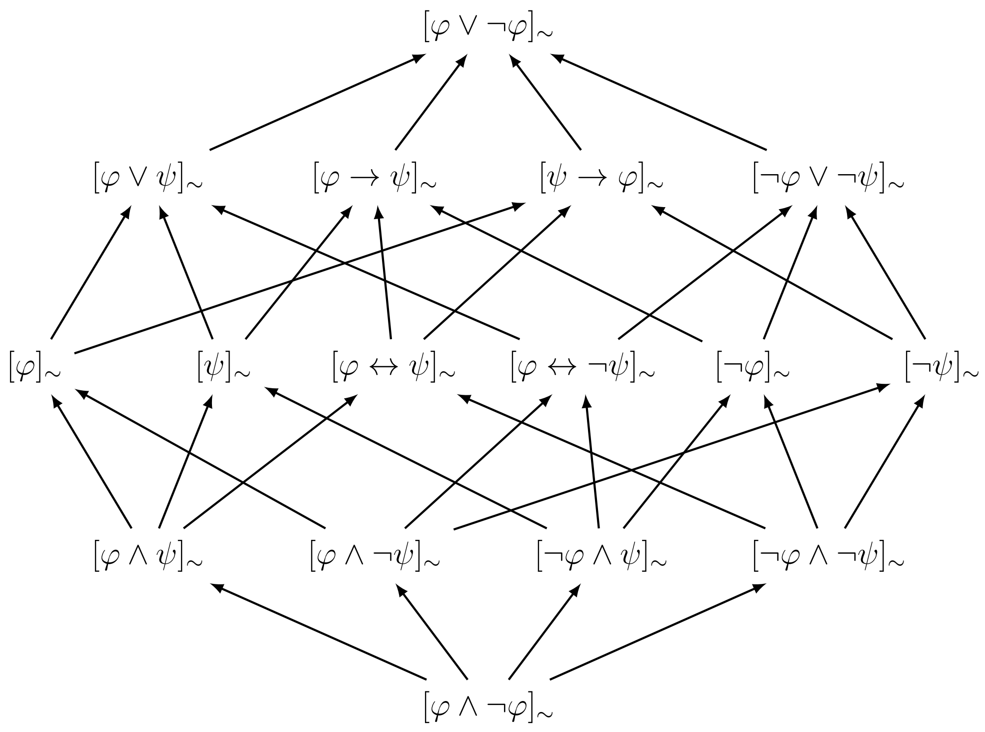

Figure 1: *Hasse diagram of* $\mathfrak{L}$

There are several interesting properties to this Hasse diagram. As usual, arrows indicate the partial order. For example, $[\varphi \wedge \psi]_\sim \preceq [\varphi]_\sim$ represents the fact that $\varphi \wedge \psi \vDash \varphi$. The element $[\varphi \vee \neg\varphi]_\sim$ being the highest element in the order indicates that all tautologies are implied by everything and so on.

A noteworthy property is that there is a direct correspondence between the level an element is on and the number of models it has. The least element $[\varphi \wedge \neg\varphi]_\sim$ has no models

25 Note that the number of elements in $\mathcal{L}$ is already infinite for two atomic formulas.

and is on the 0-th level of the diagram. The next level contains all elements with only one model, up to the greatest element which has all interpretations as models. Therefore it is on the 2^n-th level in general and on the 4-th level in particular. Consequently, the height of any LT algebra is determined by the number of its interpretations, so it is equal to $2^n + 1$, where n is the number of atomic formulas and each element on the i-th level has exactly i models.

Another representation is explicated by investigating the atoms of this partially ordered set. Characterized as a lattice $\mathfrak{L} = \langle \mathcal{L}_\sim, \preceq \rangle$, we can define the atoms in $\mathfrak{L}$ as the minimal elements of $\mathcal{L}_\sim$ strictly above the least element. Henceforth, let $\mathcal{A}$ be the set of atoms of a partially ordered set. In this case, $\mathcal{A} = \{[\varphi \wedge \psi]_\sim, [\varphi \wedge \neg\psi]_\sim, [\neg\varphi \wedge \psi]_\sim, [\neg\varphi \wedge \neg\psi]_\sim\}$ or more generally, all elements of the form $[(\neg)\varphi \wedge (\neg)\psi]_\sim$. Excluding the distinguished element 0, the atoms are the minimal elements of the ordered set.

The following definition captures this notion:

Definition 4.3.4.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra and let $\mathcal{L}_\sim^+$ denote the set $\mathcal{L}_\sim \setminus \{0\}$. Then: $\mathcal{A} = min(\mathcal{L}_\sim^+, \preceq)$, where $\preceq$ is the partial order on $\mathfrak{L}$.

As established in chapter 3.3, we can view some formulas as representing interpretations. Each atom of the algebra represents a certain interpretation $I \in \mathcal{I}$. For example, the interpretation I, where $I(p) = I(q) = 1$, can be represented by the formula $p \wedge q$, since it is the only element of $\mathbf{Mod}(p \wedge q)$. This formula is the atom $[\varphi \wedge \psi]_\sim \in \mathcal{A}$ in $\mathfrak{L}$.

The number of atoms is $|\mathcal{A}| = 2^n$, where n is the number of atomic formulas. Let $\mathbf{At} : \mathcal{L}_\sim \mapsto \mathcal{A}$ be a function, such that $\mathbf{At}(y) = \{x \in \mathcal{A} \mid x \preceq y\}$. The function maps elements of the LT algebra to their corresponding atoms. Some properties of this function are[26]:

Theorem 4.3.5.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra, $\mathcal{A}$ be the set of atoms of $\mathcal{L}_{/\sim}$ and $\mathbf{At}$ be the function defined above. Then:

(i) $\mathbf{At}(1) = \mathcal{A}$ and $\mathbf{At}(0) = \emptyset$

(ii) $\mathbf{At}([\varphi \wedge \psi]_\sim) = \mathbf{At}([\varphi]_\sim) \cap \mathbf{At}([\psi]_\sim)$

(iii) $\mathbf{At}([\varphi \vee \psi]_\sim) = \mathbf{At}([\varphi]_\sim) \cup \mathbf{At}([\psi]_\sim)$

(iv) $\mathbf{At}([\neg\varphi]_\sim) = \mathcal{A} \setminus \mathbf{At}([\varphi]_\sim)$.

(v) $\varphi \vDash \psi$ iff $\mathbf{At}([\varphi]_\sim) \subseteq \mathbf{At}([\psi]_\sim)$

[26] Analogously defined to [7], p. 241. The proofs of these properties are omitted in their paper.

Proof.

(i) By definition $\forall x \in \mathcal{L}_{\sim} : x \preceq 1$, therefore $\forall x \in \mathcal{A} : x \preceq 1$, which is $\mathbf{At}(1) = \mathcal{A}$. Suppose any atom $A_i \in \mathbf{At}(0)$. Then by definition of atoms $A_i \preceq 0$. Since 0 is the least element, $0 \preceq A_i$, which together with the above is $0 = A_i$. Contradiction! Therefore $\mathbf{At}(0) = \emptyset$.

(ii) Since all elements in $\mathcal{A}$ are minimal elements of $\mathcal{L}_{\sim}$, it is not possible for an infimum of any two elements to be strictly less then any atom. Therefore: $\mathbf{At}([\varphi]_{\sim}) \cap \mathbf{At}([\psi]_{\sim}) = \{z \in \mathcal{A} \mid z \preceq [\varphi]_{\sim} \text{ and } z \preceq [\psi]_{\sim}\} = \{z \in \mathcal{A} \mid z \preceq [\varphi]_{\sim} \curlywedge [\psi]_{\sim}\} = \{z \in \mathcal{A} \mid z \preceq [\varphi \wedge \psi]_{\sim}\} = \mathbf{At}([\varphi \wedge \psi]_{\sim})$.

(iii) $\mathbf{At}([\varphi]_{\sim}) \cup \mathbf{At}([\psi]_{\sim}) = \{z \in \mathcal{A} \mid z \preceq [\varphi]_{\sim} \text{ or } z \preceq [\psi]_{\sim}\} = \{z \in \mathcal{A} \mid z \preceq [\varphi]_{\sim} \curlyvee [\psi]_{\sim}\} = \{z \in \mathcal{A} \mid z \preceq [\varphi \vee \psi]_{\sim}\} = \mathbf{At}([\varphi \vee \psi]_{\sim})$.

(iv) $\mathbf{At}([\neg\varphi]_{\sim}) = \{z \in \mathcal{A} \mid z \preceq [\neg\varphi]_{\sim}\} = \{z \in \mathcal{A} \mid z \preceq -[\varphi]_{\sim}\}$. Applying the equivalence of $\preceq$ and $\curlywedge$ yields: $\mathbf{At}([\neg\varphi]_{\sim}) = \{z \in \mathcal{A} \mid z \curlywedge -[\varphi]_{\sim} = z\}$. Similarly, $\mathbf{At}([\varphi]_{\sim}) = \{z \in \mathcal{A} \mid z \curlywedge [\varphi]_{\sim} = z\}$. Let x be any element of $\mathbf{At}([\neg\varphi]_{\sim})$. Suppose $x \in \mathbf{At}([\varphi]_{\sim})$. Then by the above $x \curlywedge -[\varphi]_{\sim} = x$ and $x = x \curlywedge [\varphi]_{\sim}$. Taken together this results in $[\varphi]_{\sim} \curlywedge -[\varphi]_{\sim} = 0$, which contradicts $0 \notin \mathcal{A}$. Therefore $x \notin \mathbf{At}([\varphi]_{\sim})$. Then however $\mathbf{At}([\neg\varphi]_{\sim}) = \{z \in \mathcal{A} \mid z \notin \mathbf{At}([\phi]_{\sim})\}$, which is $\mathcal{A} \setminus \mathbf{At}([\varphi]_{\sim})$.

(v) By using cut to define a subset relation, the following equivalences hold:
$\varphi \models \psi$ iff $[\varphi]_{\sim} \preceq [\psi]_{\sim}$ iff $[\varphi]_{\sim} \curlywedge [\psi]_{\sim} = [\varphi]_{\sim}$ iff $[\varphi \wedge \psi]_{\sim} = [\varphi]_{\sim}$ iff $\mathbf{At}([\varphi \wedge \psi]_{\sim}) = \mathbf{At}([\varphi]_{\sim})$ iff $\mathbf{At}([\varphi]_{\sim}) \cap \mathbf{At}([\psi]_{\sim}) = \mathbf{At}([\varphi]_{\sim})$ iff $\mathbf{At}([\varphi]_{\sim}) \subseteq \mathbf{At}([\psi]_{\sim})$.

□

We can think of the function **At** as the algebraic counterpart for the function **Mod**. The function **At** maps equivalence classes to their corresponding atoms with respect to $\preceq$. The function **Mod** maps sets of formulas to the interpretations which are models of the sets. Condition (v) of Theorem 4.3.5. states that $\varphi \models \psi$ iff $\mathbf{At}([\varphi]_{\sim}) \subseteq \mathbf{At}([\psi]_{\sim})$, coinciding with the definition of logical consequence via **Mod**.

4.4 Filters in Lindenbaum-Tarski Algebras

The filters in LT algebras play an important role in the algebraic characterization of propositional logic. The reason for this is that generated filters are deductively closed sets and algebraic considerations for filters can be used to investigate theories in propositional logic. When applying the definitions and theorems of filters to LT algebras, the sets and operators are transformed into their corresponding counterpart: The underlying set is $\mathcal{L}_{/\sim}$ instead of Π, the infimum $\wedge$ is now $\curlywedge$ and the partial order is $\preceq$ as defined above.

The following theorems show that filters generated by any set are principal filters and deductively closed:

Theorem 4.4.1.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra, and let $[\varphi]_\sim$ be any element of $\mathcal{L}_{/\sim}$. Then the principal filter $\mathcal{F}([\varphi]_\sim)$ is a theory.

Proof.

$$\begin{aligned}\mathcal{F}([\varphi]_\sim) &= \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \preceq x\} = \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \vDash x\} \\ &= \mathbf{Cn}([\varphi]_\sim) = \mathbf{Cn}(\mathbf{Cn}([\varphi]_\sim)) = \mathbf{Cn}(\mathcal{F}([\varphi]_\sim))\end{aligned}$$

□

Theorem 4.4.2.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra, and let $\Gamma \subseteq \mathcal{L}_{/\sim}$. Then the filter generated by Γ, $\mathcal{F}(\Gamma)$, is a principal filter.

Proof.

Since $\mathfrak{L}$ is a Boolean algebra, $x \preceq y$ iff $x \curlywedge y = x$ holds for all $x, y \in \mathcal{L}_{/\sim}$. Suppose without loss of generality that Γ has at least two distinct elements, such that it is not principal by definition. Let $\Gamma = \{[\varphi]_\sim, [\psi]_\sim\}$. Then there are three cases:

1.) $[\varphi]_\sim \preceq [\psi]_\sim$:
By $[\varphi]_\sim \preceq [\psi]_\sim$ iff $[\varphi]_\sim \curlywedge [\psi]_\sim = [\varphi]_\sim$ and the definition of $\mathcal{F}(\Gamma)$:
$\mathcal{F}(\Gamma) = \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \curlywedge [\psi]_\sim \preceq x\} = \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \preceq x\} = \mathcal{F}([\varphi]_\sim)$

2.) $[\psi]_\sim \preceq [\varphi]_\sim$:
Analogous to case (i). Therefore $\mathcal{F}(\Gamma) = \mathcal{F}([\psi]_\sim)$.

3.) $[\psi]_\sim \otimes [\varphi]_\sim$:
Then the filter has the form $\mathcal{F}(\Gamma) = \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \curlywedge [\psi]_\sim \preceq x\}$ and since the infimum of any two elements in $\mathcal{L}_{/\sim}$ exists, there is a unique $y \in \mathcal{L}_{/\sim}$, such that $y = [\varphi]_\sim \curlywedge [\psi]$. By definition in $\mathfrak{L}$, $y = [\varphi]_\sim \curlywedge [\psi] = [\varphi \wedge \psi]_\sim$, therefore:
$$\begin{aligned}\mathcal{F}(\Gamma) &= \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \curlywedge [\psi]_\sim \preceq x\} \\ &= \{x \in \mathcal{L}_{/\sim} \mid [\varphi \wedge \psi]_\sim \preceq x\} = \mathcal{F}([\varphi \wedge \psi]_\sim)\end{aligned}$$

In each case, the filter is generated by only one element.

□

Corollary 4.4.3.

Let $\mathfrak{L} = \langle \mathcal{L}_{/\sim}, 1, 0, \curlywedge, \curlyvee, - \rangle$ be a LT algebra, and let $\Gamma \subseteq \mathcal{L}_{/\sim}$. Then the filter generated by Γ, $\mathcal{F}(\Gamma)$, is a theory.

Proof.

A filter generated by any set is principal and as such deductively closed.

□

5 Belief Revision in Lindenbaum-Tarski Algebras

In this chapter, the connections between belief revision and LT algebras are given.[27] First formally and then an example will visualize the process step-by-step in a collection of Hasse diagrams. In the last section, the problem of overgeneralization will be addressed by investigating the combinatorics of the LT algebra.

5.1 Representation Results

Before reviewing the relationship between LT algebras and belief revision, the abstractness of this approach will be addressed. Specifically, the difference between the set of well-formed formulas $\mathcal{L}$ as the syntactic component of a formal structure and the same set partitioned into its equivalence classes, $\mathcal{L}_{/\sim}$ as the carrier set of $\mathfrak{L}$. Although the theories went in different directions starting from the set $\mathcal{L}$, they coincide.

First of all, note how the AGM framework abstracted from the syntactic structure in two ways. Most obviously, the postulate of extensionality, (P6), demands that if two formulas φ and ψ are equivalent, then the result of a revision by any one of them should be the same in each case. To give a specific example, the revision by the semantically equivalent, but syntactically different formulas $p \vee q$ and $\neg p \rightarrow q$ ought to be the same. The syntactic difference shouldn't matter. Secondly, knowledge sets are supposed to be theories. Because **Cn** is based on logical consequence, syntactic structure is again not decisive for theories. Deductive closure of equivalent, but syntactically different formulas is the same: The syntactic form of elements in a knowledge set is not the measure of its content. Rephrasing $\mathrm{K} = \{p, p \leftrightarrow q\}$ to $\{p \wedge p, ((p \rightarrow q) \wedge (\neg q \vee p))\}$ for example has no impact on either the problem of the new information $\neg q$ causing an inconsistency nor the results of the applied revision function.

The AGM framework itself abstracts the set $\mathcal{L}$ as the syntactic basis. Revision is supposed to only depend on semantic content of the new information and equivalent formulas are treated alike. This is similar to the algebraic approach. The set of well-formed formulas is partitioned into its equivalence classes. The carrier set of a LT algebra is $\mathcal{L}_{/\sim}$. By the assumptions of belief revision, the logical language considered can be seen as $\mathcal{L}$ modulo equivalence and therefore the algebra can by applied without problems concerning different syntactic bases. Both the theory of belief revision as well as LT algebra started from logical bases as sets of well-formed formulas $\mathcal{L}$ and an operator defining consequences. The theory of belief revision handles this set with respect to equivalence. Accordingly, the propositional formula φ in the context of revision is represented by its equivalence class $[\varphi]_\sim$ in the algebra. Due to adequacy of propositional logic, the syntactic consequence used to define the equivalence classes is the same as the logical consequence used in belief revision. Therefore the central relation of a logic is applicable in both theories,

[27] Some investigations into algebraic aspects of belief revision are discussed in [3], [5], [7] and [11]

the correspondence is obvious. The basic components of the theory of belief revision are representable in the LT algebra. The additional results proven in this paper are noted below and are used to characterize belief revision in a LT algebra.

We can now show how the different concepts so far interact and correspond. We start we the knowledge set K. It is a deductively closed set of known facts. Hence, $\mathrm{K} = \mathbf{Cn}(\mathrm{K})$. By Corollary 4.4.3., if a filter is generated from K, it produces a theory in the LT algebra. So $\mathcal{F}(\mathrm{K})$ is the knowledge set in the algebra. Generated filters are principal, according to Theorem 4.4.2., and can therefore be generated from a specific element of $\mathcal{L}_{/\sim}$. This captures the construction of knowledge bases by Katsuno and Mendelzon. A knowledge base is a formula κ representing a knowledge set. It is constructed by forming the conjunction of all non-equivalent formulas in K. Then κ is such that $\mathrm{K} = \{\varphi \mid \kappa \vDash \varphi\}$. The filter generated by K is: $\mathcal{F}(\mathrm{K}) = \{y \in \mathcal{L}_{/\sim} \mid x_1 \curlywedge ... \curlywedge x_n \preceq y, \text{ for } x_1, ..., x_n \in \mathrm{K}\}$. By definition of $\curlywedge$ and the partial order being the consequence relation, this set can be transformed to: $\mathcal{F}(\mathrm{K}) = \{y \in \mathcal{L}_{/\sim} \mid [x_1 \wedge ... \wedge x_n]_\sim \vDash y, \text{ for } x_1, ..., x_n \in \mathrm{K}\}$. Since $[x_1 \wedge ... \wedge x_n]_\sim$ is the principal element[28] of $\mathcal{F}(\mathrm{K})$, it is taken to be the knowledge base of K. With $[x_1 \wedge ... \wedge x_n]_\sim = [\kappa]_\sim$ the correspondence is obvious.

The representation of the expansion function follows the same path as the correspondence of knowledge sets and filters. Explicating how the expansion within the AGM formalism is definable in context of LT algebras naturally leads to the results within the KM framework. The expansion function is defined as: $\mathrm{K} + \varphi = \mathbf{Cn}(\mathrm{K} \cup \{\varphi\})$. Hence, the representatives in the algebra are $\mathrm{K} \subseteq \mathcal{L}_{/\sim}$ and the element $[\varphi]_\sim \in \mathcal{L}_{/\sim}$. The knowledge set K can be represented by the corresponding knowledge base $[\kappa]_\sim$, as established above. Since the operator **Cn** is captured by the generated filter, the following equations hold:

$$
\begin{aligned}
\mathrm{K} + [\varphi]_\sim &= \mathbf{Cn}(\mathrm{K} \cup \{[\varphi]_\sim\}) = \mathcal{F}(\mathrm{K} \cup \{[\varphi]_\sim\}) && \text{Filters are theories} \\
&= \{z \in \mathcal{L}_{/\sim} \mid \exists x \in \mathrm{K}, \text{ such that } x \curlywedge [\varphi]_\sim \preceq z\} && \text{Theorem 4.2.4.(ii)} \\
&= \{z \in \mathcal{L}_{/\sim} \mid [\kappa]_\sim \curlywedge [\varphi]_\sim \preceq z\} && [\kappa]_\sim \text{ is knowledge base} \\
&= \{z \in \mathcal{L}_{/\sim} \mid [\kappa \wedge \varphi]_\sim \preceq z\} && \text{Definition of } \curlywedge \\
&= \mathcal{F}([\kappa \wedge \varphi]_\sim)
\end{aligned}
$$

Thus, expansion of a knowledge set K by a formula φ is representable in a LT algebra by generating a filter from the infimum of the principal element $[\kappa]_\sim$ and the element $[\varphi]_\sim$. It is equal to the conjunction $\kappa \wedge \varphi$ as the new principal element. Again, this corresponds to the KM expansion in which $\mathrm{K} + \varphi = \{\psi \mid \kappa \wedge \varphi \vDash \psi\}$ and $\kappa \wedge \varphi$ is the new expanded knowledge base.

To define revision in algebraic terms, a reminder of all components of the KM revision is given. A model of any set is an interpretation which is model for all elements of the set. This holds in general, however only models of formulas are needed in the KM

[28] The principal element is the only element of the singleton set generating the filter.

framework, since both the knowledge base as well as the new information are viewed as propositional formulas. The models are defined as: $\mathbf{Mod}(\varphi) = \{I \in \mathcal{I} \mid I \vDash \varphi\}$ for any φ. Obviously, this holds true if φ is the knowledge base κ or the new information. The revision function is based on the faithful assignment, which pre-orders the set $\mathcal{I}$ with respect to the knowledge base κ. Generally speaking, all models of κ are minimal, while all interpretations which are not models are strictly greater than the models, but can then be ordered in any way. The faithful assignment is denoted by a indexed less or equal sign: $\leq_\kappa$. The revision result is the model of the new information closest to the original model of the knowledge base, defined as: $\mathbf{Mod}(\kappa \circ \varphi) = min(\mathbf{Mod}(\varphi), \leq_\kappa)$.

The function **Mod** is represented by the function **At**. While the former maps formulas to its models, the latter maps elements of $\mathcal{L}_\sim$ to their atoms, which are the elements of $\mathcal{A}$ less or equal to the equivalence class in question. Because interpretations can be characterized as formulas of the form $(\neg)p\wedge(\neg)q$, the elements of the algebra corresponding to those formulas can be seen as interpretations as well. They are the atoms of the algebra; since $\mathbf{At}([\varphi]_\sim)$ lists the collection of atoms $A_i \in \mathcal{A}$, such that $A_i \preceq [\varphi]_\sim$ and therefore $A_i \vDash \varphi$, the results are identical to those of **Mod**. Consequently, the faithful assignment in $\mathfrak{L}$ pre-orders the set $\mathcal{A}$. The conditions are transferable: Instead of models, all atoms of a knowledge base $[\kappa]_\sim$ are minimal and the remaining elements of $\mathcal{A}$ are ordered according to the agent. The faithful assignment in the algebra is also denoted by $\leq_\kappa$.

To calculate the revision result, note that $\mathbf{At}(\mathcal{F}([\kappa]_\sim))$ is well-defined, although it takes a subset of $\mathcal{L}_\sim$ instead of a single element. Since the principal element $[\kappa]_\sim$ is the generator of the filter, it holds that $[\kappa]_\sim \preceq x$ for all $x \in \mathcal{F}([\kappa]_\sim)$. Due to transitivity of the partial order all atoms of $[\kappa]_\sim$ are also atoms of the filter. Thus, $\mathbf{At}([\kappa]_\sim)$ is sufficient as the function corresponding to $\mathbf{Mod}(\mathrm{K})$. The revision function algebraically is the following: The atoms of the revised knowledge base $[\kappa]_\sim \circ [\varphi]_\sim$ are the atoms of the new information $[\varphi]_\sim$, according to the faithful assignment on $\mathcal{A}$.

All results are listed in the figure below:

	Belief Revision	Lindenbaum-Tarski Algebra
Knowledge set	Theory K, with $\mathrm{K} = \mathbf{Cn}(\mathrm{K})$	Filter generated by K: $\mathcal{F}(\mathrm{K})$
Knowledge base	κ, such that $\mathrm{K} = \{\varphi \mid \kappa \vDash \varphi\}$	Principal filter: $\mathcal{F}([\kappa]_\sim)$
Models	$\mathbf{Mod}(\Gamma) = \{I \in \mathcal{I} \mid I \vDash \Gamma\}$	$\mathbf{At}([\kappa]_\sim) = \{x \in \mathcal{A} \mid x \preceq [\kappa]_\sim\}$
Expansion	$\mathrm{K} + \varphi = \mathbf{Cn}(\mathrm{K} \cup \{\varphi\})$	$\mathrm{K} + \varphi = \mathcal{F}([\kappa \wedge \varphi]_\sim)$
KM Revision	$\mathbf{Mod}(\kappa \circ \varphi) =$ $min(\mathbf{Mod}(\varphi), \leq_\kappa)$	$\mathbf{At}([\kappa]_\sim \circ [\varphi]_\sim) =$ $min(\mathbf{At}([\varphi]_\sim), \leq_\kappa)$

Figure 2: *Table of representation results*

5.2 Visualizing a Belief Revision

Reconsider the first example of a belief revision for in chapter 3.3. An agent knows $\{p, p \leftrightarrow q\}$ and acquires $\neg q$. The process of finding a suitable revised knowledge base is defined formally above. Since the algebra can be visualized in a Hasse diagram, we can do the same for the revision. The visualization is shown in the following figures.

We start with a representation of the elements p and $p \leftrightarrow q$ in their respective equivalence class. Note, that $p \in [\varphi]_\sim$ and $p \leftrightarrow q \in [\varphi \leftrightarrow \psi]_\sim$. This yields the figure:

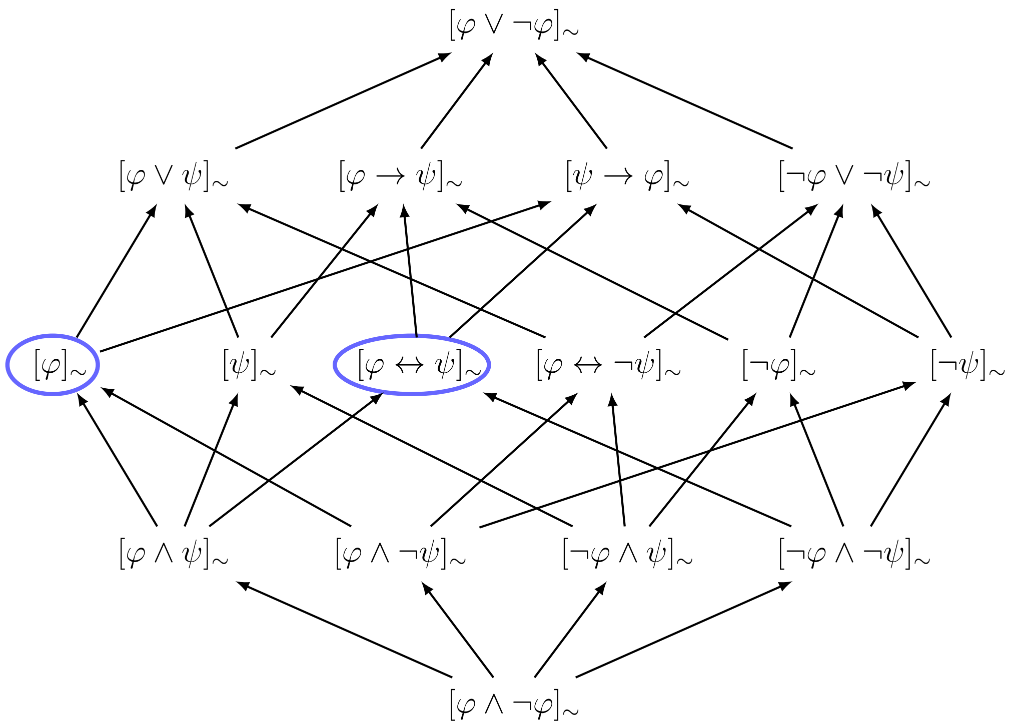

Figure 3: *Hasse diagram with elements of knowledge set*

To acquire the knowledge set with these elements, we construct the generated filter: $\mathcal{F}(\mathrm{K}) = \{x \in \mathcal{L}_{/\sim} \mid [\varphi]_\sim \curlywedge [\varphi \leftrightarrow \psi]_\sim \preceq x\} = \mathcal{F}([\varphi \wedge \psi]_\sim)$. The following figures highlight the principal element and the resulting principal filter:

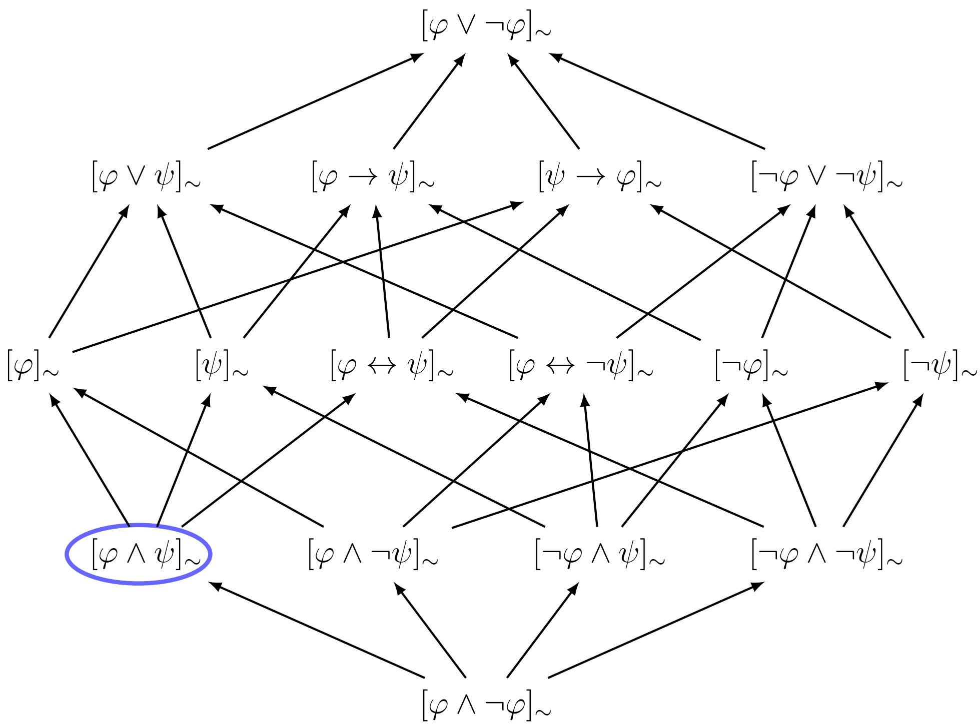

Figure 4: *Hasse diagram with knowledge base*

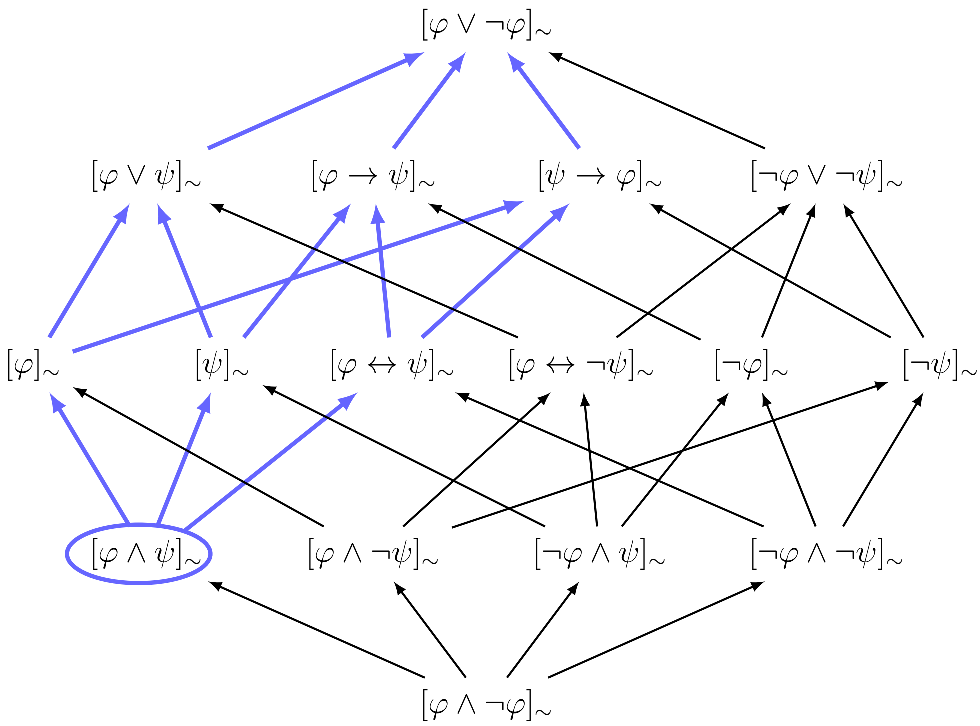

Figure 5: *Hasse diagram with generated filter*

The new information is represented by $[\neg\psi]_\sim$, shown below. The models for the formula $\neg q$ are $\mathbf{At}([\neg\psi]_\sim)$ and are needed for the revision process. These are highlighted in Figure 7:

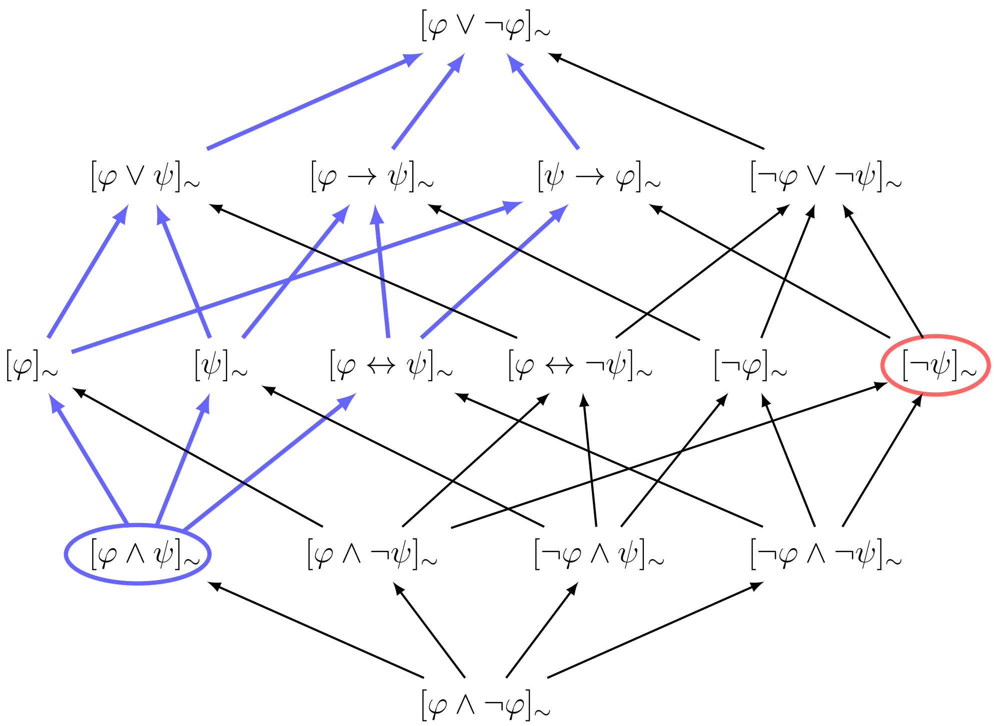

Figure 6: *Hasse diagram with filter and new information*

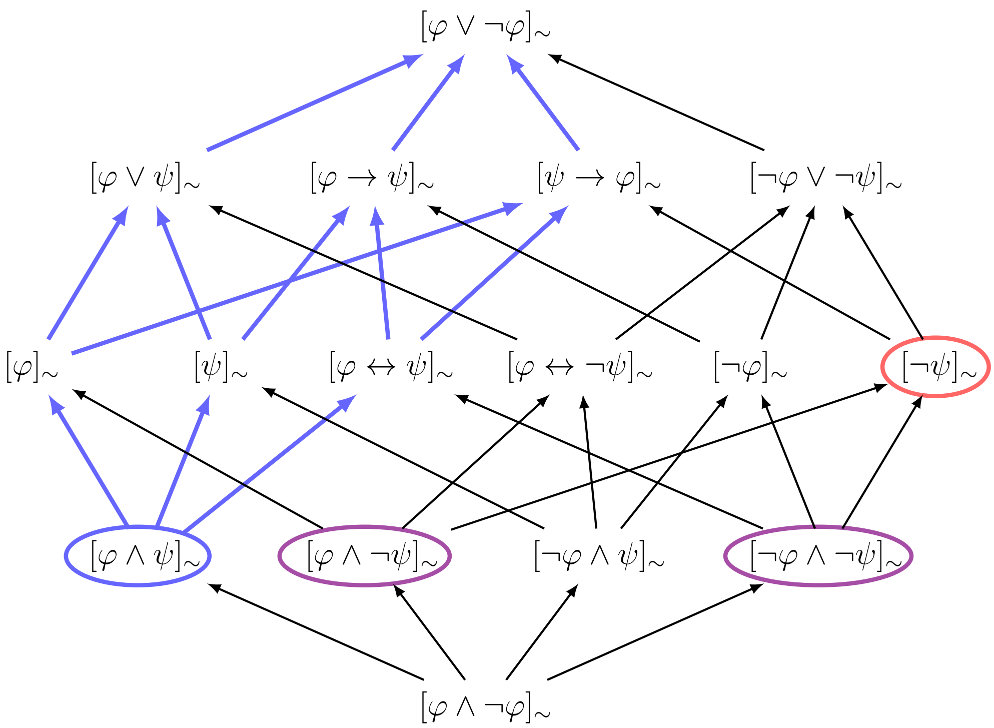

Figure 7: *Hasse diagram with filter and atoms of new information*

The results of revision are the minimal atoms according to the faithful assignment. The two result and their respective generated filters are shown below. In each case, the element remaining from the set K is still highlighted:

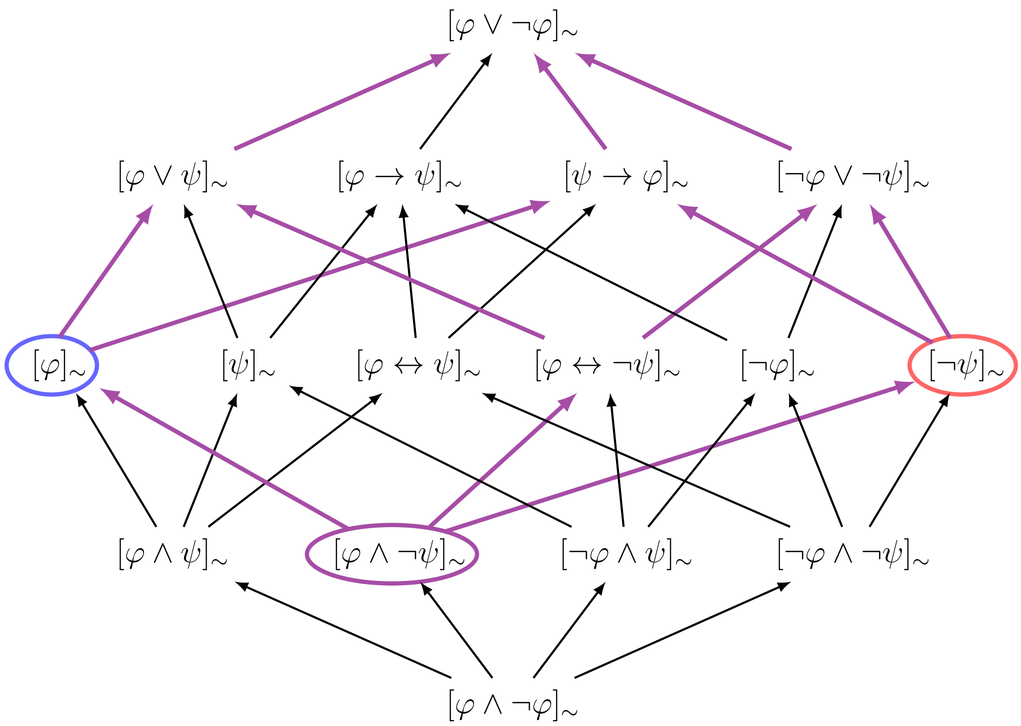

Figure 8: *Hasse diagram with revision result* A_2

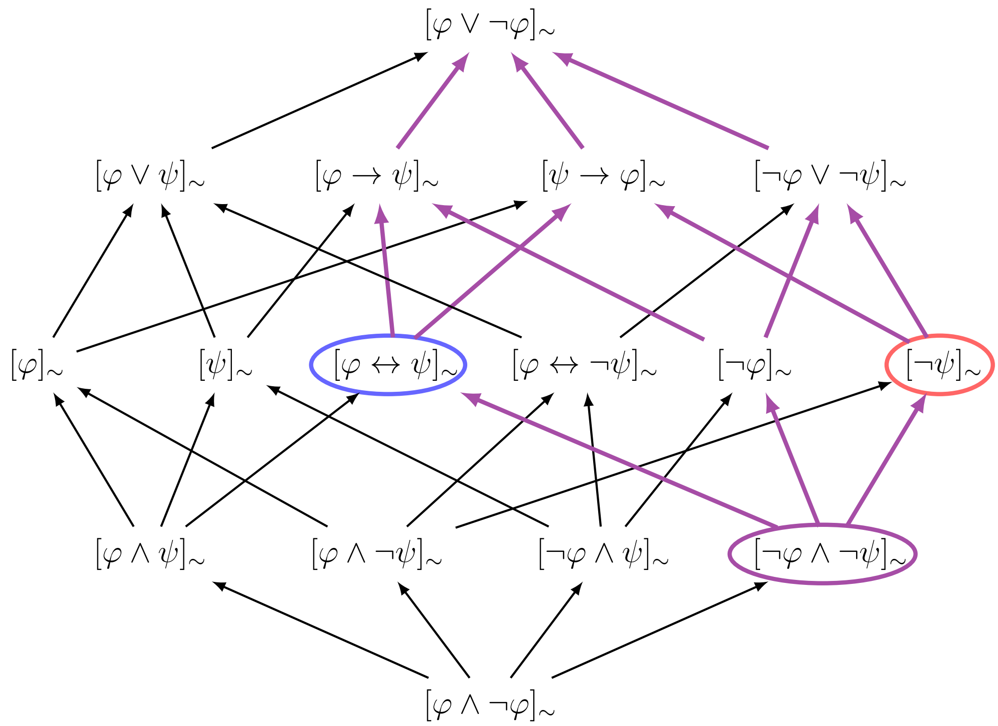

Figure 9: *Hasse diagram with revision result* A_4

5.3 Combinatorics of Partial Orders

As discussed in chapter 3.3, the model theoretic approach to belief revision overgeneralizes in the sense that a revised knowledge base is given a model based on the size of the language instead of only the available information. This model may have more information than should be possible to infer, especially if the number of atomic formulas gets bigger.

By investigating the combinatorics of partial orders, we can determine how many elements are between the new information and its corresponding atoms. Each of these elements is a viable alternative as the new knowledge base, since the filter generated by any of them contain the new information and the atoms are the same as well.

To start, we can list some mathematical facts about the components of the algebra. Since the cardinalities are dependent on the number of atomic formulas, suppose $|\mathcal{L}_0| = n$, for a $n \in \mathbb{N}$. Then it holds that $|\mathcal{I}| = 2^n$, and consequently $|\mathcal{A}| = 2^n$ as well. The carrier set of $\mathfrak{L}$ has $|\mathcal{L}_{/\sim}| = 2^{2^n}$ elements. For any formula φ, the number of models is: $|\mathbf{Mod}(\varphi)| \leq 2^n$. However, the exact number of models is known in context of belief revision. Let $|\mathbf{Mod}(\varphi)| = m$ be the number of models of the new information φ.

The solution to the problem of overgeneralization is to give the number of possible revision results not equal to a model.

Suppose the new information is the formula $p \vee q$. Then $[\varphi \vee \psi]_\sim$ is the corresponding equivalence class. The models for the new information are its atoms: $\mathbf{At}([\varphi \vee \psi]_\sim) = \{[\varphi \wedge \psi]_\sim, [\varphi \wedge \neg\psi]_\sim, [\neg\varphi \wedge \psi]_\sim\}$.

In a partial diagram, where only the elements x with $x \neq 0$ and $x \preceq [\varphi \vee \psi]_\sim$ and their respective order relations are represented, this looks like the following figure:

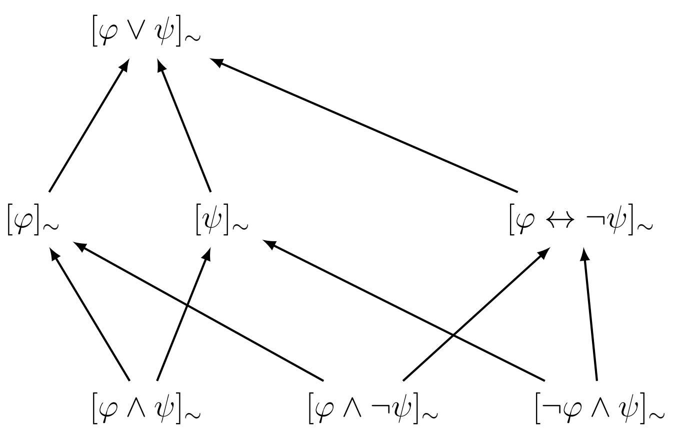

Figure 10: *Partial Hasse diagram with suitable revision results*

By revising with this formula the revision function yields one of the atoms as a model of the new knowledge base. However, note that in this simple case the result is already overgeneralized.

As visible in the figure above, the element $[\varphi]_\sim$ is easily a possible candidate for the new knowledge base, since $[\varphi \vee \psi]_\sim$ is in the filter generated by $[\varphi]_\sim$ and if, for example, $[\varphi \wedge \psi]_\sim$ is the new model, then it would also be a model for $[\varphi]_\sim$. This is the reason for overgeneralization; the model of a new knowledge base can be the knowledge base itself, but it does not have to be.

Since the KM revision yields a model of the new knowledge base, this problem always arises. However, there is a way to solve it. As above, if the revision result is any atom of the formula in question, then *each* element between the formula and its atom is a viable result. Again, in this example, if the result is the atom $[\varphi \wedge \psi]_\sim$, then both $[\varphi]_\sim$ and $[\psi]_\sim$ can also be the new knowledge bases. We can see above that there are 6 possible solutions, excluding the revision formula itself. In this simple example this can be easily visualized and counted, but there is a universal equation to calculate the number of viable solutions of the KM revision for any number of atomic formulas. The number of elements on a level of a LT algebra is calculated by the binomial coefficient: For a number of atoms m, the number of elements on the i-th level is equal to $\binom{m}{i}$. Intuitively, the reason for this is that the expression $\binom{m}{i}$ calculates the number of ways of i objects to be chosen from a set of m elements. Since any element of the i-th level has exactly i models, the question is: How many possible ways are there to distinctly choose i models of the set of models of size m? The answer is the binomial coefficient above. To give some examples, $\binom{m}{0} = 1$, as there is only one way to choose no elements of the set of models, $\binom{m}{m} = 1$, as there is only one way to choose all models simultaneously.

The following sum counts the number of elements on each level of the LT algebra:

$$\sum_{i=0}^{m} \binom{m}{i} = \binom{m}{0} + \binom{m}{1} + ... + \binom{m}{m-1} + \binom{m}{m} = 2^m \tag{1}$$

Obviously, this should just be the cardinality of $\mathcal{L}_{/\sim}$, if m is the number of all atoms. Since $m = 2^n$ for n atomic formulas, the equation $|\mathcal{L}_{/\sim}| = 2^{2^n}$ holds.

However, in the case of belief revision the number m is generally not equal to the number of atoms: $m < 2^n$. The sum then enumerates all elements on the 0-th level up to the m-th level, which have m models. This is exactly the number we need.

Some summands can be excluded. The formula φ itself is captured by $\binom{m}{m} = 1$. Since this element should not be the result of a sensible revision[29], we can subtract it. Note also that $[\varphi \wedge \neg\varphi]_\sim$ corresponds to $\binom{m}{0} = 1$. By restricting the results of the revision function to models, this element can be excluded as well. We can simplify the sum to calculate

[29] The reason for this being unsensible is that there is an enormous amount of information loss. As already stated, some content of the knowledge set is not in conflict with the new information. For example the formula p together with $\neg q$ in the running example. If we just take the formula $\neg q$ as the revision result, then the unproblematic information is lost. In this sense, it violates an implicit *minimax* principle in belief revision: A sensible revision should minimize change of knowledge necessary to avoid inconsistency, while maximizing the amount of maintained knowledge.

the number of relevant results:

$$\binom{m}{0} + \sum_{i=1}^{m-1} \binom{m}{i} + \binom{m}{m} = 2^m \tag{2}$$

$$1 + \sum_{i=1}^{m-1} \binom{m}{i} + 1 = 2^m \tag{3}$$

$$\sum_{i=1}^{m-1} \binom{m}{i} = 2^m - 2 \tag{4}$$

In other words, the number of distinct elements beneath an element $x \in \mathcal{L}_{/\sim}$ and its corresponding atoms as a bound is $2^m - 2$, where $m = |\mathbf{At}(x)|$. If we want to remove the atoms from this number, we can subtract the elements on the first level. The sum of elements without the atoms starts at level $i = 2$:

$$\binom{m}{1} + \sum_{i=2}^{m-1} \binom{m}{i} = 2^m - 2 \tag{5}$$

$$m + \sum_{i=2}^{m-1} \binom{m}{i} = 2^m - 2 \tag{6}$$

$$\sum_{i=2}^{m-1} \binom{m}{i} = 2^m - m - 2 \tag{7}$$

In the example, $[\varphi \vee \psi]_\sim$ is the revising formula. The number of atoms is 3. Inserting this number yields $2^3 - 2 = 8 - 2 = 6$ elements, which are possible results fulfilling the criteria of revision. If we exclude the atoms, then $2^3 - 3 - 2 = 8 - 5 = 3$ is the number of viable solutions.

To calculate the number for a single atom A_1 consider the following. We want to know how many elements on a level are comparable to this atom to modify the equation (1). The factor that indicates the ration of elements which are comparable to A_1 has to be determined. We know that any element on the i-th level has i models. To calculate how many elements depend on A_1 we just divide this number by m, since the proportion of A_1 contributing to this number is obviously depending on the number of models overall and its dependence is distributed evenly over all models.

Thus, to enumerate the number of elements starting from an atom up to the formula, the following sum has to be calculated:

$$\sum_{i=1}^{m} \frac{i}{m} \binom{m}{i} \tag{8}$$

Since the identity $\binom{m}{i} = \frac{m}{i}\binom{m-1}{i-1}$ holds, we can simplify the sum:

$$\sum_{i=1}^{m} \frac{i}{m}\binom{m}{i} = \sum_{i=1}^{m} \frac{i}{m}\frac{m}{i}\binom{m-1}{i-1} = \sum_{i=1}^{m}\binom{m-1}{i-1} \tag{9}$$

Note that we can shift the indices in (9):

$$\sum_{i=1}^{m}\binom{m-1}{i-1} = \binom{m-1}{0} + \binom{m-1}{1} + ... + \binom{m-1}{m-2} + \binom{m-1}{m-1} \tag{10}$$

$$= \sum_{i=0}^{m-1}\binom{m}{i} = 2^{m-1} = \frac{2^m}{2} \tag{11}$$

The equation (11) therefore counts the number of possible revision results. Again, if we want to exclude the atom and the revising formula, we have to subtract their corresponding summands. These are $\binom{m-1}{0}$ and $\binom{m-1}{m-1}$ in the simplified sum, since they are the first and last summand, respectively:

$$\sum_{i=0}^{m-1}\binom{m}{i} = 2^{m-1} \tag{12}$$

$$\binom{m-1}{0} + \sum_{i=1}^{m-2}\binom{m}{i} + \binom{m-1}{m-1} = 2^{m-1} \tag{13}$$

$$1 + \sum_{i=1}^{m-2}\binom{m}{i} + 1 = 2^{m-1} \tag{14}$$

$$\sum_{i=1}^{m-2}\binom{m}{i} = 2^{m-1} - 2 \tag{15}$$

The revision result for any formula φ depends on the number of models it has. Let $|\mathbf{Mod}(\varphi)| = m$. Trivially, there are m models possible as a result of the revision function. However, there are $2^m - 2$ possible formulas between the atoms and the formula. They are equally valid as new knowledge bases. Not counting the atoms yields the equation: $2^m - m - 2$.

If a single model is determined, the same procedure is possible. The number of elements between a fixed atom and the element in question is 2^{m-1}. If we don't count the atom and the element itself, the number is reduced to $2^{m-1} - 2$.

These combinatorical results limit the overgeneralization. If revision yields an model as a result, we can calculate how many possible alternative knowledge bases are available. To acquire the specific formulas we can use the properties of atoms to determine their suprema and the corresponding formulas. Alternatively, a visualization in a partial Hasse diagram can show what elements lie between the revising formula and its atoms.

6 Conclusion

In this paper, I have shown that belief revision is representable in a LT algebra.

The correspondence is based on the argument that belief revision handles propositional formulas effectively modulo equivalence. The knowledge sets are algebraically definable as filters generated by these sets. The generated filters are principal filters, which corresponds to the knowledge base construction of Katsuno and Mendelzon.

Algebraically, expansion of a knowledge set by a formula is a generated filter, where the principal element is a conjunction of the knowledge base and the formula.

The model based approach to revision is transformable to an atom based revision in the LT algebra. The atoms of the algebra are the minimal non-zero elements and the partial order indicates which formulas are their logical consequences. The faithful assignment is transferable by ordering the set of atoms instead of the set of interpretations. The revision function is similarly definable as a minimum with respect to the faithful assignment, the result being an atom of the revised knowledge base such that it is closest to the original atoms. A summary of results is Figure 2.

The contraction function is definable via the Harper identity. The result of a knowledge set K contracted with a formula φ is identical to the cut of the knowledge set and the revision result of K revised by $\neg\varphi$. Theorem 4.2.4. shows how we can cut both filters. The result is still a filter, where the smallest element is the disjunction of elements of both sets. An algebraic investigation into contraction functions is therefore deeply connected to the concepts discussed in this paper. Details on contraction in this sense can be found in [5] and [11].

The combinatorical investigations resulted in equations that enumerate the number of possible revision results for a given number of models. The calculations yield the number of suitable knowledge bases. If the number of models of a new information is m, then there are:

$2^m - 1$	possible results,
$2^m - 2$	possible results excluding the new information,
$2^m - m - 2$	possible results excluding the new information and its models.

If the model is a specific one, then there are:

2^{m-1}	possible results,
$2^{m-1} - 1$	possible results excluding the new information,
$2^{m-1} - 2$	possible results excluding the new information and its model.

The results of this paper are of overarching generality. A preference based revision process can be found in [7], in this paper each atom is assigned a value to determine preferred revision results. Future directions are the specification of the supplementary AGM postulates in this framework and the algebraization of other operators for belief revision, such as the update operator. The LT algebra in this paper is based on propositional logic. Investigations into other logics, for example description logics, are possible by changing the underlying equivalence relation. An example for this process are Heyting algebras, which are LT algebras for intuitonistic logics.

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